Random Locations of Periodic Stationary Processes

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Basic settings

- $\{X(t)\}_{t\in\mathbb{R}}$: A real-valued stationary stochastic process with some path property (continuity, upper-semicontinuity, *etc.*)
- Random locations:
  - Location of the path supremum over interval $[0, T]$:

$$\tau_{\mathbf{X},T} = \inf\{t \in [0, T] : X(t) = \sup_{s\in[0,T]} X(s)\}$$

  - First hitting time in $[0, T]$ to a fixed level a:

$$T^a_{\mathbf{X},T} = \inf\{t \in [0, T] : X(t) = a\}$$

  ...

- We want to know the distributional properties of these random locations.

Intrinsic location functionals

Definition 1

A mapping $L = L(f, I)$ *from* $H \times \mathcal{I}$ *to* $\mathbb{R} \cup \{\infty\}$ *is called an intrinsic location functional, if*

1. $L(\cdot, I)$ *is measurable for* $I \in \mathcal{I}$;
2. *For each function* $f \in H$, *there exists a subset* $S(f)$ *of* $\mathbb{R}$, *equipped with a partial order* $\preceq$, *satisfying:*
   - a *For any* $c \in \mathbb{R}$, $S(f) = S(\theta_c f) + c$;
   - b *For any* $c \in \mathbb{R}$ *and any* $t_1, t_2 \in S(f)$, $t_1 \preceq t_2$ *implies* $t_1 - c \preceq t_2 - c$ *in* $S(\theta_c f)$,

   such that for any $I \in \mathcal{I}$, *either* $S(f) \cap I = \phi$, *in which case* $L(f, I) = \infty$, *or* $L(f, I)$ *is the maximal element in* $S(f) \cap I$ *according to* $\preceq$.

Existing results for processes on $\mathbb{R}$ I

Theorem 2 (Samorodnitsky and S., 2013)

Let $\mathbf{X} = (X(t),\ t \in \mathbb{R})$ be a stationary process, and $L_{\mathbf{X},T}$ be an intrinsic location functional. Denote by $F_{\mathbf{X},T}$ the distribution of $L_{\mathbf{X},T}$. Then:

- *The restriction of the law $F_{\mathbf{X},T}$ to the interior $(0, T)$ of the interval is absolutely continuous.*
- *The density, denoted by $f_{\mathbf{X},T}$, can be taken to be equal to the right derivative of the cdf $F_{\mathbf{X},T}$, which exists at every point in the interval $(0, T)$.*
- *In this case the density is right continuous, has left limits, and has the following properties.*

Existing results for processes on $\mathbb{R}$ II

Theorem 2 (Samorodnitsky and S., 2013)

(a) The limits

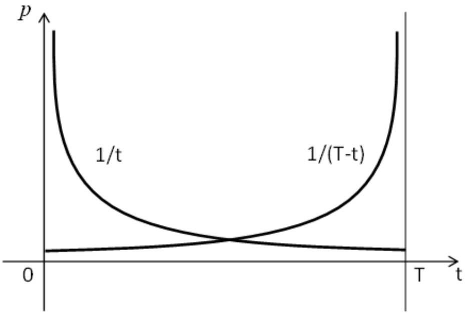
$$f_{\mathbf{X},T}(0+) = \lim_{t\to 0} f_{\mathbf{X},T}(t) \text{ and } f_{\mathbf{X},T}(T-) = \lim_{t\to T} f_{\mathbf{X},T}(t)$$

exist.

(b) The density has a universal upper bound given by

$$f_{\mathbf{X},T}(t) \leq \max\left(\frac{1}{t}, \frac{1}{T-t}\right),\ 0 < t < T\,. \quad (1)$$

Existing results for processes on $\mathbb{R}$ III



Existing results for processes on $\mathbb{R}$ IV

Theorem 2 (Samorodnitsky and S., 2013)

(c) The density has a bounded variation away from the endpoints of the interval. Furthermore, for every $0 < t_1 < t_2 < T$,

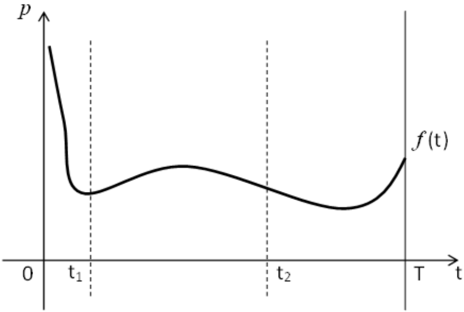
$$TV_{(t_1,t_2)}(f_{\mathbf{X},T}) \le \min\big(f_{\mathbf{X},T}(t_1), f_{\mathbf{X},T}(t_1-)\big) + \min\big(f_{\mathbf{X},T}(t_2), f_{\mathbf{X},T}(t_2-)\big), \tag{2}$$

where

$$TV_{(t_1,t_2)}(f_{\mathbf{X},T}) = \sup \sum_{i=1}^{n-1} \left| f_{\mathbf{X},T}(s_{i+1}) - f_{\mathbf{X},T}(s_i) \right|$$

is the total variation of $f_{\mathbf{X},T}$ *on the interval* (t_1, t_2)*, and the supremum is taken over all choices of* $t_1 < s_1 < \ldots < s_n < t_2$.

Existing results for processes on ℝ V



Existing results for processes on $\mathbb{R}$ VI

Theorem 2 (Samorodnitsky and S., 2013)

(d) The density has a bounded positive variation at the left endpoint and a bounded negative variation at the right endpoint, and similar bounds apply.

(e) The limit $f_{\mathbf{X},T}(0+) < \infty$ *if and only if* $TV_{(0,\varepsilon)}(f_{\mathbf{X},T}) < \infty$ *for some (equivalently, any)* $0 < \varepsilon < T$*, in which case*

$$TV_{(0,\varepsilon)}(f_{\mathbf{X},T}) \le f_{\mathbf{X},T}(0+) + \min\big(f_{\mathbf{X},T}(\varepsilon), f_{\mathbf{X},T}(\varepsilon-)\big)\,. \quad (3)$$

A similar statement applies to the right endpoint.

Structure of A_T

Theorem 3 (Samorodnitsky and S., 2013)

Let A_T be the set of all possible distributions of intrinsic location functionals for stationary processes on $[0, T]$. Then A_T is the set of all probability measures on $[0, T]$ the density function of which in $(0, T)$ is càdlàg and satisfies the total variation constraints. More over, A_T is the convex hull generated by:

(1) *the measures μ_t with density functions $f_{\mu_t} = \frac{1}{t}\mathbf{1}_{(0,t)}$, $0 < t < T$;*

(2) *the measures ν_t with density functions $f_{\nu_t} = \frac{1}{T-t}\mathbf{1}_{(t,T)}$, $0 < t < T$;*

(3) *the point masses δ_0, δ_T and δ_∞.*

Periodic framework

- Further assume that the process $\mathbf{X}$ has a fixed period 1:

$$X(t) = X(t+1), \quad t \in \mathbb{R}.$$

- Equivalently, think $\mathbf{X}$ as a stationary process defined on a circle with perimeter 1.
- Motivation: commutative Lie group
- Question: what can we say about the distribution of an intrinsic location functional over the interval $[0, T]$, $T \leq 1$?
- All the properties for general stationary processes on $\mathbb{R}$ still holds.
- And what else?

Structure of the set of possible densities

Theorem 4

Let I_T be the set of all possible càdlàg densities on $(0, T)$ of intrinsic location functionals for periodic stationary processes with period 1. Then I_T is the convex hull generated by the density functions f on $(0, T)$ s.t.

1. $\int_0^T f(t)dt \leq 1$
2. $f(t) \in \mathbb{Z}^+$, $t \in (0, T)$;
3. *If* $f(0+) \geq 1$, $f(T-) \geq 1$, *then* $f(t) \geq 1$ *for all* $t \in (0, T)$;
4. f *satisfies the total variation constraints.*

Ergodic decomposition

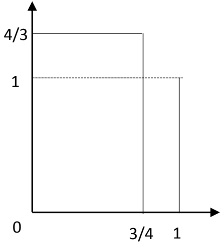
- Why are the extreme points of I_T integer-valued?
- The periodic ergodic processes with period 1 must be of a specific form

Proposition 5

Let $\mathbf{X}$ be a periodic ergodic process with period 1. Then there exists a deterministic function g with period 1, such that $X(t) = g(t+U)$ for $t \in \mathbb{R}$, where U follows a uniform distribution on $[0,1]$.

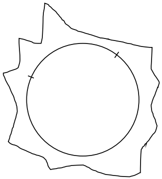
Structure of the set of possible densities

- Define A_T as the set of all the càdlàg density functions f on $(0, T)$ s.t.:
  1. $\int_0^T f(s)\,ds \leq 1$;
  2. f satisfies the total variation constraints.
- We see $I_T \subseteq A_T$.
- However, $I_T \neq A_T$!
- Example: $T = 1$, $f = \frac{4}{3}\mathbf{1}_{(0,\frac{3}{4})}$.



Invariant intrinsic location functionals

- An intrinsic location functional L is called invariant, if
  1. $L(f, I) \neq \infty$ for any compact interval I and any function f;
  2. $L(f, [0,1]) = L(f, [a, a+1]) \bmod 1$ for any $a \in \mathbb{R}$ and function f.
- Intuitively, the location does not change with the starting/ending point of the interval on the circle.



- A generalization of the location of the path supremum, also including
  - the location of the largest jump/largest drawdown
  - the location of the largest derivative
  - ...

Properties of invariant ILFs

- A natural lower bound for the density:

Proposition 6

Let L be an invariant intrinsic location functional and $\mathbf{X}$ be a periodic stationary process with period 1. Then the density $f_{L,T}^{\mathbf{X}}$ satisfies

$$f_{L,T}^{\mathbf{X}} \geq 1 \quad \text{for all } t \in (0, T).$$

- The set of all possible densities becomes the convex hull generated by the density functions f on $(0, T)$ s.t.

1. $\int_0^T f(t)dt \leq 1$
2. $f(t) \in \mathbb{N}$, $t \in (0, T)$;
3. f satisfies the total variation constraints.

Upper bound of the density function

Correspondingly, the upper bound is:

Proposition 7

Let L be an invariant intrinsic location functional, $T \in (0,1]$, and $\mathbf{X}$ be a periodic stationary process with period 1. Then $f_{L,T}^{\mathbf{X}}$ satisfies

$$f_{L,T}^{\mathbf{X}}(t) \leq \max\left(\lfloor\frac{1-T}{t}\rfloor, \lfloor\frac{1-T}{T-t}\rfloor\right) + 2.$$

This is an improvement of the general upper bound $\max\left(\frac{1}{t}, \frac{1}{T-t}\right)$ on $\mathbb{R}$.

First-time intrinsic location functionals

Definition 8

An intrinsic location functional L is called a first-time intrinsic location functional, if it has a partially ordered random set representation $(S(\mathbf{X}), \preceq)$ such that for any $t_1, t_2 \in S(\mathbf{X})$, $t_1 \leq t_2$ implies $t_2 \preceq t_1$.

- A generalization of the first hitting time to a fixed level. (First anything...)
- The density is decreasing.
- The structure of the set of all possible densities is closely related to a problem in risk measure called "joint mixability".

Joint mixability

- Random variables $X_1, ..., X_N$ is said to be a joint mix, if

$$\sum_{i=1}^{N} X_i = C \quad a.s.$$

  for some constant C.

- Distributions $F_1, ..., F_N$ is said to be jointly mixable, if there exists a joint mix $X = (X_1, ..., X_N)$, such that $X_i \sim F_i, i = 1, ..., N$.

- Financial applications

Distributions and joint mixability

- The set of all possible densities becomes the convex hull generated by the density functions f on $(0, T)$ s.t.
  1. $\int_0^T f(t)dt \leq 1$
  2. $f(t) \in \mathbb{Z}^+$, $t \in (0, T)$;
  3. f is decreasing on (0,T).

Proposition 9

Let f be a non-negative, càdlàg, decreasing function on $(0, T)$ s.t. $\int_0^T f(t)dt \leq 1$. Define the distribution functions

$$F_i(x) := \min\{(i - f(x))_+, 1\}\mathbf{1}_{\{x>0\}}, \quad i = 1, ..., N.$$

Then f is the density of a first-time intrinsic location functional for some stationary periodic process with period 1 if $(F_1, ..., F_N)$ is jointly mixable.

Proposition 10

Let f be a non-negative, càdlàg, decreasing function on $(0, T)$ s.t. $\int_0^T f(t)dt \leq 1$. Define the distribution functions

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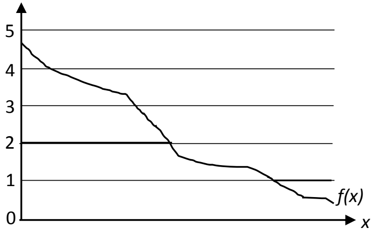
Then f is the density of a first-time intrinsic location functional for some stationary periodic process with period 1 if $(F_1, ..., F_N)$ is jointly mixable.



Thank You!

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