Joint sum-max stability and continuous time random maxima

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Definition of the Coupled CTRM

For $i = 1, 2, \ldots$

(W_i, J_i)	iid random variables on $\mathbb{R}_+ \times \mathbb{R}$,
	W_i and J_i can be dependent;
J_i	**jumps** of a particle;
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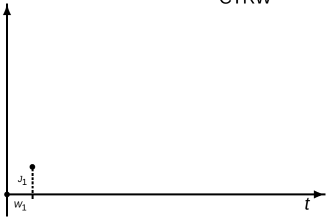

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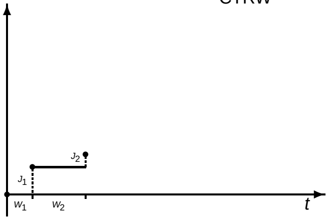

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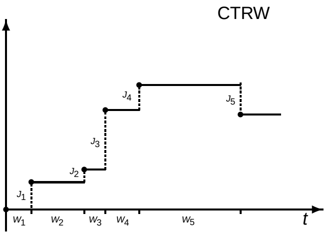

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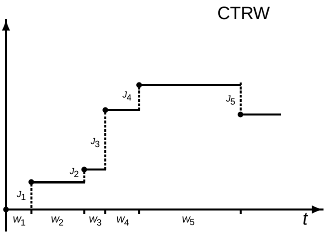

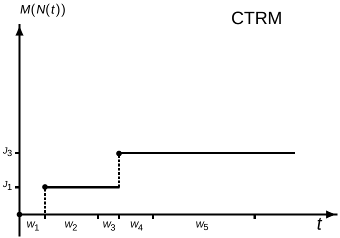

Definition of the Coupled CTRM

- $$N(t) = \max\{n \geq 0 : \sum_{i=1}^{n} W_i \leq t\}$$

  Number of observations by time t (renewal process)
- $M(n) := \bigvee_{i=1}^{n} J_i$ **maximum jump size** after n jumps

We call the process $\{M(N(t))\}_{t>0}$ with

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We are interested in the long-time behaviour of the process:

$$\left\{\tilde{b}(c)M(N(ct))\right\}_{t>0} \to \text{?} \text{ as } c \to \infty.$$

Scaling limit

Classical extreme value theory

- Observations collected at regular intervals in time
- $F_J \in \text{MDA}(\Phi_\alpha) \Rightarrow \left\{ b(c) \bigvee_{i=1}^{\lfloor ct \rfloor} J_i \right\}_{t>0} \xrightarrow{J_1} \{A(t)\}_{t>0}$ as $c \to \infty$
- $\{A(t)\}_{t>0}$ α-Fréchet-extremal process, $P(A(t) \leq x) = \Phi_\alpha^t(x)$

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CTRM with finite mean waiting times

- $N(ct) \sim ct \cdot 1/\mu$ as $c \to \infty$, where $\mu = E(W) < \infty$
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Interesting case

The **waiting times** have **infinite mean**!

Applications:

- **Computer Science**: [Resnick, Stărică, 1995]
  - W periods between transmissions for a networked computer terminal
  - heavy tailed with $P(W > t) \approx Ct^{-\beta}$ with $\beta \approx 0.6$

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- **Geophysics**: [Benson et al, 2007]
  - raindrop release and arrival on the ground
  - earthquakes

Definition 1 (Sum-max domain of attraction)

The random vector (W, J) is in the **sum-max domain of attraction** of (D, A) if there exist $a_n, b_n > 0$ such that

$$\left(a_n \sum_{i=1}^{n} W_i, b_n \bigvee_{i=1}^{n} J_i\right) \Longrightarrow (D, A) \text{ as } n \to \infty.$$

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It follows by projection on either coordinate:

- D is strictly β-stable with $0 < \beta < 1$,
- A Fréchet or Weibull distributed.

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$\longrightarrow$ For analysing these questions we use Harmonic Analysis on Semigroups.

Harmonic Analysis on semigroup $(\mathbb{R}_+, +)$

$(\mathbb{R}_+, +)$ is an **abelian semigroup** with neutral element $e = 0$.

For all $t \geq 0$ let $\rho_t(s) := e^{-st}$. Then:

- $\rho_t(0) = 1$;
- $\rho_t(s_1 + s_2) = \rho_t(s_1)\rho_t(s_2)$.

We call the functions ρ_t **semigroup characters** on $(\mathbb{R}_+, +)$.

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For a probabaility measure μ on $\mathbb{R}_+$ we get

$$L(\mu)(s) = \int_{[0,\infty)} \rho_s(t)\mu(dt) = \int_{[0,\infty)} e^{-st}\mu(dt).$$

the usual Laplace transform on $\mathbb{R}_+$.

Harmonic Analysis on the semigroup $(\overline{\mathbb{R}}, \vee)$

Let $\overline{\mathbb{R}} = [-\infty, \infty]$ denote the two-point compactification of $\mathbb{R}$.
$(\overline{\mathbb{R}}, \vee)$ is an **abelian semigroup** with neutral element $e = -\infty$.

For all $y \in \overline{\mathbb{R}} = (-\infty, \infty]$ let $\tau_y(x) := 1_{[-\infty, y]}(x)$. Then:

- $\tau_y(-\infty) = 1$;
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For a probability measure μ on $\overline{\mathbb{R}}$ we get

$$F_\mu(y) = \int_{[-\infty,\infty]} \tau_y(x)\mu(dx) = \int_{[-\infty,\infty]} 1_{[-\infty,y]}(x)\mu(dx) = \mu([-\infty, y]).$$

CDF = Laplace transformation on the max-semigroup.

Harmonic Analysis on the semigroup $(\mathbb{R}_+ \times \overline{\mathbb{R}}, \overset{+}{\vee})$

$(\mathbb{R}_+ \times \overline{\mathbb{R}}, \overset{+}{\vee})$ is an **abelian semigroup** with neutral element $e = (0, -\infty)$.
The semigroup operation $\overset{+}{\vee}$ is defined by

$$(t_1, x_1) \overset{+}{\vee} (t_2, x_2) := (t_1 + t_2, x_1 \vee x_2).$$

For $(s, y) \in \mathbb{R}_+ \times \overline{\mathbb{R}}$ let $\rho_{s,y}(t, x) := e^{-st} 1_{[-\infty,y]}(x)$. Then

- $\rho_{s,y}(0, -\infty) = 1$;
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We get for a probability measure μ on $\mathbb{R}_+ \times \overline{\mathbb{R}}$:

$$\begin{aligned}\mathcal{L}(\mu)(s, y) &= \int_{[0,\infty)} \int_{[-\infty,\infty]} \rho_{s,y}(t, x)\mu(dt, dx) \\ &= \int_{[0,\infty)} \int_{[-\infty,\infty]} e^{-st} 1_{[-\infty,y]}(x)\mu(dt, dx).\end{aligned}$$

Definition 2

For a probability measure μ on $\mathbb{R}_+ \times \bar{\mathbb{R}}$ we call

$$\mathcal{L}(\mu)(s,y) = \int_{[0,\infty)} \int_{[-\infty,\infty]} e^{-st} 1_{[-\infty,y]}(x)\mu(dt,dx), \ (s,y) \in \mathbb{R}_+ \times \bar{\mathbb{R}};$$

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the **CDF-Laplace transform** (C-L transform) of μ.

Let μ be the distribution of the random vector (W, J).

- Letting $s = 0$ we get
  $$\mathcal{L}(\mu)(0,y) = \mu(\mathbb{R}_+ \times [-\infty, y]) = P\{J \le y\} = F_J(y),$$
  the **distribution function** of J.
- Letting $y = \infty$ we get
  $$\mathcal{L}(\mu)(s,\infty) = \int_0^\infty e^{-st}\mu(dt,\overline{\mathbb{R}}) = E\left[e^{-sW}\right],$$
  the **Laplace transform** of W.

Some properties of the C-L transform

The semigroup operation $\overset{+}{\vee}$ induces a **convolution** $\star$ on $\mathcal{M}^1(\mathbb{R}_+ \times \overline{\mathbb{R}})$. If $\mu_1 := P_{(W_1,J_1)}$, $\mu_2 := P_{(W_2,J_2)}$ where (W_1, J_1) and (W_2, J_2) are independent random vectors on $\mathbb{R}_+ \times \overline{\mathbb{R}}$ we have

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Properties 1

Let μ_n be probability measures on $\mathbb{R}_+ \times \overline{\mathbb{R}}$.

(i) ***Convolution:***

$$\mathcal{L}(\mu_1 \star \mu_2)(s,y) = \mathcal{L}(\mu_1)(s,y) \cdot \mathcal{L}(\mu_2)(s,y) \text{ for all } (s,y) \in \mathbb{R}_+ \times \overline{\mathbb{R}}.$$

(ii) ***Uniqueness Theorem**:*

$$\mu_1 = \mu_2 \iff \mathcal{L}(\mu_1)(s,y) = \mathcal{L}(\mu_2)(s,y) \text{ for all } (s,y) \in \mathbb{R}_+ \times \overline{\mathbb{R}}.$$

(iii) ***Continuity Theorem**:*

$$\mu_n \stackrel{w}{\to} \mu \iff \mathcal{L}(\mu_n)(s,y) \to \mathcal{L}(\mu)(s,y) \text{ for all continuity points } (s,y) \in \mathbb{R}_+ \times \overline{\mathbb{R}} \text{ of the limit.}$$

Sum-max infinite divisibility

Definition 3

We call a $\mathbb{R}_+ \times \overline{\mathbb{R}}$-valued random vector (D, A) resp. the distribution μ $\overset{+}{\vee}$ **-infinite divisible**, if for all $n \geq 1$ there exist a probability measure μ_n on $\mathbb{R}_+ \times \overline{\mathbb{R}}$, such that

$$\mu = \mu_n^{\star n}.$$

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Lemma 4

Let μ_n, μ be probability measures on $\mathbb{R}_+ \times \overline{\mathbb{R}}$ for all $n \geq 1$. If $\mu_n^{\star n} \xrightarrow{w} \mu$ as $n \to \infty$, then μ is $\overset{+}{\vee}$ -infinite divisible.

Our sum-max stable distributions are $\overset{+}{\vee}$ -infinite divisible.

C-L-Exponent

Definition 5

We call the function $\Psi : \mathbb{R}_+ \times \overline{\mathbb{R}} \to \mathbb{R}$ with

$$\mathcal{L}(\mu)(s, y) = \exp(-\Psi(s, y)),$$

C-L-Exponent.

Remark

Let μ be the distribution of the random vector (D, A), then we call x_0 the **left endpoint** of A, that is $x_0 = \inf\{x \in [-\infty, \infty) : F_A(x) > 0\}$. In the following we only consider the case $x_0 = 0$ and $F_A(0) = 0$.

Lévy-Khintchine Representation

Theorem 6

A function $\varphi : \mathbb{R}_+ \times \overline{\mathbb{R}}_+ \to \mathbb{R}$ *is the C-L-Transform of a* $\overset{+}{\vee}$ *-infinite divisible probability measure* μ, *if and only if there exists an* $a \in \mathbb{R}_+$ *and a Radon measure* η *on* $\mathbb{R}_+ \times [0, \infty]$ *with* $\eta(\{(0,0)\}) = 0$ *and*

$$\int_{\mathbb{R}_+} \min(1, t)\eta(dt, [0, \infty]) < \infty \quad \textit{and} \quad \eta(\mathbb{R}_+ \times (y, \infty]) < \infty \ \forall y > 0,$$

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$$\Psi(s, y) = \begin{cases} as + \int_{\mathbb{R}_+} \int_{[0,\infty]} \left(1 - e^{-st} 1_{[0,y]}(x)\right) \eta(dt, dx) & \forall y > 0 \\ \infty & \forall y \leq 0. \end{cases}$$

and $s \geq 0$*. We write* $\mu \sim [a, \eta]$ *and call* η *the* ***Lévy measure of*** μ**.**

Characterization of the sum-max domain of attraction

Theorem 7

Let (W_i, J_i) *be i.i.d.* $\mathbb{R}_+ \times \mathbb{R}$*-valued random vectors.*

Then there exist $a_n, b_n > 0$ *such that*

$$\left(a_n \sum_{i=1}^{n} W_i, b_n \bigvee_{i=1}^{n} J_i\right) \Longrightarrow (D, A) \text{ as } n \to \infty \text{ with } (D, A) \sim [0, \eta]$$

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"$\mu_n \xrightarrow{v} \mu$" means:
$\mu_n(B) \to \mu(B)$ as $n \to \infty$ for all $B \in \mathcal{B}(\mathbb{R}^2_+)$ with $\mu(\partial B) = 0$ that are bounded away from $(0, 0)$.

Characterization of the sum-max domain of attraction

Properties 1

Let (D_1, A_1) *be i.i.d. copies of the limit* (D, A) *in Theorem 7 above. Then there exist* $0 < \beta < 1$ *and* $\alpha > 0$ *such that*

$$(D_1, A_1) \overset{+}{\vee} \cdots \overset{+}{\vee} (D_n, A_n) \stackrel{d}{=} (n^{1/\beta} D, n^{1/\alpha} A)$$

for all $n \geq 1$. (D, A) *is called* **sum-max stable**.

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Properties 2

Let η *be the Lévy measure of* (D, A)*. Moreover, let* $E = \mathrm{diag}(1/\beta, 1/\alpha)$ *so that* $t^{-E} = \mathrm{diag}(t^{-1/\beta}, t^{-1/\alpha})$*. Then we have for all Borel sets* $B \subset \mathbb{R}^2_+$ *which are bounded away from* $(0, 0)$ *that*

$$t \cdot \eta(B) = \eta(t^{-E} B) \quad \text{for all } t > 0.$$

Representation of the Lévy measure

Theorem 8

Let η be the Lévy measure of (D, A), where D is β sum-stable ($0 < \beta < 1$) and A has an α-Fréchet distribution ($\alpha > 0$). Then there exists $C \geq 0, K > 0$ and $\omega \in \mathcal{M}^1(\mathbb{R})$ with

$$\int_0^\infty x^\alpha \, \omega(dx) < \infty$$

such that

$$\eta(dt, dx) = \varepsilon_0(dt) C\alpha x^{-\alpha-1} dx + 1_{(0,\infty)\times\mathbb{R}_+}(t, x) \cdot \left(t^{\beta/\alpha}\omega\right)(dx) K\beta t^{-\beta-1} dt.$$

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Corollary 9

The random variables A and D are independent ***if and only if*** *we have $C > 0$ and $\omega = \epsilon_0$ in the representation of the Lévy measure.*

Representation of the Lévy measure

Example 10

Recall that the Lévy measure of (D, A) is given by

$$\eta(dt, dx) = \varepsilon_0(dt) C\alpha x^{-\alpha-1} dx + 1_{(0,\infty)\times\mathbb{R}_+}(t, x) \cdot \left(t^{\beta/\alpha}\omega\right)(dx) K\beta t^{-\beta-1} dt.$$

Assume that $J_i = W_i$. Then

$$(a_n \sum_{i=1}^{n} W_i, a_n \bigvee_{i=1}^{n} W_i) \Longrightarrow (D, A) \text{ as } n \to \infty \text{ with } (D, A) \sim [0, \eta]$$

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$$\eta(dt, dx) = 1_{(0,\infty)\times\mathbb{R}_+}(t, x) \cdot \varepsilon_t(dx) K\beta t^{-\beta-1} dt,$$

that is $C = 0, \alpha = \beta$ and $\omega = \varepsilon_1$. $\longrightarrow$ **complete dependence**

Theorem 11 (J_1-convergence of the joint sum-max process)

Assume that there exist $a_n > 0, b_n > 0$ with

$$\left(a_n \sum_{i=1}^{n} W_i, b_n \bigvee_{i=1}^{n} J_i\right) \Longrightarrow (D, A) \text{ as } n \to \infty.$$

Then

$$\left\{\left(a(c) \sum_{i=1}^{\lfloor ct \rfloor} W_i, b(c) \bigvee_{i=1}^{\lfloor ct \rfloor} J_i\right)\right\}_{t>0} \xrightarrow{J_1} \{(D(t), A(t))\}_{t>0} \text{ as } c \to \infty,$$

where the C-L-Transform of the fdds of $\{(D(t), A(t))\}_{t>0}$ *are given by*

$$\mathcal{L}(P_{(D(t_j),A(t_j))_{j=1,\ldots,m}})(\boldsymbol{s}, \boldsymbol{y}) = \prod_{j=1}^{m} \varphi_{(D,A)}(\Sigma_{k=j}^{m} s_k, \min(v_j, \ldots, v_m))^{(t_j - t_{j-1})}$$

$\boldsymbol{s} := (s_1, \ldots, s_m), \boldsymbol{y} := (y_1, \ldots, y_m).$

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- $\{D(t)\}_{t>0}$ β – **stable subordinator.**
- $\{A(t)\}_{t>0}$ **F-Extremal process** , $P\{A(t) \leq x\} = F(x)^t$ and F is the CDF of an α-Fréchet distribution.

Limit Theorem for the CTRM

Theorem 12

Assume $(W_i, J_i)_{i\in\mathbb{N}}$ *are i.i.d.* $\mathbb{R}_+ \times \mathbb{R}$*-valued rv's and there exist* $a_n, b_n > 0$ *such that*

$$n \cdot P_{(a_n W, b_n J)}(B) \xrightarrow{v} \eta(B) \text{ as } n \to \infty.$$

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Then

$$\left\{ \tilde{b}(c) \bigvee_{i=1}^{N(ct)} J_i \right\}_{t>0} \xrightarrow{J_1} \left\{ A(E(t)-)^+ \right\}_{t>0} \text{ as } c \to \infty$$

where $\tilde{b}(c) \in RV(-\alpha/\beta)$.

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where

- $N(t) = \max\{n \geq 0 : \sum_{i=1}^{n} W_i \leq t\}$
- $\{A(t)\}_{t>0}$ α-Fréchet extremal process,
- $E(t) := \inf\{x \geq 0 : D(x) > t\}$ inverse of stable subordinator.

Distribution of the limit

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Assume $(W_i, J_i)_{i\in\mathbb{N}}$ *are i.i.d.* $\mathbb{R}_+ \times \mathbb{R}$*-valued random vectors and*

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Then

$$P\left\{A(E(t)-)^+ \le x\right\} = \int_0^\infty \int_0^t \Phi_D(t-u,\infty) P_{(D(s),A(s))}(du, [0,x]) ds$$

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is the distribution function of the limit $\left\{A(E(t)-)^+\right\}_{t>0}$ and for all $\xi > 0$ we have

$$\int_0^\infty e^{-\xi t} P\left\{A(E(t)-)^+ \le x\right\} dt = \frac{1}{\xi}\frac{\Psi_D(\xi)}{\Psi(\xi,x)}.$$

- $\Phi_D(dt)$ is the (usual) Lévy measure of $\{D(t)\}_{t\ge 0}$
- Ψ is the C-L-Exponent of $\{(D(t), A(t))\}_{t>0}$
- Ψ_D denotes the (usual) Laplace-Exponent of $\{D(t)\}_{t\ge 0}$.

Example

$(W_i, J_i)_{i\in\mathbb{N}}$ i.i.d. as (W, J) with

- $W \in DONA(D)$, D β-stable $(0 < \beta < 1)$ with $E(e^{-sD}) = \exp(-s^\beta)$.
- $J := W^{1/\gamma} Z$ with Z γ-Fréchet, Z and W **independent**.

Then we have

$$\left(n^{-1/\beta}\sum_{i=1}^{n} W_i, n^{-1/(\beta\gamma)}\bigvee_{i=1}^{n} J_i\right) \Longrightarrow (D, A) \sim [0, \eta]$$

as $n \to \infty$, where the Lévy measure is

$$\eta(dt, dx) = (t^{1/\gamma} P_Z)(dx)\frac{\beta}{\Gamma(1-\beta)} t^{-\beta-1}dt,$$

that is we have $C = 0$, $\alpha = \beta\gamma$ and $\omega = P_Z$ in the representation of the Lévy measure.

Example

The C-L exponent in this example is given by

$$\Psi(\xi, x) = (\xi + x^{-\gamma})^{\beta}.$$

If we set $G(t,x) = P\{A(E(t)-)^{+} \leq x\}$ we have

$$L(G(\cdot, x))(\xi) = \int_0^{\infty} e^{-\xi t} G(t,x)\, dt = \frac{1}{\xi} \frac{\psi_D(\xi)}{\psi(\xi, x)} = \frac{\xi^{\beta-1}}{(\xi + x^{-\gamma})^{\beta}}.$$

Inverting the Laplace transform yields

$$G(t,x) = \int_0^t e^{-x^{-\gamma} u} \frac{u^{\beta-1}}{\Gamma(\beta)} \cdot \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)} du$$

Observe that

$$A(E(t)-)^{+} \stackrel{d}{=} (tB)^{1/\gamma} Y$$

where B has a β-distribution, Y is standard γ-Fréchet, independent of B.

Example

Moreover

$$(\xi + x^{-\gamma})^{\beta} L(G(\cdot, x))(\xi) = \xi^{\beta-1}.$$

and an application of the inverse Laplace transform on both sides yields the **governing equation**

$$\partial_t^{\beta}\left[e^{tx^{-\gamma}} G(t,x)\right] = e^{tx^{-\gamma}} \frac{t^{-\beta}}{\Gamma(1-\beta)};$$

where we have used that

- $L(\partial_t^{\beta} f(s))(t) = t^{\beta} L(f(s))(t)$ (Riemann-Liouville fractional derivative)
- $L(e^{-as} f(s))(t) = L(f(s))(t+a)$.

$G(\cdot, x)$ is called the **mild solution** of this equation.

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