Measuring dependence in heavy tailed processes

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How does one measure dependence in a stochastic process?

Let $\mathbf{X} = (X_n,\ n \in \mathbb{Z})$ be a stationary process. How long is the memory in this process?

If $EX_n^2 < \infty$ we can use the covariance function.

In heavy tailed processes we often have

$$P(|X_n| > x) \in Reg(-\alpha) \ \text{ as } x \to \infty, \text{ some } \alpha > 0.$$

- Covariances are not defined if $\alpha < 2$.
- Even if covariances are defined, should we use them?
- The key question: why do we want to measure memory?

The original motivation, probably, came from statistics.

How should we modify a statistical procedure designed for i.i.d. observations to account for dependent data?

Question 1: how does dependence change the behaviour of sample statistics?

Question 2: do covariances provide a way to answer **Question 1**?

The most important statistics are

- the sum of the observations;
- the largest observation.

- Let $S_n = X_1 + \ldots + X_n$, $n \geq 1$ be the partial sums;
- $M_n = \max(X_1, \ldots, X_n)$, $n \geq 1$ the partial maxima.

A useful notion of dependence in a stationary process $\mathbf{X}$ should have an impact on the "size" of S_n and M_n.

What is the "size" of the partial sums and maxima as a function of n?

Balanced regular variation assumption:

- the random variable $|X_0|$ is regularly varying with exponent α;
- for some $0 \leq p, q \leq 1$, $p + q = 1$,

$$\lim_{x \to \infty} \frac{P(X_0 > x)}{P(|X_0| > x)} = p, \quad \lim_{x \to \infty} \frac{P(X_0 < -x)}{P(|X_0| > x)} = q\,.$$

Suppose first that (X_n) are i.i.d.

What is the "size" of the partial sums and partial maxima with no memory?

The quantile sequence:

$$a_n = \inf\big\{x > 0 : \ P(|X_0| > x) \le 1/n\big\}, \ \ n = 1, 2, \ldots .$$

The sequence (a_n) is $\mathrm{Reg}(1/\alpha)$.

Case 1: $0 < \alpha < 1$.

- $S_n/a_n \Rightarrow Z_\alpha$, a standard α-stable random variable, with skewness p.
- $M_n/a_n \Rightarrow Y_\alpha$, an α- Fréchet random variable with scale $p^{1/\alpha}$.

Case 2: $1 < \alpha < 2$.

- $S_n/n \to EX_0$ and $(S_n - nEX_0)/a_n \Rightarrow Z_\alpha$, a standard zero mean α-stable random variable, with skewness p.
- $M_n/a_n \Rightarrow Y_\alpha$, an α- Fréchet random variable with scale $p^{1/\alpha}$.

Case 3: $\alpha > 2$.

- $S_n/n \to EX_0$ and $(S_n - nEX_0)/\sqrt{n} \Rightarrow G$, a zero mean normal random variable, with the same variance as X_0.
- $M_n/a_n \Rightarrow Y_\alpha$, an α- Fréchet random variable with scale $p^{1/\alpha}$.

If $\alpha > 2$ the variance is finite.

Covariances carry information about S_n.

$$\mathrm{Var}(S_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} R_{i-j}$$

$$= \sum_{i=-(n-1)}^{n-1} (n - |i|) R_i$$

$$\sim n \sum_{i=-\infty}^{\infty} R_i, \ n \to \infty$$

$$\text{if} \quad \sum_{i=-\infty}^{\infty} |R_i| < \infty \, .$$

$$\frac{S_n}{\sqrt{\mathrm{Var}(S_n)}} \text{ forms a tight sequence.}$$

If the covariances are summable, then

$$\frac{S_n}{\sqrt{n}} \text{ forms a tight sequence.}$$

- This does NOT guarantee that $n^{1/2}$ is the distributional "size" of S_n.
- This does not say what is the distribution of $\frac{S_n}{\sqrt{\mathrm{Var}(S_n)}}$.

Even if the variance is finite, covariances carry very little information about partial maxima.

If $\mathbf{X}$ is a stationary Gaussian sequence with standard normal marginals, then

$$\frac{M_n}{\sqrt{2 \log n}} \to 1 \text{ in probability}$$

as long as $R_n \to 0$ as $n \to \infty$.

The extremal index is much more informative than covariances for the “size” of the partial maxima.

Let $\mathbf{X} = \big(X_n,\ n \in \mathbb{Z}\big)$ be a stationary process with a continuous marginal distribution F.

Let $\mathbf{Y} = \big(Y_n,\ n \in \mathbb{Z}\big)$ be an i.i.d. sequence with the same marginal distribution F.

Suppose that there is $\theta > 0$ with the property:

for every $\tau > 0$ there is a sequence (u_n) such that

$$P\Big(\max_{i=1,\dots,n} Y_i \le u_n\Big) \to e^{-\tau} \,,$$

$$P\Big(\max_{i=1,\dots,n} X_i \le u_n\Big) \to e^{-\theta\tau} \,.$$

Then θ is **the extremal index** of $\mathbf{X}$. We always have $\theta \le 1$.

We have both

$$P\Big(\max_{i=1,\ldots,[n\theta]} Y_i \leq u_n\Big) \to e^{-\theta\tau} \,,$$

$$P\Big(\max_{i=1,\ldots,n} X_i \leq u_n\Big) \to e^{-\theta\tau} \,.$$

Dependence measured through extremal index effectively reduces the sample size.

Covariances do not provide this type of information

- If $\alpha < 2$, variance does not exist.
- Covariances cannot be used.
- Some substitutes for covariances have been suggested.
- In the stable case *covariation* and *codifference* have been used.

Typically, substitutes for covariance carry even less information than covariance does.

However: codifference is useful to characterize *mixing* in infinitely divisible processes.

Codifference can be defined for any stationary process $\mathbf{X}$.

$$\begin{aligned} \tau_n &= Ee^{i(X_n - X_0)} - Ee^{iX_n} Ee^{-iX_0} \\ &= Ee^{i(X_n - X_0)} - \left| Ee^{iX_0} \right|^2, \ n \in \mathbb{Z}, \\ \tau_{-n} &= \bar{\tau}_n . \end{aligned}$$

For any mixing process

$$\lim_{n\to\infty}\left(Ee^{i(\theta_1 X_n+\theta_2 X_0)} - Ee^{i\theta_1 X_n}Ee^{i\theta_2 X_0}\right) = 0$$

for any real θ_1, θ_2.

In particular, mixing implies $\tau_n \to 0$ as $n \to \infty$.

If $\mathbf{X}$ is a stationary infinitely divisible process with marginal Lévy measure not charging the set $\{2\pi k,\ k \in \mathbb{Z}\}$, then

$\tau_n \to 0$ as $n \to \infty$ implies mixing.

Mixing and mixing coefficients

For any stationary stochastic process $\mathbf{X} = \big(X_n,\ n \in \mathbb{Z}\big)$ one can use ergodic properties and strong mixing to measure dependence.

- The process $\mathbf{X}$ is mixing if for any $k \geq 1$ and Borel sets $C, D \in \mathbb{R}^k$,

$$
\begin{aligned}
&P\big((X_1, \ldots, X_k) \in C,\ (X_{n+1}, \ldots, X_{n+k}) \in D\big) \\
\to &P\big((X_1, \ldots, X_k) \in C\big) P\big((X_1, \ldots, X_k) \in D\big).
\end{aligned}
$$

- The process $\mathbf{X}$ is ergodic if for any $k \geq 1$ and Borel sets $C, D \in \mathbb{R}^k$,

$$\frac{1}{n}\sum_{i=1}^{n} P\big((X_1, \ldots, X_k) \in C, \, (X_{i+1}, \ldots, X_{i+k}) \in D\big)$$
$$\to P\big((X_1, \ldots, X_k) \in C\big) P\big((X_1, \ldots, X_k) \in D\big) .$$

Both properties are asymptotic properties and do not provide "numerical" measures of dependence.

- Numerical measures can be obtained by taking "worst" sets at a distance n.
- This leads to notions of **strong mixing**.
- All notions of strong mixing are different ways of quantifying the proximity to independence of the past and future.

Denote

$$\mathcal{F}_{-\infty}^{0} = \sigma(X_k,\, k \le 0), \quad \mathcal{F}_n^{\infty} = \sigma(X_k,\, k \ge n),\; n \ge 1\,.$$

The α-mixing coefficient

$$\alpha_X(n) = \sup_{A \in \mathcal{F}_{-\infty}^{0}, B \in \mathcal{F}_n^{\infty}} \left| P(A \cap B) - P(A)P(B) \right| \in [0, 1/4] .$$

The β-mixing coefficient

$$\beta_X(n) = \frac{1}{2} \sup_{\mathcal{I}, \mathcal{J}} \sum_{i=1}^{I} \sum_{j=1}^{J} \left| P(A_i \cap B_j) - P(A_i)P(B_j) \right| \in [0, 1] ,$$

the supremum is over all finite partitions $\mathcal{I} = \{A_1, \ldots, A_i\}$ and $\mathcal{J} = \{B_1, \ldots, B_J\}$ of Ω into $\mathcal{F}_{-\infty}^{0}$ sets and into $\mathcal{F}_n^{\infty}$ sets.

The ϕ-mixing coefficient

$$\phi_X(n) = \sup_{A \in \mathcal{F}^0_{-\infty}, P(A)>0,\, B \in \mathcal{F}_n^\infty} \left| P(B|A) - P(B) \right| \in [0,1]\,.$$

Ordering:

$$2\alpha_X(n) \leq \beta_X(n) \leq \phi_X(n)\,.$$

If $\alpha_X(n) \to 0$ ($\beta_X(n) \to 0$, $\phi_X(n) \to 0$)

then $\mathbf{X}$ is α-mixing (β-mixing, ϕ-mixing).

- The sequences $(\alpha_X(n))$, $(\phi_X(n))$,$(\phi_X(n))$ are numerical measures of dependence.
- It is not easy to connect these sequences to the distribution of the statistics of the observations.
- There are some connections to approximate normality of the partial sums
- Weak connections to the partial maxima.

Long range dependence in heavy tailed processes

Suppose a stationary process $\mathbf{X} = \big(X_n,\ n \in \mathbb{Z}\big)$ has a finite variance,

It is common to define long range dependence (LRD) via second order characteristics. One can look at:

- rate of decay of covariances;
- rate of increase of the variance of the partial sums;
- the behaviour of the spectrum at the origin.

1. Rate of decay of covariances

A common requirement for LRD is

$$\sum_{n=-\infty}^{\infty} |R_X(n)| = \infty \, .$$

Sometimes a more specific requirement is imposed:

$\big(R_X(n)\big)$ is regularly varying with exponent $-d$, $0 < d < 1$.

$$R_X(n) = n^{-d} L(n), \quad L \text{ slowly varying} \, .$$

2. Rate of increase of the variance of partial sums

A common requirement for LRD is

$$\limsup_{n\to\infty} \frac{\mathrm{Var}\,(X_1 + \ldots + X_n)}{n} = \infty \,.$$

Sometimes a more specific requirement is imposed:

$$\big(\mathrm{Var}\,(X_1 + \ldots + X_n)\big) \text{ is regularly varying}$$

$$\text{with exponent } 2 - d,\ 0 < d < 1.$$

3. Behaviour of the spectrum at the origin

The spectral measure of a process $\mathbf{X}$ is a is a symmetric measure F_X on $(-\pi, \pi]$ such that

$$
\begin{aligned}
R_X(n) &= \int_{(-\pi,\pi]} e^{inx} \, F_X(dx) \\
&= \int_{(-\pi,\pi]} \cos nx \, F_X(dx), \;\; n = 0, 1, \ldots .
\end{aligned}
$$

If a spectral density exists, it is denoted by f_X.

A common requirement for LRD is

a spectral density exists, and is unbounded near 0.

Sometimes a more specific requirement is imposed:

a spectral density exists, and is regularly varying at 0

with exponent $-(1-d)$, $0 < d < 1$.

1. Regular variation of the covariances implies regular variation of the variance of the partial sums.
2. Regular variation of the spectral density implies regular variation of the variance of the partial sums.
3. Regular variation of the variance of the partial sums does not imply anything.

Is regular variation of the covariances equivalent to regular variation of the spectral density?

The answer is “no” without additional assumptions on the slowly varying functions.

The general definition of a slowly varying function: a measurable function $L: [0, \infty) \to (0, \infty)$ is slowly varying if for each $b > 0$

$$\lim_{x \to \infty} \frac{L(bx)}{L(x)} = 1\,.$$

A function of the Zygmund class: a measurable function $L : [0, \infty) \to (0, \infty)$ belongs to the Zygmund class if for any $\delta > 0$ both

- the function $\left(x^{\delta} g(x),\, x > 0\right)$ is eventually non-decreasing,
- the function $\left(x^{-\delta} g(x),\, x > 0\right)$ is eventually non-increasing.

Any function of the Zygmund class is slowly varying (but not vice versa).

Theorem If the covariances satisfy

$$R_X(n) = n^{-d} L(n),\ n = 1, 2, \ldots,\ 0 < d < 1\,,$$

and the slowly varying function L belongs to the Zygmund class, then a spectral density exists and

$$f_X(x) \sim c_d |x|^{-(1-d)} L\big(1/|x|\big),\ x \to 0\,.$$

Theorem Suppose that the spectral density satisfies

$$f_X(x) = |x|^{-(1-d)} L\big(1/|x|\big),\ x \to 0,\ 0 < d < 1\,,$$

and the function L belongs to the Zygmund class. Assume also that the spectral density has a bounded variation on the interval (ε, π) for any $0 < \varepsilon < \pi$. Then the covariance function satisfies

$$R_X(n) \sim c_d^{-1} n^{-d} L(n),\ n \to \infty\,.$$

- Second-order notions of LRD provide limited information on the partial sums.
- They provide even less information on the partial maxima.
- For heavy tailed processes second-order notions may not even be well defined.
- Alternative ways to look at LRD are needed.
- It is useful to look at LRD as at a phase transition.

Long range dependence as a phase transition

Suppose that

- $\left(P_\theta, \theta \in \Theta\right)$: laws of a stationary stochastic process $\mathbf{X} = \left(X_n, n \in \mathbb{Z}\right)$.
- The one-dimensional marginal distributions of the process $\mathbf{X}$ do not change significantly as θ varies.
- The behaviour of important functionals of $\mathbf{X}$ may change as θ varies.

Typical important functionals are: partial sums and partial maxima.

Typical examples of families of laws $\big(P_\theta,\, \theta \in \Theta\big)$: infinite moving averages and infinitely divisible processes.

Θ_0: the subset of Θ corresponding to sequences of i.i.d. random variables.

Θ_0 may or may not be a singleton.

LRD is a phase transition between two parts of the parameter space: $\Theta = \Theta_1 \cup \Theta_2$.

- $\Theta_0 \subset \Theta_1$, and for $\theta \in \Theta_1$ the functional of interest behaves similarly to the i..i.d. case, $\theta \in \Theta_0$.
- For $\theta \in \Theta_2$, there is a change "in nature and the order of magnitude" in the behaviour of the functional.

Model 1: infinite moving averages

$$X_n = \sum_{j=-\infty}^{\infty} \varphi_{n-j}\, \varepsilon_j = \sum_{j=-\infty}^{\infty} \varphi_j\, \varepsilon_{n-j}\,, \quad n = 1, 2, \ldots\,,$$

- (ε_n) are i.i.d. *noise variables*, or *innovations*,
- (φ_n) are deterministic coefficients.

Conditions for convergence:

Suppose that $E|\varepsilon|^p < \infty$ for some $p > 0$.

Case 1 If $0 < p \leq 1$, then the condition

$$\sum_{j=-\infty}^{\infty} |\varphi_j|^p < \infty \tag{1}$$

is sufficient for convergence of the series $\mathbf{X}$.

Case 2 If $1 < p \leq 2$ and $E\varepsilon = 0$, then condition (1) is sufficient for convergence of the series $\mathbf{X}$.

If $E\varepsilon \neq 0$, then (1) and the condition

$$\text{the series} \quad \sum_{j=-\infty}^{\infty} \varphi_j \quad \text{converges} \tag{2}$$

are sufficient for convergence of the series $\mathbf{X}$.

Case 3 If $p > 2$ and $E\varepsilon = 0$, then the condition

$$\sum_{j=-\infty}^{\infty} \varphi_j^2 < \infty \tag{3}$$

is sufficient for convergence of the series $\mathbf{X}$.

If $E\varepsilon \neq 0$, then conditions (3) and (2) are sufficient for convergence of the series $\mathbf{X}$.

The parameter θ is a sequence of coefficients (φ_j)

Stationary infinitely divisible processes

A stochastic process $\mathbf{X} = \big(X_n,\ n \in \mathbb{Z}\big)$ is infinitely divisible if for every $n = 1, 2, \ldots$ there is a process $\mathbf{Y} = \big(Y_n,\ n \in \mathbb{Z}\big)$ such that

$$\big(X_n,\ n \in \mathbb{Z}\big) \stackrel{\mathrm{d}}{=} \left(\sum_{j=1}^{n} Y_n^{(j)},\ n \in \mathbb{Z} \right),$$

where $\mathbf{Y}^{(j)} = \big(Y_n^{(j)},\ n \in \mathbb{Z}\big)$ are i.i.d. copies of $\mathbf{Y} = \big(Y_n,\ n \in \mathbb{Z}\big)$.

A stationary infinitely divisible process $\mathbf{X}$ is characterized by two parameters:

1. The mean μ and the covariance function R of a stationary Gaussian process.

2. A σ-finite shift invariant measure ν on $\mathbb{R}^{\mathbb{Z}}$ such that

$$\int_{\mathbb{R}^{\mathbb{Z}}} \min(1, x_0^2)\, \nu(d\mathbf{x}) < \infty\,.$$

   ν: the Lévy measure of $\mathbf{X}$.

Here $\mathbf{x} = \big(\ldots, x_1, x_0, x_1, x_2, \ldots\big)$.

The joint characteristic function of $\mathbf{X}$: with

$$\mathbb{R}^{(T)} = \left\{ \mathbf{x} \in \mathbb{R}^T : \ x(t) = 0 \ \text{ for all but finitely many } t \in T \right\},$$

$$E \exp\left\{ i \sum_{t \in T} \theta(t) X(t) \right\}$$

$$= \exp\left\{ -\frac{1}{2} \boldsymbol{\theta}^T R \boldsymbol{\theta} + ib(\mathbf{e}, \boldsymbol{\theta}) + \int_{\mathbb{R}^{\mathbb{Z}}} \left(e^{i(\boldsymbol{\theta}, \mathbf{x})} - 1 - i\big(\boldsymbol{\theta}, [\![\mathbf{x}]\!]\big) \right) \nu(d\mathbf{x}) \right\} .$$

$$\mathbf{e} = (\ldots, 1, 1, 1, \ldots) .$$

Truncation:

- for $x \in \mathbb{R}$,

$$[\![x]\!] = \begin{cases} x & \text{if } |x| \leq 1 \\ -1 & \text{if } x < -1 \\ 1 & \text{if } x > 1 \end{cases} ;$$

- for a function $\mathbf{x} = (x(t),\ t \in \mathbb{R})$,

$$[\![\mathbf{x}]\!] = \left([\![x(t)]\!],\ t \in \mathbb{R}\right).$$

Phase transitions in behaviour of the partial sums

Let $\mathbf{X} = \big(X_n,\ n \in \mathbb{Z}\big)$ be a stationary process.

- the partial sum sequence

$$S_n = X_1 + \ldots + X_n,\ n = 1, 2, \ldots\ ,$$

- the partial sum process

$$S_n(t) = S_{[nt]},\ t \geq 0\,.$$

Case 1: the finite variance case

Let $\mathbf{X} = \big(X_n,\ n \in \mathbb{Z}\big)$ be i.i.d. zero mean, finite variance σ^2.

Central Limit Theorem

$$\frac{1}{n^{1/2}} S_n \Rightarrow \sigma G,\ \ G \sim N(0,1)\,.$$

Invariance Principle

$$\left(\frac{1}{n^{1/2}} S_n(t),\ t \geq 0\right) \Rightarrow \big(\sigma B(t),\ t \geq 0\big)$$

weakly in the J_1 topology on $D[0,\infty)$.

B is the standard Brownian motion.

Let $\mathbf{X} = \big(X_n,\ n \in \mathbb{Z}\big)$ be stationary, zero mean, finite variance.

Indications of LRD from the point of view of partial sums:

- The order of magnitude a_n of S_n is different from $n^{1/2}$.
- The limiting process $\mathbf{Y}$ in

$$\left(\frac{1}{a_n} S_n(t),\ t \geq 0\right) \Rightarrow \big(Y(t),\ t \geq 0\big)$$

  is different from a Brownian motion.

The limiting process **Y** in the functional limit theorem must have two properties.

- It must be self-similar: for any $c > 0$,

$$\big(Y(ct),\, t \geq 0\big) \overset{\mathrm{d}}{=} \big(c^H Y(t),\, t \geq 0\big), \quad \text{some } H \geq 0\,.$$

- It must have stationary increments: for any $c > 0$,

$$\big(Y(t+c) - Y(c),\, t \geq 0\big) \overset{\mathrm{d}}{=} \big(Y(t),\, t \geq 0\big)\,.$$

- Fractional Brownian motion is the only self-similar Gaussian process with stationary increments.
- It has Hurst exponent $0 < H < 1$.
- $\big(B_H(t),\ t \geq 0\big)$ is a zero mean Gaussian process with incremental variance

$$E\big(B_H(t) - B_H(s)\big)^2 = |t - s|^{2H},\ t, s \in \mathbb{R}\,.$$

- There exist other finite variance self-similar processes with stationary increments.

A finite variance infinite moving average model:

$$X_n = \sum_{j=-\infty}^{\infty} \varphi_{n-j}\, \varepsilon_j\,, \quad n = 1, 2, \ldots\,,$$

(ε_n) i.i.d. zero mean, finite variance σ_ε^2.

$$\sum_{j=-\infty}^{\infty} \varphi_j^2 < \infty\,.$$

How do the partial sums behave?

Theorem Suppose that

$$\sum_{j=-\infty}^{\infty} |\varphi_j| < \infty\,,$$

$$a_\varphi = \sum_{j=-\infty}^{\infty} \varphi_j \neq 0\,.$$

Then

$$\left(n^{-1/2} S_n(t),\ t \geq 0\right) \Rightarrow \left(a_\varphi \sigma_\varepsilon B(t),\ \ t \geq 0\right) \text{ as } n \to \infty$$

weakly in the J_1 topology on $D[0, \infty)$.

There is no LRD in this case

Balanced regular variation assumption: for some regularly varying with exponent $-\beta \in (-1, -1/2)$ sequence (b_n)

$$\lim_{n\to\infty} \frac{\varphi_n}{b_n} = c_+, \quad \lim_{n\to\infty} \frac{\varphi_{-n}}{b_n} = c_- \,,$$

$c_+,\, c_- \geq 0$, not both zero.

Theorem With $H = 3/2 - \beta$,

$$\left(\frac{1}{n^{3/2} b_n} S_n(t),\ t \geq 0 \right) \Rightarrow \left(c_\varphi \sigma_\varepsilon B_H(t),\ t \geq 0 \right)$$

weakly in the J_1 topology on $D[0, \infty)$.

This process has LRD

- Similarly, if the coefficients are summable, but $a_\varphi = 0$:
- regular variation of coefficients with $-\beta \in (-3/2, -1)$ leads to a Fractional Brownian limit with $H = 3/2 - \beta$.
- The process is still long range dependent.
- The rate of decay of coefficients mostly determines the memory in moving average processes.

Memory in stationary infinitely divisible processes

LRD in stationary infinitely divisible processes is strongly related to ergodic-theoretical properties of the Lévy measures.

Ergodic-theoretical setup

- $(E, \mathcal{E}, m)$ a σ-finite measure space.
- $\phi : E \to E$ one-to-one, both ϕ and ϕ^{-1} measurable.
- ϕ preserves the measure m.

Dissipative and conservative maps

A set $W \in \mathcal{E}$ is called wandering if $\left(\phi^{-n}(W),\, n = 1, 2, \ldots\right)$ are pairwise disjoint mod(m).

Any set of measure 0 is wandering.

A map ϕ on $\left(E, \mathcal{E}, m\right)$ is conservative if it does not admit a wandering set of a positive measure.

Example

- Let $E = \mathbb{Z}$ and m the counting measure.
- The right shift $\phi(x) = x + 1$ is measure preserving.
- The set $W = \{0\}$ is a wandering set of a positive measure.
- Hence, the right shift is not conservative.

Hopf decomposition: there is a partition of E into ϕ-invariant sets $\mathcal{C}(\phi)$ and $\mathcal{D}(\phi)$ such that

(i) there is no wandering set of a positive measure which is a subset of $\mathcal{C}(\phi)$;

(ii) there is a wandering set W such that $\mathcal{D}(\phi) = \cup_{n=-\infty}^{\infty} \phi^n(W)$ mod(m).

Terminology:

- $\mathcal{C}(\phi)$: the conservative part of ϕ.
- $\mathcal{D}(\phi)$: the dissipative part of ϕ.

A map ϕ is dissipative if $\mathcal{C}(\phi) = \emptyset \bmod(m)$.

A map ϕ is conservative if $\mathcal{D}(\phi) = \emptyset \bmod(m)$.

Examples The right shift on $\mathbb{Z}$ is dissipative.
Any map preserving a finite measure is conservative.

Let $\mathbf{X} = \big(X_n,\ n \in \mathbb{Z}\big)$ be a stationary infinitely divisible process.

Assume no Gaussian component: $R = 0$.

The left shift:

$$\phi\big((\ldots, x_{-1}, x_0, x_1, \ldots)\big) = (\ldots, x_0, x_1, x_2, \ldots)$$

preserves the Lévy measure ν.

The memory in the process $\mathbf{X}$ is strongly affected by whether ϕ is dissipative or conservative.

It is easier to analyze the situation if **X** is represented as

$$X_n = \int_E f \circ \phi^n(s)\, M(ds),\ \ n \in \mathbb{Z}\,.$$

- M a homogeneous infinitely divisible random measure on E.
- Control measure m.
- The local 1-dimensional Lévy measure ρ.
- f is integrable with respect to M.
- ϕ preserves the measure m.

The Lévy measure of the process **X** is

$$\nu = (m \times \rho) \circ H^{-1} \, ,$$

with $H : \, E \times \mathbb{R} \to \mathbb{R}^{\mathbb{Z}}$ given by

$$H(s, x)(n) = xf \circ \phi^n(s), \;\; n \in \mathbb{Z}, \quad \text{for } s \in E, \, x \in \mathbb{R}.$$

The ergodic-theoretical properties of the map ϕ with respect to the control measure m translate into the properties of the left shift with respect to the Lévy measure ν.

A typical example of a stationary infinitely divisible process corresponding to a dissipative map is a moving average process

$$X_n = \int_{-\infty}^{\infty} f(s-n)\, M(ds), \; n \in \mathbb{Z}, \quad m = \text{Leb}.$$

Dissipative maps contribute to short memory of stationary infinitely divisible processes.

Theorem Let $\mathbf{X}$ be a finite variance stationary infinitely divisible process. Suppose that the map ϕ is dissipative, and that

$$\int_E |f(s)| \sum_{k=-\infty}^{\infty} |f| \circ \phi^k(s)\, m(ds) < \infty \,.$$

Then

$$\big(n^{-1/2} S_n(t),\ t \geq 0\big) \Rightarrow \big(\sigma_{\mathbf{X}} B(t),\ \ t \geq 0\big) \ \text{ as } n \to \infty$$

in finite-dimensional distributions, where

$$\sigma_{\mathbf{X}}^2 = \int_{-\infty}^{\infty} x^2\, \rho(dx) \left(\sum_{k=-\infty}^{\infty} \int_E f(s) f \circ \phi^k(s)\, m(ds) \right) \,.$$

- A dissipative map ϕ and a "small" kernel f lead to short memory from the point of view of partial sums if $\sigma_X \neq 0$.
- If the kernel f is not sufficiently "small", or if $\sigma_X = 0$, one can have long memory.
- In the limit one will obtain a Fractional Brownian motion with $H \neq 1/2$.
- If the map ϕ is conservative, it is hard to have short memory from the point of view of partial sums even if f is "small".

Theorem Assume that the map ϕ in is conservative, and that $f \geq 0$ m-a.e. Unless $f = 0$ m-a.e., there is no $\sigma \geq 0$ such that

$$\left(n^{-1/2} S_n(t),\ t \geq 0\right) \Rightarrow \left(\sigma B(t),\ t \geq 0\right) \text{ as } n \to \infty$$

in terms of convergence of the finite-dimensional distributions.

Example of a process with a conservative map

$(p_{ij},\ i,j \in \mathbb{Z})$: transition probabilities of an irreducible null recurrent Markov chain on $\mathbb{Z}$.

$(\pi_i,\ i \in \mathbb{Z})$: a σ-finite invariant measure.

$$m(A) = \sum_{i\in\mathbb{Z}} \pi_i P_i\big(\text{the trajectory of the Markov chain is in } A\big)$$

for a measurable A of $E = \mathbb{Z}^{\mathbb{Z}}$: a σ-finite measure on E, invariant under the left shift ϕ on E.

The kernel:

$$f(\mathbf{x}) = 1(x_0 = 0) \ \text{ for } \mathbf{x} = (\dots, x_{-1}, x_0, x_1, x_2, \dots) \in \mathbb{Z}^{\mathbb{Z}}.$$

First return time:

$$\tau_1 = \tau_1(\mathbf{x}) = \inf\left\{ n \geq 1 : \, x_n = 0 \right\}.$$

How fast the Markov chain returns determines the length of the memory of the infinitely divisible process!

Theorem Assume that $P_0(\tau_1 > n)$, $n = 1, 2, \ldots$ is regularly varying with exponent $-\beta \in (-1, 0)$. Then

$$\left(\left(\frac{P_0(\tau_1 > n)}{n} \right)^{1/2} S_n(t),\ t \geq 0 \right) \Rightarrow \big(c_\beta B_H(t),\ \ t \geq 0 \big) \text{ as } n \to \infty$$

in the Skorohod J_1 topology on $D[0, \infty)$, where $\mathbf{B}_H$ is the standard Fractional Brownian motion with $H = (1 + \beta)/2$,

Long memory with respect to partial sums when the variance is infinite

Let $\mathbf{X} = \big(X_n,\ n \in \mathbb{Z}\big)$ be i.i.d. symmetric, regularly tails, $0 < \alpha < 2$.

The quantile sequence:

$$a_n = \inf\big\{x > 0:\ P(X_0 > x) \leq 1/n\big\},\ n = 1, 2, \ldots .$$

Central Limit Theorem

$$\frac{1}{a_n} S_n \Rightarrow Y_\alpha,\ \text{symmetric } \alpha\text{-stable}.$$

Invariance Principle

$$\left(\frac{1}{a_n} S_n(t),\ t \geq 0\right) \Rightarrow \big(Y_\alpha(t),\ t \geq 0\big)$$

weakly in the J_1 topology on $D[0, \infty)$.

$\mathbf{Y}_\alpha$: a symmetric symmetric α-stable Lévy motion.

A heavy tailed infinite moving average model:

$$X_n = \sum_{j=-\infty}^{\infty} \varphi_{n-j}\, \varepsilon_j\,, \quad n = 1, 2, \ldots\,.$$

- $(\varepsilon_n,\ n \in \mathbb{Z})$: i.i.d. symmetric ,
- balanced regularly varying tails, $0 < \alpha < 2$,
- the coefficients $(\varphi_n,\ n \in \mathbb{Z})$ satisfy

$$\sum_{n=-\infty}^{\infty} |\varphi_n|^{\alpha-\varepsilon} < \infty \ \text{ for some } 0 < \varepsilon < \alpha\,.$$

Short memory with respect to partial sums

Theorem If $1 < \alpha < 2$, assume that the coefficients are absolutely summable. Then

$$\left(a_n^{-1} S_n(t),\ t \geq 0\right) \Rightarrow \left(c\, a_\varphi Y_\alpha(t),\ t \geq 0\right) \quad \text{as } n \to \infty$$

in finite dimensional distributions, $a_\varphi = \sum \varphi_i$.

The memory is short unless $a_\varphi = 0$.

Long memory with respect to partial sums

Theorem Let $1 < \alpha < 2$, and (φ_i) balanced regularly varying with $-1 < \beta < -1/\alpha$. Then,

$$\left(\frac{1}{na_n b_n} S_n(t),\ t \geq 0 \right) \Rightarrow \left(cX^{(\mathrm{L})}(t),\ \ t \geq 0 \right)$$

weakly in the Skorohod J_1 topology on $D[0, \infty)$.

$\mathbf{X}^{(\mathrm{L})}$: linear fractional symmetric α-stable stable motion, $H = 1 + 1/\alpha + \beta$.

It is a moving average process.

A heavy tailed infinitely divisible model: assume α-stable.

Again: a dissipative map ϕ tends to lead to short memory.

Theorem If $1 < \alpha < 2$, assume that

$$\int_E |f(s)| \left(\sum_{k=-\infty}^{\infty} |f| \circ \phi^k(s) \right)^{\alpha-1} m(ds) < \infty \,.$$

Then

$$\big(n^{-1/\alpha} S_n(t),\ t \geq 0\big) \Rightarrow \big(b\, Y_\alpha(t),\ t \geq 0\big) \quad \text{as } n \to \infty$$

in finite-dimensional distributions.

- Unless $b = 0$, the memory with respect to partial sums is short.
- If $1 < \alpha < 2$, only "smallness " of kernel f is required.
- There exist processes with dissipative map ϕ and long memory.
- A whole class of examples is dilated fractional stable noises of Pipiras and Taqqu.

If the map ϕ is conservative, it is hard for $\mathbf{X}$ to have short memory with respect to partial sums.

Example of a process with a conservative map

The Markov chain setup as above.

The random measure M is now symmetric α-stable.

The limit is no longer a Linear Fractional Stable motion

Theorem Assume that $P_0(\tau_1 > n)$, $n = 1, 2, \ldots$ is regularly varying with $-\beta \in (-1, 0)$. Then for (c_n) regularly varying with exponent $(1-\beta)/\alpha + \beta$,

$$\big(c_n^{-1} S_n(t),\ t \geq 0\big) \Rightarrow \big(c_{\alpha,\beta} Y_{\alpha,\beta}(t),\ t \geq 0\big) \text{ as } n \to \infty$$

in the Skorohod J_1 topology on $D[0, \infty)$, where $\mathbf{Y}_{\alpha,\beta}$ is the β-Mittag-Leffler Fractional symmetric α-stable motion.

The limit is self-similar with $H = (1-\beta)/\alpha + \beta$.

- $\big(S_\beta(t), t \geq 0\big)$ a β-stable subordinator.
- Its inverse

$$M_\beta(t) = S_\beta^{\leftarrow}(t) = \inf\big\{u \geq 0 : \ S_\beta(u) \geq t\big\}, \ t \geq 0\,,$$

  is the Mittag-Leffler process.
-

$$\nu(dx) = (1-\beta)x^{-\beta}\,dx\,,$$

  a σ-finite measure on $[0,\infty)$.

$M_{\alpha,\beta}$: a symmetric α-stable random measure on $\Omega' \times [0,\infty)$ with control measure $P' \times \nu$,

The β-Mittag-Leffler Fractional symmetric α-stable motion:

$$Y_{\alpha,\beta}(t) = \int_{\Omega' \times [0,\infty)} M_\beta\big((t-x)_+, \omega'\big)\, M_{\alpha,\beta}(d\omega',\, dx), \quad t \geq 0\,.$$

It is self-similar with $H = (1-\beta)/\alpha + \beta$.

Long range dependence with respect to partial maxima

- Long memory with respect to partial maxima differs from long memory with respect to partial sums.
- With the same marginal tails, the partial maxima grow the fastest when the sequence is i.i.d.
- The long range dependence can only mean smaller partial maxima.

In the Gaussian case partial maxima are barely affected by correlations.

Theorem Let $\mathbf{X} = (X_n,\ n \in \mathbb{Z})$ be 0 mean variance 1 stationary Gaussian process with $R_X(n) \to 0$. Then

$$\frac{1}{\sqrt{2\log n}}\, M_n \to 1 \text{ in probability as } n \to \infty.$$

An even more precise result can be obtained if $R_X(n)/\log n \to 0$.

Partial maxima for stationary symmetric α-stable processes

The nature of the map ϕ determines whether long range dependence is present or not.

- If the map ϕ is dissipative, the memory is short.
- If the map ϕ is conservative, the memory is long.

Theorem Suppose that the map ϕ is dissipative. Then

$$\big(n^{-1/\alpha} M_n(t),\ t \geq 0\big) \Rightarrow \big(c\, Y_{\Phi_\alpha}(t),\ t \geq 0\big) \quad \text{as } n \to \infty,$$

$c > 0$, in the sense of finite-dimensional distributions.

$\mathbf{Y}_{\Phi_\alpha}$: the extremal process corresponding to the standard Fréchet distribution.

Theorem Suppose that the map ϕ is conservative. Then

$$n^{-1/\alpha} \max_{i=1,\dots,n} |X_i| \to 0 \text{ in probability}$$

as $n \to \infty$.

With different normalization one can obtain different interesting limits in the functional extremal theorem.