Standardization of upper-semicontinuous processes applications in Extreme Value Theory

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Dependence, Stability and Extremes workshop.

This talk

- Standardization of stochastic processes
- Upper-semicontinuous processes
- Applications in extreme value theory (EVT)
  (this is the initial motivation)

Standardizing stochastic processes: why?

Collection of random variables $\boldsymbol{\xi} = (\xi_s)_{s \in \mathbb{D}}$, $\mathbb{D}$ finite or compact $\subset \mathbb{R}^p$.
Margins: $F_s(x) = \mathbb{P}(\xi_s \leq x)$.

Clément's talk this morning: Max-stable continuous process.

- Choose a (nice) target cdf Φ : (uniform, Fréchet(1), Pareto(1), ...)
- **Standardization map** $\mathbf{U} : \boldsymbol{\xi} \mapsto \boldsymbol{\xi}^* = \Big(\Phi^{-}(F_s(\xi_s))\Big)_{s \in \mathbb{D}}$.

$\boldsymbol{\xi}$ is max-stable

$\Longleftrightarrow$

$\boldsymbol{\xi}^*$ is simple max-stable and the margins are max-stable.

Work-flow for **model construction / statistical inference:**
Choose a model for $\boldsymbol{\xi}^*$, fit the standardized data to/simulate from it, then apply the inverse standardization map.

Under which conditions can we do that?

Finite case $\mathbb{D} = \{1, \ldots, d\}$: standardizing makes sense

$\Phi = \mathcal{U}_{[0,1]}$; $\boldsymbol{\xi}^* = \mathbf{U}(\boldsymbol{\xi}) = (F_1(\xi_1), \ldots F_d(\xi_d))$.

Sklar's theorem:

(I) For all { copula $\mathbf{C}$ + margins $(F_j, 1 \leq j \leq d)$ },
$\exists F$ a d-variate cdf with margins (F_j) and copula C.

(II) Every cdf F may be decomposed this way, *i.e.*

$$\exists \mathbf{C} : \quad \mathbf{F}(x_1, \ldots, x_d) = \mathbf{C}(F_1(x_1), \ldots, F_d(x_d)).$$

$$\underbrace{\mathcal{L}(\boldsymbol{\xi})}_{\mathbf{F}} \text{ is characterized by } \Big\{ \underbrace{\mathcal{L}(\boldsymbol{\xi}^*)}_{\mathbf{C}} + \text{margins } F_s \Big\}.$$

Continuous processes: standardizing makes sense

- $(\mathcal{C}(\mathbb{D},\mathbb{R}), \|.\|_\infty)$, continuous functions $\mathbb{D} \to \mathbb{R}$.
- "continuous process": a random continuous function, *i.e.* a measurable map $\Omega \mapsto \mathcal{C}(\mathbb{D},\mathbb{R})$.
- $\mathcal{L}(\boldsymbol{\xi})$ characterized by the fidis $\mathcal{L}(\xi_{s_1}, \ldots, \xi_{s_d})$
  $\to$ **back to the d-variate case.**

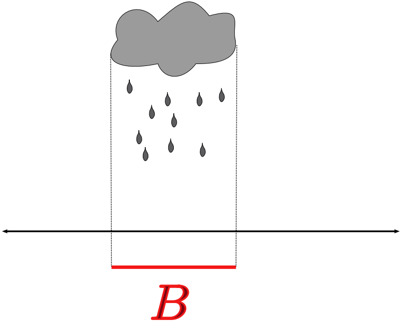
$$\underbrace{\mathcal{L}(\boldsymbol{\xi})}_{\text{fidis } F_{s_1,\ldots,s_d}(\cdot)} \quad \text{characterized by} \quad \Big(\underbrace{\mathcal{L}(\boldsymbol{\xi}^*)}_{\text{fidi copulas } \mathbf{C}_{s_1,\ldots,s_d}(\cdot)} + \text{margins } F_s \Big).$$

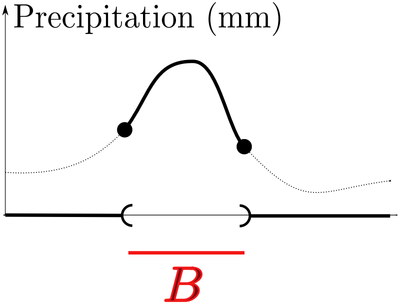
Question: Similar decomposition with semicontinuous processes?

Why care about semicontinuity? spatial statistics

Truncating rainstorms outside a random closed patch (Schlather (2002), Huser and Davison (2014))

$\longrightarrow$ long-range independence (in space) of very heavy rainstorms (difficult to achieve without truncation).

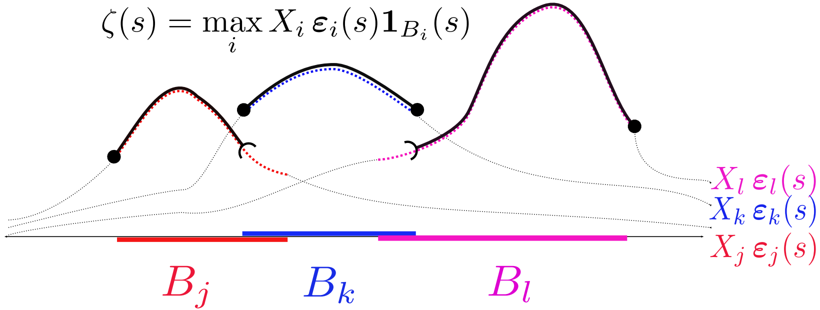




$$\text{Rain}(s) = X\,\epsilon(s)\,\mathbb{1}_B(s)$$

$\longrightarrow$ Upper semicontinuous (usc) process

(Pointwise) maximum of truncated rainstorms



$\longrightarrow$ Again, *usc* process.

Semicontinuity in EVT

Theory for **continuous** processes: well established.
Giné, Hahn and Vatan (1990), de Haan and Lin (2001), de Haan and Ferreira (2007), Einmahl and Lin (2006)

Practice some **semicontinuous** models are also used ('truncated storms', Voronoï fields) Schlather (2002), Davison and Gholamrezaee (2011), Huser and Davison (2014), Robert (2013)

Question **Does standard EVT still apply to semicontinuous processes, and how?**

Upper / Lower semicontinuous functions and processes

Roots:

- in variational analysis and random set theory:
  Choquet (47), Matheron (75), Norberg (86, 87), Salinetti and Wets (86), Rockafellar and Wets (98), Molchanov (2005), ...
- Mentioned occasionally in extreme-value analyis
  Vervaat (1981, 1988), Norberg (1987), Resnick and Roy (1991), ...

EVT for semicontinuous processes so far:

- Mostly restricted to **simple** max-stable processes
- Open problems (to our knowledge)
  Standardization, domains of attraction (asymptotics for maxima), Parallels with multivariate / continuous EVT, vague convergence (law of excesses), **Statistical inference!**

Upper semicontinuous functions

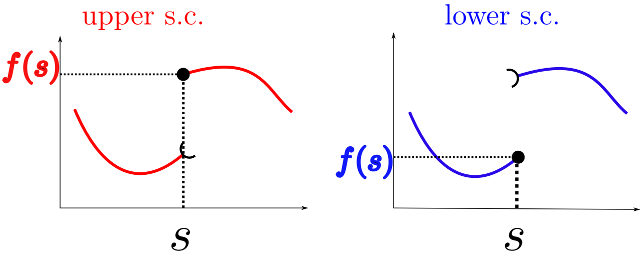
$(\mathbb{D}, d)$ a compact metric space. (Think $\mathbb{D} = [0,1]$.)

A function $f : \mathbb{D} \to \overline{\mathbb{R}}$ is **upper semicontinuous (*usc*)** if

$$\forall s \in \mathbb{D}, \quad f(s) = \lim_{\varepsilon \to 0} \sup_{t : d(s,t) \leq \varepsilon} f(t).$$

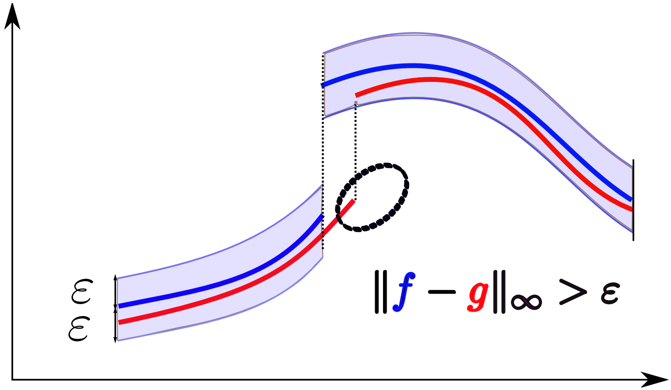
this is equivalent to

$$\forall y \in \mathbb{R}, \quad A = \{s : f(s) \geq y\} \text{ is closed.}$$



$\mathrm{USC}(\mathbb{D}) = \{f : \mathbb{D} \to [-\infty, +\infty] : \ f \text{ is upper semicontinuous}\}$

Semicontinuous functions: uniform topology inadequate



Locations of discontinuities don't match exactly: no proximity

Try hypo-topology!

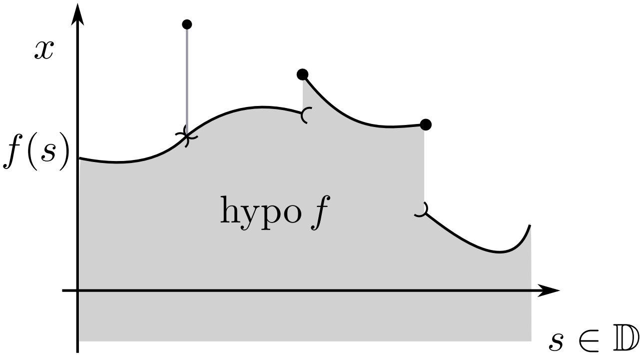
Key: identify a function with its hypograph

The **hypograph** of $f : \mathbb{D} \to \overline{\mathbb{R}}$ is a subset of $\mathbb{D} \times \mathbb{R}$:

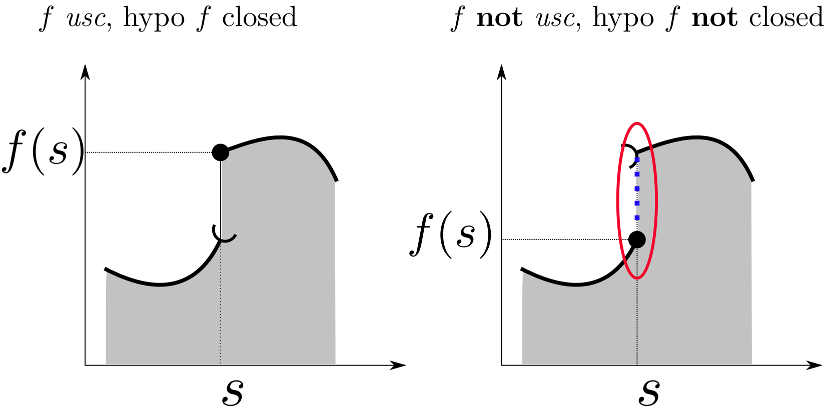
$$\operatorname{hypo} f = \{(s, x) \in \mathbb{D} \times \mathbb{R} : x \leq f(s)\}$$

Clearly, a function can be reconstructed from its hypograph:

$$f(s) = \sup\{x \in \mathbb{R} : (s, x) \in \operatorname{hypo} f\}$$



Upper semicontinuous function $\iff$ closed hypograph



f is upper semicontinuous $\iff$ hypo f is closed in $\mathbb{D} \times \mathbb{R}$

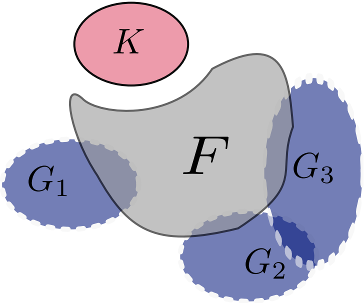
$\mathrm{USC}(\mathbb{D}) \sim \mathrm{HYPO}(\mathbb{D})$: family of closed hypographs $\subset \mathcal{F}$.

Fell topology on the family of closed sets

Painlevé/Kuratowski/Fell topology on $\mathcal{F} = \mathcal{F}(\mathbb{D} \times \mathbb{R})$

$$\text{subbase: } \left\{ \begin{array}{ll} \mathcal{F}_G = \{F \in \mathcal{F} : F \cap G \neq \emptyset\}, & G \text{ open}, \\ \mathcal{F}^K = \{F \in \mathcal{F} : F \cap K = \emptyset\}, & K \text{ compact} \end{array} \right\}$$

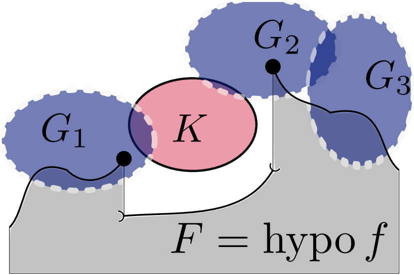
Base for the Fell topology: $\{\mathcal{F}^K_{G_1,\ldots,G_n} = \mathcal{F}^K_{G_1} \cap \ldots \cap \mathcal{F}^K_{G_n}\}$.



$F \in \mathcal{F}^K_{G_1,G_2,G_3}$.

Hypo-topology

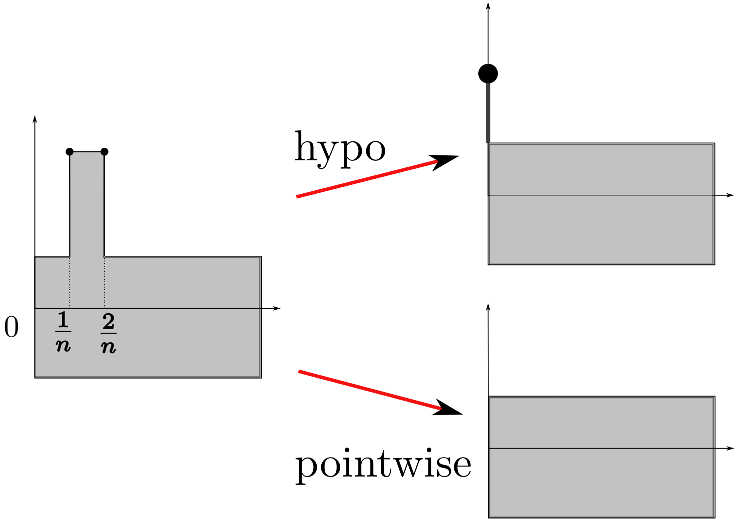
Topology on USC($\mathbb{D}$):**trace** of Fell's topology onto HYPO($\mathbb{D}$).
Open sets $= \{U \cap \mathrm{HYPO}(\mathbb{D}), U \in \mathcal{F}\}$



$$F \in \mathcal{F}^K_{G_1,G_2,G_3} \cap \mathrm{HYPO}(\mathbb{D}).$$

N.B. : $(\mathrm{USC}(\mathbb{D}), \mathrm{HYPO}(\mathbb{D}))$ is compact, metric!

Hypo and pointwise convergence are different

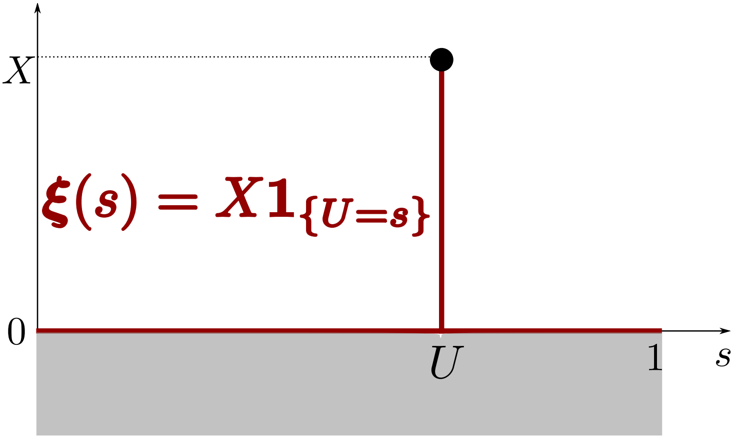


Upper semicontinuous process = random *usc* function

By definition, an *usc* process is a random element in $\mathrm{USC}(\mathbb{D})$, *i.e.* a map

$$\xi : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathrm{USC}(\mathbb{D}), \mathrm{HYPO}(\mathbb{D})).$$

The law of an *usc* process is *not* determined by its fidis



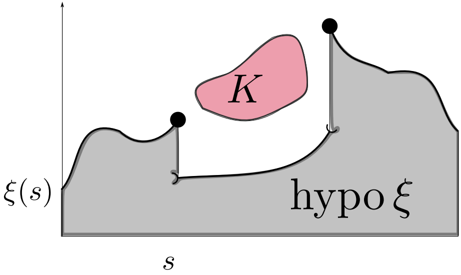
$\mathbb{D} = [0,1], \quad U \sim \text{Uniform}(0,1), \quad X$ any random variable $\Omega \to \mathbb{R}^+$

$\forall (s_1, \ldots, s_k) \in [0,1] : \xi(s_1) = \cdots = \xi(s_k) = 0$ a.s., although $\xi \neq 0$

The law of an *usc* process: determined by the capacity functional

Capacity functional of a random closed set F:

$$T_F(K) = \mathbb{P}\left(F \cap K \neq \varnothing\right), \quad K \text{ compact.}$$



For an *usc* process:

$$1 - T_\xi(K) = \mathbb{P}(\text{hypo}\,\xi \cap K = \varnothing), \qquad K \subset \mathbb{D} \times \mathbb{R} \text{ compact}$$

Max-stable processes

Definition: *usc* max-stable process

An *usc* process $\boldsymbol{\xi}$ with non-degenerate margins is **max-stable** if $\forall n$ there exist functions $\alpha_n > 0$ and β_n such that, for $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n \overset{iid}{\sim} \boldsymbol{\xi}$,

$$\bigvee_{i=1}^{n} \xi_i \overset{d}{=} a_n \xi + b_n$$

- The margins $(G_s)_{s \in \mathbb{D}}$ are necessarily max-stable (one-to-one $(x_G^-, x_G^+) \to (0,1)$).

- Definition implicitly assumes that the r.h.s. is a *usc* process.

- **Simple max-stable**: Fréchet(1) margins $\Phi(x) = \mathbf{1}_{x>0}\ e^{-1/x}$, then $\alpha_n(s) = n$, $\beta_n(s) = 0$;

Can we 'reduce' to the simple max-stable case?

Sklar's theorems for max-stable processes?

Questions

Sklar I Given a simple max-stable process $\boldsymbol{\xi}^*$, and max-stable margins G_s, $s \in \mathbb{D}$, is the stochastic process

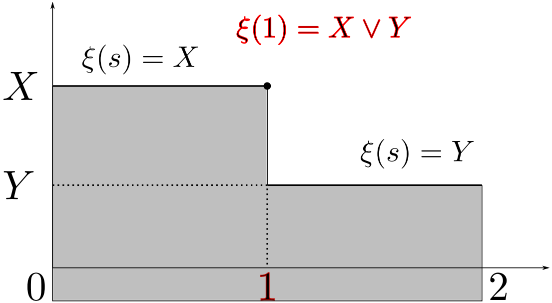
$$\left\{\xi_s = G_s^-(\Phi(\xi_s^*))\right\}_{s\in\mathbb{D}}$$

a (max-stable) *usc* process?

Sklar II Given a max-stable process $\boldsymbol{\xi}$, $\exists$**?** a simple max-stable *usc* process $\boldsymbol{\xi}^*$ such that

$$\boldsymbol{\xi} \stackrel{d}{=} \left\{G_s^- \circ \Phi(\boldsymbol{\xi}_s^*)\right\}_{s\in\mathbb{D}} \quad \text{in } \mathrm{USC}(\mathbb{D}) \quad ?$$

Standardization to simple max-stable processes: not always possible



$$X, Y \overset{i.i.d.}{\sim} \Phi \text{ (Fréchet(1))}$$

Standardization to Fréchet(1) requires halving $\xi(1)$: no longer *usc*

An admissible class of transformations for *usc* processes

- $\boldsymbol{\mathcal{U}}$ = family of functions $U : \mathbb{D} \times [-\infty, +\infty] \to [-\infty, +\infty]$ s.t.
  (a) For every s, $x \mapsto U(s, x)$ is non-decreasing, right-continuous.
  (b) For every x, $s \mapsto U(s, x)$ is *usc*
  (think $U(s, x) = F(s, x)$ for now)

- For $U \in \boldsymbol{\mathcal{U}}$, define the mapping

$$U^* : z \in \mathrm{USC}(\mathbb{D}) \mapsto U^*(z) := \{U(s, z(s)\}_{s \in \mathbb{D}}.$$

- Let $\boldsymbol{\mathcal{U}}^* = \{U^* : \quad U \in \boldsymbol{\mathcal{U}}\}$.

Proposition: usc preserving transformations

Every $U^* \in \boldsymbol{\mathcal{U}}^*$ is a hypo-measurable mapping from $\mathrm{USC}(\mathbb{D})$ to itself.

Lemma: Composition

If $U, V \in \boldsymbol{\mathcal{U}}$, then $U \circ V : (s, x) \mapsto U(s, V(s, x))$ belongs to $\boldsymbol{\mathcal{U}}$

Examples of usc-preserving maps

- Fix $y \in \mathrm{USC}(\mathbb{D})$. Then
  $U_\vee : (s,x) \mapsto x \vee y(s)$ and $U_\wedge : (s,x) \mapsto x \wedge y(s)$ belong to $\boldsymbol{\mathcal{U}}$.
  Associated maps: $U_\vee^* : z \mapsto z \vee y$ and $U_\wedge^* : z \mapsto z \wedge y$.
- If $a : \mathbb{D} \to (0,\infty)$ is continuous, then $U_{a^*} : (s,x) \mapsto a(s)\,x \ \in \boldsymbol{\mathcal{U}}$.
  If $b : \mathbb{D} \to \mathbb{R}$ is usc, then $U_{b^+}(s,x) \mapsto x + b(x) \in \boldsymbol{\mathcal{U}}$.
  Associated maps: $U_{a^*}^* : z \mapsto az$ and $U_{b^+}^* : z \mapsto z + b$.

Lemma (the right-continuous inverse is usc-preserving)

$\boldsymbol{\xi} = (\xi(s) : s \in \mathbb{D})$ an *usc* process, $\xi(s) \in [-\infty,\infty]$, $F_s(x) = \mathbb{P}\left(\xi(s) \le x\right)$.
Define

$$F_s^{\rightarrow}(p) = \sup\{y \in \mathbb{R} : F_s(y) \le p\}, \qquad (s,p) \in \mathbb{D} \times [0,1].$$

Then $U_\xi : (s,x) \mapsto F_s^{\rightarrow}((x \vee 0) \wedge 1)$ belongs to $\boldsymbol{\mathcal{U}}$

Building a non-standard usc process from a standard one

Necessary and sufficient condition: on marginal distribution functions.

- $\mathbf{Z}$: an *usc* process with standard uniform margins.
- $(F_s)_{s\in\mathbb{D}}$: a family of cdf's, right-continuous inverses $F_s^{\rightarrow}(\,\cdot\,)$.
- Define a stochastic process $\boldsymbol{\xi} : \xi(s) = F_s^{\rightarrow}((Z(s) \vee 0) \wedge 1)$

Proposition (Sklar I for usc processes)

The following are equivalent:

1. $\boldsymbol{\xi}$ is an *usc* process (with margins F_s).
2. For every $p \in [0,1]$, the function $s \mapsto F_s^{\rightarrow}(p)$ is usc.

Building a max-stable $\boldsymbol{\xi}$ from a simple max-stable $\boldsymbol{\xi}^*$

- Let $\boldsymbol{\xi}^*$ be a simple max-stable *usc* process, Fréchet(1) margins Φ.
- Let $G_s(\cdot), s \in \mathbb{D}$ be GEV distributions, $G_s^{\rightarrow}$: right-inverse.
- Define a stochastic process $\boldsymbol{\xi}$: $\quad \xi_s = G_s^{\rightarrow}(\Phi(\xi_s^*)), s \in \mathbb{D}$.

Proposition (à la Sklar I for max-stable usc processes)

The following are equivalent

1. $\boldsymbol{\xi}$ is an *usc* process (with margins G_s).
2. $\forall p \in [0,1]$, the function $s \mapsto G_s^{\rightarrow}(p)$ is *usc*.

In such a case $\boldsymbol{\xi}$ is max-stable with norming functions a_n, b_n determined by the margins G_s.

Sklar II: Can every $\boldsymbol{\xi}$ be represented this way ?

Not clear without additional assumptions on the margins

$\boldsymbol{\xi}$: an *usc* process, margins F_s, right inverses $F_s^{\rightarrow}$.

à la Sklar II for usc processes

Suppose the following two conditions hold:

(a) For every $s \in \mathbb{D}$, $F_s(\,\cdot\,)$ has no atoms in $[-\infty, \infty]$.

(b) For every $x \in \mathbb{R} \cup \{+\infty\}$, the function $s \mapsto F_s(x)$ is *usc*.

Then, indeed:

(i) $\mathbf{Z} : Z(s) = F_s(\xi(s))$ is an *usc* process with uniform margins.

(ii) $\tilde{\boldsymbol{\xi}} : \tilde{\xi}(s) = F_s^{\rightarrow}(Z(s))$ is an *usc* process, and

$$\forall s \in \mathbb{D}, \text{ almost surely }, \tilde{\xi}(s) = \xi(s).$$

In particular, $\tilde{\xi}$ and ξ have identical fidis.

! same fidis $\neq$ same distribution in $\mathrm{USC}(\mathbb{D})$!

Regurity properties of a usc process with GEV margins

$\boldsymbol{\xi}$: a *usc* process with GEV margins $G_s(\cdot) = G(\cdot, \theta(s))$,
$\theta(s) = (\mu(s), \sigma(s), \gamma)$.

Question: Are the sufficient conditions for standardization met ?

Answer:

(a) For every $s \in \mathbb{D}$, $G_s(\cdot)$ has no atoms in $[-\infty, \infty]$.

(b) For every $x \in \mathbb{R} \cup \{+\infty\}$,

$$\text{The function } s \mapsto G_s(x) \text{ is } \textit{usc}$$
$$\Longleftrightarrow$$
$$\boldsymbol{\theta} \textbf{ is continuous}.$$

Standardizing a max-stable *usc* process

$\boldsymbol{\xi}$: an *usc* process with GEV margins $G_s(\cdot) = G(\cdot, \theta(s))$.

à la Sklar II for usc processes with GEV margins

If the GEV parameter $\theta : \mathbb{D} \to \Theta$ is continuous, then:

(a) $\boldsymbol{\xi}^* : \xi^*(s) = \frac{-1}{\log G_s(\xi(s))}$ is an *usc* process with Fréchet(1) margins.

(b) $\tilde{\boldsymbol{\xi}} : \tilde{\xi}(s) = G_s^{\rightarrow}(\Phi(\xi^*(s)))$ is an *usc* process and, with proba. 1,

$$\forall s \in \mathbb{D}, \quad \tilde{\xi}(s) = \begin{cases} \xi(s) & \text{if } G_s(\xi(s)) < 1, \\ \infty & \text{if } G_s(\xi(s)) = 1. \end{cases}$$

If in addition, $\sup_{\mathbf{s}\in\mathbb{D}} \mathbf{G_s}(\xi(\mathbf{s})) < \mathbf{1}$ **(a.s.)**, then:

with probability 1, $\boldsymbol{\xi} = \tilde{\boldsymbol{\xi}}$ and the following are equivalent:

(i) The *usc* process $\boldsymbol{\xi}$ is max-stable.

(ii) The *usc* process $\boldsymbol{\xi}^*$ is simple max-stable.

Conclusion

- **Hypo-topology is well-adapted** to extremes of *usc* processes.

- Standardization of ***usc*** processes with GEV margins is possible if the **marginal *c.d.f.*'s are continuous w.r.t. space** variable
- Max-stability is preserved if the **upper end-point is almost never reached anywhere**.

First step towards statistically grounded modeling within classical EVT framework

Further topics:

Standardization in the **max-domain of attraction** is possible too, and the **limit is max-stable** under mild conditions (in progress)

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