Isomorphism identities for infinitely divisible processes with some applications

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Outline

Recall the Cameron-Martin Formula: Given a centered Gaussian process $G = (G_t)_{t \in T}$ over an arbitrary set T and a random variable ξ in L^2_G, the L^2-closure of the subspace spanned by G, we have for any measurable functional $F : \mathbb{R}^T \mapsto \mathbb{R}$

$$\mathbb{E}\left[F\left((G_t + \phi(t))_{t \in T}\right)\right] = \mathbb{E}\left[F\left((G_t)_{t \in T}\right) e^{\xi - \frac{1}{2}\mathbb{E}\xi^2}\right] \tag{1}$$

where $\phi(t) = \mathbb{E}(\xi G_t)$.

This formula has many applications, including SDEs and SPDEs driven by Gaussian random fields.

The C-M formula can also be viewed an isomorphism identity expressing a translated Gaussian process in terms of the untranslated process, but the latter is under the changed probability measure.

The set of all translation functions

$$\mathcal{H}_G = \{\phi : T \to \mathbb{R} : \phi(t) = \mathbb{E}(\xi G_t) \text{ for some } \xi \in L^2_G\}$$

forms a Hilbert space, called the Cameron-Martin space (or the reproducing kernel Hilbert space).

It is well-known that (1) does not extend to the Poissonian case.

Indeed, it is easy to see that if $Y = (Y_t)_{t\in[0,1]}$ is a Poisson process, then there is no function $v : [0,1] \to \mathbb{R}$, $v \not\equiv 0$ such that

$$\mathbb{E}\left[F\left((Y_t + v(t))_{t\in[0,1]}\right)\right] = \mathbb{E}\left[F\left((Y_t)_{t\in[0,1]}\right)Z\right]$$

for all functionals F and some random variable $Z \geq 0$ with $\mathbb{E}Z = 1$.

We propose isomorphism identities based on random translations instead.

We will say that a stochastic process is Poissonian infinitely divisible is its finite dimensional distributions are infinitely divisible without Gaussian part.

Any infinitely divisible process $X = (X_t)_{t \in T}$ can be written as

$$X \stackrel{d}{=} G + Y$$

where $G = (G_t)_{t \in T}$ and $Y = (Y_t)_{t \in T}$ are independent processes, G is centered Gaussian and Y is Poissonian infinitely divisible.

Isomorphism identities for infinitely divisible processes come from combining the corresponding identities for Gaussian and Poissonian infinitely divisible processes.

What kind of functionals can be of interest? A few examples:

- $F\left((Y_t)_{t\in T}\right) = f(Y_{t_1},\ldots,Y_{t_n})$ cylindrical functional;
- $F\left((Y_t)_{t\in T}\right) = \sup_{t\in T} Y_t$ extremum;
- $F\left((Y_t)_{t\in T}\right) = \int_T |Y_t|^p \,\mu(dt)$ path integral;
- $F\left((Y_t)_{t\in[0,u]}\right) = \int_0^u \delta_y(Y_t)\,dt$ local time;
- $F\left((Y_t)_{t\in T}\right) = \int_0^\infty e^{-\eta_{t-}}\,d\xi_t$ exponential functional, where (η_t,ξ_t), $t\geq 0$ is a Lévy process, $T = R_1\cup R_2$ the union of two disjoint copies of $\mathbb{R}_+$ and $Y_t = \eta_t$ if $t\in R_1$, $Y_t = \xi_t$ if $t\in R_2$.

Let $Y = (Y_t)_{t \in T}$ be a Poissonian infinitely divisible process over a general set T. Let ν be the Lévy measure of Y on the path space $\mathbb{R}^T$ and assume that ν is σ-finite.

Let $V = (V_t)_{t \in T}$ be an arbitrary process, which is independent of the process Y, and whose distribution on $\mathbb{R}^T$ is absolutely continuous with respect to ν.

We will show that for any measurable functional $F : \mathbb{R}^T \mapsto \mathbb{R}$,

$$\mathbb{E}\left[F\left((Y_t + V_t)_{t \in T}\right)\right] = \mathbb{E}\left[F\left((Y_t)_{t \in T}\right) Z\right], \tag{2}$$

for some random variable $Z \geq 0$, $\mathbb{E}Z = 1$, such that $((Y_t, Z)_t)_{t \in T}$ is Poissonian infinitely divisible.

Returning to our example of Poisson process $(Y_t)_{t\in[0,1]}$, if we take

$$V_t = \mathbf{1}_{[0,t]}(\eta)$$

where $\eta \in [0,1]$ is a random variable with absolutely continuous density f_η and independent of Y then (2) holds:

$$\mathbb{E}\left[F\left((Y_t + \mathbf{1}_{[0,t]}(\eta))_{t\in[0,1]}\right)\right] = \mathbb{E}\left[F\left((Y_t)_{t\in[0,1]}\right) Z\right]$$

with $Z = \lambda^{-1}\int_0^1 f_\eta(t)\, dY_t$ and λ being the rate of Y.

2. Lévy measures on path spaces

Notation: Path space $\mathbb{R}^T = \{x : T \to \mathbb{R}\}$; $\mathcal{B}^T$ the cylindrical (product) σ-algebra of $\mathbb{R}^T$; 0_T the origin of $\mathbb{R}^T$.

Definition

A measure ν on $(\mathbb{R}^T, \mathcal{B}^T)$ is said to be a Lévy measure if

(L1) for each $t \in T$

$$\int_{\mathbb{R}^T} |x(t)|^2 \wedge 1 \, \nu(dx) < \infty,$$

(L2) for every $A \in \mathcal{B}^T$

$$\nu(A) = \nu_*(A \setminus 0_T),$$

where ν_* denotes the inner measure.

Remark

(A) When T is countable, then $0_T \in \mathcal{B}^T$ and (L2) is equivalent to $\nu(0_T) = 0$. This is the usual condition guaranteeing the uniqueness of a Lévy measure.

When T is uncountable, then $0_T \notin \mathcal{B}^T$ and so $\nu(0_T)$ is undefined. Condition (L2) plays the role of $\nu(0_T) = 0$ in this case.

Theorem

Let $Y = (Y_t)_{t \in T}$ be a Poissonian infinitely divisible process. Then there exist unique Lévy measure ν on $(\mathbb{R}^T, \mathcal{B}^T)$ and $b \in \mathbb{R}^T$ such that for every finite set $I \subset T$ and $a \in \mathbb{R}^I$

$$\mathbb{E} \exp i \sum_{t \in I} a_t Y_t \tag{3}$$

$$= \exp \left\{ \int_{\mathbb{R}^T} (e^{\langle a, x_I \rangle} - 1 - i \langle a, [\![x_I]\!] \rangle) \, \nu(dx) + i \langle a, b_I \rangle \right\}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^I$ and $[\![\cdot]\!]$ denotes a truncation function.

Conversely, given a Lévy measure ν on $(\mathbb{R}^T, \mathcal{B}^T)$ and $b \in \mathbb{R}^T$ there exists a unique in distribution process $Y = (Y_t)_{t \in T}$ satisfying (3).

Lévy measures for large Lévy systems can be not σ-finite.
Hence we have the following:

Theorem

Measure ν on $(\mathbb{R}^T, \mathcal{B}^T)$ is a σ-finite Lévy measure if and only if one of the following equivalent conditions holds:

(i) *ν satisfies (L1) and $\nu^*(0_T) = 0$, where ν^* is the outer measure;*

(ii) *ν satisfies (L1) and there exists a countable set $T_0 \subset T$ such that*

$$\nu\{x \in \mathbb{R}^T : x_{T_0} \equiv 0\} = 0 \,.$$

Theorem

Let $Y = (Y_t)_{t \in T}$ be a Poissonian process with a σ-finite Lévy measure ν and let $V = (V_t)_{t \in T}$ be an arbitrary process such that $\mathcal{L}(V) \preceq \nu$. Let N be a Poisson random measure on $(\mathbb{R}^T, \mathcal{B}^T)$ with intensity ν and independent of V. Then for some $b \in \mathbb{R}^T$

$$\bar{Y}_t := \int_{\mathbb{R}^T} x(t) \left[N(dx) - \mathbf{1}_{\{|x(t)| \le 1\}} \nu(dx) \right] + b_t, \quad t \in T,$$

is a version of the process Y. Put

$$N(q) := \int_{\mathbb{R}^T} q(x)\, N(dx),$$

where $q(x) = \frac{d\mathcal{L}(Y)}{d\nu}(x)$, $x \in \mathbb{R}^T$.

Theorem (continue)

Then for any measurable functional $F : \mathbb{R}^T \mapsto \mathbb{R}$

$$\mathbb{E}F\left((Y_t + V_t)_{t\in T}\right) = \mathbb{E}\left[F\left(\left(\bar{Y}_t\right)_{t\in T}\right) N(q)\right] .$$

Conversely, for any F *as above,*

$$\mathbb{E}\left[F\left(\left(\bar{Y}_t\right)_{t\in T}\right); N(q) > 0\right]$$
$$= \mathbb{E}\left[F\left(\left(\bar{Y}_t + V_t\right)_{t\in T}\right)(N(q) + q(V))^{-1}; q(V) > 0\right] .$$

Furthermore, if $\nu\{x \in \mathbb{R}^T : q(x) > 0\} = \infty$, *then*

$$\mathbb{E}\left[F\left(\left(\bar{Y}_t\right)_{t\in T}\right)\right] = \mathbb{E}\left[F\left(\left(\bar{Y}_t + V_t\right)_{t\in T}\right)(N(q) + q(V))^{-1}\right] .$$

Theorem

Let $Y = (Y_t)_{t \in T}$ *be a Poissonian process having the spectral representation*

$$Y_t = \int_S f_t(s)\Big[N(ds) - \mathbf{1}_{\{|x(t)|\le 1\}} n(ds)\Big] + b_t\,,$$

where N *is a Poisson random measure on a* σ*-finite measure space* $(S, \mathcal{S}, n)$*. Let* $q : S \mapsto \mathbb{R}_+$ *be such that* $\int_S q(s)\, n(ds) = 1$*. Then for any measurable functional* $F : \mathbb{R}^T \mapsto \mathbb{R}$

$$\mathbb{E}\int_S F\left((Y_t + f_t(s))_{t\in T}\right)\, q(s)\, n(ds) = \mathbb{E}[F\left((Y_t)_{t\in T}\, N(q)\right)]\,,$$

where

$$N(q) = \int_S q(s)\, N(ds)\,.$$

Theorem (Continue)

Conversely, for any F as above,

$$\mathbb{E}[F((Y_t)_{t\in T})\,;N(q)>0] \tag{4}$$
$$= \int_{\{q(s)>0\}} \mathbb{E}\left[F\left((Y_t+f_t(s))_{t\in T}\right)\left(N(q)+q(s)\right)^{-1}\right] q(s)\,n(ds)\,.$$

If $n\{s : q(s) > 0\} = \infty$*, then*

$$\mathbb{E}[F((X_t)_{t\in T})]$$
$$= \int_S \mathbb{E}\Big[F((X_t+f_t(s))_{t\in T})(N(q)+q(s))^{-1}\Big]\, q(s)\,n(ds)\,.$$

Trying to understand the following isomorphism theorem inspired the present study:

Example (Dynkin isomorphism theorem)

Let $X = \{X_t\}_{t\geq 0}$ be a strongly symmetric transient Markov process in a state space E and a Green function $g(x,y)$.

Since g is positive definite, there is a centered Gaussian process $G = \{G_x\}_{x\in E}$ with a covariance $g(x,y) = \mathrm{Cov}(G_x, G_y)$, whose squared process $G^2 = \{G_x^2\}_{x\in E}$ is infinitely divisible.

Let $\{L_t^x : x \in E, t \geq 0\}$ be local time of X normalized to satisfy $\mathbb{E}_x(L_\infty^y) = g(x,y)$. Fix $a \in E$ with $g(a,a) > 0$ and let $\tilde{P}_a$ will be appropriately changed P_a. (Under $\tilde{P}_a$, the process X starts at a and is killed at its last visit to a.)

Example (Dynkin isomorphism theorem (cont.))

Following some calculations of Eisenbaum and Kaspi (2009) we infer that the law $\mathcal{L}(V)$ on $\mathbb{R}_+^E$ of the process $V_x := L_\infty^x$, $x \in E$ under $\tilde{P}_a$ is absolutely continuous with respect to the Lévy measure ν of $Y_x := \frac{1}{2} G_x^2$, $x \in E$ and the Radon-Nikodym derivative

$$q(y) = \frac{d\mathcal{L}(V)}{d\nu}(y) = \frac{2y(a)}{g(a,a)}, \quad y \in \mathbb{R}_+^E$$

so that

$$q(N) = \int_{\mathbb{R}_+^E} \frac{2y(a)}{g(a,a)} N(dy) = \frac{2}{g(a,a)} \frac{G_a^2}{2} = \frac{G_a^2}{g(a,a)}$$

Example (Dynkin isomorphism theorem (cont.))

Taking processes V and Y independent from each other, we get

$$\mathbb{E}\left[F\left((Y_t + V_t)_{t\in T}\right)\right] = \mathbb{E}\left[F\left(\left(\bar{Y}_t\right)_{t\in T}\right)N(q)\right].$$

or

$$\mathbb{E}\otimes\tilde{\mathbb{E}}_a\left[F\left(\left(\frac{1}{2}G_x^2 + L_\infty^x\right)_{x\in E}\right)\right] = \mathbb{E}\left[F\left(\left(\frac{1}{2}G_x^2\right)_{x\in E}\frac{G_a^2}{g(a,a)}\right)\right].$$

for any measurable mapping $F : \mathbb{R}^E \mapsto \mathbb{R}$.

Example (Lévy processes)

Let $Y = \{Y_t\}_{t\in[0,1]}$ be a Lévy process such that $\mathbb{E}e^{iuX_t} = e^{tC(u)}$, where

$$C(u) = \int_{\mathbb{R}} (e^{iux} - 1 - iu[\![x]\!])\,\rho(dx) + iuc.$$

Let $q_0 : \mathbb{R} \mapsto [0,\infty]$ be a measurable function such that $\int_{\mathbb{R}} q_0(v)\,\rho(dv) = 1$. Fix $h > 0$. By Lévy-Itô

$$X_t = \int_{[0,h]\times\mathbb{R}} \mathbf{1}_{[0,t]}(r)v\,(N(dr,dv) - \chi(v)dr\rho(dv)) + ct,\ \ t \in [0,h].$$

Thus $S = [0,h] \times \mathbb{R}$ and $n = Leb \otimes \rho$. Let $q(r,v) = h^{-1}q_0(v)$, so that $\int_{[0,h]\times\mathbb{R}} q\,dn = 1$. We have

$$N(q) = \int_{[0,h]\times\mathbb{R}} q(r,v)\,N(dr,dv) = h^{-1}\int_{[0,h]\times\mathbb{R}} q_0(v)\,N(dr,dv)$$
$$= \sum_{r\le h,\,\Delta X_r\neq 0} h^{-1}q_0(\Delta X_r) := h^{-1}Z_h.$$

Example (Lévy processes, continue)

Z_h, $h \geq 0$ is a subordinator.
We got that for any measurable functional $F : \mathbb{R}^{[0,h]} \mapsto \mathbb{R}$

$$\mathbb{E}\int_{[0,h]\times\mathbb{R}} F\left(\left(X_t + \mathbf{1}_{[0,t]}(r)v\right)_{t\in[0,h]}\right) q_0(v)\, dr\rho(dv)$$
$$= \mathbb{E}[F\left((X_t)_{t\in[0,h]}\right) Z_h],$$

and from (4)

$$\mathbb{E}[F\left((X_t)_{t\in[0,h]}\right); Z_h > 0] \tag{5}$$
$$= \int_0^h \int_{\{q_0(v)>0\}} \mathbb{E}[F\left(\left(X_t + \mathbf{1}_{[0,t]}(r)v\right)_{t\in[0,h]}\right) \frac{q_0(v)}{Z_h + q_0(v)}]\, dr\rho(dv).$$

Let's reinstate a generic form of the isomorphism:

$$\mathbb{E}\left[F\left((Y_t + V_t)_{t\in T}\right)\right] = \mathbb{E}\left[F\left((Y_t)_{t\in T}\right) Z\right], \qquad (6)$$

where $Y \perp\!\!\!\perp V$, $Y = (Y_t)_{t\in T}$ is a Poissonian infinitely divisible process with a σ-finite Lévy measure xn and $V = (V_t)_{t\in T}$ is an arbitrary process such that $\mathcal{L}(V) \preceq \nu$.

There are two basic directions of applying (6). The first one is to start with a process $V = (V_t)_{t\in T}$ of interest, associate with it possibly easier to handle infinitely divisible process $Y = (Y_t)_{t\in T}$ as above, and transfer properties of Y to V via isomorphism (6). This is the direction of applying of Dynkin's isomorphism, successfully followed by Marcus and Rosen.

The second direction is to derive information about Y utilizing V. As a toy example, we consider a Lévy process.

Example

Let $Y = (Y_t)_{t \geq 0}$ be a Poissonian Lévy process with Lévy measure ρ. Then for any continuous ρ-a.e. function $f : \mathbb{R} \mapsto \mathbb{R}$ satisfying $|f(x)| \leq C \min\{x^2, 1\}$ for some $C > 0$,

$$\lim_{h \to 0} h^{-1} \mathbb{E} f(Y_h) = \int_{\mathbb{R}} f(v)\, \rho(dv)\,. \tag{7}$$

If Y is a subordinator, then (7) holds for any right-continuous ρ-a.e. $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$.

Proof: Given $h > 0$, let $F : \mathbb{R}^{[0,h]} \mapsto \mathbb{R}$ be defined by $F(x) = f(x(h))$. By (5)

$$\begin{aligned} & h^{-1} \mathbb{E}[f(Y_h); Z_h > 0] \\ & = \int_{\{q_0(v) > 0\}} \mathbb{E}\Big[f\left(Y_h + v\right) \frac{q_0(v)}{Z_h + q_0(v)} \Big]\, \rho(dv)\,. \end{aligned}$$

Example (Continue)

Let $q_0 = \lambda^{-1}\mathbf{1}_{\{|v|>\delta\}}$, where $\lambda = \rho\{|v| > \delta\} > 0$ and $\delta \in (0,1)$ is fixed. We have

$$h^{-1}\mathbb{E}[f(Y_h)] = h^{-1}\mathbb{E}[f(Y_h); Z_h = 0] + \int_{\{|v|>\delta\}} \mathbb{E}\Big[f\,(Y_h + v)\,\frac{\lambda^{-1}}{Z_h + \lambda^{-1}}\Big]\,\rho(dv)$$

The last term converges to $\int_{\{|v|>\delta\}} f(v)\,\rho(dv)$ as $h \downarrow 0$ and the middle one can be made small when δ is small. $\square$

The following result shows that many "nice" path properties of Y can be transferred to Y.

Theorem

Let $Y = \{Y_t\}_{t \in T}$ be a Poissonian process with a σ-finite Lévy measure ν. Assume that paths of Y lie in a set U that is Borel for the σ-algebra $\mathcal{U} = \mathcal{B}^T \cap U$ and such that U is a subgroup of $\mathbb{R}^T$ under addition.
If $V = \{V_t\}_{t \in T}$ is a process whose distribution is absolutely continuous with respect to ν, then V has a version with paths in the set U.

Thank you!