Multivariate Regular Variation of In- and Out-Degree in a Network Growth Model

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1. Growing preferential attachment networks

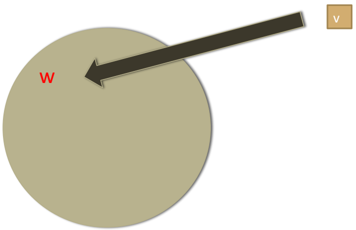
See Bollobás, Borgs, Chayes, and Riordan (2003) and Krapivsky and Redner (2001).

1.1. Model description

- Model parameters: $\alpha, \beta, \gamma, \delta_{\text{in}}, \delta_{\text{out}}$ with $\alpha + \beta + \gamma = 1$.
- $G(n)$ is a directed random graph with n edges, $N(n)$ nodes.
- Set of nodes of $G(n)$ is V_n; so $|V_n| = N(n)$.
- Set of edges of $G(n)$ is $E_n = \{(u, v) \in V_n \times V_n : (u, v) \in E_n\}$.
- In-degree of v is $D_{\text{in}}(v)$; out-degree of v is D_{out}. Dependence on n is suppressed.
- Obtain graph $G(n)$ from $G(n-1)$ in a Markovian way as follows:

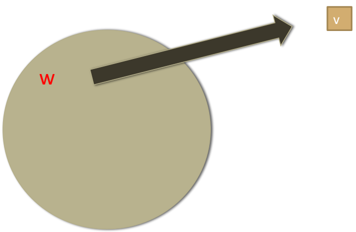
1. With probability α, append to $G(n-1)$ a new node $v \notin V_{n-1}$ and create directed edge $v \mapsto w \in V_{n-1}$ with probability

$$\frac{D_{\text{in}}(w) + \delta_{\text{in}}}{n - 1 + \delta_{\text{in}} N(n-1)}.$$



2. With probability γ, append to $G(n-1)$ a new node $v \notin V_{n-1}$ and create directed edge $w \in V_{n-1} \mapsto v \notin V_{n-1}$ with probability

$$\frac{D_{\text{out}}(w) + \delta_{\text{out}}}{n - 1 + \delta_{\text{out}} N(n-1)}.$$

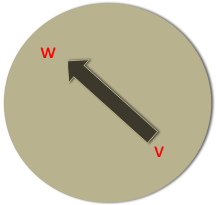


3. With probability β, create new directed edge between existing nodes

$$v \in V_{n-1} \mapsto w \in V_{n-1}$$

with probability

$$\Big(\frac{D_{\text{out}}(v) + \delta_{\text{out}}}{n - 1 + \delta_{\text{out}} N(n-1)}\Big)\Big(\frac{D_{\text{in}}(w) + \delta_{\text{in}}}{n - 1 + \delta_{\text{in}} N(n-1)}\Big)$$



1.2. Background: What's known.

Notation:

$$N(n) = \#\text{ nodes in } V_n.$$
$$n = \#\text{ edges in } E_n.$$
$$N_{ij}(n) = \#\text{ nodes with in-degree}=i \text{ and out-degree}=j \text{ in } G(n).$$

Then (eg, Bollobás, Borgs, Chayes, and Riordan (2003)) the limiting proportion of nodes with in-degree=i and out-degree=j is

$$\lim_{n\to\infty} \frac{N_{ij}(n)}{N(n)} = p_{ij} = \text{a prob mass function.}$$

Marginally, the limiting degree frequency (p_{ij}) has power-law tails: For some finite positive constants C_{in} and C_{out},

$$p_i(\text{in}) := \sum_{j=0}^{\infty} p_{ij} \sim C_{\text{in}} i^{-\alpha_{\text{in}}} \quad \text{as } i \to \infty, \text{ as long as } \alpha\delta_{\text{in}} + \gamma > 0,$$

$$p_j(\text{out}) := \sum_{i=0}^{\infty} p_{ij} \sim C_{\text{out}} j^{-\alpha_{\text{out}}} \quad \text{as } j \to \infty, \text{ as long as } \gamma\delta_{\text{out}} + \alpha > 0,$$

where

$$\alpha_{\text{in}} = 1 + \frac{1+\delta_{\text{in}}(\alpha+\gamma)}{\alpha+\beta}, \quad \alpha_{\text{out}} = 1 + \frac{1+\delta_{\text{out}}(\alpha+\gamma)}{\gamma+\beta}.$$

Conclude that:

- $\alpha_{in} > 1$, $\alpha_{out} > 1$, and
- Manufacture random pair

$$(I, 0) \sim \{p_{ij}\},$$

and then

$$P[I = i] \sim C_{\text{in}} i^{-\alpha_{\text{in}}}, \quad i \to \infty; \quad P[O = j] \sim C_{\text{out}} j^{-\alpha_{\text{out}}}, \quad j \to \infty.$$

- So, as $x \to \infty$,

$$P[I > x] \sim k_{\text{in}} x^{-(\alpha_{\text{in}} - 1)}, \quad P[O > x] \sim k_{\text{out}} x^{-(\alpha_{\text{out}} - 1)}.$$

- Statisticians note: First we let $n \to \infty$ and then $x \to \infty$. Data usually node based without letting $n \to \infty$.

Question: Is (I, O) jointly regularly varying? (Hint: Yes.)

1.3. Methods of attack.

1. Full frontal assault: Prove directly $P[(I,O) \in \cdot]$ is a regularly varying measure. (Samorodnitsky, Resnick, Towsley, Davis, Willis, and Wan, 2016).
2. Sneaky Resnicki: Multivariate Tauberian theorem after computing the generating function. (Resnick and Samorodnitsky, 2015)
3. Analysis of the mass function $\{p_{ij}\}$. (Wang and Resnick, 2016)

1.4. The Generating Function of $(I, O) \sim \{p_{ij}\}$.

- Because of the Markovian nature of the model construction, p_{ij} satisfies difference relation in (i, j). (Bollobás, Borgs, Chayes, and Riordan, 2003).

- Solve the difference equations using pde methods (Samorodnitsky, Resnick, Towsley, Davis, Willis, and Wan, 2016) to get explicit formula for

$$\varphi(x,y) = E\left(x^I y^O\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x^i y^j p_{ij}, \ 0 \le x, y \le 1, \tag{1}$$

where

$$\varphi(x,y) = \frac{\gamma}{\alpha+\gamma} x\varphi_1(x,y) + \frac{\alpha}{\alpha+\gamma} y\varphi_2(x,y), \tag{2}$$

with (constants c_1, c_2, a are functions of $\alpha_{\text{in}}, \alpha_{\text{out}}$)

$$\varphi_1(x,y) = c_1^{-1}\int_1^{\infty} z^{-(1+1/c_1)}\big(x+(1-x)z\big)^{-(\delta_{\text{in}}+1)}\big(y+(1-y)z^a\big)^{-\delta_{\text{out}}}\,dz, \tag{3}$$

$$\varphi_2(x,y) = c_1^{-1}\int_1^{\infty} z^{-(1+1/c_1)}\big(x+(1-x)z\big)^{-\delta_{\text{in}}}\big(y+(1-y)z^a\big)^{-(\delta_{\text{out}}+1)}\,dz \tag{4}$$

for $0 \leq x, y \leq 1$. Further:

$$c_1 = \frac{1}{\alpha_{in} - 1}, \quad c_2 = \frac{1}{\alpha_{out} - 1}, \quad a = c_2/c_1.$$

- Hmmmmmm. . . !
- Analyze the generating functions φ_1, φ_2 separately.

2. Skip the blood and guts! What's the answer?

Theorem. The random vector (I, O) with joint mass function $\{p_{ij}\}$ satisfies as $t \to \infty$,

$$tP\left[\left(\frac{I}{t^{1/(\alpha_{in}-1)}}, \frac{O}{t^{1/(\alpha_{out}-1)}}\right) \in \cdot\,\right] \xrightarrow{v} \frac{\gamma}{\alpha+\gamma}\nu_1(\cdot) + \frac{\alpha}{\alpha+\gamma}\nu_2(\cdot),$$

vaguely in $M_+([0,\infty]^2 \setminus \{\mathbf{0}\})$ and ν_1 and ν_2 concentrate on $(0,\infty)^2$ and have Lebesgue densities f_1, f_2 given by,

$$\begin{aligned} f_1(x,y) =& c_1^{-1}\big(\Gamma(\delta_{\text{in}}+1)\Gamma(\delta_{\text{out}})\big)^{-1} x^{\delta_{\text{in}}} y^{\delta_{\text{out}}-1} \\ &\times \int_0^\infty z^{-(2+1/c_1+\delta_{\text{in}}+a\delta_{\text{out}})} e^{-(x/z+y/z^a)}\, dz, \end{aligned}$$

and

$$\begin{aligned} f_2(x,y) =& c_1^{-1}\big(\Gamma(\delta_{\text{in}})\Gamma(\delta_{\text{out}}+1)\big)^{-1} x^{\delta_{\text{in}}-1} y^{\delta_{\text{out}}} \\ &\times \int_0^\infty z^{-(1+a+1/c_1+\delta_{\text{in}}+a\delta_{\text{out}})} e^{-(x/z+y/z^a)}\, dz\,. \end{aligned}$$

3. Full frontal direct assault.

Samorodnitsky, Resnick, Towsley, Davis, Willis, and Wan (2016)
From the form of the generating function:

- The pair (I, O) has representation

$$(I, O) \stackrel{d}{=} B(1 + X_1, Y_1) + (1 - B)(X_2, 1 + Y_2),$$

where B is a Bernoulli switching variable independent of X_j, Y_j, $j = 1, 2$ with

$$P(B = 1) = 1 - P(B = 0) = \frac{\gamma}{\alpha + \gamma}.$$

and

$$(X_1, Y_1) \sim \varphi_1 \quad \text{and} \quad (X_2, Y_2) \sim \varphi_2.$$

- Form of φ_i $i = 1, 2$:
  - The negative binomial distribution is id;
  - Suppose $\{T_\lambda(p), p \in (0,1)\}$ and $\{\tilde{T}_\mu(p), p \in (0,1)\}$ are two independent families of negative binomial random variables for any choice of λ, μ.
  - Write X_j, Y_j, $j = 1, 2$

$$(X_1, Y_1) = (T_{\delta_{\text{in}}+1}(Z^{-1}), \tilde{T}_{\delta_{\text{out}}}(Z^{-a})),$$
$$(X_2, Y_2) = (T_{\delta_{\text{in}}}(Z^{-1}), \tilde{T}_{\delta_{\text{out}}+1}(Z^{-a})),$$

    where
  - Z is Pareto on $[1, \infty)$ with index c_1^{-1}, independent of the negative binomial random variables.
- Prove multivariate regular variation of (X_1, Y_i), $i = 1, 2$ is inherited from the Pareto Z. [But the negative binomials smear the limit measure mass over whole first quadrant.]

4. Sneaky: Tauberian method

Resnick and Samorodnitsky (2015)

- Abel-Tauberian theorems relate power law behavior of distributions and their transforms in $\mathbb{R}_+^p$.

  Antecedents:

  - $p = 1$, Bingham, Goldie, and Teugels (1987), Feller (1971), Karamata (1931).
  - $p > 1$ for standard regular variation: Resnick (1991, 2007), Stadtmüller (1981), Stadtmüller and Trautner (1979, 1981), Stam (1977), Yakimiv (2005).

- Assume $U \in M_+(\mathbb{R}_+^p)$ with distribution function $U(\mathbf{x}) = U([\mathbf{0}, \mathbf{x}])$ and the Laplace transform $\hat{U}(\boldsymbol{\lambda})$ of U exists:

$$\hat{U}(\boldsymbol{\lambda}) = \int_{\mathbb{R}_+^p} \exp\{-\sum_{i=1}^{p} \lambda_i x_i\} U(d\mathbf{x}) < \infty, \quad \boldsymbol{\lambda} \in \mathbb{R}_+^p.$$

4.1. First 1/2 Taub

Assume

1. $\exists\, b_i \in RV_{1/\gamma_i}$, $\gamma_i > 0$, $i = 1, \dots, p$; set

$$\boldsymbol{b}(t) = (b_1(t), \dots, b_p(t)).$$

2. U is regularly varying infinite measure with limit measure U_∞ on $\mathbb{R}_+^p$:

$$U_t(\cdot) := \frac{1}{t} U(\boldsymbol{b}(t)\cdot\,) \xrightarrow{v} U_\infty(\cdot).$$

3. Regularity condition:

$$\lim_{y\to\infty} \limsup_{t\to\infty} \int_{\cup_{i=1}^p [v_i > y]} e^{-\sum_{i=1}^p v_i/x_i} U_t(d\boldsymbol{v}) = 0. \qquad ([\mathrm{UR}])$$

Then

1. The Laplace transforms $\hat{U}(\mathbf{1}/\mathbf{x})$ and $\hat{U}_\infty(\mathbf{1}/\mathbf{x})$ are distribution functions of Radon measures on $\mathbb{R}_+^p$ and

2. First 1/2 Taub: the measure corresponding to $\hat{U}(\mathbf{1}/\mathbf{x})$ inherits non-standard regular variation: for $\mathbf{x} > 0$

$$\frac{1}{t}\hat{U}\Big(\frac{\mathbf{1}}{\boldsymbol{b}(t)\mathbf{x}}\Big) \to \hat{U}_\infty\Big(\frac{\mathbf{1}}{\mathbf{x}}\Big), \qquad (5)$$

4.1.1. Proof sketch of First 1/2 Taub; map your way to happiness.

(i) Assume U is regularly varying:

$$U_t(\cdot) := \frac{1}{t} U(\boldsymbol{b}(t)\cdot) \xrightarrow{v} U_\infty(\cdot).$$

(ii) Set $\mathcal{F} = (\frac{1}{E_1}, \dots, \frac{1}{E_p})$, where $(E_1, \dots, E_p)$ are iid exp rv's so $1/E_i$ is unit Frechet. In $M_+([0,\infty]^p \times \mathbb{R}_+^p)$

$$P[\mathcal{F} \in \cdot\,] \times U_t \xrightarrow{v} P[\mathcal{F} \in \cdot\,] \times U_\infty.$$

(iii) Define $h : [0,\infty]^p \times \mathbb{R}_+^p \mapsto [0,\infty]^p \times \mathbb{R}_+^p$ by

$$h(\mathbf{x}, \boldsymbol{y}) = (\mathbf{x}\boldsymbol{y}, \boldsymbol{y}).$$

(iv) By a continuity theorem for convergence of measures:

$$\begin{aligned}&\big(P[\mathcal{F} \in \cdot\,] \times U_t\big) \circ h^{-1}\big([\mathbf{0}, \mathbf{x}] \times [\mathbf{0}, y\mathbf{1}]\big)\\ &\qquad \to \big(P[\mathcal{F} \in \cdot\,] \times U_\infty\big) \circ h^{-1}\big([\mathbf{0}, \mathbf{x}] \times [\mathbf{0}, y\mathbf{1}]\big).\end{aligned}$$

(v) Unpack: As $t \to \infty$,

$$\int_{\boldsymbol{v} \le y\mathbf{1}} e^{-\sum_{i=1}^p v_i/x_i} U_t(d\boldsymbol{v}) \to \int_{\boldsymbol{v} \le y\mathbf{1}} e^{-\sum_{i=1}^p v_i/x_i} U_\infty(d\boldsymbol{v}).$$

(vi) Let $y \to \infty$ via condition [UR]. Done!

4.2. Second 1/2 Taub.

Assume

1. The infinite measure $U \in M_+(\mathbb{R}_+^p)$ has distribution function $U(\mathbf{x}) = U([\mathbf{0}, \mathbf{x}])$ and the Laplace transform $\hat{U}(\boldsymbol{\lambda})$ of U exists:
$$\hat{U}(\boldsymbol{\lambda}) = \int_{\mathbb{R}_+^p} \exp\{-\boldsymbol{\lambda}'\mathbf{x}\} U(d\mathbf{x}) < \infty, \quad \boldsymbol{\lambda} \in \mathbb{R}_+^p.$$
2. $b_i \in RV_{1/\gamma_i}$, $\gamma_i > 0$ for $i = 1, \ldots, p$.
3. Condition [UR] holds:
$$\lim_{y\to\infty} \limsup_{t\to\infty} \int_{\cup_{i=1}^p [v_i > y]} e^{-\sum_{i=1}^p v_i/x_i} U_t(d\boldsymbol{v}) = 0. \qquad \text{([UR])}$$
4. There exists a finite-valued function $\hat{U}_\infty$ such that for $\mathbf{x} > \mathbf{0}$,
$$\frac{1}{t}\hat{U}\Big(\frac{\mathbf{1}}{\boldsymbol{b}(t)\mathbf{x}}\Big) = \frac{1}{t}\hat{U}\Big(\frac{1}{b_1(t)x_1}, \ldots, \frac{1}{b_p(t)x_p}\Big) \to \hat{U}_\infty(\mathbf{1}/\mathbf{x}). \qquad (6)$$

Then for some measure $U_\infty \in M_+(\mathbb{R}_+^p)$ whose Laplace transform is $\hat{U}_\infty$, we have as $t \to \infty$,
$$U_t(\cdot) = \frac{1}{t} U(\boldsymbol{b}(t)\cdot\,) \to U_\infty(\cdot), \qquad \text{in } M_+(\mathbb{R}_+^p).$$

4.3. Apply to random graph.

- To apply Tauberian theory, need infinite measure: So for φ_1, fix $k > \alpha_{\text{in}} - 1$ and set

$$\psi(x,y) = \frac{\partial^k \varphi_1}{\partial x^k}(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x^i y^j m_{ij}^{(k)}$$

  where the quantities $m_{ij}^{(k)}$ can be calculated in terms of p_{ij}'s.

- Define

$$U(\cdot) = \sum_{i,j} m_{ij}^{(k)} \epsilon_{(i,j)}(\cdot)$$

  as an infinite Radon measure on $\mathbb{R}_+^2$ concentrating on $\big(\{0,1,2,\dots\}\big)^2$ that puts mass $m_{ij}^{(k)}$ at (i,j).

- Compute $\hat{U}$, verify [UR], verify $\hat{U}(\mathbf{1}/\mathbf{x})$ is regularly varying.

- Voila!

5. Analysis of the mass function $\{p_{ij}\}$.

Wang and Resnick (2016)
Suppose $U(\cdot)$ is a measure on $\mathbb{R}^2_+$ with [pdf, pmf] [$f(x,y)$, $f(i,j)$].

- If U is a regularly varying measure, is f a regularly varying [function, array]?

  Not always true even in one dimension.

- If f is a regularly varying [function, array], is U a regularly varying measure.

  Shock! Not necessarily and a general analogue of Karamata's theorem on integration fails.

- Definition of regularly varying array?

CORNELL

5.1. Definitions: Regularly varying array.

- A doubly indexed function $f : \mathbb{Z}_+^2 \setminus \{\mathbf{0}\} \mapsto \mathbb{R}_+$ is regularly varying with scaling functions b_1 and b_2 and limit function $\lambda(x, y)$ if for some $h \in RV_\alpha$ for some $\alpha \in \mathbb{R}$, $b_i \in RV_{\beta_i}$, $\beta_i > 0$, we have

$$\lim_{n\to\infty} \frac{f([b_1(n)x], [b_2(n)y])}{h(n)} = \lambda(x, y) > 0, \quad \forall x, y > 0.$$

- A function $f : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ is regularly varying if the same limit holds without the square brackets [],[].

- If $f(i, j)$ is regularly varying, it is embeddable if $\exists$ regularly varying $g(x, y)$ and

$$g(x, y) = f([x], [y]).$$

5.2. Typical result.

Suppose $u(i,j) > 0$ satisfies

1. $u(i,j)$ is regularly varying.
2. u satisfies some extra condition.

Then

1. The function
$$g(x,y) := u([x],[y])$$
is regularly varying as function of continuous variables and $u(i,j)$ is embeddable.
2. If $u(i,j) = p(i,j)$ is a pmf corresponding to (X,Y), then
$$P[(X,Y) \in \cdot\ ]$$
is a regularly varying measure.

Example of extra condition: Easiest is to suppose $u(i,j)$ is eventually decreasing.

BUT: This does not hold for preferential attachment problem.

5.2.1. More flexible for the non-standard case.

Suppose

- $h(\cdot) \in RV_\rho$, $\rho < 0$, and $u : \mathbb{Z}_+^2 \mapsto \mathbb{R}_+$,
- Scaling functions are power laws: $b_i(t) = t^{1/\alpha_i}$, $i = 1, 2$. [The non-standard case is harder than the standard and we had to suppose b_i are power functions.]
- There exists a limit function $\lambda_0 > 0$ defined on

$$\mathcal{E}_0 := \{(x, y) : \|(x^{\alpha_1}, y^{\alpha_2})\| = 1\},$$

such that u satisfies

$$\lim_{t\to\infty} \frac{u([t^{1/\alpha_1}x], [t^{1/\alpha_2}y])}{h(t)} = \lambda_0(x, y), \quad \forall (x, y) \in \mathcal{E}_0. \tag{7}$$

Then

1. The doubly indexed function $u(i, j)$ is regularly varying: For all $x, y > 0$, define $\mathbf{w} = \mathbf{w}(x, y) := (x^{\alpha_1}, y^{\alpha_2})$ and

$$\lim_{n\to\infty} \frac{u([n^{1/\alpha_1}x], [n^{1/\alpha_2}y])}{h(n)} = \lambda(x, y) := \lambda_0\Big(\frac{x}{\|\mathbf{w}\|^{1/\alpha_1}}, \frac{y}{\|\mathbf{w}\|^{1/\alpha_2}}\Big)\|\mathbf{w}\|^\rho;$$

2. The doubly indexed function $u(i,j)$ is embeddable in a non-standard regularly varying function $f : \mathbb{R}_+^2 \mapsto \mathbb{R}$ with limit function $\lambda(\cdot)$ such that $f(x,y) = u([x],[y])$;
3. If convergence in (7) is uniform on $\mathcal{E}_0$, then also the measure corresponding to $u(i,j)$ is a (discretely supported) regularly varying measure.

5.3. Back to preferential attachment.

Recall the representation of

$$(I,O) \sim p_{i,j}.$$

$$(I,O) \stackrel{d}{=} B(1+X_1,Y_1) + (1-B)(X_2,1+Y_2),$$

where for Pareto Z:

$$\begin{aligned}
(X_1,Y_1) =& (T_{\delta_{\text{in}}+1}(Z^{-1}), \tilde{T}_{\delta_{\text{out}}}(Z^{-a})),\\
(X_2,Y_2) =& (T_{\delta_{\text{in}}}(Z^{-1}), \tilde{T}_{\delta_{\text{out}}+1}(Z^{-a})),
\end{aligned}$$

From the representations:

$$\begin{aligned}
\lim_{n\to\infty} \frac{p([n^{c_1}x],[n^{c_2}y])}{n^{-(1+c_1+c_2)}} =& \frac{\gamma}{\alpha+\gamma} \frac{x^{\delta_{\text{in}}} y^{\delta_{\text{out}}-1}}{c_1\Gamma(\delta_{\text{in}}+1)\Gamma(\delta_{\text{out}})} \int_0^\infty z^{-(2+1/c_1+\delta_{\text{in}}+a\delta_{\text{out}})} e^{-\left(\frac{x}{z}+\frac{y}{z^a}\right)} \mathrm{d}z\\
&+ \frac{\alpha}{\alpha+\gamma} \frac{x^{\delta_{\text{in}}-1} y^{\delta}_{\text{out}}}{c_1\Gamma(\delta_{\text{in}})\Gamma(\delta_{\text{out}}+1)} \int_0^\infty z^{-(1+a+1/c_1+\lambda+a\delta_{\text{out}})} e^{-\left(\frac{x}{z}+\frac{y}{z^a}\right)} \mathrm{d}z.
\end{aligned}$$

Remarks:

1. This convergence can be shown to be uniform on $\mathcal{E}_0$.
2. Therefore, the uniform convergence implies

$$P[(I, O) \in \cdot\ ]$$

   is a regularly varying measure.
3. This closes the loops.

6. Conclusion

- Reciprocity? Cliques? Neighborhoods?
- Inference based on
  - Tail empirical measure based on degree data–study extreme values.
  - Asymptotic normality of degree counts for undirected and directed graphs. Inference for central values. Progress:
    - Undirected case: Resnick and Samorodnitsky (2016).
    - Directed case: Wang and Resnick (2015).
- Embedding techniques in birth-death processes may illuminate properties of this model and other models.

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