The eigenvalues and eigenvectors of the sample covariance matrix of heavy-tailed multivariate time series[1]

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[1]Fields, May 2016

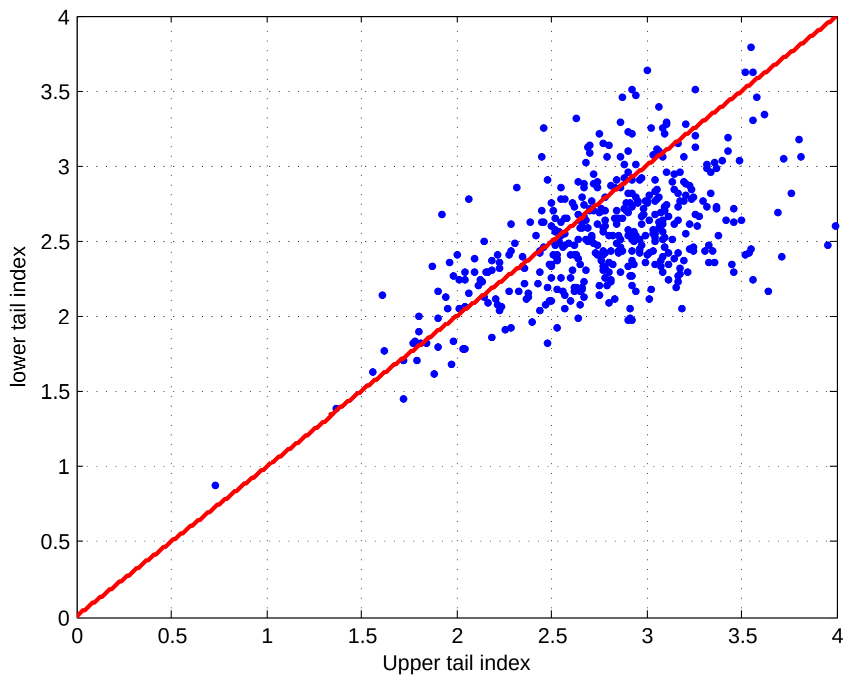

FIGURE 1. Estimated tail indices for S&P 500.

1. Heavy-tailed matrices with iid entries: Finite p

- **We consider $p \times n$ matrices**

$$\mathbf{X} = \mathbf{X}_n = \left(X_{it}\right)_{i,\ldots,p;t=1,\ldots,n},$$

where the iid entries have a distribution with heavy tails

$$\mathbb{P}(X > x) \sim p_+ \frac{L(x)}{x^\alpha} \quad \text{and} \quad \mathbb{P}(X < -x) \sim p_- \frac{L(x)}{x^\alpha}$$

for a slowly varying function L, some $\alpha > 0$, $p_+ + p_- = 1$.

- **Assume $\mathbb{E}[X] = 0$ if expectation is finite.**

- **Let X_1, X_2 be iid copies of X.**
- **Then**

$$\mathbb{P}(X^2 > x) \sim \frac{L(\sqrt{x})}{x^{\alpha/2}}$$

- **If $X > 0$** Embrechts and Goldie (1984)**:**

$$\mathbb{P}(X_1\, X_2 > x) \sim \frac{\widetilde{L}(x)}{x^{\alpha}}$$

for some slowly varying $\widetilde{L}$.

- **If $X > 0$ and some additional condition holds** Davis and Resnick (1985)

$$\frac{\mathbb{P}(X_1\, X_2 > x)}{\mathbb{P}(X > x)} \to 2\,\mathbb{E}[X^{\alpha}]\,.$$

- **If $X, \sigma > 0$ independent and $\mathbb{E}[\sigma^{\alpha+\delta}] < \infty$, then by** Breiman (1965)

$$\frac{\mathbb{P}(\sigma\, X > x)}{\mathbb{P}(X > x)} \to \mathbb{E}[\sigma^{\alpha}]\,.$$

- **Therefore** Gnedenko and Kolmogorov (1949), Feller (1971), Resnick (2007)

$$a_n^{-2}\Big(\sum_{t=1}^{n} X_{it}^2 - c_n\Big) \xrightarrow{\mathrm{d}} \xi_i^{(\alpha/2)}\,, \qquad \alpha \in (0,4)\,,$$

where $\mathbb{P}(|X| > a_n) \sim n^{-1}$ and $(\xi_i^{(\gamma)})$ are iid γ-stable, $\gamma \in (0,2]$, and for $i \neq j$,

$$b_n^{-1}\big(\textstyle\sum_{t=1}^{n} X_{it}X_{jt} - c_n\big) \xrightarrow{\mathrm{d}} \xi_i^{(2\wedge\alpha)}\,, \qquad \alpha \in (0,4)\,,$$

where $\mathbb{P}(|X_1X_2| > b_n) \sim n^{-1}$ for $\alpha \in (0,2)$ and $b_n = \sqrt{n}$ for $\alpha \in (2,4)$.

- **Since $a_n^2 \approx n^{2/\alpha}$ and $b_n \approx n^{1/(2\wedge\alpha)}$,**

$$\Big\| a_n^{-2}(\mathrm{XX}') - \mathrm{diag}(\mathrm{XX}')\Big\|_2^2 \le \frac{b_n^2}{a_n^4} \sum_{i=1}^{p} \Big(\frac{1}{b_n}\Big(\sum_{t=1}^{n} X_{it}X_{jt} - c_n\Big)\Big)^2 \xrightarrow{\mathbb{P}} 0\,.$$

- **Therefore, by Weyl's inequality,**

$$a_n^{-2} \max_{i=1,\ldots,p} \big|\lambda_{(i)} - \lambda_{(i)}(\mathrm{diag}(\mathrm{XX}'))\big| \\ \le a_n^{-2} \big\| \mathrm{XX}' - \mathrm{diag}(\mathrm{XX}') \big\|_2 \stackrel{\mathbb{P}}{\to} 0\,,$$

where $\lambda_{(1)}(A) \ge \cdots \ge \lambda_{(p)}(A)$ **for any symmetric matrix** A**,** $\lambda_i = \lambda_i(\mathrm{XX}')$**,**

- **Hence for** $\alpha \in (0,4)$**,** $\xi_{(1)}^{(\alpha/2)} \ge \cdots \ge \xi_{(p)}^{(\alpha/2)}$

$$a_n^{-2}\big(\lambda_{(i)} - c_n\big)_{i=1,\ldots,p} \stackrel{\mathrm{d}}{\to} \big(\xi_{(i)}^{(\alpha/2)}\big)_{i=1,\ldots,p}$$

In particular,

$$\frac{n}{a_n^2}\Big(\frac{\lambda_{(1)}}{n} - \mathbb{E}[X^2]\,\mathbf{1}_{(2,4)}(\alpha)\Big) \stackrel{\mathrm{d}}{\to} \max_{i=1,\ldots,p} \xi_i^{(\alpha/2)}\,.$$

- **The unit eigenvector $\mathbf{V}_j$ associated with $\boldsymbol{\lambda}_{(j)} = \boldsymbol{\lambda}_{L_j}$ can be localized:**

$$\|\mathbf{V}_j - \mathbf{e}_{L_j}\| \xrightarrow{\mathbb{P}} \mathbf{0}\,.$$

- **We have for $\boldsymbol{\alpha} \in (\mathbf{2}, \mathbf{4})$,**

$$\begin{aligned} \frac{1}{a_n^2 c_n^{p-1}}\big(\det(\mathbf{X}\mathbf{X}') - c_n^p\big) &= \sum_{i=1}^{p} a_n^{-2}(\boldsymbol{\lambda}_{(i)} - c_n) \prod_{j=1}^{i-1} \frac{\boldsymbol{\lambda}_{(j)}}{c_n} \\ &\xrightarrow{\mathrm{d}} \sum_{i=1}^{p} \xi_{(i)}^{(\alpha/2)} = \sum_{i=1}^{p} \xi_i^{(\alpha/2)} \stackrel{\mathrm{d}}{=} p^{2/\alpha} \xi_1^{(\alpha/2)}\,. \end{aligned}$$

2. Heavy-tailed matrices with regularly varying stochastic volatility entries: p fixed

- **We consider $p \times n$ matrices**

$$\mathbf{X} = \mathbf{X}_n = \left(X_{it}\right)_{i,\ldots,p;t=1,\ldots,n},$$

- **A stochastic volatility model is given by**

$$X_{it} = \sigma_{it}\, Z_{it} \text{ for independent } \sigma\text{- and } Z\text{-fields}$$

where (σ_{it}) is a strictly stationary non-negative field and (Z_{it}) has mean zero (if exists).

- **We assume either**

Case (1) $\mathbb{E}[\boldsymbol{\sigma}^{\alpha+\delta}] < \infty$ **and**

$$\mathbb{P}(Z > x) \sim p_+ \frac{L(x)}{x^\alpha} \quad \text{and} \quad \mathbb{P}(Z < -x) \sim p_- \frac{L(x)}{x^\alpha}$$

for a slowly varying function L**, some** $\alpha > 0$**,** $p_+ + p_- = 1$**.**

Then, by Breiman (1965)**,**

$$\mathbb{P}(\pm X > x) \sim \mathbb{E}[\boldsymbol{\sigma}^\alpha]\, \mathbb{P}(\pm Z > x)\,.$$

Case (2) $\mathbb{E}[|Z|^{\alpha+\delta}] < \infty$ **and**

$\log \boldsymbol{\sigma}_{it} = \sum_{kl} \psi_{kl}\eta_{i-k,t-l}$ **for an iid field** (η_{ij})**, non-neg.** (ψ_{kl})

such that e^η **is regularly varying with index** $\alpha > 0$ **and**

$\max_{kl} \psi_{kl} = 1$**. Then** $\boldsymbol{\sigma}$ **is regularly varying with index** α **and**

$$\mathbb{P}(\pm X > x) \sim \mathbb{E}[Z_\pm^\alpha]\, \mathbb{P}(\boldsymbol{\sigma} > x)\,.$$

- **Case (1) Analogous to iid case.**

  The diagonal of XX' dominates and diagonal elements converge to iid $\alpha/2$-stable random variables.

- **Case (2)**

  Case (2a) $\mathbb{E}[e^{\alpha\eta}] = \infty$. In this case

$$\frac{\mathbb{P}(e^{\eta_1}e^{\eta_2} > x)}{\mathbb{P}(e^{\eta} > x)} \to \infty$$

  Case (2b) $\mathbb{E}[e^{\alpha\eta}] < \infty$ and an additional condition. In this case

$$\frac{\mathbb{P}(e^{\eta_1}e^{\eta_2} > x)}{\mathbb{P}(e^{\eta} > x)} \to 2\,\mathbb{E}[e^{\alpha\eta}]\,.$$

- **Example.** Consider $\sigma_{it} = \mathrm{e}^{\eta_{it}+\eta_{i-1,t}}$. Then

$$\sigma_{it}^2 = \mathrm{e}^{2\eta_{it}+2\eta_{i-1,t}},$$
$$\sigma_{it}\sigma_{i-1,t} = \mathrm{e}^{\eta_{it}+2\eta_{i-1,t}+\eta_{i-2,t}}.$$

- In Case (2a),

  σ_{it}^2, $\sigma_{it}\sigma_{i-1,t}$, $\sigma_{it}\sigma_{i+1,t}$ are regularly varying with index $\alpha/2$

  but $\mathbb{P}(\sigma_{it}^2 > x))/\mathbb{P}(\sigma_{it}\sigma_{i-1,t} > x) \to \infty$

  while $\sigma_{it}\sigma_{i-k,t}$, $k > 2$, is regularly varying with index α.

  Hence the tails of σ_{it}^2 dominate and the diagonal elements of $a_n^{-2}\mathrm{XX}'$ dominate the entries off the diagonal.

  The limit behavior of XX' and its eigenvalues are analogous to the iid case.

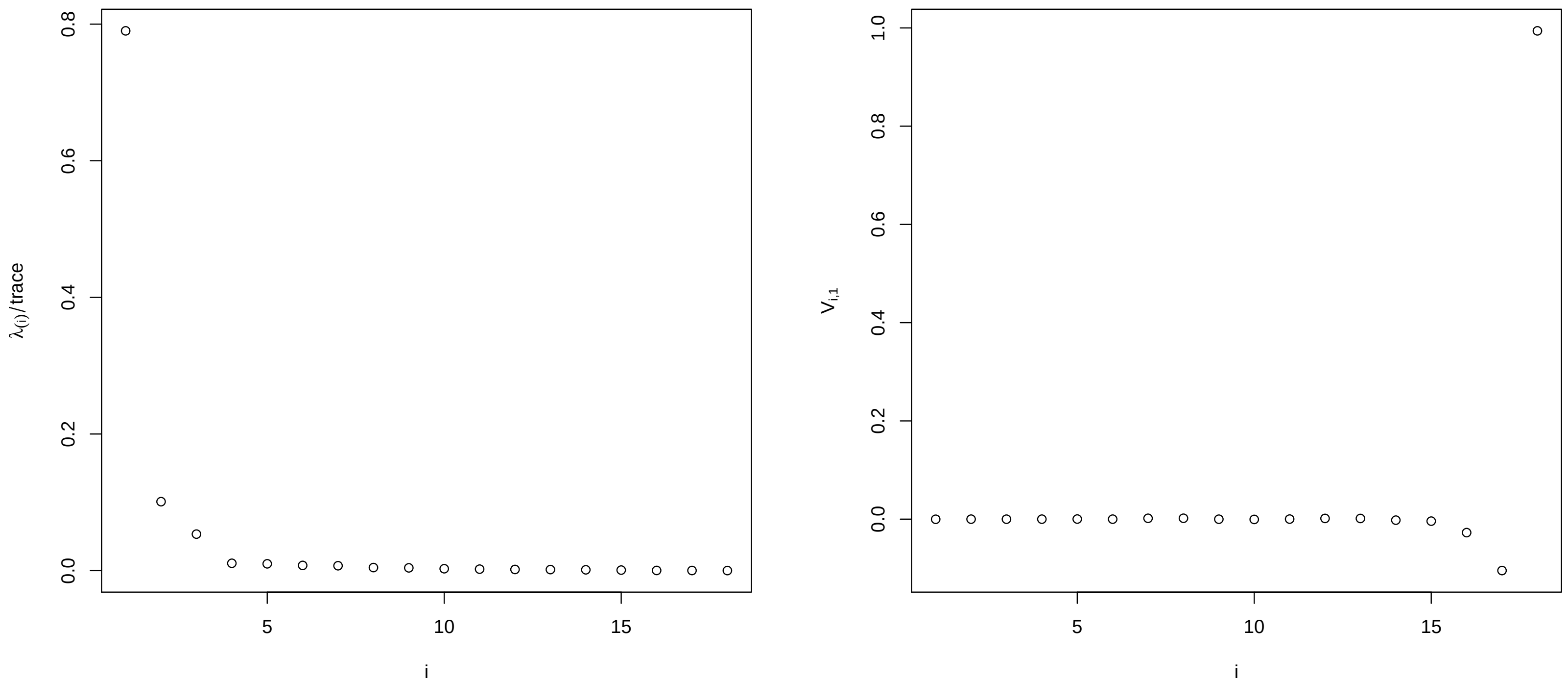

Figure 2. Eigenvalues (left) and unit eigenvector corresponding to the largest eigenvalue (right): Cases (1) and (2a)

- **In Case (2b),**

  σ_{it}^2, $\sigma_{it}\sigma_{i-1,t}$, $\sigma_{it}\sigma_{i+1,t}$ are regularly varying with index $\alpha/2$ and have equivalent tails

  while $\sigma_{it}\sigma_{i-k,t}$, $k > 2$, is regularly varying with index α.

  Hence the tails of σ_{it}^2, $\sigma_{it}\sigma_{i-1,t}$, $\sigma_{it}\sigma_{i+1,t}$ dominate $a_n^{-2}\mathrm{XX}'$ off the diagonal and first subdiagonals.

  In this case $a_n^{-2}\mathrm{XX}'$ has $\alpha/2$-stable dependent limits on the diagonal and first subdiagonals; the limiting eigenstructure is not evident.

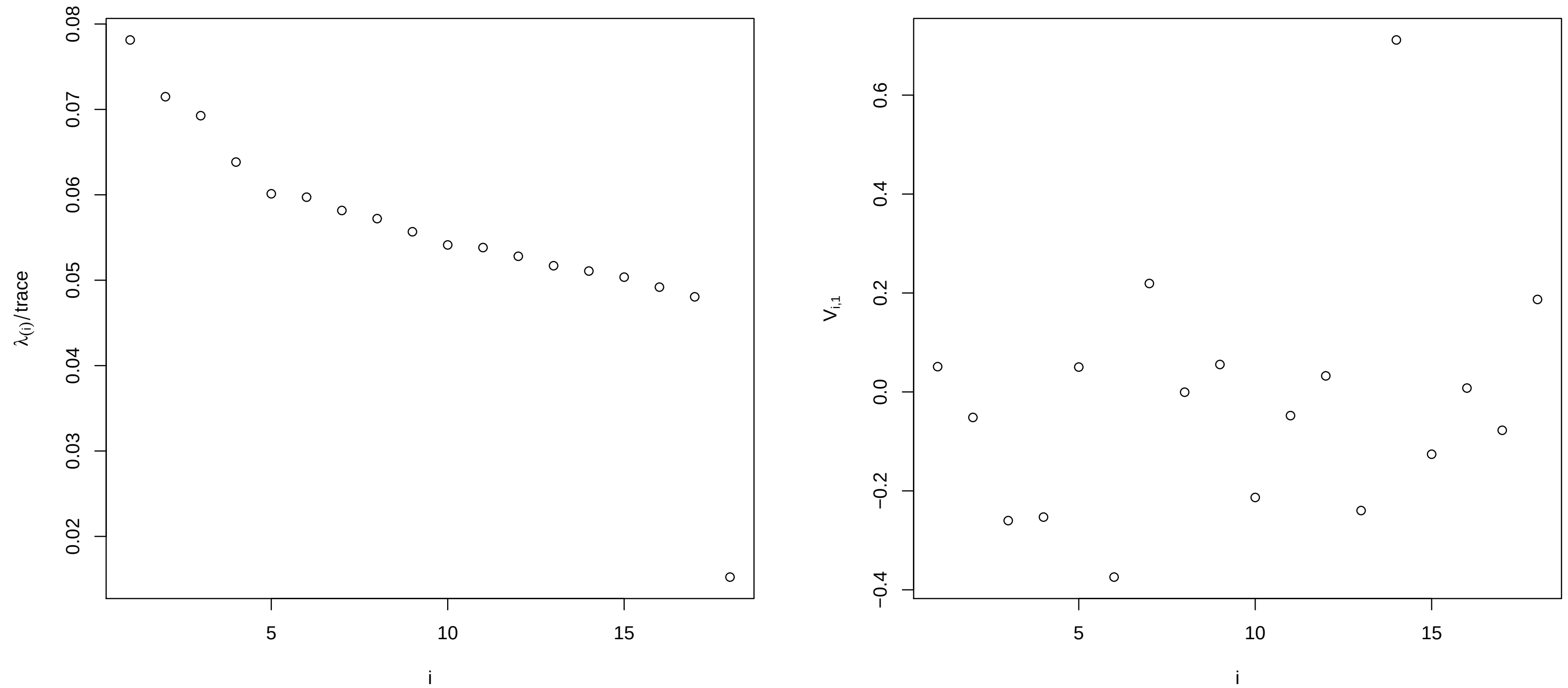

FIGURE 3. Eigenvalues (left) and unit eigenvector corresponding to the largest eigenvalue (right): Case (2b)

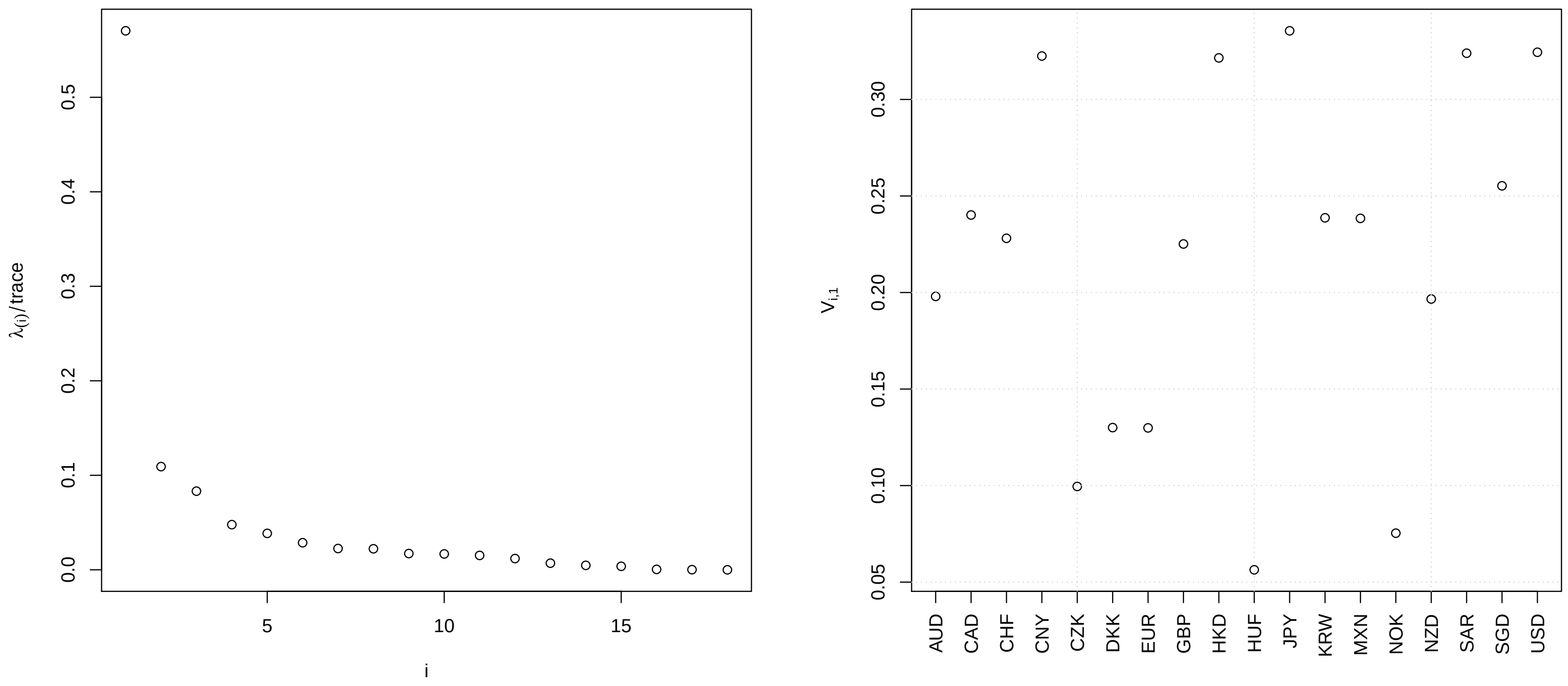

FIGURE 4. Eigenvalues (left) and unit eigenvector corresponding to the largest eigenvalue (right): FX rate daily log-return series against SEK

3. Heavy-tailed matrices with iid entries: p increases with n

- **Assume the previous iid conditions on**

$$\mathrm{X} = \left(X_{it}\right)_{i=1,\ldots,p;t=1,\ldots,n}$$

This means: X_{it} **iid, regular variation with** $\alpha \in (0,4)$, $\mathbb{E}[X] = 0$ **if expectation exists.**

- **We assume**

$$p = p_n = n^{\beta}\ell(n)\,,$$

for some $\beta > 0$ **and some slowly varying function** $\ell(n)$.

- **Then**

$$a_{np}^{-2}\big\|\mathrm{X}\mathrm{X}' - \mathrm{diag}(\mathrm{X}\mathrm{X}')\big\|_2 \stackrel{\mathbb{P}}{\to} 0\,, \qquad \beta \in (0,1]\,,$$

$$a_{np}^{-2}\big\|\mathrm{X}'\mathrm{X} - \mathrm{diag}(\mathrm{X}'\mathrm{X})\big\|_2 \stackrel{\mathbb{P}}{\to} 0\,, \qquad \beta \in (1,\infty)\,.$$

Note: **XX' and $X'X$ have the same positive eigenvalues.**

- **One needs Nagaev-type large deviations: for $i \neq j$,**

$$p^2\,\mathbb{P}\big(a_{np}^{-2}\big|\sum_{t=1}^{n} X_{it}X_{jt}\big| > \varepsilon\big) \approx p^2\,n\,\mathbb{P}(|X_1X_2| > \varepsilon a_{np}^2) \stackrel{\mathbb{P}}{\to} 0\,,$$

provided $\beta \leq 1$.

- **For $\beta > 1$, consider $X'X$, interchange the roles of n and p, and observe that $n = p^{1/\beta}\widetilde{\ell}(n)$ for some slowly varying $\widetilde{\ell}$.**
- **Write $\lambda_1, \ldots, \lambda_p$ for the eigenvalues of XX' and**

$$\lambda_{(p)} \leq \cdots \leq \lambda_{(1)}\,.$$

We also write

$$D_i^{\to} = \sum_{t=1}^{n} X_{it}^2 = \lambda_i(\mathbf{XX}'), \qquad i = 1, \ldots, p,$$
$$D_t^{\downarrow} = \sum_{i=1}^{p} X_{it}^2 = \lambda_t(\mathbf{X}'\mathbf{X}), \qquad t = 1, \ldots, n.$$

and for the ordered values

$$D_{(1)}^{\to/\downarrow} \geq D_{(2)}^{\to/\downarrow} \geq \cdots$$

- **By Weyl's inequality,**

$$\frac{1}{a_{np}^2} \max_{i=1,\ldots,p} \left|\lambda_{(i)} - D_{(i)}^{\to}\right| \leq \frac{1}{a_{np}^2} \left\|\mathbf{XX}' - \mathrm{diag}(\mathbf{XX}')\right\|_2 \xrightarrow{\mathbb{P}} 0, \quad \beta \in (0, 1]$$
$$\frac{1}{a_{np}^2} \max_{t=1,\ldots,n} \left|\lambda_{(t)} - D_{(t)}^{\downarrow}\right| \leq \frac{1}{a_{np}^2} \left\|\mathbf{X}'\mathbf{X} - \mathrm{diag}(\mathbf{X}'\mathbf{X})\right\|_2 \xrightarrow{\mathbb{P}} 0, \quad \beta > 1$$

- **Limit theory for the order statistics of $(D_i^{\to/\downarrow})$ can be handled by point process convergence: for example,**

$$\sum_{i=1}^{p} \varepsilon_{a_{np}^{-2}(D_i^{\to}-c_n)} \stackrel{\mathrm{d}}{\to} \sum_{i=1}^{\infty} \varepsilon_{\Gamma_i^{-2/\alpha}} = N_\Gamma$$

for $\Gamma_i = E_1 + \cdots + E_i$, (E_i) iid standard exponential, if and only if Resnick (2007)

$$p\,\mathbb{P}(a_{np}^{-2}(D_i^{\to} - c_n) > x) \to x^{-\alpha/2}, \qquad x > 0,$$

$$p\,\mathbb{P}(a_{np}^{-2}(D_i^{\to} - c_n) < -x) \to 0, \qquad x > 0.$$

This follows by Nagaev-type large deviations.

- **Centering with c_n is necessary only for $\alpha \in [2,4)$ and if $(n \vee p)/a_{np}^2 \not\to 0$. Centering is not needed if**

$$\min(\beta, \beta^{-1}) \in \big((\alpha/2 - 1)_+, 1\big].$$

- **Under the latter condition one also has the alternative approximation**

$$a_{np}^{-2} \max_{i=1,\ldots,p} \left|\lambda_{(i)} - Z^2_{(i),np}\right| \xrightarrow{\mathbb{P}} 0\,.$$

- **Under the latter condition,**

$$\sum_{i=1}^{p} \varepsilon_{a_{np}^{-2}\lambda_i} \xrightarrow{\mathrm{d}} \sum_{i=1}^{\infty} \varepsilon_{\Gamma_i^{-2/\alpha}}\,.$$

This was proved by Soshnikov (2004,2006) **for $\alpha \in (0,2)$ and by** Auffinger, Ben Arous, Péché (2009) **for $\alpha \in (0,4)$ in the case when**

$$p/n \to \gamma \in (0,\infty)\,. \qquad (3.1)$$

- **Partial results (for particular choices of p) were proved in** Davis, Pfaffel and Stelzer (2014), Davis, Mikosch, Pfaffel (2016) **for (X_{it}) satisfying some linear dependence conditions.**

- **In the light-tailed case, limit results for eigenvalues of $\mathbf{XX}'$ are very sensitive with respect to the growth rate of p.**

- Johnstone (2001) **showed for iid standard normal X_{it} under (3.1) that**

$$\frac{\sqrt{n}+\sqrt{p}}{\left(1/\sqrt{n}+1/\sqrt{p}\right)^{1/3}}\left(\frac{\lambda_{(1)}}{(\sqrt{n}+\sqrt{p})^2}-1\right)\stackrel{\mathrm{d}}{\rightarrow}\text{Tracy-Widom distr.}$$

- **This result remains valid for iid entries X_{it} whose first four moments match those of the standard normal.** Tao and Vu (2011).

- **In the iid regular variation case, using a.s. continuous mappings, folklore** (Resnick (1987,2007)) **applies to prove**

  the joint convergence of the order statistics (possibly with centering c_n)

$$a_{np}^{-2}\big(\lambda_{(1)},\ldots,\lambda_{(k)}\big) \xrightarrow{d} \big(\Gamma_1^{-2/\alpha},\ldots,\Gamma_k^{-2/\alpha}\big)\,.$$

  the joint convergence of the ratios of successive order statistics

$$\Big(\frac{\lambda_{(2)}}{\lambda_{(1)}},\ldots,\frac{\lambda_{(k-1)}}{\lambda_{(k)}}\Big) \xrightarrow{d} \Big(\Big(\frac{\Gamma_1}{\Gamma_2}\Big)^{2/\alpha},\ldots,\Big(\frac{\Gamma_{k-1}}{\Gamma_k}\Big)^{2/\alpha}\Big)\,.$$

  the convergence of ratio of largest eigenvalue to trace of $\mathbf{X}\mathbf{X}'$ for $\alpha \in (0,2)$

$$\frac{\lambda_{(1)}}{\lambda_1+\cdots+\lambda_p} \xrightarrow{d} \frac{\Gamma_1^{-2/\alpha}}{\Gamma_1^{-2/\alpha}+\Gamma_2^{-2/\alpha}+\cdots}$$

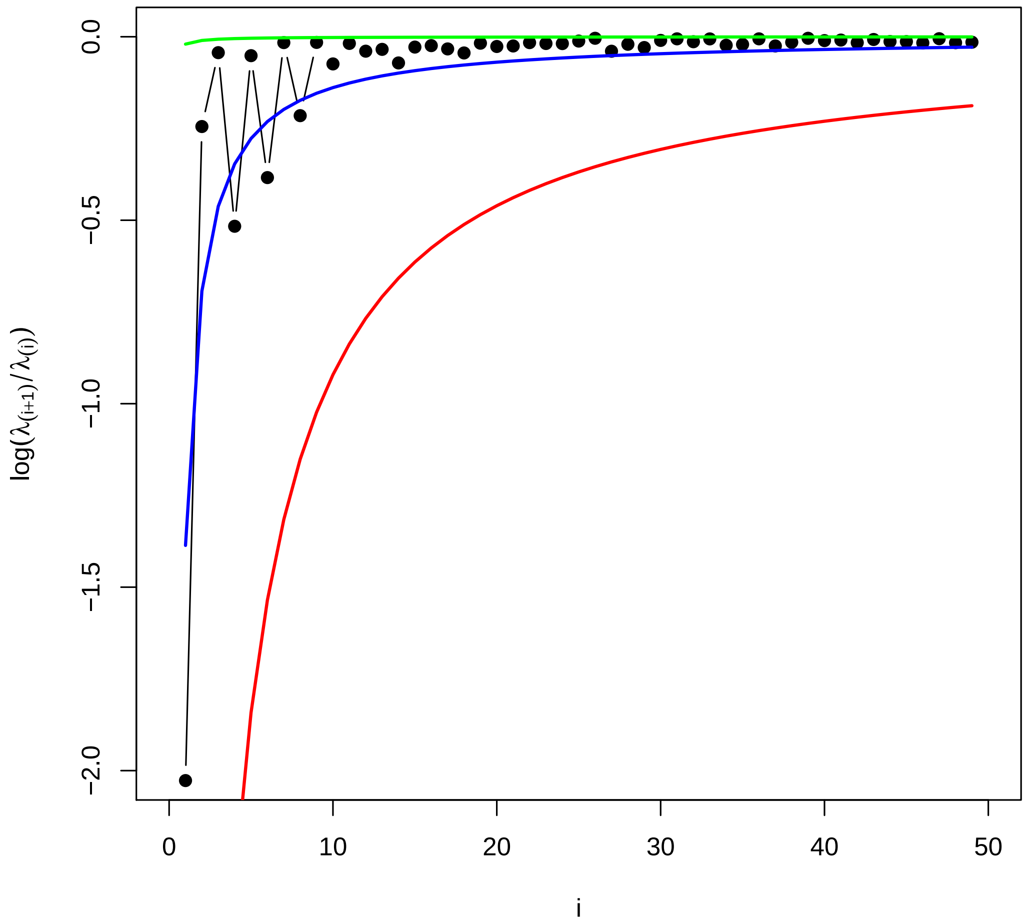

FIGURE 5. The logarithms of the ratios $\boldsymbol{\lambda_{(i+1)}/\lambda_{(i)}}$ for the S&P 500 series after rank transform. We also show the 1, 50 and 99% quantiles (bottom, middle, top lines, respectively) of the variables $\mathbf{log((\Gamma_{\boldsymbol{i}}/\Gamma_{\boldsymbol{i+1}})^2)}$.

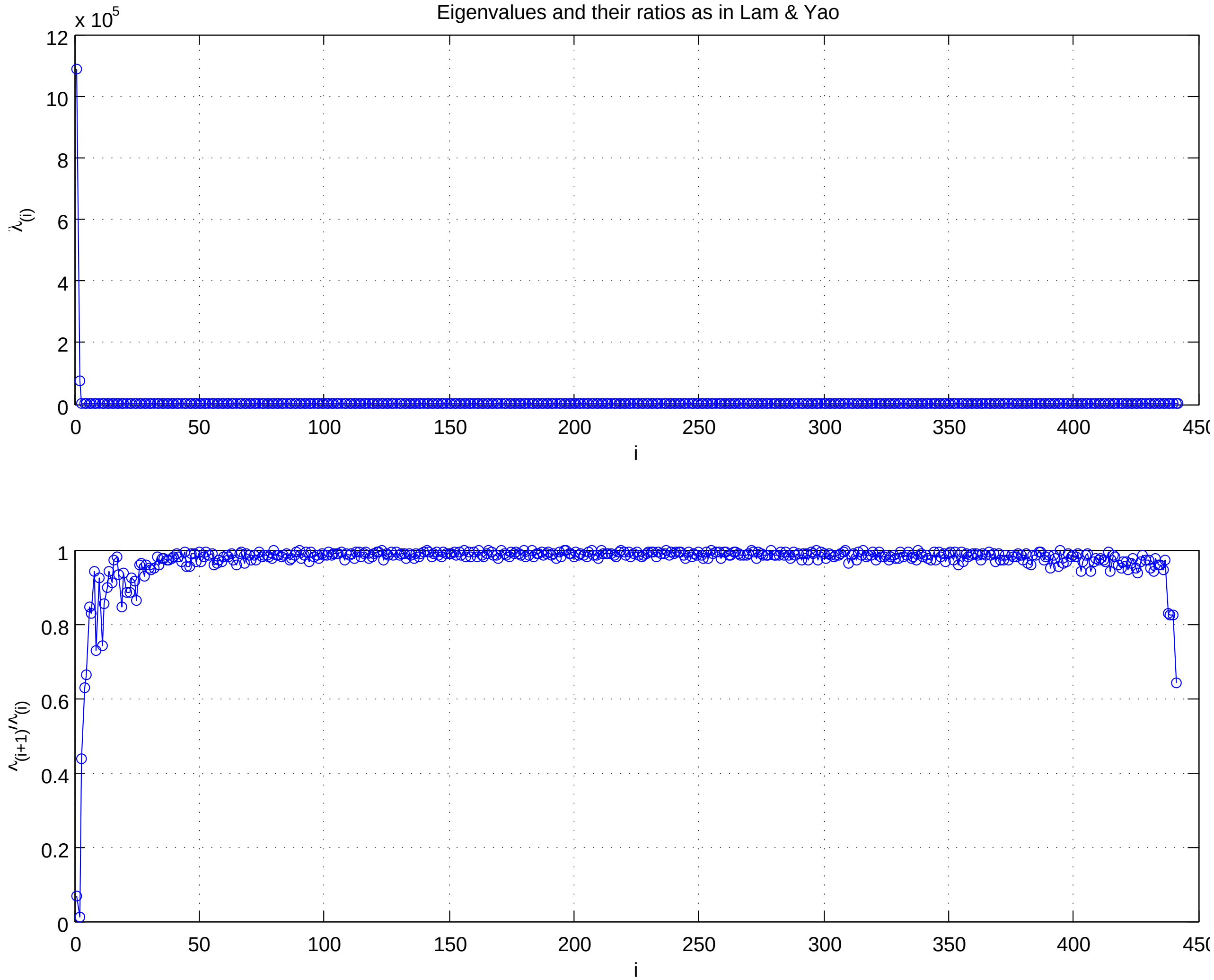

FIGURE 6. Eigenvalues and ratios $\lambda_{(i+1)}/\lambda_{(i)}$ for S&P 2002-2014, $p = 442$

- **With these techniques one can handle results involving finitely many of the largest eigenvalues of $\mathbf{XX'}$.**
- **One cannot handle the smallest eigenvalues, determinants,. . . .**
- **The largest eigenvalues have different sizes and the eigenvectors are localized: let $\mathbf{V}_k$ be the unit eigenvector corresponding to $\boldsymbol{\lambda}_{(k)} = \boldsymbol{\lambda}_{L_k}$. Then**

$$\|\mathbf{V}_k - \mathbf{e}_{L_k}\| \xrightarrow{\mathbb{P}} \mathbf{0}\,.$$

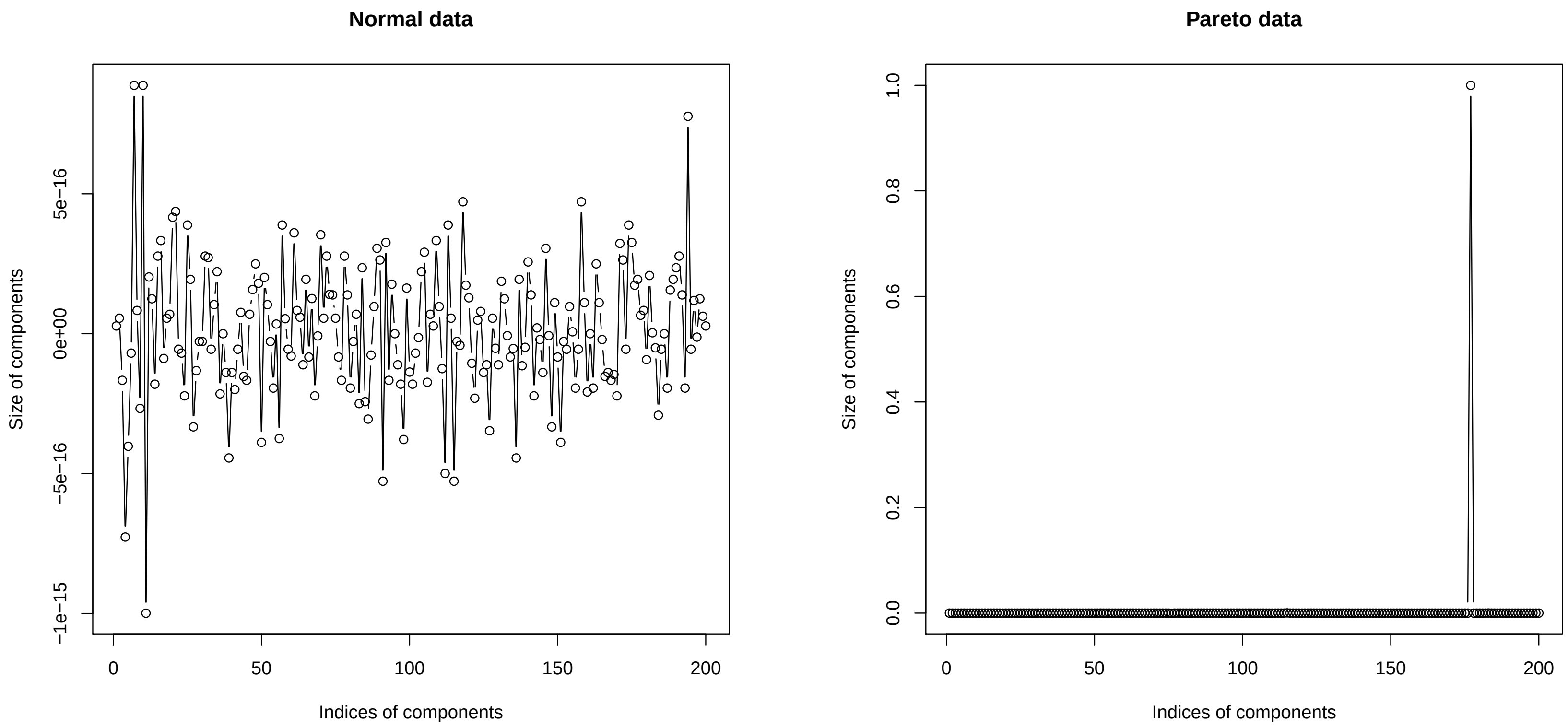

FIGURE 7. The components of the eigenvector $\mathbf{V_1}$. Right: The case of iid Pareto(0.8) entries. Left: The case of iid standard normal entries. We choose $\boldsymbol{p = 200}$ and $\boldsymbol{n = 1,000}$.

4. Extensions

- **The main idea in the iid case is to exploit the heavier tails of X^2 in comparison with those of X_1X_2.**
- **Similar behavior can be observed for regularly varying stochastic volatility models**

$$X_{it} = \sigma_{it} Z_{it}\,, \qquad i,t \in \mathbb{Z}\,.$$

with index $\alpha \in (0,4)$.

- **Nagaev-type large deviation for stochastic volatility models** Mikosch and Wintenberger (2016) **exist and are likely to ensure**

$$a_{np}^{-2} \big\| \mathrm{XX}' - \mathrm{diag}(\mathrm{XX}') \big\|_2 \stackrel{\mathbb{P}}{\to} 0\,.$$

- **Similar techniques apply when $\sigma_{it} = \sigma_{it}^{(n)}$ assumes zero with positive probability, e.g. σ_{it} Bernoulli distributed and $\mathbb{P}(\sigma^{(n)} = 0) \to 1$; see Auffinger et al. (2016). (thinning of X)**
- **The largest eigenvalue of sample correlation matrices of an iid heavy-tailed sequence has similar behavior as in the light-tailed case ask Johannes Heiny.**

5. Another structure where the squares dominate: linear processes. Davis, Pfaffel, Stelzer (2014), Davis, Mikosch, Pfaffel (2016)

- **Special case:**

$$X_{it} = \theta_0 Z_{i,t} + \theta_1 Z_{i-1,t}$$

for iid (Z_{it}) with regularly varying Z with index $\alpha \in (0,4)$, real coefficients θ_i.

- **Observe that**

$$\sum_{t=1}^{n} X_{i,t}^2 = \sum_{t=1}^{n} \big(\theta_0^2 Z_{i,t}^2 + \theta_1^2 Z_{i-1,t}^2\big) + 2\theta_0\theta_1 \sum_{t=1}^{n} Z_{i,t} Z_{i-1,t}$$
$$= \theta_0^2 D_i^{\rightarrow} + \theta_1^2 D_{i-1}^{\rightarrow} + o_P(a_n^2)\,.$$

- **Here we used Nagaev-type large deviations and the fact that Z^2 has tail index $\alpha/2$, while $Z_1 Z_2$ has tail index α.**
- **Similarly,**

$$\begin{aligned} \sum_{t=1}^{n} X_{i,t} X_{i+1,t} &= \theta_0 \theta_1 \sum_{t=1}^{n} Z_{i,t}^2 + o_P(a_n^2) \\ &= \theta_0 \theta_1 D_i^{\rightarrow} + o_P(a_n^2)\,. \end{aligned}$$

- **This leads to the approximation**

$$\begin{aligned} &\begin{pmatrix} \sum_{t=1}^{n} X_{i,t}^2 & \sum_{t=1}^{n} X_{i,t} X_{i+1,t} \\ \sum_{t=1}^{n} X_{i,t} X_{i+1,t} & \sum_{t=1}^{n} X_{i+1,t}^2 \end{pmatrix} \\ \approx &\begin{pmatrix} \theta_0^2 & \theta_0\theta_1 \\ \theta_0\theta_1 & \theta_1^2 \end{pmatrix} D_i^{\rightarrow} + \begin{pmatrix} \theta_1^2 & 0 \\ 0 & 0 \end{pmatrix} D_{i-1}^{\rightarrow} + \begin{pmatrix} 0 & 0 \\ 0 & \theta_0^2 \end{pmatrix} D_{i+1}^{\rightarrow}. \end{aligned}$$

- The sample covariance matrix can be approximated by

$$\left\| \mathrm{XX}' - \sum_{i=1}^{p} D_i^{\rightarrow} \mathrm{M}_i \right\|_2 = o_P(a_{np}^2)\,,$$

where

$$\mathrm{M}_1 = \begin{pmatrix} \theta_0^2 & \theta_0\theta_1 & 0 & \dots & 0 \\ \theta_0\theta_1 & \theta_1^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \mathrm{M}_2 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \theta_0^2 & \theta_0\theta_1 & \dots & 0 \\ 0 & \theta_0\theta_1 & \theta_1^2 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \dots$$

- Denote the order statistics of the $D_i^{\rightarrow}$ by $D_{(1)}^{\rightarrow} \geq \cdots \geq D_{(p)}^{\rightarrow}$ and $D_{L_i}^{\rightarrow} = D_{(i)}^{\rightarrow}$.

- Then

$$a_{np}^{-2} \left\| \mathrm{XX}' - \sum_{i=1}^{p} D_{L_i}^{\rightarrow} \mathrm{M}_{L_i} \right\|_2 \xrightarrow{\mathbb{P}} 0\,.$$

- For $k = k_n \to \infty$ slowly,

$$a_{np}^{-2} \Big\| \mathrm{XX}' - \sum_{i=1}^{k} D_{L_i}^{\to} \mathrm{M}_{L_i} \Big\|_2 \xrightarrow{\mathbb{P}} 0\,.$$

- Since $(D_i^{\to})$ is iid, $(L_1, \ldots, L_p)$ is a random permutation of $(1, \ldots, p)$, hence the event

$$A_k = \{|L_i - L_j| > 1, i \neq j = 1, \ldots, k\}$$

has probability close to one provided $k^2 = o(p)$.

- On the set A_k, the matrix $\sum_{i=1}^{k} D_{L_i}^{\to} \mathrm{M}_{L_i}$ is block-diagonal with non-zero eigenvalues $D_{L_i}^{\to}(\theta_0^2 + \theta_1^2)$, $i = 1, \ldots, k$.

- Here we used that M_{L_i} has rank 1 with non-zero eigenvalue equal to $\theta_0^2 + \theta_1^2$.

- By **Weyl's inequality**,

$$a_{np}^{-2} \max_{i=1,\dots,k} \left| \lambda_{(i)} - D_{L_i}^{\rightarrow}(\theta_0^2 + \theta_1^2) \right| \le a_{np}^{-2} \left\| \mathrm{XX}' - \sum_{i=1}^{k} D_{L_i}^{\rightarrow} \mathrm{M}_{L_i} \right\|_2 \xrightarrow{\mathbb{P}} 0 \,.$$

- **Extension to general linear structure:**

$$X_{it} = \sum_{k,l} h_{kl} \, Z_{k-i,l-t}$$

- Use truncation of the coefficient matrix $H = (h_{kl})$ of the linear process.
- Then

$$a_{np}^{-2} \max_{i=1,\dots,p} \left| \lambda_{(i)} - \delta_{(i)} \right| \xrightarrow{\mathbb{P}} 0 \,, \qquad n \to \infty \,,$$

where $\delta_{(1)}, \dots, \delta_{(p)}$ are the p ordered values (with respect to absolute value) of the set

$\{(D_i^{\to} - \mathbb{E}[D_1^{\to}])v_j, i = 1, \ldots, k; j = 1, 2, \ldots\}$ for $\alpha \in (0, 2)$, $(\alpha \in (2, 4))$ where (v_j) are the eigenvalues of

$$H\,H' = \Big(\sum_{l=0}^{\infty} h_{il}h_{jl}\Big)_{i,j=1,2,\ldots}$$

- The mapping theorem implies for suitable real or complex numbers (v_j)

$$\sum_{j=1}^{\infty}\sum_{i=1}^{p} \varepsilon_{a_{np}^{-2}(D_i - ED_1)v_j} \xrightarrow{\text{d}} \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \varepsilon_{\Gamma_i^{-2/\alpha}v_j}\,.$$

  The limit is a Poisson cluster process.

- An example: The separable case: We assume $h_{kl} = \theta_k c_l$. The matrix $HH' = \sum_{l=0}^{\infty} c_l^2\,\big(\theta_i\theta_j\big)_{i,j\geq 0}$ has rank $r = 1$ and

$$v_{(1)} = \sum_{l=0}^{\infty} c_l^2 \sum_{k=0}^{\infty} \theta_k^2\,.$$

The limit point process is Poisson as in the iid case. For $\alpha < 2$,

$$a_{np}^{-2}\big(\lambda_{(1)}, \ldots, \lambda_{(k)}\big) \xrightarrow{\mathrm{d}} v_{(1)}\big(\Gamma_1^{-2/\alpha}, \ldots, \Gamma_k^{-2/\alpha}\big)\,,$$

$$\frac{\lambda_{(1)}}{\lambda_{(1)} + \cdots + \lambda_{(k)}} \xrightarrow{\mathrm{d}} \frac{\Gamma_1^{-2/\alpha}}{\Gamma_1^{-2/\alpha} + \cdots + \Gamma_k^{-2/\alpha}}\,,$$

$$\frac{\lambda_{(1)}}{\lambda_1 + \lambda_2 + \cdots + \lambda_p} \xrightarrow{\mathrm{d}} \frac{\Gamma_1^{-2/\alpha}}{\Gamma_1^{-2/\alpha} + \Gamma_2^{-2/\alpha} + \cdots}\,,$$

Concluding remarks

- **Extensions to non-linear heavy-tailed multivariate time series where squares do not dominate are difficult: even the definition of X is not straightforward if one wants to model dependence between rows/columns.**
- **Heavy-tailed multivariate models with iid rows:** Davis, Pfaffel, Stelzer (2014). **Limit behavior of eigenvalues as in iid case.**
- **Multivariate models with different tail indices in rows ?**