Non-Skorokhodian functional convergence for dependent heavy-tailed models

Workshop on Dependence, Stability and Extremes
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Stable limits

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- Suppose also that

$$\lim_{x\to\infty} \frac{P(Y_j > x)}{P(|Y_j| > x)} = c_+, \quad \lim_{x\to\infty} \frac{P(Y_j < -x)}{P(|Y_j| > x)} = c_-.$$

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- Then
$$Z_n = \frac{1}{a_n}\sum_{i=1}^{n} Y_i \underset{\mathcal{D}}{\longrightarrow} Z,$$
where $\{a_n\}$ is such that $nP(|Y_1| > a_n) \to 1$.

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- Here Z has the strictly p-stable distribution $\mathrm{Pois}(\nu(p, c_+, c_-))$, with the Lévy measure $\nu = \nu(p, c_+, c_-)$ given by the density

$$f_\nu(x) = (pc_+ I(x > 0) + pc_- I(x < 0))|x|^{-(1+p)}.$$

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- Notice that we mention two different papers by Skorokhod.

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- Thus linear processes form the simplest model for phenomena of clustering of big values, what is important in insurance - see e.g. Mikosch & Samorodnitsky, Ann. Appl. Probab. **10 (2000)**, 1025–1064.

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- Thus linear processes form the simplest model for phenomena of clustering of big values, what is important in insurance - see e.g. Mikosch & Samorodnitsky, Ann. Appl. Probab. **10 (2000)**, 1025–1064.
- Clustering of big values is especially seen in the case, when Y_j's have really heavy tails.

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showed the following theorem.

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Let $X_i = Y_i - Y_{i-1}$, i.e. $c_0 = 1$ and $c_1 = -1$, then $A = \sum_j c_j = 0$ and we have

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In particular, none of Skorokhod's J_1, J_2, M_1 and M_2 topologies is applicable.

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- More in Bojan's talk this afternoon (hopefully).

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If $c_j = 0$, for $j \leqslant 0$ and $c_1, c_2, \ldots \in \mathbb{R}^1$ are such that

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then on $(\mathbb{D}, M_2)$

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- Basrak and Krizmanić confirmed this conjecture in 2014,
- But we saw by example that in the general case none of Skorokhod's J_1, J_2, M_1 and M_2 topologies is applicable.

Linear processes converge in S

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Theorem (Balan, J. & Louhichi (2012, 2014, 2016))

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Theorem (Zhang, Sin &Ling (2015))

Under some additional technical conditions linear processes with GARCH(1,1) stationary noise converge to $A \cdot Z(t)$ on the Skorokhod space $\mathbb{D}([0,1])$ equipped with the S topology.

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A direct application of the linear structure of S

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$$\frac{1}{n a_n^\beta} \sum_{k=1}^{n} \Big| \sum_{i=1}^{k} \big(\big(\sum_j c_{i-j} Y_j \big) - A Y_i \big) \Big|^\beta \xrightarrow[\mathcal{P}]{} 0.$$

- There exist more advanced examples of the use of the S topology, mainly related to the convergence of stochastic integrals.

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- There exist more advanced examples of the use of the S topology, mainly related to the convergence of stochastic integrals.
- For statistical examples of this type - see e.g. Chen & Zhang (2010), (2013).

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- It is possible to give an explicit expression for $x_n \longrightarrow_S x_0$ (see J. (1997)).

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$$\|x_n - v_{n,\varepsilon}\|_\infty \leqslant \varepsilon, \qquad n = 0, 1, 2, \dots, \tag{1}$$

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- S is weaker than M_1 (and J_1) but is incomparable with M_2.

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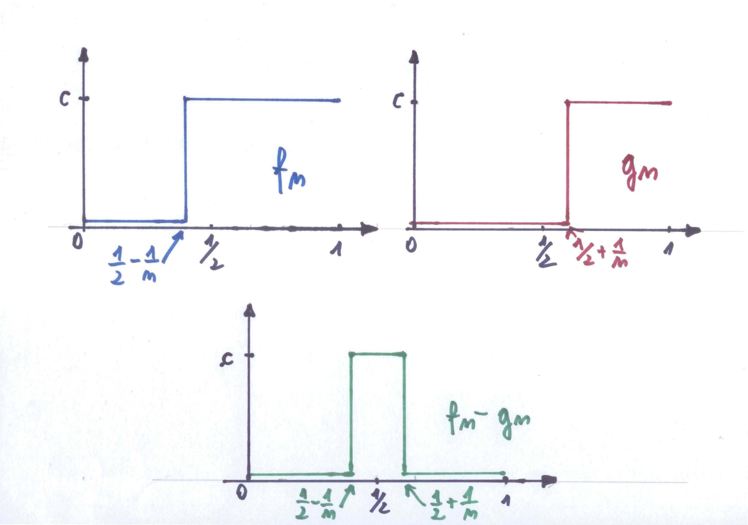

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- S is a submetric topology, for there exists a countable family of S-continuous functions which separate points in $\mathbb{D}$.
- In particular, compact subsets of $(\mathbb{D}, S)$ are metrisable.
- Notice that we arrive to S starting from criteria of compactness!

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  a.s. convergence $\longleftrightarrow$ convergence in probability.

- The question is: can we provide a "compact" characterization of $\stackrel{*}{\longrightarrow}_S$?

Compact definition of $\xrightarrow{*}_S$

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- Let $A_n \in \mathbb{A}$, $n = 0, 1, 2, \ldots$. We say that $A_n \longrightarrow_\tau A_0$, if

$$\sup_{t \in [0,1]} |A_n(t) - A_0(t)| \to 0,$$

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Theorem

$x_n \xrightarrow{*}_S x_0$ if, and only if, $x_n(1) \to x_0(1)$ and

$$\int_0^1 x_n(u)\, dA_n(u) \to \int_0^1 x_0(u)\, dA_0(u),$$

for each sequence $A_n \longrightarrow_\tau A_0$.

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- If $\{x_n(1)\}$ is bounded and

$$\int_0^1 x_n(u)\,dA_n(u) \to \int_0^1 x_0(u)\,dA_0(u)$$

  for each sequence $A_n \longrightarrow_\tau A_0$, then $\{x_n\}$ is relatively S-compact, hence contains a subsequence $x_{n_k} \longrightarrow_S x_0$.

Equivalent form of $\xrightarrow{*}_S$

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$x_n \xrightarrow{*}_S x_0$ if, and only if, $x_n(1) \to x_0(1)$ and for each relatively τ-compact set $\mathcal{A} \subset \mathbb{A}$

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- Let σ be the (locally convex) topology on $\mathbb{D}$ given by the seminorm $\rho_1(x) = |x(1)|$ and the seminorms

$$\rho_{\mathcal{A}}(x) = \sup_{A \in \mathcal{A}} \left| \int_0^1 x(u)\, dA(u) \right|,$$

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  where $\mathcal{A}$ runs over relatively τ-compact subsets of $\mathbb{A}$.
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- Then $x_n \xrightarrow{*}_S x_0$ if, and only if, $x_n \longrightarrow_\sigma x_0$.
- Corollary: $S \supset \sigma$.
- Conjecture: $S \equiv \sigma$. In other words, $(\mathbb{D}, S)$ is a linear topological space (in fact: locally convex LTS).

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- $(\mathbb{H}, \tau_s)$ is a linear topological space!
- Because τ_s coincides with so called bounded $\mathbb{H}$-topology on $\mathbb{H}$!
- The space $\mathcal{D}$ (Schwartz's sample functions) with the topology of the inductive limit does not have this property!

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- Addition is not sequentially J_1-continuous!
- A discontinuous function cannot be approximated by continuous functions in the J_1 topology.
- OBSERVATION: the S topology is weaker than J_1.
- QUESTION: Can we find a position for S in the hierarchy of topologies on $\mathbb{D}$?

Addition is not continuous in J_1, **but** is sequentially continuous **in** S

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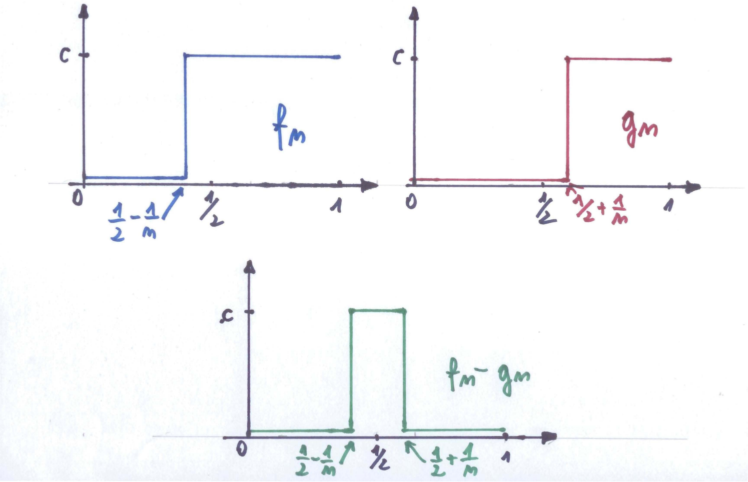

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Corollary

Were $(\mathbb{D}, S)$ a linear topological space, S would be the finest linear topology on $\mathbb{D}$ "below" J_1.

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Remark

S is incomparable with Skorokhod's M_2!