Stability of point process, regular variation and branching random walk

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- *Each individual in the n-th generation produce independently of each other and everything else.*

Growth process

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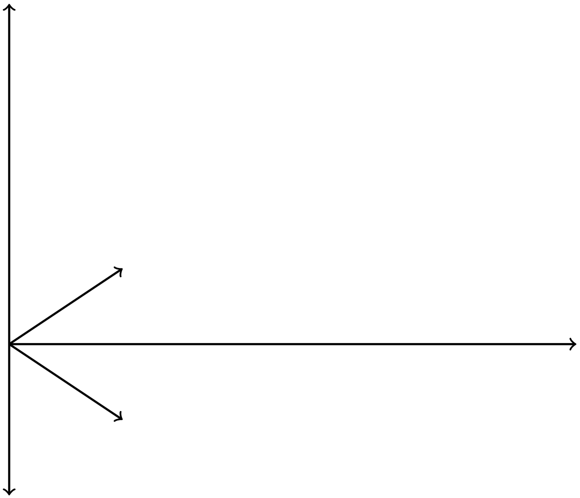

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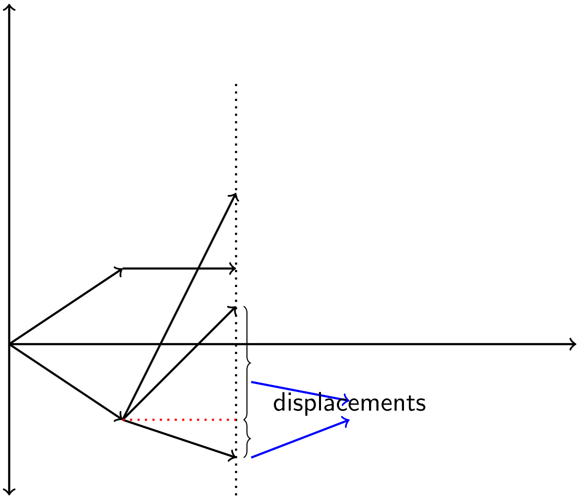

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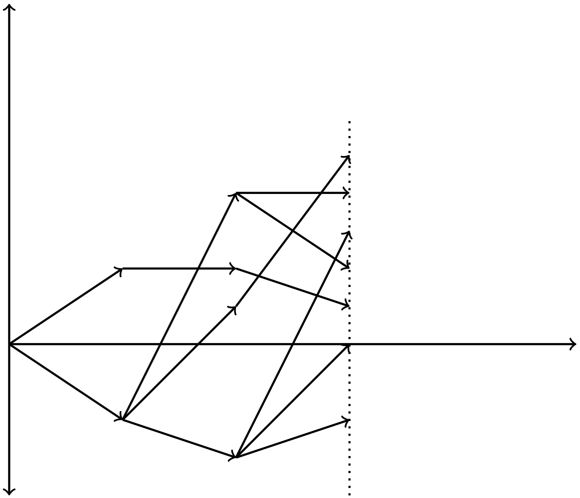

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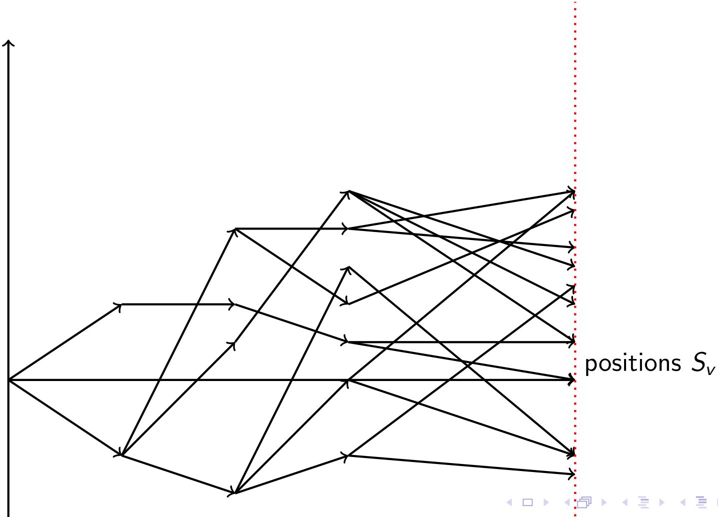


Questions?

- *The underlying tree is a Galton-Watson tree.*
- *Various assumptions on displacements and positions can be assumed.*
- *Questions of interest: If S_v denotes the position of a particle v then the behaviour as $n \to \infty$ of*

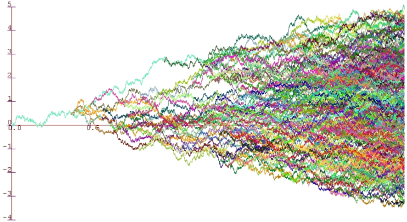
$$N_n = \sum_{|v|=n} \delta_{a_n^{-1}(S_v - b_n)}.$$

- *Position of the top most particle in the n-th generation and scaling limits.*

How did it begin? and state of the art!

Branching Brownian motion (BBM):

- *At time* 0*, particle at* $0 \in \mathbb{R}$*.*
- *Particle moves by a Brownian motion until for exponential time.*
- *After the step, particle splits into two. Repeat independently.*
- $N(t) \sim e^{-t}$ *number of particles in time* t *and positions be denoted by* $S_1(t), \cdots, S_{N(t)}(t)$*.*



[Picture by Matt Roberts]

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- **Bramson (1978)** *showed*

$$u(t, x+m(t)) \to w(x) \qquad m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t.$$

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- **Arguin-Bovier-Kistler** *(2013),* **Aidekon-Brunet-Berestycki-Shi** *(2013) showed the point process*

$$L_t = \sum_{1 \le i \le N(t)} \delta_{S_i(t)-m(t)} \to L, \text{ where } L \text{ is superposable.}$$

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- **Madaule** *(2015) : Point process convergence of the position in n-th generation (seen from the tip).*
- *Non-boundary, heavy tails:* **Durrett** *(1979, 1983),***Bhattacharya, H., Roy** *(2015, 2016),* **Bhattacharya, Maulik, Palmowski, Roy** *(2016+).*

Assumptions on Branching Mechanism

- *Underlying tree is a Galton-Watson tree.*
- Z_n *denotes the number of particles at n-th generation and* $\mu := \mathrm{E}(Z_1) \in (1, \infty)$.
- *We shall assume that* $\mathrm{P}(Z_1 = 0) = 0$ *(no leaves).*
- *Using martingale convergence theorem,*

$$\frac{Z_n}{\mu^n} \to W (\geq 0) \text{ almost surely.}$$

- *Kesten-Stigum condition :*

$$\mathrm{E}(Z_1 \log Z_1) < \infty \Leftrightarrow \mathrm{P}(W > 0) = 1.$$

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for all $A \subset \mathbb{K}_0 = \mathbb{K} \setminus \{0_\infty\}$ *such that* $0_\infty \notin \bar{A}$ *and* $\lambda(\partial A) = 0$ *and* $\lambda(\cdot)$ *is a measure on* $\mathbb{K}_0$ *such that for all* $\epsilon > 0$, $\lambda(\mathbb{K} \setminus B(0_\infty, \epsilon)) < \infty$.

- *This convergence is introduced by* **Hult and Lindskog(2006)** *and have been extended by* **Das, Mitra, Resnick (2013)**, **Lindskog, Resnick and Roy(2014)**.

First main result

Let us denote the random point process of the positions of the particles by

$$N_n = \sum_{|v|=n} \delta_{c_n^{-1} S_v}$$

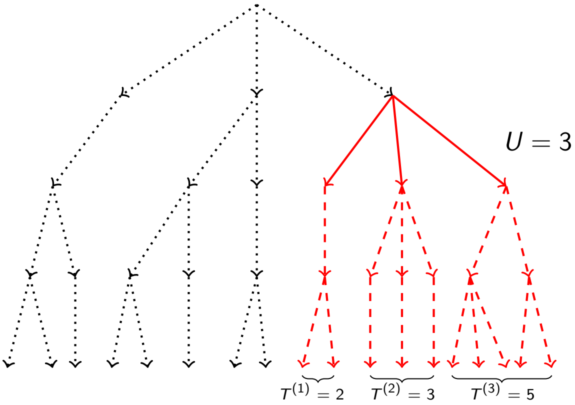
where $c_n \approx \mu^{n/\alpha}$.

Theorem (Bhattacharya, H. and Roy (2016))

Under our assumptions, the random point configuration converges in distribution to the Cox cluster process N_* *where*

$$N_* \stackrel{d}{=} \sum_{l=1}^{\infty} \sum_{k=1}^{U_l} T_l^{(k)} \delta_{W^{1/\alpha} j_l^{(k)}}$$

Description of N_*: One Large Bunch Phenomenon



$$\boxed{N_* \stackrel{d}{=} \sum_{l=1}^{\infty} \sum_{k=1}^{U_l} T_l^{(k)} \delta_{j_l^{(k)} W^{\frac{1}{\alpha}}}}, \text{ where } \sum_{l=1}^{\infty} \delta_{\left(j_l^{(1)}, j_l^{(2)}, \dots\right)} \sim PRM(\mathbb{K}_0, \lambda)$$

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- *This is an extension of main result of* **Durrett(1983)**.
- *Extensions of point process result to multi-type in forth coming article by* **Bhattacharya, Maulik, Palmowski, Roy (2016+)**

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where N_1 and N_2 are two independent copies of N.

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- *We have shown that BD1 and BD2 are equivalent (heavy-tailed extension of* **Subag and Zeitouni (2014)***).*

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Theorem (Bhattacharya, H., Roy (2015))

Let $\mathcal{L}$ be a point process on S. Suppose $\mathcal{L}$ is $RV_{-\alpha}$, that is,

$$nP(b_n^{-1} \circ \mathcal{L} \in \cdot) \overset{HL}{\to} \mu_\alpha(\cdot).$$

Then

$$b_n^{-1} \circ \sum_{i=1}^{n} \mathcal{L}_i \Rightarrow \text{ Strictly } \alpha\text{-stable Point Process}$$

Thank You

- **Point process convergence for branching random walks with regularly varying steps:** *arXiv:1411.5646, to appear in Annales de l'Institut Henri Poincaré.*
- **Branching random walks, stable point processes and regular variation:** *arXiv:1601.01656, submitted.*