A guided tour in the theory of max-stable processes

Clément Dombry

Based on joint work with
S.Engelke, F.Eyi-Minko, Z.Kabluchko, M.Oesting, M.Ribatet, S.Stoev.

Laboratoire de Mathématiques de Besançon,
Université de Franche-Comté, France.

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Acknowledgement

I discover extreme value theory at the workshop Spatial Extremes and Applications organized by A. Davison at EPFL in July 2009.



- Laurens de Haan : theoretical foundations, non parametric statistics.
- Martin Schlather : models and simulations, RandomFields package.
- Anthony Davison : parametric estimation, applications.

Motivations

- Modeling extremes in environmental sciences :
  - maximal temperatures in a heat wave,
  - intensity of winds during a storm,
  - water heights in a flood ...
- Annual maxima method :
  - no need to model seasonality,
  - no need to model dependence or clustering of extremes,
  - suitable to compute long term return levels.
- Max-stable random fields play a crucial role in this approach and have known many developments in the last decades :
  - theoretical properties,
  - statistics and inference,
  - modeling and applications.

Structure of the talk

Univariate EVT

$X_1, X_2, \ldots$ i.i.d. random variables.
Convergence of the normalized maximum

$$\frac{\max(X_1,\ldots,X_n) - b_n}{a_n} \xrightarrow{d} G, \quad a_n > 0,\ b_n \in \mathbb{R} \text{ ?}$$

Fisher and Typett's Theorem (1928)

The possible non-degenerate limits are the GEV distributions

$$\mathbb{P}[G \leq x] = \exp\Big(-(1+\gamma\frac{x-\mu}{\sigma})^{-1/\gamma}\Big), \quad 1+\gamma\frac{x-\mu}{\sigma} > 0.$$

Notation $G \sim \mathrm{GEV}(\gamma,\mu,\sigma)$ with $\gamma,\mu \in \mathbb{R}$, $\sigma > 0$.

Parametrization of the 3 parameter GEV distribution by Von Mises ('36) and Jenkinson ('55).

With a possible change of the norming constant, one can always assume that the limit is $\mathrm{GEV}(\gamma, 0, 1)$.

Univariate EVT

- Max-stability : for (G_i) i.i.d. $\mathrm{GEV}(\gamma, \mu, \sigma)$

$$\tilde{a}_n^{-1}(\max(G_1, \ldots, G_n) - \tilde{b}_n) \sim \mathrm{GEV}(\gamma, \mu, \sigma)$$

with $\tilde{a}_n = n^\gamma$ and $\tilde{b}_n = (\sigma - \gamma\mu)\frac{n^\gamma - 1}{\gamma}$.

- Standardization : if $G \sim \mathrm{GEV}(\gamma, \mu, \sigma)$, then

$$F = \left(1 + \gamma \frac{G - \mu}{\sigma}\right)^{1/\gamma} \sim \mathrm{Frechet}(1).$$

Univariate EVT

- Domain of attraction : under which condition does $X \sim F$ belong to the max-domain of attraction of $G \sim \mathrm{GEV}(\gamma, 0, 1)$?
- Introduce the return level of period $t > 1$ defined by

$$U(t) = F^{\leftarrow}(1 - 1/t).$$

First order condition

F belong to the max-domain of attraction of $\mathrm{GEV}(\gamma, 0, 1)$ if and only if U satisfies the extended regular variation condition

$$\lim_{t \to \infty} \frac{U(tx) - U(t)}{a(t)} = \frac{x^\gamma - 1}{\gamma}, \quad x > 0,$$

for some positive function a.

- If $\gamma > 0$, one can take $a(t) = \gamma U(t)$ and the condition rewrites

$$\lim_{t \to \infty} \frac{U(tx)}{U(t)} = x^\gamma, \quad x > 0.$$

Univariate EVT

Regular variations, maxima, peak over threshold and point processes.

Proposition

Let $(X_i)_{i\geq 1} \overset{i.i.d.}{\sim} F$ be non-negative random variables and $a_n = U(n)$.
The following statements are equivalent :

i) $1 - F(x) = x^{-\alpha} L(x)$ with $\alpha > 0$ and L slowly varying ;

ii) $a_n^{-1} \max(X_1, \ldots, X_n) \xrightarrow{d} \text{Frechet}(\alpha)$, as $n \to \infty$;

iii) $\mathbb{P}[X/u \in \cdot \mid X > u] \xrightarrow{d} \text{Pareto}(\alpha)$, as $u \to \infty$;

iv) $\{a_n^{-1} X_1, \ldots, a_n^{-1} X_n\} \xrightarrow{d} \mathrm{PPP}(\mathrm{d}x^{-\alpha})$ on $(0, +\infty]$, as $n \to \infty$,

[Gnedenko, Balkema and de Haan, Resnick, ...]

Maxima of i.i.d. processes

- Let $\mathcal{C}(S)$: space of continuous functions on a compact metric space S.
- Let $X_1, X_2, \ldots$ i.i.d. stochastic processes in $\mathcal{C}(S)$ with marginal cdf F_s and associated $U_s(t) = F_s^{\leftarrow}(1 - 1/t)$.
- Under which condition does the normalized maximum converge ?

$$\left(\max_{1 \leq i \leq n} \frac{X_i(s) - b_n(s)}{a_n(s)} \right)_{s \in S} \xrightarrow{d} (Z(s))_{s \in S} \quad \text{in } \mathcal{C}(S)?$$

We require the limit Z has non-degenerate margins.

Maxima of i.i.d. processes

- From univariate theory, $Z(s) \sim \mathrm{GEV}(\gamma(s), \mu(s), \sigma(s))$.
  Continuity of the path impose continuity of γ, μ, σ.
  W.l.o.g. one can assume $\mu \equiv 0$ and $\sigma \equiv 1$.
- Necessary condition from univariate EVT :

$$\lim_{n\to\infty} \frac{U_s(nx) - U_s(n)}{a_n(s)} = \frac{x^{\gamma(s)} - 1}{\gamma(s)} \quad s \in S, x > 0.$$

- Questions :
  - Is the condition sufficient ?
  - How to characterize the limit process Z ?

Maxima of i.i.d. processes

Assume the margins F_s are continuous.
Note that $\xi(s) = \frac{1}{1-F_S(X(s))}$ has 1-Pareto distribution.

Theorem (de Haan and Lin '01)

The following are equivalent :

i) there are continuous normalisation functions $a_n(\cdot) > 0$, $b_n(\cdot)$ such that

$$\left(\max_{1 \le i \le n} \frac{X_i(s) - b_n(s)}{a_n(s)} \right)_{s \in S} \xrightarrow{d} (Z(s))_{s \in S} \quad \text{in } \mathcal{C}(S)$$

with $Z(s) \sim \mathrm{GEV}(\gamma(s), 0, 1)$;

ii) we have, for all $x \in (0, +\infty)$,

$$\lim_{n \to \infty} \frac{U_s(nx) - U_s(n)}{a_n(s)} = \frac{x^{\gamma(s)} - 1}{\gamma(s)} \quad \text{uniformly for } s \in S$$

and convergence of the standard-Pareto normalized process

$$\left(\frac{1}{n} \max_{1 \le i \le n} \frac{1}{1 - F_s(X_i(s))} \right)_{s \in S} \xrightarrow{d} (\eta(s))_{s \in S} \quad \text{in } \mathcal{C}(S).$$

Then $\{\eta(s)\}_{s \in s} \stackrel{d}{=} \{(1 + \gamma(s) Z(s))^{1/\gamma(s)}\}_{s \in S}$.

Max-stability

- The limit process η is simple max-stable, i.e. it has unit Fréchet margins and

$$\frac{1}{n}\bigvee_{i=1}^{n}\eta_i \stackrel{d}{=} \eta, \quad \eta_1,\eta_2,\ldots \text{ i.i.d. copies of } \eta.$$

- The limit process Z in $\mathcal{C}(S)$ is max-stable, i.e. it has non degenerate margins and there are functions $\tilde{a}_n > 0$ and $\tilde{b}_n$ in $\mathcal{C}(S)$ such that

$$\bigvee_{i=1}^{n}\frac{Z_i-\tilde{b}_n}{\tilde{a}_n} \stackrel{d}{=} Z, \quad Z_1,Z_2,\ldots \text{ i.i.d. copies of } Z.$$

- Conclusion : the class of max-stable processes is equal to the class of possible non degenerate limits for maxima of i.i.d. stochastic processes under affine normalisation.

Standardization

- Standardization : if Z is max-stable in $\mathcal{C}(S)$ with margins $Z(s) \sim \mathrm{GEV}(\gamma(s), \mu(s), \sigma(s))$, then

$$\{\eta(s)\}_{s\in S} = \left\{ \left(1 + \gamma(s) \frac{Z(s) - \mu(s)}{\sigma(s)} \right)^{1/\gamma(s)} \right\}_{s\in S}$$

  is simple max-stable.
- For the theory, we focus on simple max-stable processes. Marginal distributions important for applications and statistics.

Conditions for convergence

Introduce the polar decomposition $f \leftrightarrow (\|f\|_\infty, f/\|f\|_\infty)$ and consider

$$\overline{\mathcal{C}}^+(S) = (0, +\infty] \times \mathcal{C}_1^+(S)$$

with $\mathcal{C}_1^+(S) = \{f \in \mathcal{C}(S) : f \geq 0, \|f\|_\infty = 1\}$.

Theorem (de Haan and Lin '01)

Let $\xi, \xi_1, \xi_2, \ldots$ be i.i.d. processes in $\mathcal{C}^+(S)$ with standard Pareto margins. The following are equivalent :

i) $\frac{1}{n}\bigvee_{i=1}^n \xi_i \xrightarrow{d} \eta$ in $\mathcal{C}(S)$;

ii) $n\mathbb{P}(n^{-1}\xi \in A) \to \mu(A)$ for all bounded $A \in \overline{\mathcal{C}}^+(S)$ such that $\mu(\partial A) = 0$;

iii) the point process $\Phi_n = \{n^{-1}\xi_1, \ldots, n^{-1}\xi_n\}$ converges to a Poisson point process Φ with intensity μ on $\overline{\mathcal{C}}^+(S)$.

Exponent measure

- The limit measure μ is called the exponent measure.
- It is homogeneous of order -1, i.e.

$$\mu(uA) = u^{-1}\mu(A), \quad u > 0, A \subset \overline{\mathcal{C}}^{+}(S) \text{ Borel.}$$

- The f.d.d. of μ are linked to those of η by the relation

$$\mathbb{P}[\eta(s_j) \leq z_j,\ j = 1, \ldots, k] = \exp\left(-\mu\{f : f(s_i) > z_i \text{ for some } j = 1, \ldots, k\}\right).$$

- In particular, due to standard 1-Fréchet margins

$$\mu(\{f : f(s) > 1\}) = -\log \mathbb{P}(\eta(s) \leq 1) = 1, \quad s \in S,$$

and due to continuity

$$\mu(\{f : \|f\|_\infty > 1\}) = -\log \mathbb{P}(\|\eta\|_\infty \leq 1) < \infty.$$

Spectral measure

- By homogeneity, the spectral measure σ on $\mathcal{C}_1^+(S)$ defined by

$$\sigma(A) = c^{-1}\mu(\{f : \|f\|_\infty > 1, f/\|f\|_\infty \in A\}) \quad \text{with } c = \mu(\{f : \|f\|_\infty > 1)$$

  characterizes the exponent measure.
- In polar coordinate $f \leftrightarrow (\|f\|_\infty, f/\|f\|_\infty)$, we have

$$\mu(\{f : \|f\|_\infty > r, f/\|f\|_\infty \in A\}) = cr^{-1}\sigma(A)$$

  so that the limit point process $\Phi \sim \mathrm{PPP}(\mathrm{d}\mu)$ can be constructed as

$$\Phi \stackrel{d}{=} \{U_i Y_i, i \geq 1\} \quad \text{with } \{U_i\} \sim \mathrm{PPP}(u^{-2}\mathrm{d}u),\ \{Y_i/c\} \stackrel{i.i.d.}{\sim} \sigma(\mathrm{d}f).$$

- Note that $\|Y\|_\infty \equiv c$ and $\mathbb{E}[Y(s)] \equiv 1$.

Structure of simple max-stable processes

Theorem (de Haan 1984, Giné et al. 1990)

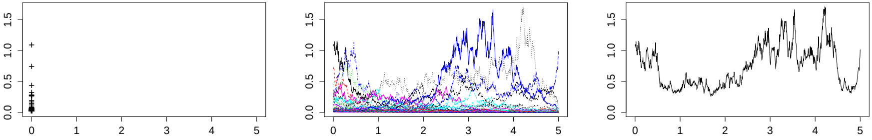
Let η be a simple max-stable process in $\mathcal{C}(S)$. Then η can be represented as

$$\Big(\eta(s)\Big)_{s\in S} \stackrel{d}{=} \Big(\bigvee_{i\geq 1} U_i Y_i(s)\Big)_{s\in S}$$

with

- $U_1 > U_2 > U_3 > \cdots$ the points of a PPP on $(0,+\infty)$ with intensity $u^{-2}\mathrm{d}u$;
- $Y_1, Y_2, Y_3, \ldots$ i.i.d. copies of a process Y in $\mathcal{C}(S)$ such that $\mathbb{E}[Y(s)] = 1$, $s \in \mathcal{S}$, and $\mathbb{E}[\|Y\|_\infty] < \infty$.
- $(Y_i)_{i\geq 1}$ and $\{U_i, i \geq 1\}$ independent.

Example : "historical Brown-Resnick" process $Y(s) = e^{B(s)-|s|/2}$



Finite dimensional distributions

Finite dimensional distributions

For $k \geq 1$, $s_1, \ldots, s_k \in S$ and $z_1, \ldots, z_k > 0$

$$\mathbb{P}[\eta(s_j) \leq z_j, 1 \leq j \leq k] = \exp\left(-V(z_1, \ldots, z_k)\right)$$

with

$$V(z_1, \ldots, z_k) = \mathbb{E}\left[\bigvee_{j=1}^{k} \frac{Y(s_j)}{z_j}\right]$$

- Exercise : prove that max-linear combinations $\bigvee_{i=1}^{k} a_i \eta(s_i)$, $a_1, \ldots, a_k > 0$, are 1-Fréchet . What is the scale parameter ?

Pair extremal coefficient

In particular for $k = 2$ and $z_1 = z_2 = z$

$$\mathbb{P}[\eta(s_1) \leq z, \eta(s_2) \leq z] = \exp\left(-\mathbb{E}[Y(s_1) \vee Y(s_2)]/z\right)$$

i.e. $\eta(s_1) \vee \eta(s_2)$ has a 1-Fréchet distribution with scale parameter

$$\theta(s_1, s_2) = \mathbb{E}[Y(s_1) \vee Y(s_2)].$$

Pair extremal coefficient

- $\theta(s_1, s_2) \in [1, 2]$ is called the pair extremal coefficient.
- It gives some insight into the bivariate dependence structure :
  - $\theta(s_1, s_2) = 2$ iff $\eta(s_1)$ and $\eta(s_2)$ are independent.
  - $\theta(s_1, s_2) = 1$ iff $\eta(s_1)$ and $\eta(s_2)$ are equal a.s.

Structure of the talk

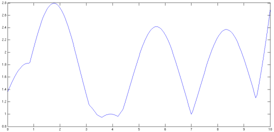
Stationary spatial models

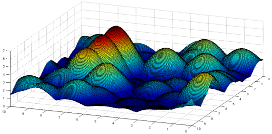
M. Schlather, Models for stationary max-stable random fields. *Extremes*. 2002.

- Early construction of the moving maximum process by Deheuvels (82) :

$$\eta(s) = \bigvee_{i \geq 1} U_i h(s - X_i), \quad s \in \mathbb{R}^d,$$

with h a density on $\mathbb{R}^d$ and $\{U_i, X_i\}_{i \geq 1} \sim \mathrm{PPP}(u^{-2}\mathrm{d}u\mathrm{d}x)$.





- Interpretation as a storm process by Smith (90).

Storm process

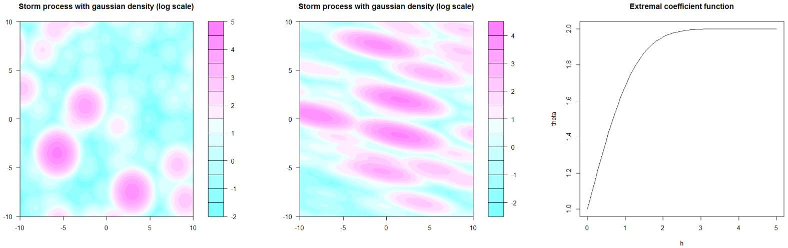
Proposition

For the storm process on $\mathbb{R}^d$ with gaussian density $h \sim \mathcal{N}(0, \Sigma)$,

$$\mathbb{P}[\eta(s_1) \leq z_1, \eta(s_2) \leq z_2] = \exp\left\{ -\frac{1}{z_1}\Phi\left(\frac{a}{2} + \frac{1}{a}\log\frac{z_2}{z_1}\right) - \frac{1}{z_2}\Phi\left(\frac{a}{2} + \frac{1}{a}\log\frac{z_1}{z_2}\right) \right\}$$

with $a^2 = (s_2 - s_1)^T \Sigma^{-1}(s_2 - s_1)$ the Mahalanobis distance.
Hence ,

$$\theta(h) = 2\Phi\left(\frac{\sqrt{h^T\Sigma^{-1}h}}{2}\right), \quad h \in \mathbb{R}^d.$$



Mixed moving maximum process

- The construction can be generalized to the class of mixed moving maximum process by making the density h random.

Definition : M3-process

A continuous simple max-stable process is a M3-process if

$$\eta(s) = \bigvee_{i \geq 1} U_i Y_i(s - X_i), \quad s \in \mathbb{R}^d,$$

with

- $\{U_i, X_i\}_{i\geq 1} \sim \mathrm{PPP}(u^{-2}\mathrm{d}u\mathrm{d}x)$
- $\{Y_i\}_{i\geq 1}$ i.i.d. non-negative continuous random process such that

$$\mathbb{E}\left[\int Y(x)\mathrm{d}x\right] = 1 \quad \text{and} \quad \mathbb{E}\left[\int \sup_{h\in K} Y(x+h)\mathrm{d}x\right] < \infty.$$

- Interesting class because of its good ergodic/mixing properties.

Extremal Gaussian process

- Alternatively, stationarity obviously obtained when starting from a stationary spectral process Y in the representation

$$\eta(s) = \bigvee_{i \geq 1} U_i Y_i(s), \quad s \in \mathbb{R}^d.$$

- Extremal Gaussian model :

$$Y(s) = \sqrt{2\pi} W(s)_+, \quad s \in \mathbb{R}^d$$

  with W stationary continuous Gaussian random fields with unit variance and correlation ρ. [Schlather 2012]
- Classical geostatistics at our disposal for the choice of ρ !

Extremal Gaussian process

Proposition

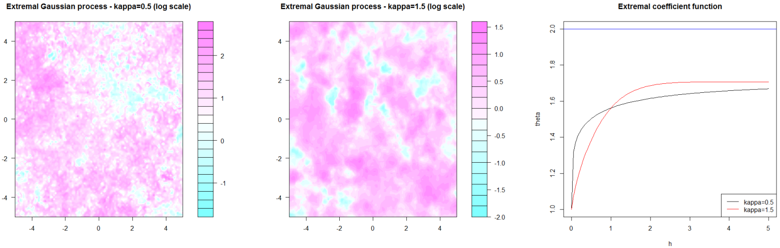
For the extremal Gaussian model :

$$\mathbb{P}[\eta(s_1) \leq z_1, \eta(s_2) \leq z_2] = \exp\left\{-\frac{1}{2}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)\left(1+\sqrt{1-2(1+\rho(s_2-s_1))\frac{z_1 z_2}{(z_1+z_2)^2}}\right)\right\}$$

and

$$\theta(h) = 1 + \sqrt{\frac{1-\rho(h)}{2}}, \quad h \in \mathbb{R}^d.$$

- Example : isotropic model with $\rho(h) = e^{-\|h\|^\kappa}$, $\kappa = 0.5, 1.5$.



Extremal-t process

- Extremal-t model with $\nu > 0$ degrees of freedom :

$$Y(s) = c_\nu W(s)_+^\nu, \quad c_\nu = \frac{\sqrt{\pi} 2^{1-\nu/2}}{\Gamma((1+\nu)/2)},$$

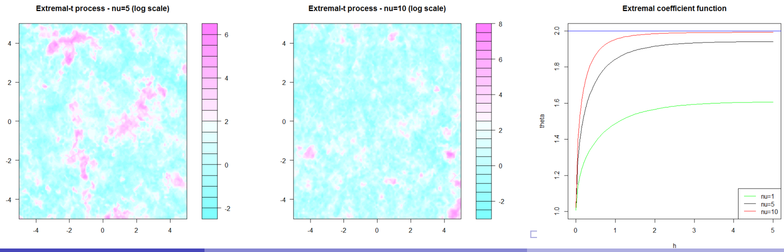
with W stationary continuous Gaussian random fields with unit variance and correlation ρ. [Nikoloulopoulos et al. 2009, Opitz 2013]

Proposition

For the extremal-t model :

$$\theta(h) = 2T_{\nu+1}\left(\sqrt{\frac{1-\rho(h)}{(1+\nu)(1+\rho(h))}}\right), \quad h \in \mathbb{R}^d.$$

- Example : isotropic model with $\rho(h) = e^{-\|h\|}$, $\nu = 5, 10$.



Brown-Resnick model

Proposition (Kabluchko et al. 2009)

Let W be a continuous stationary increments centered Gaussian process on $\mathbb{R}^d$ and define the lognormal process

$$Y(s) = e^{W(s)-\sigma^2(s)/2}, \quad s \in \mathbb{R}^d.$$

Then the simple max-stable process $\eta = \bigvee_{i\geq 1} U_i Y_i$ is a stationary max-stable process.
Its law depends only on the semi-variogram

$$\gamma(h) = \frac{1}{2}\mathbb{E}[(W(t+h) - W(t))^2], \quad h \in \mathbb{R}^d.$$

Furthermore, we have

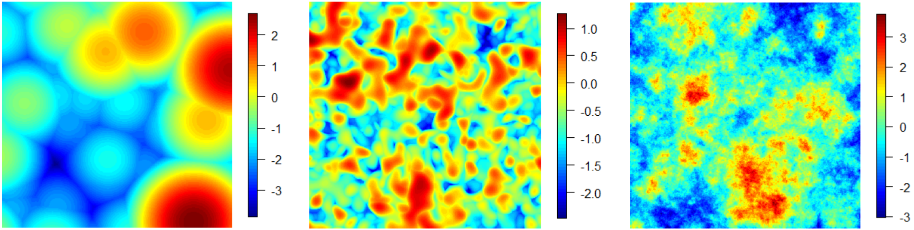
$$\mathbb{P}[\eta(s_1) \leq z_1, \eta(s_2) \leq z_2] = \exp\left\{-\frac{1}{z_1}\Phi\left(\frac{a}{2} + \frac{1}{a}\log\frac{z_2}{z_1}\right) - \frac{1}{z_2}\Phi\left(\frac{a}{2} + \frac{1}{a}\log\frac{z_1}{z_2}\right)\right\}$$

with $a = \sqrt{2\gamma(s_2 - s_1)}$ and hence

$$\theta(h) = 2\Phi\left(\sqrt{\frac{\gamma(h)}{2}}\right).$$

Brown-Resnick model

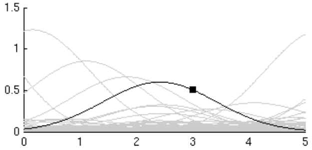
$$\gamma(h) = \|h\|^2, \qquad \gamma(h) = 1 - e^{-\|h\|^2}, \qquad \gamma(h) = \|h\|$$



- Surprisingly, the first example corresponds to the Storm process !
- Independance at long distance if $\gamma(h) \to \infty$ (examples 1 and 3)

Structure of the talk

Notion of extremal function



$$\eta(s) = \bigvee_{i\geq 1} U_i Y_i(s) = \bigvee_{i\geq 1} \phi_i(s), \quad \phi_i = U_i Y_i.$$

- Recall that $\Phi = \{\phi_i\}_{i\geq 1} \sim \mathrm{PPP}(\mu(df))$ on $\mathcal{C}_0 = \mathcal{C}(S) \setminus \{0\}$ with

$$\mu(A) = \int_0^\infty \mathbb{P}[uY \in A] u^{-2}\,\mathrm{d}u, \quad A \subset \mathcal{C}_0 \text{Borel}\ .$$

Lemma/Definition

Let $s_0 \in S$. The maximum $\eta(s_0) = \bigvee_{i\geq 1} \phi_i(s_0)$ is a.s. uniquely reached. The function that reaches the maximum is called extremal function and denoted by $\varphi_{s_0}^+$.

- How to compute the distribution of $\varphi_{s_0}^+$?

Slyvniak-Mecke Theorem

Slyvniak-Mecke Theorem

Let $\Phi \sim \mathrm{PPP}(\mu)$ on E and $F : E \times \mathcal{M}_p(E) \to [0, +\infty)$. Then,

$$\mathbb{E}\left[\sum_{x\in\Phi} F(x, \Phi \setminus \{x\})\right] = \int_E \mathbb{E}[F(x, \Phi)]\mu(\mathrm{d}x).$$

More generally, for $F : E^k \times \mathcal{M}_p(E) \to [0, +\infty)$

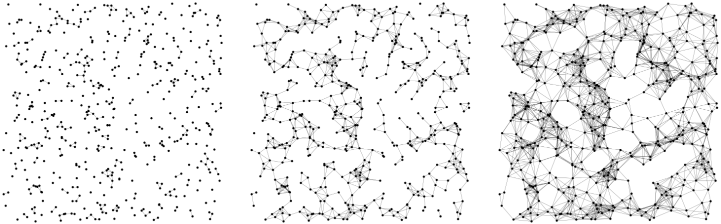
$$\mathbb{E}\left[\sum_{(x_1,\ldots,x_k)\in\Phi^k_{\neq}} F(x_1, \ldots, x_k, \Phi \setminus \{x_1, \ldots, x_k\})\right] = \int_{E^k} \mathbb{E}[F(x_1, \ldots, x_k, \Phi)]\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_k).$$

- Link with the Palm measure of the Poisson point process :

$$\text{"}\ \mathcal{L}(\Phi \mid\ x \in \Phi) = \mathcal{L}(\Phi \cup \{x\})\ \text{"}.$$

A standard application from stochastic geometry

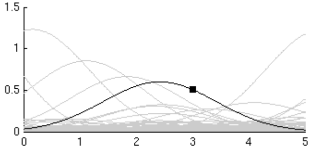
- Random geometric graph :
  - Vertices $V \sim \mathrm{PPP}(\lambda \mathrm{d}x)$ on $[0,1]^2$
  - Edges between points at distance less than $r > 0$.



[Pictures taken from Tobias Muller webpage, Utrecht university]

- Exercise : let N be the number of isolated points.
  Can you compute $\mathbb{E}[N]$, $\mathrm{Var}[N]$?

Distribution of extremal function



The extremal function $\varphi^+_{s_0}$ realizes the maximum at s_0.

Proposition (D., Eyi-Minko (2013))

The random variables $\eta(s_0)$ and $\varphi^+_{s_0}/\eta(s_0)$ are independent, $\eta(s_0)$ has a unit Fréchet distribution and the distribution of $\varphi^+_{s_0}/\eta(s_0)$ is

$$P_{s_0}(A) = \mathbb{P}(\varphi^+_{s_0}/\eta(s_0) \in A) = \mathbb{E}[Y(s_0)1_{\{Y/Y(s_0)\in A\}}]$$

Proof on blackboard.

Examples

Moving-maximum model : $\eta(s) = \bigvee_{i \geq 1} U_i h(s - X_i)$.

Proposition

For the moving-maximum model, the distribution P_{s_0} is equal to the distribution of the random function

$$\frac{h(\cdot + X - s_0)}{h(X)}, \quad X \sim h(x)\lambda(\mathrm{d}x).$$

Examples

Brown-Resnick type model : $Y(s) = \exp\left(W(s) - \sigma^2(s)/2\right)$ with W centered Gaussian process.

Proposition

In the Brown-Resnick type model, for all $s_0 \in S$, the distribution P_{x_0} is equal to the distribution of the log-normal process

$$\widetilde{Y}(s) = \exp\left(W(s) - W(s_0) - \frac{1}{2}\mathrm{Var}[W(s) - W(s_0)]\right), \quad s \in S.$$

Examples

Extremal-t model : $Y(s) = c_\nu W(s)_+^\nu$ with W continuous centered Gaussian process with unit variance and covariance ρ.

Proposition

In the extremal-t model, the distribution P_{s_0} is equal to the distribution of T_+^ν, where $T = (T(s))_{s\in S}$ is a Student process with $\nu + 1$ degrees of freedom, location and scale functions given respectively by

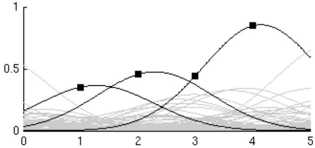
$$\mu(s) = \rho(s_0, s), \quad \hat{\rho}(s_1, s_2) = \frac{\rho(s_1, s_2) - \rho(s_0, s_1)\rho(s_0, s_2)}{(\nu + 1)}.$$

Decomposition of the point process

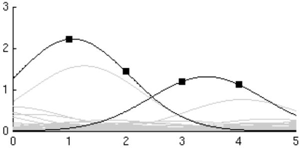
Definition

Consider k locations $s_1, \ldots, s_k \in S$.

- extremal point process : $\Phi^+ = \{\varphi^+_{s_1}, \ldots, \varphi^+_{s_k}\}$ (disregard repetitions).
- hitting scenario : random partition $\theta = (\theta_1, \ldots, \theta_\ell)$ accounting for possible repetitions within $\varphi^+_{s_1}, \ldots, \varphi^+_{s_k}$
- subextremal point process : $\Phi^- = \Phi \setminus \Phi^+$.

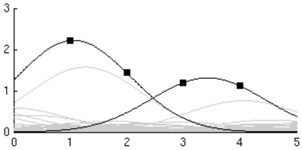


$\theta = \{\{1\}, \{2\}, \{3, 4\}\}$



$\theta = \{\{1, 2\}, \{3, 4\}\}$

Obvious constraints



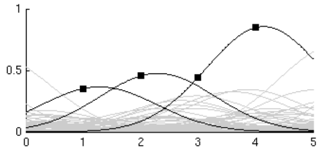
$$\varphi_1^+ = \varphi_2^+ = \varphi_{\{1,2\}}^+$$
$$\varphi_3^+ = \varphi_4^+ = \varphi_{\{3,4\}}^+$$

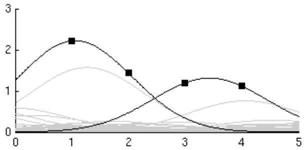
- If $\theta = (\tau_1, \dots, \tau_\ell)$:

$$\Phi^+ = \{\varphi_{\tau_1}^+, \dots, \varphi_{\tau_\ell}^+\},$$
$$\eta(s) = \bigvee_{1 \le j \le \ell} \varphi_{\tau_j}^+(s), \quad s \in \{s_1, \dots, s_k\},$$
$$\eta(s) = \varphi_{\tau_j}^+(s) > \varphi_{\tau_{j'}}^+(s), \quad s \in \tau_j, j' \neq j.$$

- For all $s \in \{1, \dots, k\}$ and $\phi \in \Phi^-$, $\eta(s) > \phi(s)$.

Joint law of (θ, Φ^+, Φ^-)





Theorem

- For all partition $\tau = (\tau_1, \ldots, \tau_\ell) \in \mathcal{P}_k$ and Borel set $A \subset \mathcal{C}^\ell$

$$\begin{aligned} &\mathbb{P}[\theta = \tau, (\varphi_1^+, \ldots, \varphi_\ell^+) \in A] \\ = \;& \int_A 1_{\{f_j >_{\tau_j} f_{j'},\ 1 \le j \neq j' \le \ell\}} \mathbb{P}[\eta <_{\tau_j} f_j, j = 1, \ldots, \ell] \mu(\mathrm{d}f_1) \cdots \mu(\mathrm{d}f_\ell). \end{aligned}$$

- Given θ and Φ_+ (and hence $\eta(s_1), \ldots, \eta(s_k)$), the conditional distribution of Φ^- is equal to the distribution of a PPP with intensity

$$1_{\{f(s) < \eta(s),\ s = s_1, \ldots, s_k\}} \mu(\mathrm{d}f).$$

Density of finite dimensional distributions

Let $\mathbf{s} = (s_1, \dots, s_k)$ and let $\mu_{\mathbf{s}}(\mathrm{d}\mathbf{z}) = \mu(f(\mathbf{s}) = \mathrm{d}\mathbf{z})$ be the intensity measure of the Poisson point process $\{U_i Y_i(\mathbf{s}), i \geq 1\} \subset [0, +\infty)^k \setminus \{0\}$.

Proposition

Assume the regularity condition $\mu_{\mathbf{s}}(\mathrm{d}\mathbf{z}) = \lambda(\mathbf{z})\mathrm{d}\mathbf{z}$ on $[0, +\infty)^k \setminus \{0\}$.

- Joint law of hitting scenario and extremal functions :

$$\mathbb{P}(\theta = (\tau_1, \dots, \tau_\ell), \varphi^+_{\tau_1}(\mathbf{s}) = \mathrm{d}\mathbf{z}_1, \dots, \varphi^+_{\tau_\ell}(\mathbf{s}) = \mathrm{d}\mathbf{z}_\ell) = \exp(-V(\vee_{j=1}^{\ell} \mathbf{z}_j)) \prod_{j=1}^{\ell} \lambda(\mathbf{z_j})\mathrm{d}\mathbf{z}_j$$

  if $\mathbf{z}_1, \dots, \mathbf{z}_\ell$ is compatible with $\tau_1, \dots, \tau_\ell$ and 0 otherwise.

- Joint law of θ and $\eta(\mathbf{s})$:

$$\mathbb{P}(\theta = (\tau_1, \dots, \tau_\ell), \eta(\mathbf{s}) = \mathrm{d}\mathbf{z}) = \exp(-V(\mathbf{z})) \left(\prod_{j=1}^{\ell} \int_{\mathbf{u}_j < \mathbf{z}_{\tau_j^c}} \lambda(\mathbf{z}_{\tau_j}, \mathbf{u}_j)\mathrm{d}\mathbf{u}_j \right) \mathrm{d}\mathbf{z}.$$

- Density of $\eta(\mathbf{s})$:

$$\mathbb{P}(\eta(\mathbf{s}) = \mathrm{d}\mathbf{z}) = \exp(-V(\mathbf{z})) \sum_{\tau \in \mathcal{P}_k} \left(\prod_{j=1}^{\ell} \int_{\mathbf{u}_j < \mathbf{z}_{\tau_j^c}} \lambda(\mathbf{z}_{\tau_j}, \mathbf{u}_j)\mathrm{d}\mathbf{u}_j \right) \mathrm{d}\mathbf{z}.$$

Brown-Resnick model

Brown-Resnick model : $Y(s) = \exp\Big(W(s) - \sigma^2(s)/2\Big)$ with W centered Gaussian process with stationary increments.

Proposition

Let $\mathbf{s} = (s_1, \dots, s_k)$ and assume the variance matrix $\Sigma_{\mathbf{s}} = \mathrm{Var}(W(\mathbf{s}))$ is non-singular. Then $\mu_{\mathbf{s}}(\mathrm{d}\mathbf{z})$ has density on $[0, +\infty)^k \setminus \{0\}$:

$$\lambda(\mathbf{z}) = C_{\mathbf{s}} \exp\left(-\frac{1}{2}\log \mathbf{z}^T Q_{\mathbf{s}} \log \mathbf{z} + L_{\mathbf{s}} \log \mathbf{z}\right) \prod_{i=1}^{d} z_i^{-1}$$

with $1_k = (1)_{1\le i\le k}$, $\gamma_{\mathbf{s}} = (\gamma(s_i))_{1\le i\le k}$ and

$$Q_{\mathbf{s}} = \Sigma_{\mathbf{s}}^{-1} - \frac{\Sigma_{\mathbf{s}}^{-1} 1_k 1_k^T \Sigma_{\mathbf{s}}^{-1}}{1_k^T \Sigma_{\mathbf{s}}^{-1} 1_k},$$

$$L_{\mathbf{s}} = \left(\frac{1_k^T \Sigma_{\mathbf{s}}^{-1}\gamma_{\mathbf{s}} - 1}{1_k^T \Sigma_{\mathbf{s}}^{-1} 1_k} 1_k - \gamma_{\mathbf{s}}\right)^T \Sigma_{\mathbf{s}}^{-1},$$

$$C_{\mathbf{s}} = (2\pi)^{(1-k)/2} \det(\Sigma_{\mathbf{s}})^{-1/2} (1_k^T \Sigma_{\mathbf{s}}^{-1} 1_k)^{-1/2} \exp\left\{\frac{1}{2}\frac{(1_k^T \Sigma_{\mathbf{s}}^{-1}\gamma_{\mathbf{s}} - 1)^2}{1_k^T \Sigma_{\mathbf{s}}^{-1} 1_k} - \frac{1}{2}\gamma_{\mathbf{s}}^T \Sigma_{\mathbf{s}}^{-1}\gamma_{\mathbf{s}}\right\}.$$

Extremal-t model

- Extremal-t model : $Y(s) = c_\nu (W(s))_+^\nu$ with W centered stationary Gaussian process with unit variance.
- The intensity measure $\mu_s(\mathrm{d}\mathbf{z})$ does not have a density on

$$E = [0, +\infty)^k \setminus \{0\}$$

because it puts mass on all the subsets of the form

$$E_I = \{\mathbf{z} \in [0, +\infty)^k ; z_i > 0,\ i \in I \text{ and } z_i = 0,\ i \notin I\}$$

for $\emptyset \neq I \subseteq \{1, \ldots, k\}$.

Proposition

Let $\mathbf{s} = (s_1, \ldots, s_k)$. Provided the covariance matrix $\Sigma_{\mathbf{s}} = \mathrm{Var}(W(\mathbf{s}))$ is non-singular, then $\mu_s(\mathrm{d}\mathbf{z})$ has a density on $(0, +\infty)^k$:

$$\lambda_I(\mathbf{z}) \quad = \quad c_\nu \nu^{-k+1} 2^{(\nu-2)/2} \pi^{-k/2} \det(\Sigma_{\mathbf{s}})^{-1/2} \left((\mathbf{z}^{1/\nu})^T \Sigma_{\mathbf{s}}^{-1} \mathbf{z}^{1/\nu} \right)^{-(k+\nu)/2} \Gamma\left(\frac{k+\nu}{2} \right) \prod_{j=1}^k z_j^{(1-\nu)/\nu}.$$

Existence of a density

Proposition

Let $\mathbf{s} = (s_1, \ldots, s_k)$. The following statements are equivalent :

- $\eta(\mathbf{s})$ has a density on $[0, +\infty)^k$;
- For all $\emptyset \neq I \subseteq \{1, \ldots, k\}$, the restriction of $\mu_s(\mathrm{d}\mathbf{z})$ to E_I has a density $\lambda_I(\mathrm{d}\mathbf{z})$.

- Extremal-t model : existence of density provided $W(\mathbf{s})$ non degenerate ;
- Moving-maximum-process : existence of density for low dimension only (e.g. if $S \subset \mathbb{R}^k$, density exists up to dimension $k + 1$).

Structure of the talk

Structure of simple max-stable processes

Theorem (de Haan 1984, Giné et al. 1990)

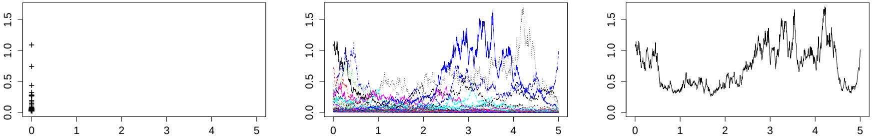
Let η be a simple max-stable process in $\mathcal{C}(S)$. Then η can be represented as

$$\big(\eta(s)\big)_{s\in S} \stackrel{d}{=} \Big(\bigvee_{i\geq 1} U_i Y_i(s)\Big)_{s\in S}$$

with

- $U_1 > U_2 > U_3 > \cdots$ the points of a PPP on $(0,+\infty)$ with intensity $u^{-2}\mathrm{d}u$;
- $Y_1, Y_2, Y_3, \ldots$ i.i.d. copies of a process Y in $\mathcal{C}(S)$ such that $\mathbb{E}[Y(s)] = 1$, $s \in S$, and $\mathbb{E}[\|Y\|_\infty] < \infty$.
- $(Y_i)_{i\geq 1}$ and $\{U_i, i \geq 1\}$ independent.

Example : "historical Brown-Resnick" process $Y(s) = e^{B(s)-|s|/2}$



Naive simulation of max-stable processes

- Naive algorithm : truncated maximum $\eta_n(s) = \bigvee_{i=1}^n U_i Y_i(s)$ satisfies

$$\mathbb{P}\left[\eta_n(s) \equiv \eta(s) \text{ on } S\right] \to 1 \quad \text{as } n \to \infty$$

but rate of convergence are often difficult to get.

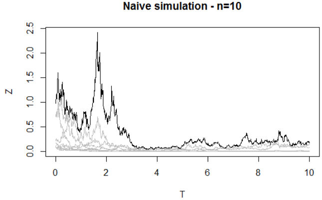
- De Haan's spectral representation is not unique :

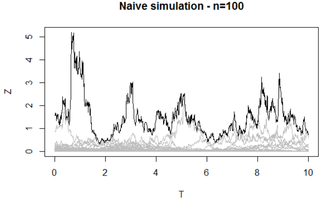
$$\eta(s) = \bigvee_{i \geq 1} U_i Y_i(s) \stackrel{d}{=} \bigvee_{i \geq 1} \tilde{U}_i \tilde{Y}_i(s)$$

and performance of the naive truncation algorithm relies on the choice of a "good representation".

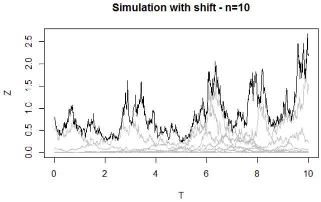
Example : naive simulation

- Brown-Resnick processes $\eta(s) = \bigvee_{i\geq 1} U_i e^{B_i(s)-|s|/2}$:





- Simulation with random shift $\eta(s) = \bigvee_{i\geq 1} U_i e^{B_i(s-X_i)-|s-X_i|/2}$ with $X_i \overset{i.i.d.}{\sim} \mathcal{U}_{[0,10]}$:



Exact simulation by Oesting, Schlather and Zhou

- Oesting, Schlather and Zhou (2013) recommend to use the normalized spectral representation, i.e.

$$\eta(s) = \bigvee_{i\geq 1} U_i \tilde{Y}_i(s) \quad \text{with } \|\tilde{Y}\|_\infty \equiv c.$$

- Exact simulation based on a simple stopping rule (Schlather 2002) :

$$\eta(x) = \max\Big(\underbrace{\bigvee_{i=1}^{n} U_i \tilde{Y}_i(s)}_{\eta_n(s)} , \underbrace{\bigvee_{i\geq n+1} U_i \tilde{Y}_i(s)}_{\leq U_{n+1} c} \Big)$$

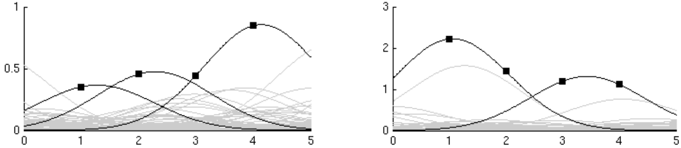
$$n^* = \min_{n\geq 1} \left\{ U_{n+1} c < \min_{s\in S} \eta_n(s) \right\}.$$

- Practical issue : how to simulate exactly $\tilde{Y}$?

$$\mathbb{P}\left[\widetilde{Y} \in A\right] = \frac{1}{c}\mathbb{E}\left[\|Y\|_\infty 1_{\{cY/\|Y\|_\infty \in A\}}\right], \quad c = \mathbb{E}\left[\|Y\|_\infty\right].$$

Exact simulation via extremal functions

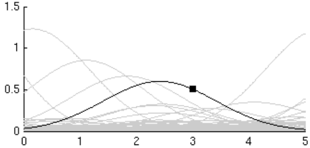
- **Goal :** simulate max-stable process η exactly at points $s_1, \ldots, s_N$.
- **Observations :** only a finite number of functions play a role.



We call these functions extremal (in black) and we denote by φ_s^+ the extremal function at point s.
Other functions are called subextremals (in grey).

- **Idea** : simulate only the extremal functions and no subextremals functions ! More precisely, simulate successively $\varphi_{s_1}^+, \varphi_{s_2}^+, \ldots, \varphi_{s_N}^+$.

Notion of extremal function



The extremal function $\varphi^+_{s_0}$ realizes the maximum at s_0.

Proposition

The random variables $\eta(s_0)$ and $\varphi^+_{s_0}/\eta(s_0)$ are independent, $\eta(s_0)$ has a unit Fréchet distribution and the distribution of $\varphi^+_{s_0}/\eta(s_0)$ is

$$P_{s_0}(A) = \mathbb{P}(\varphi^+_{s_0}/Z(s_0) \in A) = \mathbb{E}[Y(s_0)1_{\{Y/Y(s_0)\in A\}}]$$

Example : Brown-Resnick

- Brown-Resnick type model : $Y(s) = \exp\left(W(s) - \sigma^2(s)/2\right)$ with W centered Gaussian process.

Proposition

In the Brown-Resnick type model, for all $s_0 \in S$, the distribution P_{s_0} is equal to the distribution of the log-normal process

$$\widetilde{Y}(x) = \exp\left(W(s) - W(s_0) - \frac{1}{2}\mathrm{Var}[W(s) - W(s_0)]\right), \quad s \in S.$$

- Similar results for moving maximum process, extremal Gaussian process, extremal-t process.

Joint law of extremal functions

Theorem

The distribution of $(\varphi^+_{s_i})_{1\leq i\leq N}$ is given as follows :

- Initial distribution :

$$\varphi^+_{s_1} \stackrel{d}{=} F_1 Y_1 \quad \text{with } F_1 \sim \mathrm{Frechet}(1,1) \text{ and } Y_1 \sim P_{x_1}.$$

- Conditional distribution :
$\mathcal{L}(\varphi^+_{s_{n+1}} \mid (\varphi^+_{s_i})_{1\leq i\leq n})$ is equal to the distribution of

$$\tilde{\varphi}^+_{s_{n+1}} = \begin{cases} \mathrm{argmax}_{\phi\in\tilde{\Phi}_{n+1}\cap\mathcal{C}_n}\, \phi(s_{n+1}) & \text{if } \tilde{\Phi}_{n+1}\cap\mathcal{C}_n \neq \emptyset \\ \mathrm{argmax}_{\phi\in(\varphi^+_{s_i})_{1\leq i\leq n}}\, \phi(s_{n+1}) & \text{if } \tilde{\Phi}_{n+1}\cap\mathcal{C}_n = \emptyset \end{cases}$$

where

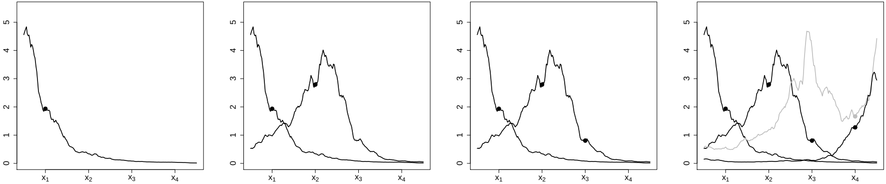
  - $\tilde{\Phi}_{n+1} = \{U_i\tilde{Y}_i,\ i\geq 1\}$ with $\tilde{Y}_i \sim P_{s_{n+1}}$;
  - $\mathcal{C}_n = \{f\in\mathcal{C}_l; f(s_i) < \eta_n(s_i),\ 1 = 1,\ldots,n$ and $f(s_{n+1}) > \eta_n(s_{n+1})\}$ with $\eta_n(s) = \vee_{i=1}^n \varphi^+_{s_i}(s)$.

Exact simulation algorithm 1

Algorithm 1 (simulation of η exactly at $s = s_1, \ldots, s_N$)

Simulate $U^{-1} \sim \mathrm{Exp}(1)$ and $Y \sim P_{s_1}$.
Set $\eta(s) = UY(s)$.
For $n = 2, \ldots, N$:
  Simulate $U^{-1} \sim \mathrm{Exp}(1)$.
  while $(U > \eta(s_n))$ {
    Simulate $Y \sim P_{s_n}$.
    If $UY(s_i) < \eta(s_i)$ for all $i = 1, \cdots, n-1$,
      update $\eta(s)$ by the componentwise $\max(\eta(s), UY(s))$.
    Simulate $E \sim \mathrm{Exp}(1)$ and update U^{-1} by $U^{-1} + E$.
  }

Return Z.



ℓ^1-spectral representation

- Change of spectral representation as in Oesting, Schlather and Zhou but with a different norm :

$$\eta(s) = \bigvee_{i\geq 1} U_i \tilde{Y}_i(s), \quad s = (s_1, \ldots, s_N)$$

with

$$\|\tilde{Y}\|_1 = \frac{1}{N}\sum_{i=1}^{N} |\tilde{Y}(s_i)| \equiv 1.$$

- Since $\|\tilde{Y}\|_\infty \leq N$, we can use the simple stopping rule.
- The distribution of $\tilde{Y}$ is obtained from the original spectral process Y by the transformation

$$\mathbb{P}\left[\widetilde{Y} \in A\right] = \mathbb{E}\left[\|Y\|_1 1_{\{Y/\|Y\|_1 \in A\}}\right]$$

Link with the extremal function distribution

Using

$$\mathbb{P}\left[\widetilde{Y} \in A\right] = \mathbb{E}\left[\|Y\|_1 1_{\{Y/\|Y\|_1 \in A\}}\right]$$
$$= \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left[Y(s_i) 1_{\{Y/\|Y\|_1 \in A\}}\right]$$

we deduce easily how to sample exactly according $\tilde{Y}$:

i) Sample T uniformly in $\{1, \dots, N\}$;
ii) Sample Y with distribution P_{s_T} ;
iii) Set $\tilde{Y} = Y/\|Y\|_1$.

Exact simulation algorithm 2

Algorithm 2 (simulation of η exactly at $s = s_1, \ldots, s_N$)

Simulate $U^{-1} \sim \mathrm{Exp}(1)$ and set $\eta(s) \equiv 0$.
While ($NU > \min(\eta(s_1), \ldots, \eta(s_N))$) {
  Simulate T uniform on $\{1, \ldots, N\}$ and Y according to the law P_{s_T}.
  Update $\eta(s)$ by the componentwise $\max(\eta(s), UY(s)/\|Y\|_1)$.
  Simulate $E \sim \mathrm{Exp}(1)$ and update U^{-1} by $U^{-1} + E$.
}
Return η.

In the Brown-Resnick case, we recover the algorithm proposed by Diecker and Mikosch (2015).

Complexity of both algorithms

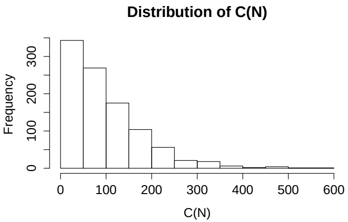
Proposition

Let $C_i(N)$ be the number of simulated functions Y's during the exact sampling procedure of η on $\{s_1, \ldots, s_N\}$.

Algorithm 1 : $\mathbb{E}(C_1(N)) = N$

Algorithm 2 : $\mathbb{E}(C_2(N)) = N\mathbb{E}\left(\max_{i=1}^{N} \eta(s_i)^{-1}\right) \geq N.$

- Algorithm 1 should be prefered and is particularly efficient for moderate N ($N < 100$), for instance to simulate exactly extreme value copulas or 1*d*-process.
- Distribution of $C(100)$ in the BR-model with semi-variogram $\gamma(h) = \|h\|$ on a regular 10×10 grid on $[0, 10]^2$.



1000 replications of $C(100)$

Empirical mean : 98.75.
Standard deviation is large : 83.3. !

Min.	1st Qu.	Median	3rd Qu.	Max.
1.00	35.00	77.50	140.00	555.00

Adaptive design

- In dimension 2, simulation are often performed on a grid so that N can be large ($N = 10^4$ for a 100×100 grid).
- We may want to produce exact simulation on a subset $s_1, \cdots, s_n$, $n < N$ and hope that simulation is exact on $s_1, \cdots, s_N$.
- Ordering of the points is then crucial.
- We study numerically the random time

$$n_{opt} = \min\{n \geq 1; \eta_n \equiv \eta_N\}.$$

with two designs :

  - deterministic lexicographic ordering of points $s_1, \cdots, s_N$;
  - adaptive design with random ordering of points

$$s_{(i+1)} = \underset{s \in \{s_1, \cdots, s_N\} \setminus \{s_{(1)}, \cdots, s_{(i)}\}}{\operatorname{argmin}} \eta(s).$$

Simulation study

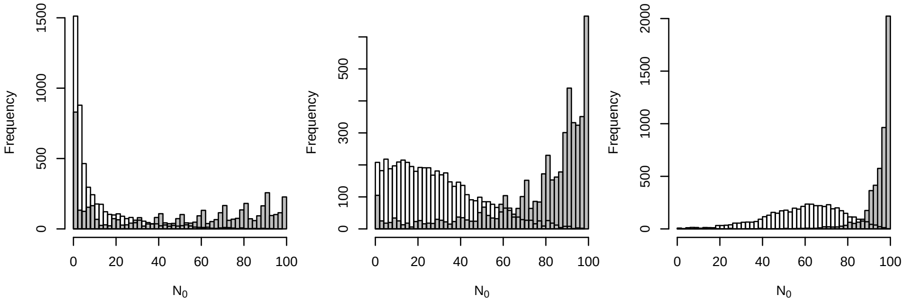


FIGURE: Histogram of n_{opt} based on 5000 realizations of a Brown–Resnick process associated to the semi-variogram $\gamma(h) = c\|h\|^{\alpha}$ with $c = 1$ and $\alpha = 1.5$ (left), $c = 2.5$ and $\alpha = 1$ (middle) and $c = 5$ and $\alpha = 0.5$ simulated with the deterministic design (grey) and the adaptive design (white), respectively.

Structure of the talk

Motivations

- Observations of a max-stable process η at some stations only :

$$\eta(s_i) = z_i, \quad i = 1, \ldots, k.$$

  How to predict what happens at other locations ?
- We are naturally lead to consider the conditional distribution of η given the observations.
- Different goals :
  - theoretical formulas for the conditional distribution,
  - sample from the conditional distribution,
  - compute (numerically) the conditional median or quantiles ...
- Results for spectrally discrete max-stable processes :

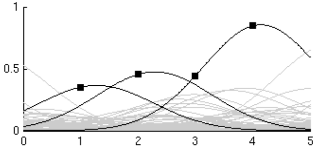
Y. Wang and S. Stoev, Conditional sampling of spectrally discrete max-stable processes. *Adv. in Applied Probab.*, 2011.

Reminder : decomposition of the point process

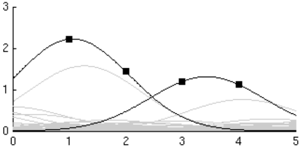
Definition

Consider k locations $s_1, \ldots, s_k \in S$.

- extremal point process : $\Phi^+ = \{\varphi^+_{s_1}, \ldots, \varphi^+_{s_k}\}$ (disregard repetitions).
- hitting scenario : random partition $\theta = (\theta_1, \ldots, \theta_\ell)$ accounting for possible repetitions within $\varphi^+_{s_1}, \ldots, \varphi^+_{s_k}$
- subextremal point process : $\Phi^- = \Phi \setminus \Phi^+$.



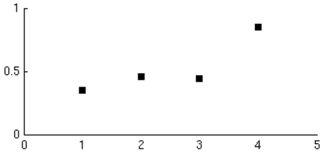
$\theta = \{\{1\}, \{2\}, \{3, 4\}\}$

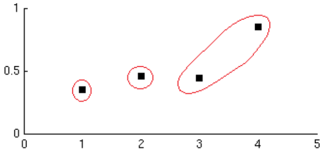


$\theta = \{\{1, 2\}, \{3, 4\}\}$

Heuristic for conditional sampling

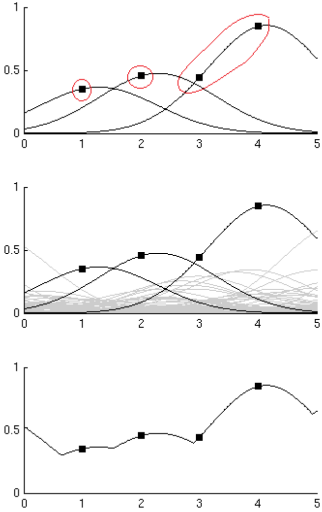
A three step procedure for the conditional law of η given $\eta(\mathbf{s}) = \mathbf{z}$:





Step 1 : sample θ from the conditional law w.r.t. $\eta(\mathbf{s}) = \mathbf{z}$.

Conditional distribution



Step 2 : sample (φ_j^+) from the conditional law w.r.t. $\eta(\mathbf{s}) = \mathbf{z}$, $\theta = \tau$.

Step 3 : sample Φ^- from the conditional law w.r.t. $\eta(\mathbf{s}) = \mathbf{z}$, $\theta = \tau$, $(\varphi_j^+) = (f_j)$.

Finally, set
$\eta(s) = \bigvee_{j=1}^{\ell} \varphi_j^+(s) \bigvee_{\phi \in \Phi^-} \phi(s)$.

Main theorem (regular case)

Theorem

Assume the model is regular at $\mathbf{s}$ with intensity

$$\mu_{\mathbf{s}}(\mathrm{d}\mathbf{z}) = \lambda_{\mathbf{s}}(\mathbf{z})\mathrm{d}\mathbf{z}.$$

1. For all $\tau \in \mathcal{P}_k$,

$$\mathbb{P}[\theta = \tau \mid \eta(\mathbf{s}) = \mathbf{z}] = \frac{1}{C(\mathbf{s}, \mathbf{z})} \prod_{j=1}^{\ell} \int_{\{\mathbf{u}_j < \mathbf{z}_{\tau_j^c}\}} \lambda_{(\mathbf{s}_{\tau_j}, \mathbf{s}_{\tau_j^c})}(\mathbf{z}_{\tau_j}, \mathbf{u}_j) d\mathbf{u}_j;$$

2. Conditionally on $\eta(\mathbf{s}) = \mathbf{z}$ and $\theta = \tau$, the extremal functions $(\varphi_j^+)_{1 \le j \le \ell}$ are independent and φ_j^+ has distribution

$$\mu\big(df \mid f(\mathbf{s}_{\tau_j}) = \mathbf{z}_{\tau_j}, f(\mathbf{s}_{\tau_j^c}) < \mathbf{z}_{\tau_j^c} \big).$$

3. The conditional distribution of Φ^- given $\eta(\mathbf{s}) = \mathbf{z}$, $\theta = \tau$ and $(\varphi_j^+) = (f_j)$ is a PPP with intensity $1_{\{f(\mathbf{s}) < \mathbf{z}\}} \mu(df)$.

Application to Brown-Resnick processes

- If $\mathrm{Var}[W(\mathbf{s})]$ is nonsingular :

$$\lambda_{\mathbf{s}}(\mathbf{z}) = C_{\mathbf{s}} \exp\left(-\frac{1}{2} \log \mathbf{z}^T Q_{\mathbf{s}} \log \mathbf{z} + L_{\mathbf{s}}^T \log \mathbf{z}\right) \prod_{i=1}^{k} z_i^{-1}$$

  with $C_{\mathbf{s}}$, $Q_{\mathbf{s}}$, $L_{\mathbf{s}}$ depending on $\mathrm{Var}[W(\mathbf{s})]$.
- Step 1 : explicit formula for conditional hitting scenario distribution (with log-Gaussian integrals) but combinatorial explosion of the state space $\mathcal{P}_k$ for large k.
- Step 2 : extremal functions are conditioned log-normal processes.

Gibbs sampler for the conditional hitting scenario

- The conditional hitting scenario distribution is of the form

$$p(\tau \mid \mathbf{z}) = \frac{1}{C_{\mathbf{z}}} \prod_{j=1}^{\ell} \omega_{\tau_j} \quad \text{with} \quad \omega_{\tau_j} = \int_{u_j < \mathbf{z}_{\tau_j^c}} \lambda(\mathbf{z}_{\tau_j}, u_j) \mathrm{d}u_j$$

  and $C_{\mathbf{z}}$ the normalization constant

$$C_{\mathbf{z}} = \sum_{\tau \in \mathcal{P}_k} \prod_{j=1}^{\ell} \omega_{\tau_j}.$$

- Combinatorial explosion of the state space $\mathcal{P}_k$:

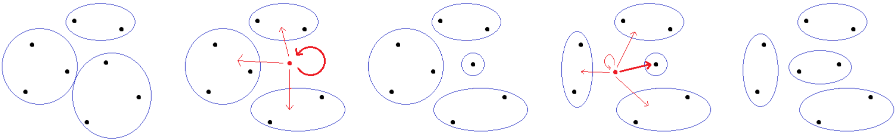
$$B_5 = 52, \quad B_{10} = 115975, \quad B_{20} \approx 5,17 \times 10^{13} \quad \text{(Bell numbers)}.$$

- MCMC simulation in order to avoid the computation of the normalization constant $C_{\mathbf{z}}$: Metropolis-Hasting algorithm or Gibbs sampler.
- Gibbs sampler : update $\tau \in \mathcal{P}_k$ at one randomly chosen component j using the conditional distribution

$$\mathbb{P}[\theta \in \cdot \mid \theta_{-j} = \tau_{-j}], \tag{1}$$

  where $\theta \sim p(\tau \mid \mathbf{z})$ has the target distribution and θ_{-j}, τ_{-j} denote restrictions to $\{s_1, \ldots, s_k\} \setminus \{x_j\}$.

Gibbs sampler for the conditional hitting scenario



- Combinatorial explosion is avoided because the number of possible updates $\tau^* \in \mathcal{P}_k$ such that $\tau^*_{-j} = \tau_{-j}$ is

$$\begin{cases} \ell & \text{if } \{s_j\} \text{ is a partitioning set of } \tau, \\ \ell + 1 & \text{if } \{s_j\} \text{ is not a partitioning set of } \tau. \end{cases}$$

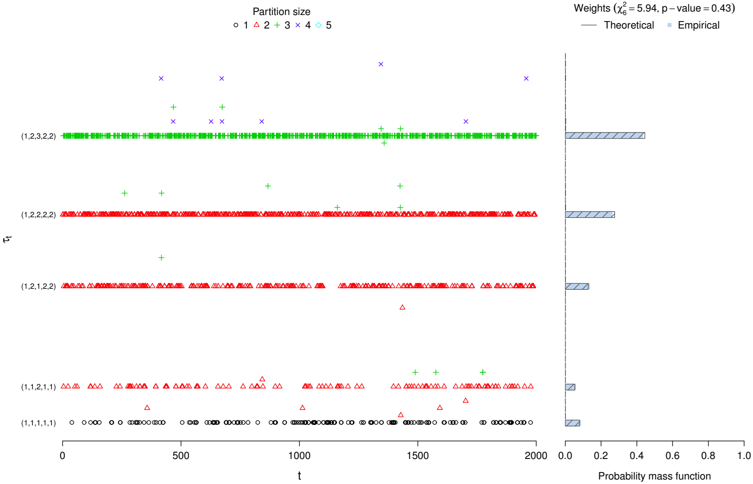
- Proposal distribution is computed easily :

$$\mathbb{P}[\theta = \tau^* \mid \theta_{-j} = \tau_{-j}] = \frac{p(\tau^*, z)}{\sum_{\tilde\tau \in \mathcal{P}_d} p(\tilde\tau, z) 1_{\{\tilde\tau_{-j} = \tau_{-j}\}}} \propto \frac{\prod_{k=1}^{|\tau^*|} \omega_{\tau^*_k}}{\prod_{k=1}^{|\tau|} \omega_{\tau_k}}. \qquad (2)$$

where many terms cancel out except at most four of them.

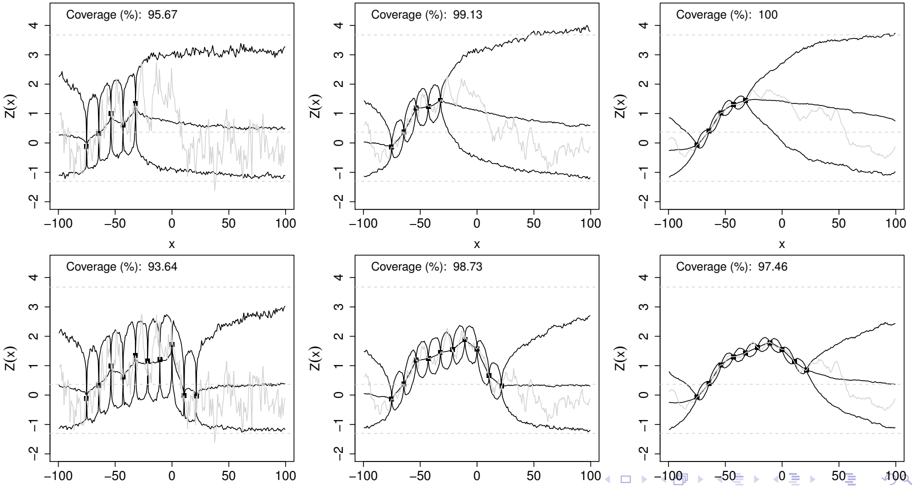
Gibbs sampler

Verification in the Brown-Resnick case with $k = 5$, $B_5 = 52$



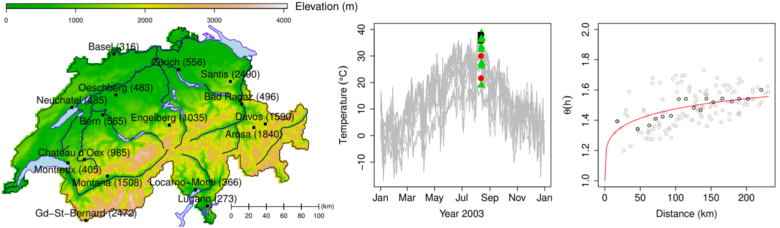
Conditional sampling

- Brown-Resnick process driven by FMB ($H = 1/4$, $1/2$ or $3/4$).
- Conditional sampling of a path (given $k = 5$ or 10 conditioning points).
- Conditional median and quantiles evaluated numerically (95% confidence interval).



Application : temperature in Switzerland

- Annual maxima for temperatures in Switzerland at 16 stations :



- Particular interest for the heatwave from Summer 2003.
  Can we "interpolate" the observed maxima at 16 locations to other locations ?

Application : temperature in Switzerland

- Max-stable model $\eta(s)$:

  A. Davison and Gholamrezaee, Geostatistics of extremes. *Proc. Roy. Soc. A.*, 2011.

  Marginal distributions

$$\eta(x) \sim \mathrm{GEV}(\gamma(s), \mu(s), \sigma(s)) \quad \text{with} \begin{cases} \gamma(s) = \beta_{0,\gamma} \\ \mu(s) = \beta_{0,\mu} + \beta_{1,\mu}\mathrm{alt}(s) \\ \sigma(s) = \beta_{0,\sigma} + \beta_{1,\sigma}\mathrm{alt}(s) \end{cases} .$$

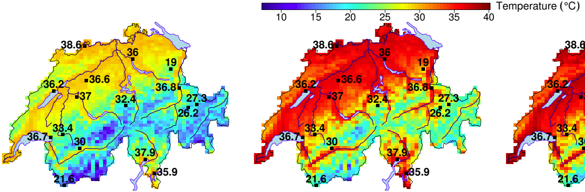
  Dependance structure of type "extremal Gaussian process" with correlation

$$\rho(s_1, s_2) = \exp\Big[- \Big(\frac{\|s_2 - s_1\|}{\lambda} \Big)^{\kappa} \Big].$$

- Model fitted by the "pairwise likelihood method" using annual maxima over the period 1965-2005.
- Can we compute conditional distribution within this model ?

Application : temperature in Switzerland

- Conditional sampling with values observed at the 16 stations in 2003.
- Estimation of conditional quantiles (0.025, 0.5, 0.975) :



References

L. de Haan and A. Ferreira
Extreme Value Theory - an introduction.
Springer Series in Operations Research and Financial Engineering, 2006.

C. Dombry and F. Eyi-Minko.
Regular conditional distributions of max infinitely divisible processes.
Electronic Journal of Probability, 18 :1-21, 2013

C. Dombry, F. Eyi-Minko and M. Ribatet.
Conditional simulation of max-stable processes.
Biometrika, 100(1) :111–124, 2013.

C. Dombry, S. Engelke, M. Oesting,
Exact simulation of max-stable random fields.
Biometrika, to appear, 2016.

M. Schlather
Models for stationary max-stable random fields.
Extremes 5(1) :33-44, 2002.