Applications of Distance Correlation to Time Series

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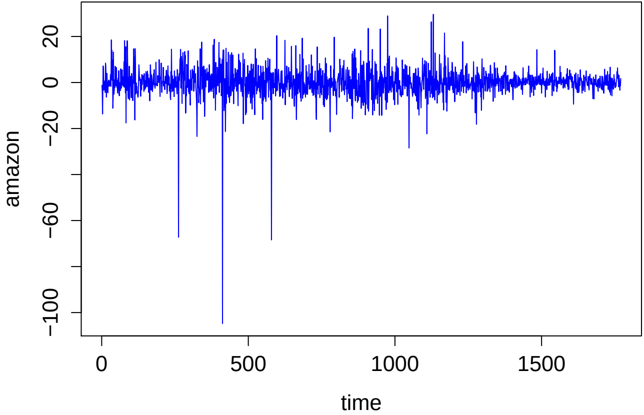
Workshop on Dependence, Stability, and Extremes
The Fields Institute

Warm-up example: Amazon-returns (5-16-97 to 6-16-04)



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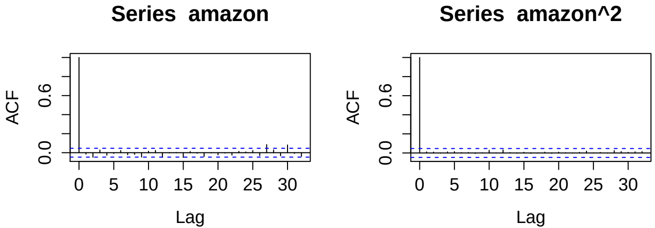

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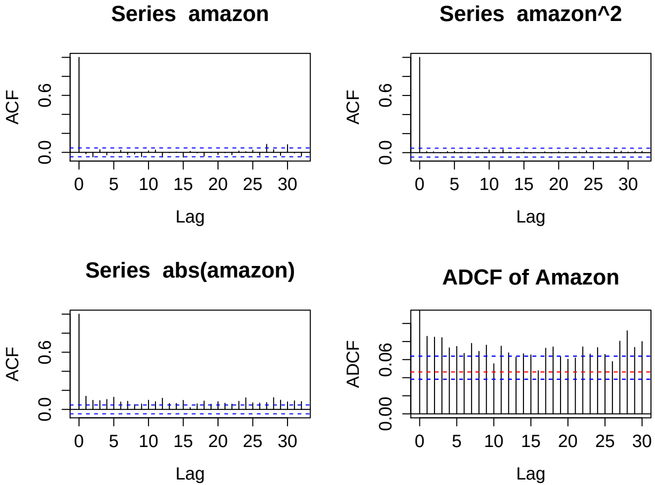
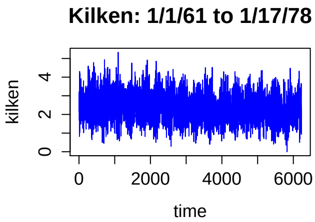
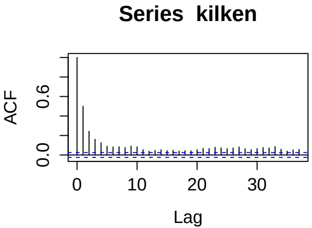


Example: Kilkenny wind speed time series Bonus (teaser)





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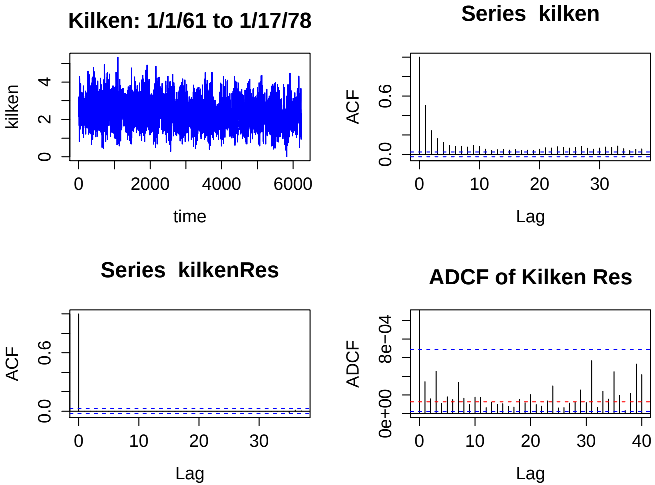

Figure: Auto-distance correlation function (ADCF) of residuals from AR(9) model applied to Kilkenny daily wind speed.

Distance Covariance: a measure of dependence

- Distance covariance: random vectors $X \in \mathbb{R}^p$ and $Y \in \mathbb{R}^q$,

$$T(X, Y; \mu) = \int_{\mathbb{R}^{p+q}} \left|\varphi_{X,Y}(s,t) - \varphi_X(s)\,\varphi_Y(t)\right|^2 \mu(ds, dt)\,,$$

where $\varphi_{X,Y}, \varphi_X, \varphi_Y$ denote the respective characteristic functions of $(X, Y), X, Y$, and μ is a measure.

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- Distance correlation: $R(X,Y;\mu) = \frac{T(X,Y;\mu)}{\sqrt{T(X,X;\mu)T(Y,Y;\mu)}}$

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- Distance correlation: $R(X, Y; \mu) = \frac{T(X,Y;\mu)}{\sqrt{T(X,X;\mu)T(Y,Y;\mu)}}$
- Sample distance covariance: Observations $(X_1, Y_1), \ldots, (X_n, Y_n)$ from a statioonary time series (X_t, Y_t). Then

$$T_n^{X,Y}(0) = \int_{\mathbb{R}^{p+q}} |\hat{\varphi}_{X,Y}(s,t) - \hat{\varphi}_X(s)\hat{\varphi}_Y(t)|^2 \cdot \mu(ds, dt),$$

  where $\hat{\varphi}$ denotes the empirical characteristic function, e.g.,

$$\hat{\varphi}_{X,Y}(s,t) = n^{-1} \sum_{j=1}^{n} e^{i\langle s, X_j, \rangle + i\langle t, Y_j \rangle}$$

Background

- Feuerverger et al. (1981,...), Feuerverger and Mureika (1977); applications of the empirical characteristic function, inference using Fourier methods, etc. Feuerverger (1993), a bivariate test for independence, $\text{cor}(\cos(sX), cos(tY))$, etc.
- S. Csörgő (1981a,b,c) Limit behavior of characteristic functions.
- Meintannis et al. (2008), 2015, Fourier methods for testing multivariate independence
- Székely, Rizzo, Bakirov (2007), Székely and Rizzo (2009), (2014), special choice of weight function, Brownian distance covariance
- Dueck et al. (2014) affinely invariant distance correlation
- Zhou (2012) application to time series.
- Fokianos and Pitsillou (2016). Testing pairwise dependence in time series.

Distance Covariance: choice of weight function

$$T^{X,Y}(0) = \int_{\mathbb{R}^{p+q}} |\varphi_{X,Y}(s,t) - \varphi_X(s)\varphi_Y(t)|^2 \cdot \mu(ds,dt)$$

- If $\mu = \mu_1 \times \mu_2$, let $\tilde{\mu}_i$ be the Fourier transform of μ_i, i.e.,

$$\tilde{\mu}_i(x) = \int_{\mathbb{R}^p} e^{i\langle s,x\rangle} d\mu_i(s),$$

  and assuming Fubination is okay,

$$\begin{aligned} T(X,Y;\mu) &= \mathbf{E}[\hat{\mu}_1(X-X')\,\hat{\mu}_2(Y-Y')] + \mathbf{E}[\hat{\mu}_1(X-X')]\mathbf{E}[\hat{\mu}_2(Y-Y')] \\ &\quad -2\,\mathbf{E}[\hat{\mu}_1(X-X')\,\hat{\mu}_2(Y-Y'')]\,. \end{aligned}$$

  where $(X,Y),(X',Y'),(X'',Y'')$ are iid copies.

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- Choose μ_i so that $\tilde{\mu}_i$ have an explicit and easy to compute form.

Distance Covariance: computing T_n

$T^{X,Y}(0) = \int_{\mathbb{R}^{p+q}} |\varphi_{X,Y}(s,t) - \varphi_X(s)\varphi_Y(t)|^2 \cdot \mu(ds,dt)$

Fourier transform:

$$\tilde{\mu}_i(x) = \int_{\mathbb{R}^p} e^{i\langle s,x\rangle} d\mu_i(s),$$

Computing T_n.

$$\begin{aligned} T_n^{X,Y}(0) &= n^{-2} \sum_{s=1}^{n} \sum_{t=1}^{n} \tilde{\mu}_1(X_s - X_t)\tilde{\mu}_2(Y_s - Y_t) \\ &+ n^{-2} \sum_{s,t=1}^{n} \tilde{\mu}_1(X_s - X_t) n^{-2} \sum_{s,t=1}^{n} \tilde{\mu}_2(Y_s - Y_t) \\ &- 2n^{-3} \sum_{s,t,u=1} \tilde{\mu}_1(X_s - X_t)\tilde{\mu}_2(Y_s - Y_u) \end{aligned}$$

Distance Covariance: choice of weight function

- finite (probability) measures:
  - normal density: $\tilde{\mu}(x) = \exp\{-1/2x'\Sigma x\}$
  - sub-Gaussian $\alpha/2$-stable: $\tilde{\mu}(x) = \exp\{-|(x, y)'\Sigma(x, y)|^{\alpha/2}\}, \quad \alpha \in (0, 2)$.

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- infinite measures:
  - Lévy measure corresponding to an infinitely divisible random vector.
  - Székely and Rizzo (2009):

$$w(s,t) = \frac{1}{c_p c_q |t|_p^{\alpha+p} |s|_q^{\alpha+q}}.$$

with $\alpha \in (0,2)$ and $\tilde{\mu}(x) = |x|^{\alpha}$

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- Distance correlation is scale and rotational invariant relative to $w(s,t)$.

Results—consistency

- Existence of $T(X, Y; \mu) = \int_{\mathbb{R}^{p+q}} |\varphi_{X,Y}(s,t) - \varphi_X(s)\varphi_Y(t)|^2 \cdot w(s,t) ds dt,.$

  1. μ a finite measure.
  2. μ is infinite in a neighborhood of the origin and for some $\alpha \in (0, 2]$, $\mathbf{E}[|X|^\alpha] + \mathbf{E}[|Y|^\alpha] < \infty$ and

$$\int_{\mathbb{R}^{p+q}} 1 \wedge |(s,t)|^\alpha \, \mu(ds, dt) < \infty \, .$$

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- Consistency: If (X_t, Y_t) is a stationary ergodic sequence satisfying 1 or 2 above, then

$$T_n^{X,Y}(h) \stackrel{a.s.}{\rightarrow} T^{X,Y}(h).$$

Results—weak convergence

Assume that $X_0 \perp\!\!\!\perp Y_0$, α-mixing ($\sum_{h=1}^{\infty} \alpha_h^{1/r} < \infty$, $r > 1$) + moment condition,

$$\mathbf{E}[|X|^\alpha + |Y|^\alpha] < \infty, \qquad \mathbf{E}\Big[\prod_{l=1}^{p} |X^{(l)}|^\alpha\Big] < \infty\,, \quad \mathbf{E}\Big[\prod_{l=1}^{q} |Y^{(l)}|^\alpha\Big] < \infty\,, \tag{1}$$

and

$$\int_{\mathbb{R}^{p+q}} (1 \wedge |s|^{\alpha'(1+\epsilon)/u})(1 \wedge |t|^{\alpha'(1+\epsilon)/u})\,\mu(ds, dt) < \infty \tag{2}$$

where $u = 2r/(r-1)$, $\alpha' \leq \min(2, \alpha)$. Then

$$n\,T_n(X, Y; \mu) \xrightarrow{d} \|G\|_\mu^2 = \int_{\mathbb{R}^{p+q}} |G(s,t)|^2\,\mu(ds, dt)\,, \tag{3}$$

where G is a complex-valued mean-zero Gaussian process.

Results—weak convergence

Assume that X_0 and Y_0 are dependent and for some $\alpha \in (u/2, u]$ and for $\alpha' \le \min(2, \alpha)$ the following hold:

$$\mathbf{E}[|X|^{2\alpha} + |Y|^{2\alpha}] < \infty, \quad \mathbf{E}\Big[(1 \vee \prod_{l=1}^{p} |X^{(l)}|^{\alpha})(1 \vee \prod_{k=1}^{q} |Y^{(k)}|^{\alpha})\Big] < \infty\,, \tag{4}$$

and

$$\int_{\mathbb{R}^{p+q}} (1 \wedge |s|^{\alpha'/u})(1 \wedge |t|^{\alpha'/u})\,\mu(ds, dt) < \infty\,. \tag{5}$$

Then

$$\sqrt{n}\,(T_n(X, Y; \mu) - T(X, Y; \mu)) \xrightarrow{d} G'_\mu = \int_{\mathbb{R}^{p+q}} G'(s, t)\,\mu(ds, dt)\,, \tag{6}$$

where $G'(s, t) = 2\mathrm{Re}\{G(s, t)C(s, t)\}$ is a mean-zero Gaussian process.

Distance correlation and AR(p) models

Let (X_t) be the causal AR(p) process given by

$$X_t = \sum_{k=1}^{p} \phi_k X_{t-k} + Z_t\,, \quad (Z_t) \sim IID(0, \sigma^2)\,.$$

- Least squares estimate: observations $X_1, \ldots, X_n$

$$\hat{\boldsymbol{\phi}} - \boldsymbol{\phi} \quad = \quad \Gamma_{n,p}^{-1} \frac{1}{n} \sum_{t=p+1}^{n} \mathbf{X}_{t-1}\, Z_t\,,$$

where $\mathbf{X}_{t-1} = (X_{t-1}, \ldots, X_{t-p})'$

$$\hat{\Gamma}_{n,p} \quad = \quad \frac{1}{n} \sum_{t=p+1}^{n} \mathbf{X}_{t-1}^{T} \mathbf{X}_{t-1}\,.$$

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- $$n^{1/2}(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}) \stackrel{d}{\rightarrow} Q \sim N(0,\sigma^2 \Sigma_p^{-1})\,.$$

Distance correlation and AR(p) residuals

Fitted residuals

$$\begin{aligned}\hat{Z}_k &= X_k - \hat{\boldsymbol{\phi}}'\mathbf{X}_{k-1}\,, \quad k = p+1\ldots,n\,.\\ &= Z_k + (\boldsymbol{\phi} - \hat{\boldsymbol{\phi}})'\mathbf{X}_{k-1}\end{aligned}$$

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Difference in characteristic functions between noise and residuals:

$$\begin{aligned}n^{-1/2} \quad & \sum_{k=p+1}^{n} e^{is\hat{Z}_k + it\hat{Z}_{k+h}} - n^{-1/2}\sum_{k=p+1}^{n} e^{isZ_k + itZ_{k+h}}\\ &= \quad n^{-1/2}\sum_{k=p+1}^{n} e^{isZ_k + itZ_{k+h}}\left(e^{is(\boldsymbol{\phi}-\hat{\boldsymbol{\phi}})'\mathbf{X}_{k-1} + it(\boldsymbol{\phi}-\hat{\boldsymbol{\phi}})'\mathbf{X}_{k+h-1}} - 1\right)\end{aligned}$$

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ADCF of AR(p) residuals – Z_t finite and infinite variance

$$\tilde{R}_n(h) := R_n(\hat{\mathbf{Z}}_1, \hat{\mathbf{Z}}_{h+1}; \mu),$$

where $\hat{Z}_t = X_t - \sum_{k=1}^{p} \hat{\phi}_k X_{t-k}$.

Theorem

Assume $\int [(1 \wedge |s|^2)(1 \wedge |t|^2) + (s^2 + t^2)\, 1_{\{|s| \wedge |t| > 1\}}] \mu(ds, dt) < \infty$

1. *If* $\mathbf{E} Z_t^2 < \infty$, *then*

$$n \tilde{R}_n(h) \xrightarrow{d} \frac{\int |G_Z(s,t) + \xi_h(s,t)|^2 \mu(ds, dt)}{T(0)}.$$

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2. *If* Z_t *~DOA(α) with index* $\alpha \in (0, 2)$, *then*

$$n\, \tilde{R}_n(h) \xrightarrow{d} \frac{\int |G_Z(s,t)|^2 \mu(ds, dt)}{T(0)}.$$

Example: ADCF of AR(10) residuals: $Z_t \sim N(0,1)$ vs. $Z_t \sim t(1.5)$

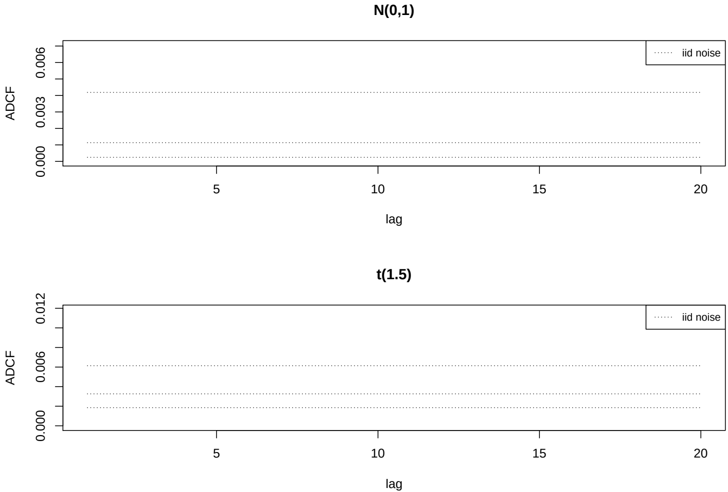

Figure: Empirical 5%, 50%, 95% quantiles of R_n for $Z_t \sim N(0,1)$ (upper panel) and $Z_t \sim t(1.5)$ (lower panel)

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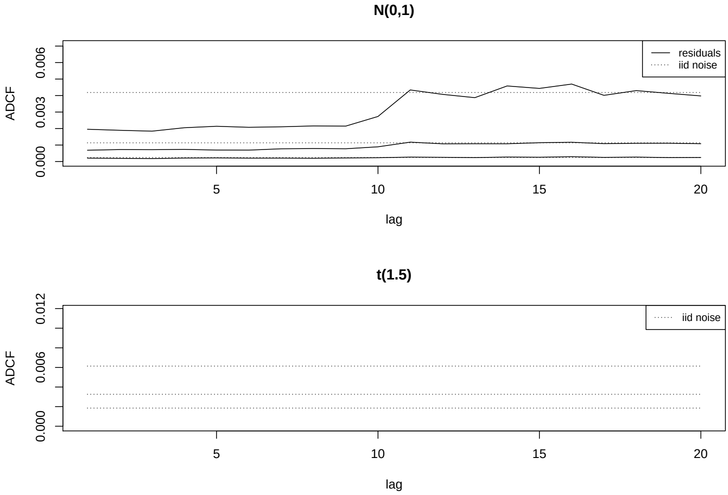

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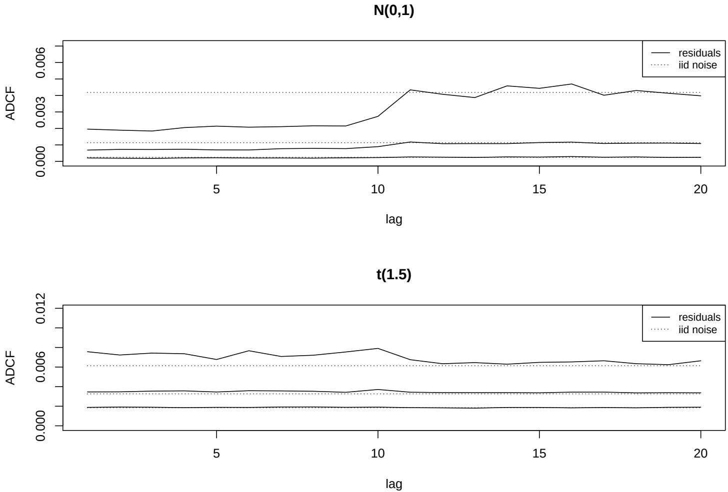

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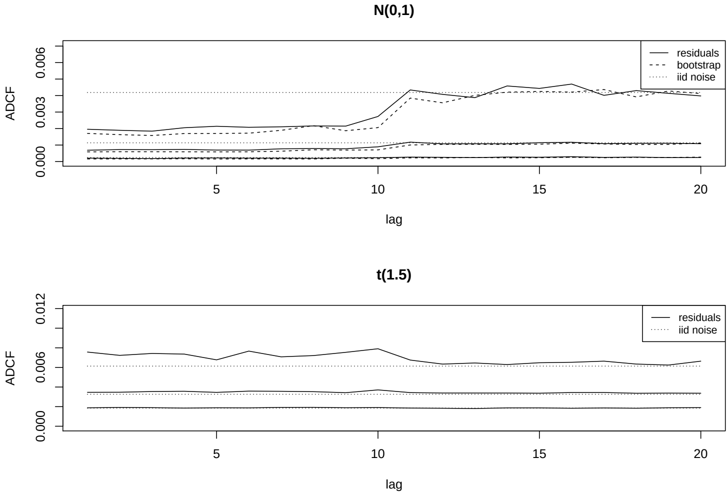

Figure: Empirical 5%, 50%, 95% quantiles of R_n and $\tilde{R}_n$ for $Z_t \sim N(0,1)$ (upper panel) and $Z_t \sim t(1.5)$ (lower panel)

Example: ADCF of AR(10) residuals: $Z_t \sim N(0,1)$ vs. $Z_t \sim t(1.5)$

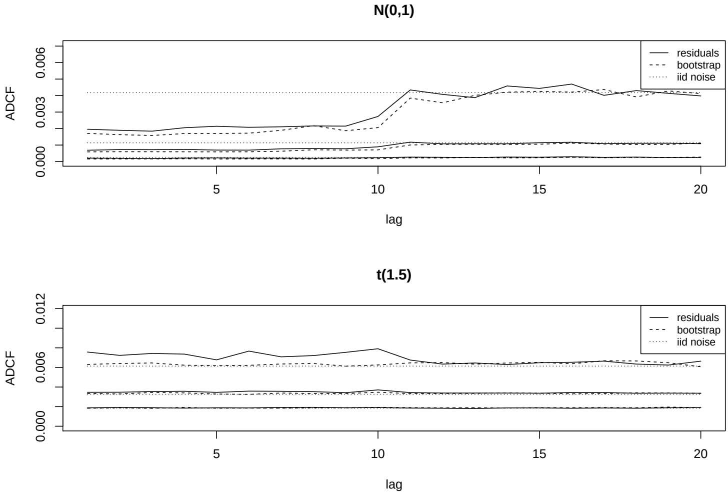

Figure: Empirical 5%, 50%, 95% quantiles of R_n and $\tilde{R}_n$ for $Z_t \sim N(0,1)$ (upper panel) and $Z_t \sim t(1.5)$ (lower panel)

Example: ADCF of AR(10) residuals: symmetric Gamma(0.2,0.5) noise

Székely and Rizzo weight function,

$$w(s,t) = \frac{1}{c_p c_q |t|_p^{1+p} |s|_q^{1+q}} ,$$

need not work.

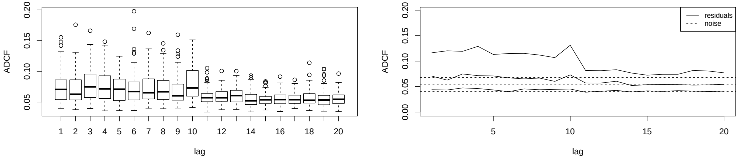

Figure: Left: Box-plots from 500 independent replications. Right panel: empirical 5%, 50%, 95% quantiles from simulated residuals and from iid noise.

Example: Kilkenny wind speed time series

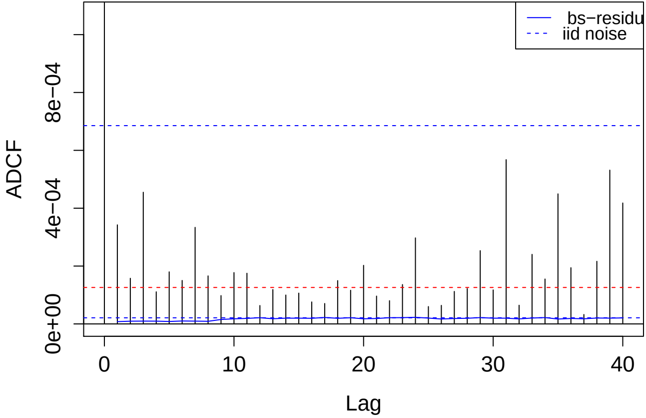

Figure: Auto-distance correlation function (ADCF) of residuals from AR(9) model applied to Kilkenny daily wind speed.

Example: Kilkenny wind speed time series

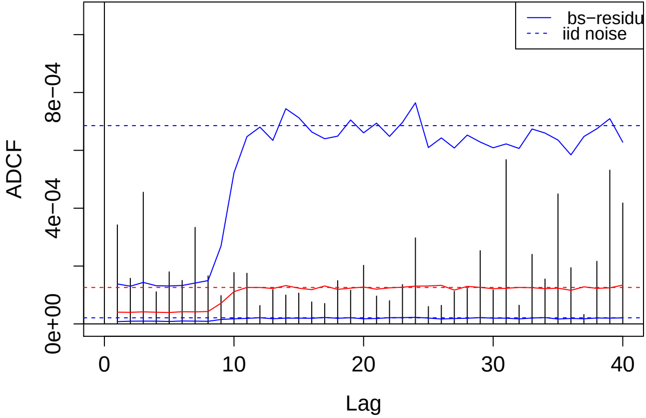

Figure: Auto-distance correlation function (ADCF) of residuals from AR(9) model applied to Kilkenny daily wind speed.

Example: Kilkenny wind speed time series

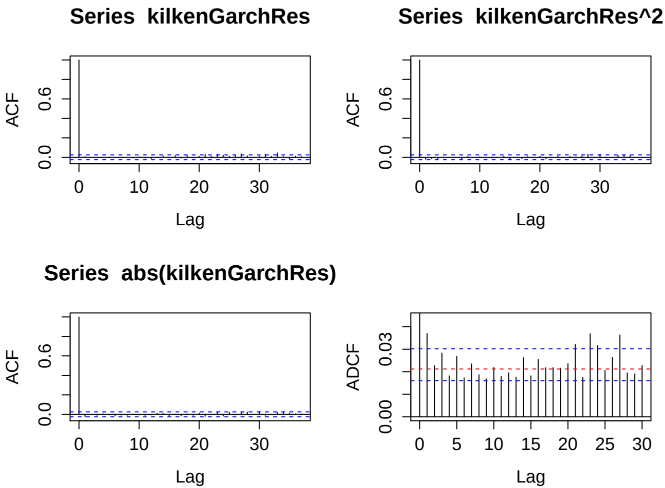

Figure: Auto-distance correlation function (ADCF) of residuals from AR(9) followed by a GARCH(1,1) model applied to Kilkenny daily wind speed.