Operator-scaling random ball model

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Anisotropic Random ball model

We start with a family of grains $X_j + B_E(0, R_j)$ in $\mathbb{R}^d$ where

$$(X_j, R_j)_j \text{ is a Poisson point process in } \mathbb{R}^d \times \mathbb{R}^+,$$

with intensity measure $n(\mathrm{d}x, \mathrm{d}r) = \mathrm{d}x F(\mathrm{d}r)$ where

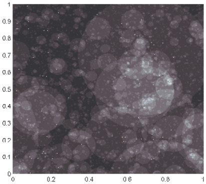
- E is a $d \times d$ matrix with real parts of eigenvalues given by $a_1 \geq \ldots \geq a_d > 0$;
- $B_E(0, r) = r^E B$ for some ball B with finite volume $v_B = \mathcal{L}_d(B)$;
- F is a σ-finite non-negative measure on $\mathbb{R}^+ = (0, +\infty)$ such that

$$\int_{\mathbb{R}^+} \mathcal{L}_d(B_E(0, r)) F(\mathrm{d}r) < +\infty.$$

Let N be the associated Poisson random measure on $\mathbb{R}^d \times \mathbb{R}^+$.
For $A \in \mathcal{B}\left(\mathbb{R}^d \times \mathbb{R}^+\right)$ with $n(A) < \infty$

$$N(A) = \#\left\{j; (X_j, R_j) \in A\right\} \sim \mathcal{P}(n(A)).$$

Anisotropic Random ball model



Associated shot-noise random field

One can define the random field X on $\mathbb{R}^d$ by

$$\begin{aligned} X(t) &= \int_{\mathbb{R}^d \times \mathbb{R}^+} \mathbf{1}_{B_E(x,r)}(t) N(\mathrm{d}x, \mathrm{d}r) \\ &= \#\ \text{grains containing } t \in \mathbb{R}^d. \end{aligned}$$

Isotropic scaling case $E = I_d$ considered in [HB, Estrade 06] and [Kaj et al 07]

Shot noise random field

Examples

$\mathbf{d} = 1 \quad \longrightarrow \quad X(t) =$ **numbers of connections to a server at time** t

$\mathbf{d} = 2 \quad \longrightarrow \quad X(t) =$ **discretized gray level** at point t in a picture

$\mathbf{d} = 3 \quad \longrightarrow \quad X(t) =$ **mass density** of a 3D granular media in t

Main properties

- X is stationary and second-order
- $\mathbb{E}X(t) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \mathbf{1}_{B_E(x,r)}(t) n(\mathrm{d}x, \mathrm{d}r) = \int_{\mathbb{R}^+} \mathcal{L}_d(B_E(0,r))F(\mathrm{d}r)$ with

$$\mathcal{L}_d(B_E(0,r)) = \mathcal{L}_d(B)r^q, \text{ for } q = \mathrm{tr}(E).$$

- $\mathrm{Cov}(X(t), X(t')) = \int_{\mathbb{R}^+} \mathcal{L}_d \left(B_E(t,r) \cap B_E(t',r)\right) F(\mathrm{d}r)$

Towards α stable law : weighted random balls

Let $\alpha \in (1,2]$, $\sigma > 0$, M a $S\alpha S(\sigma)$ r.v. with probability distribution G_α and $(M_j)_j$ iid M

We consider the independently marked point process

$$(X_j, R_j, M_j)_j \text{ in } \mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}.$$

It is a Poisson point process with intensity measure

$$n_\alpha(\mathrm{d}x, \mathrm{d}r, \mathrm{d}m) = n(\mathrm{d}x, \mathrm{d}r) G_\alpha(\mathrm{d}m) = \mathrm{d}x F(\mathrm{d}r) G_\alpha(\mathrm{d}m).$$

Associated shot noise random field X on $\mathbb{R}^d$

$$X(t) = \int_{\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}} m \mathbf{1}_{B_E(x,r)}(t) N_\alpha(\mathrm{d}x, \mathrm{d}r, \mathrm{d}m)$$

Ideas from [Breton & Dombry, 09]

Main properties

- X is stationary, integrable $(\alpha > 1)$
- $\mathbb{E}X(t) = \mathbb{E}(M)\left(\int_{\mathbb{R}^+} \mathcal{L}_d(B_E(0,r))F(\mathrm{d}r)\right) = 0$
- the characteristic function of $X(t)$ is given by

$$\begin{aligned}
\mathbb{E}\left(e^{iuX(t)}\right) &= \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R}} \left[e^{ium\mathbf{1}_{B_E(x,r)}(t)} - 1\right] \mathrm{d}xF(\mathrm{d}r)G_\alpha(\mathrm{d}m)\right) \\
&= \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}^+} \left[\hat{G}_\alpha(u\mathbf{1}_{B_E(x,r)}(t)) - 1\right] \mathrm{d}xF(\mathrm{d}r)\right) \\
&= \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}^+} \left[\hat{G}_\alpha(u) - 1\right] \mathbf{1}_{B_E(x,r)}(t)\mathrm{d}xF(\mathrm{d}r)\right) \\
&= \exp\left(\int_{\mathbb{R}^+} \mathcal{L}_d(B_E(0,r))F(\mathrm{d}r)[e^{-\sigma^\alpha|u|^\alpha} - 1]\right)
\end{aligned}$$

Generalized anisotropic random balls

We consider the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of all infinitely differentiable rapidly decreasing functions on $\mathbb{R}^d$ equipped with the family of seminorms

$$\pi_{N,j}(f) := \sup_{z\in\mathbb{R}^d} (1+|z|)^N \left|D^j(f)(z)\right|, \forall N \in \mathbb{N}, j \in \mathbb{N}^d$$

that induces its usual nuclear Fréchet topology and the closed subspace

$$\mathcal{S}_n(\mathbb{R}^d) = \left\{ f \in \mathcal{S}(\mathbb{R}^d); \int_{\mathbb{R}^d} z^j f(z)dz = 0, \forall j \in \mathbb{N}^d \text{ with } |j| < n \right\}, \text{ for } n \in \mathbb{N}$$

Note that

- a closed subspace of a nuclear space is a nuclear space
- the topological dual of a nuclear Fréchet space is a nuclear space

Definition

A generalized random field on $\mathcal{S}_n(\mathbb{R}^d)$ defined on $(\Omega, \mathcal{A}, \mathbb{P})$ is a measurable map : $(\Omega, \mathcal{A}) \to (\mathcal{S}_n(\mathbb{R}^d)', \mathcal{B}(\mathcal{S}_n(\mathbb{R}^d)'))$.

Generalized anisotropic random balls

For all $f \in \mathcal{S}(\mathbb{R}^d)$ we define the r.v. as "$\int_{\mathbb{R}^d} X(t)f(t)\mathrm{d}t$"

$$X(f) := \int_{\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}} m\left(\int_{\mathbb{R}^d} \mathbf{1}_{B_E(x,r)}(t)f(t)\mathrm{d}t\right) N_\alpha(\mathrm{d}x, \mathrm{d}r, \mathrm{d}m).$$

Proposition

a.s. $X \in \mathcal{S}(\mathbb{R}^d)' \subset \mathcal{S}_n(\mathbb{R}^d)'$ *for all* $n \geq 1$.

It follows from the fact that X is linear and $|X(f)| \leq C_N \pi_{N,0}(f)$, with

$$C_N = \int_{\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}} |m| \left(\int_{\mathbb{R}^d} \mathbf{1}_{B_E(x,r)}(t)(1+|t|)^{-N}\mathrm{d}t\right) N_\alpha(\mathrm{d}x, \mathrm{d}r, \mathrm{d}m),$$

with finite expectation as soon as $N > d$.

Generalized anisotropic random balls

The characteristic functional of a generalized field Y on $\mathcal{S}_n(\mathbb{R}^d)$ is defined as

$$\Psi_Y(f) = \mathbb{E}\left(e^{iY(f)}\right) = \int_{\mathcal{S}_n(\mathbb{R}^d)'} e^{iu(f)} d\mathbb{P}_Y(u), \forall f \in \mathcal{S}_n(\mathbb{R}^d)$$

Theorem (Minlos-Bochner Theorem)

A functional $\Psi : \mathcal{S}_n(\mathbb{R}^d) \to \mathbb{C}$ is the characteristic functional of a generalized field Y on $\mathcal{S}_n(\mathbb{R}^d)$ iff Ψ is continuous, $\Psi(0) = 1$ and Ψ is positive definite on $\mathcal{S}_n(\mathbb{R}^d)$.

We write $T_r^E f(x) = \int_{\mathbb{R}^d} \mathbf{1}_{B_E(x,r)}(t) f(t) \mathrm{d}t$ so that

$$X(f) := \int_{\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}} m T_r^E f(x) N_\alpha(\mathrm{d}x, \mathrm{d}r, \mathrm{d}m).$$

Then $\Psi_X(f) = \mathbb{E}\left(e^{iX(f)}\right) = \exp\left(\int_{\mathbb{R}^d \times \mathbb{R}^+} \left[e^{-\sigma^\alpha |T_r^E f(x)|^\alpha} - 1\right] \mathrm{d}x F(\mathrm{d}r)\right)$ and

$$|\Psi_X(f)| \leq \exp\left(\sigma^\alpha \int_{\mathbb{R}^+} \| T_r^E f\|_\alpha^\alpha F(\mathrm{d}r)\right).$$

Scaling behavior

Let us multiply the radii by $\rho > 0$ and the intensity measure by $\lambda(\rho) > 0$:

$$n(\mathrm{d}x, \mathrm{d}r) = \mathrm{d}xF(\mathrm{d}r) \curvearrowright n_{\lambda(\rho),\rho}(\mathrm{d}x, \mathrm{d}r) = \lambda(\rho)\mathrm{d}xF_\rho(\mathrm{d}r)$$

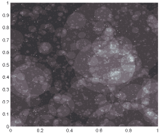
$$F_\rho(\mathrm{d}r) = \text{image measure of } F(\mathrm{d}r) \text{ by the change of scale } r \mapsto \rho r.$$

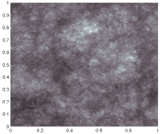
$$\boxed{X_\rho(f) = \int_{\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}} m T_r^E f(x)\ N_{\alpha,\lambda(\rho),\rho}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}m)}$$

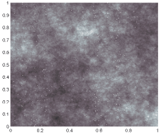
with $N_{\alpha,\lambda(\rho),\rho}$ Poisson measure with intensity

$$n_{\alpha,\lambda(\rho),\rho}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}m) = n_{\lambda(\rho),\rho}(\mathrm{d}x, \mathrm{d}r) G_\alpha(\mathrm{d}m).$$

$$\begin{cases} \text{zoom-in}: & \rho \to +\infty \quad \text{(small grain assumption)} \\ \text{zoom-out}: & \rho \to 0 \quad \text{(large grain assumption)} \end{cases}$$







Convergence in law

Theorem (Lévy Theorem, [Meyer, 1964, Fernique, 1968])

Let $(Y_m)_m$ *be a sequence of generalized field : TFAE*

- $Y_m \xrightarrow[m\to+\infty]{d} Y$
- $(\mathbb{P}_{Y_m})_m$ *weakly converges to* $(\mathbb{P}_Y)_m$: $\forall \varphi \in \mathcal{C}_b(\mathcal{S}'_n(\mathbb{R}^d))$

$$\int_{\mathcal{S}'_n(\mathbb{R}^d)} \varphi(u) \mathrm{d}\mathbb{P}_{Y_m}(u) \underset{m\to+\infty}{\longrightarrow} \int_{\mathcal{S}'_n(\mathbb{R}^d)} \varphi(u) \mathrm{d}\mathbb{P}_Y(u).$$

- *the characteristic functionals converge pointwize* : $\forall f \in \mathcal{S}_n(\mathbb{R}^d)$, *one has*

$$\Psi_{Y_m}(f) \underset{m\to+\infty}{\longrightarrow} \Psi_Y(f).$$

Convergence in law

We are looking for a normalization term $n(\rho)$ s.t. $Y_\rho := X_\rho / n(\rho)$ converges in law.
Note that $\Psi_{Y_\rho}(f) = \mathbb{E}\left(e^{iX_\rho(f)/n(\rho)}\right)$

$$= \exp\left(\int_{\mathbb{R}^d \times \mathbb{R}^+} \lambda(\rho)\left[e^{-\sigma^\alpha n(\rho)^{-\alpha}|T_r^E f(x)|^\alpha} - 1\right] \mathrm{d}x F_\rho(\mathrm{d}r)\right)$$

Power law assumptions : for $\beta \neq q$, assume $F(\mathrm{d}r) = f(r)\mathrm{d}r$ with

$$f(r) \sim C_\beta r^{-\beta-1}, \text{ as } r \to 0^{q-\beta} = \begin{cases} r \to +\infty, & \beta > q \text{ (zoom-out)} \\ r \to 0, & \beta < q \text{ (zoom-in)} \end{cases}.$$

Recall that $\int_{\mathbb{R}^+} r^q F(\mathrm{d}r) < +\infty$ with $q = \mathrm{tr}(E)$.
Lemma :[HB, Estrade, Kaj, 2010] if g is a continuous function on $\mathbb{R}^+$ such that $|g(r)| \leq C \min(r^{p_1}, r^{p_2})$, for some $0 < p_1 < \beta < p_2$, then

$$\int_{\mathbb{R}^+} g(r) F_\rho(\mathrm{d}r) \underset{\rho \to 0^{\beta - q}}{\sim} C_\beta \rho^\beta \int_{\mathbb{R}^+} g(r) r^{-\beta-1} \mathrm{d}r.$$

Operator properties

Recall that $T_r^E f(x) = \int_{B_E(x,r)} f(t)\mathrm{d}t$, we need assumptions on $\beta \neq q$ and f s.t.

$$\int_{\mathbb{R}^+} \| T_r^E f \|_\alpha^\alpha r^{-\beta-1} \mathrm{d}r < +\infty.$$

Proposition

For $\alpha \in [1,2]$, and $f \in \mathcal{S}(\mathbb{R}^d)$

- $\| T_r^E f \|_\alpha^\alpha \leq c_{B,1} r^{q\alpha} \| f \|_\alpha^\alpha$
- $\| T_r^E f \|_\alpha^\alpha \leq c_{B,2} r^{q} \| f \|_1^\alpha$
- *if moreover $\int f = 0$ ie $f \in \mathcal{S}_1(\mathbb{R}^d)$, then for $r > 1$,*

$$\| T_r^E f \|_\alpha^\alpha \leq c_{B,3} r^{q-a_d} \max(|\log r|, 1)^{\ell_d - 1} \| f \|_1^{\alpha-1} \int_{\mathbb{R}^d} |t||f(t)|\mathrm{d}t,$$

where a_d is the minimal real part of the eigenvalues of E and ℓ_d the size of its associated Jordan block.

Generalized operator scaling α stable field

Let $\alpha \in (1,2]$ and $\beta \in \begin{cases} (q, \alpha q) & n = 0 \\ (q - a_d, q) & n = 1 \end{cases}$, we can define the $S\alpha S$ r.v.

$$Z_\alpha^E(f) := \int_{\mathbb{R}^d \times \mathbb{R}^+} T_r^E f(x) W_{\alpha,\beta}(\mathrm{d}x\mathrm{d}r),$$

where $W_{\alpha,\beta}$ is a $S\alpha S$ random measure with intensity $\sigma^\alpha C_\beta r^{-1-\beta} \mathrm{d}x\mathrm{d}r$ for $f \in \mathcal{S}_n(\mathbb{R}^d)$.

Proposition

The random field $(Z_\alpha^E(f))_{f \in \mathcal{S}_n(\mathbb{R}^d)}$ admits a regularization $(\tilde{Z}_\alpha^E(f))_{f \in \mathcal{S}_n(\mathbb{R}^d)}$ in $\mathcal{S}_n(\mathbb{R}^d)'$ meaning that there exists a generalized field $(\tilde{Z}_\alpha^E(f))_{f \in \mathcal{S}_n(\mathbb{R}^d)}$ defined on some probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$ such that $\forall k \in \mathbb{N}, f_1, \ldots, f_k \in \mathcal{S}_n(\mathbb{R}^d), A_1, \ldots A_k \in \mathcal{B}(\mathbb{R})$

$$\mathbb{P}\left(Z_\alpha^E(f_1) \in A_1, \ldots, Z_\alpha^E(f_k) \in A_k\right) = \tilde{\mathbb{P}}\left(\tilde{Z}_\alpha^E(f_1) \in A_1, \ldots, \tilde{Z}_\alpha^E(f_k) \in A_k\right).$$

Functional scaling limit

Theorem

Assume that $\beta \in \begin{cases} (q, \alpha q) & n = 0 \\ (q - a_d, q) & n = 1 \end{cases}$. *As* $n(\rho) := \rho^{\beta}\lambda(\rho) \to +\infty$ *with* $\rho \to 0^{\beta - q}$, *then*

$$Y_\rho := \frac{X_\rho}{n(\rho)} \xrightarrow{d} \tilde{Z}_\alpha^E,$$

for the convergence in law in $\mathcal{S}_n(\mathbb{R}^d)'$.

Let us quote that by linearity fdd convergence follows once we have proven that

$$\Psi_{Y_\rho}(f) \to \Psi_{Z_\alpha^E}(f)$$

This follows the same lines as in [Breton, Dombry, 09].
Since $\Psi_{Z_\alpha^E} : \mathcal{S}_n(\mathbb{R}^d) \to \mathbb{C}$ is positive definite and continuous, Bochner-Minlos Theorem gives the existence of $\tilde{Z}_\alpha^E$. But $\Psi_{Z_\alpha^E}(f) = \Psi_{\tilde{Z}_\alpha^E}(f)$, such that Lévy's Theorem directly gives us the functional convergence !

Some remarks

Functional convergence results previously obtained for

- isotropic scaling $E = I_d$;
- zoom out case : $\rho \to 0$;
- $\beta \in (\alpha, \alpha d)$;

1. In [Breton, Dombry, 2011], for $\mathcal{D}(\mathbb{R}^d)'$ with $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ the space of smooth compactly supported functions, using a different approach with a tighness criterion
2. In [Breton, Gobard, 2015], for $\mathcal{C}(\mathcal{A})$ the space of continuous function on a compact subset $\mathcal{A}$ of finite signed measure for the $\|\cdot\|_{TV}$ topology $[(\mu(\mathrm{d}t)$ instead of $f(t)\mathrm{d}t]$

Properties of the limit

Following [Dobrushin, 1979], define

- the group shift transformation $\mathcal{T} = \{\tau_h; h \in \mathbb{R}^d\}$ on the space $\mathcal{S}_n(\mathbb{R}^d)$:

$$\tau_h f(t) = f(t-h), f \in \mathcal{S}_n(\mathbb{R}^d), h \in \mathbb{R}^d, t \in \mathbb{R}^d;$$

- the group of E-operator scaling transformations $\Delta^E = \{\delta_c^E; c \in (0,+\infty)\}$ on the space $\mathcal{S}_n(\mathbb{R}^d)$:

$$\delta_c^E f(t) = c^{-q} f(c^{-E} t), f \in \mathcal{S}_n(\mathbb{R}^d), c \in (0,+\infty), q = \mathrm{tr}(E), t \in \mathbb{R}^d;$$

and their analogous $\mathcal{T}$, Δ^E on $\mathcal{S}_n(\mathbb{R}^d)'$ with

$$\tau_h(L(f)) := L(\tau_h f), \text{ and } \delta_c^E L(f) := L(\delta_c^E f) \text{ for } L \in \mathcal{S}_n(\mathbb{R}^d)'.$$

Note that for L given by a function g, one has

$$\tau_h g(x) = g(x+h) \text{ and } \delta_c^E g(x) = g(c^E x).$$

Properties of the limit

Let $\alpha \in (1,2]$ and $\beta \in (q - a_p, \alpha q) \setminus \{q\}$. The generalized random field $\tilde{Z}_\alpha^E$ is

- with stationary nth increments : $\forall h \in \mathbb{R}^d$,

$$\tau_h \tilde{Z}_\alpha^E \stackrel{d}{=} \tilde{Z}_\alpha^E,$$

  equivalently

$$\forall f \in \mathcal{S}_n(\mathbb{R}^d), \Psi_{\tilde{Z}_\alpha^E}(\tau_h f) = \Psi_{\tilde{Z}_\alpha^E}(f).$$

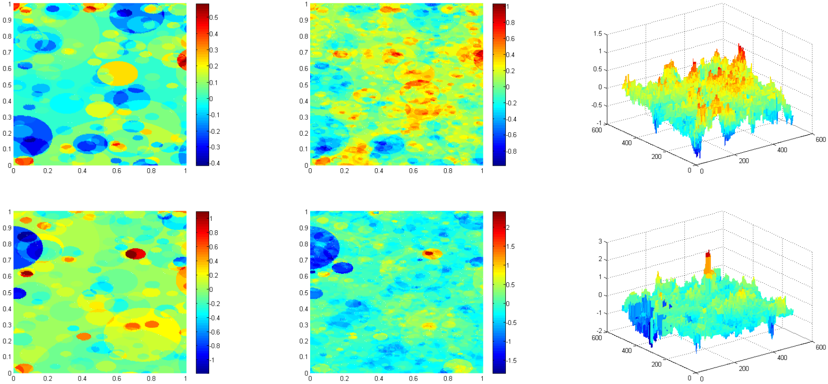
- (E, H)-operator scaling for $H = \frac{q-\beta}{\alpha} \in (-q(1-1/\alpha), 0) \cup (0, a_d/\alpha)$, $\forall c > 0$,

$$\delta_c^E \tilde{Z}_\alpha^E \stackrel{d}{=} c^H \tilde{Z}_\alpha^E,$$

  equivalently $\forall f \in \mathcal{S}_n(\mathbb{R}^d), \Psi_{\tilde{Z}_\alpha^E}(\delta_c^E f) = \Psi_{c^H \tilde{Z}_\alpha^E}(f)$.

- Self-similarity exponents are defined by $H_i := H/a_i \leq H/a_d < 1/\alpha$ in the direction of the eigenvector when a_i is an eigenvalue of E.

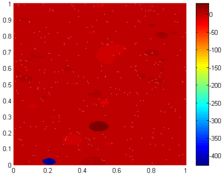
Illustration



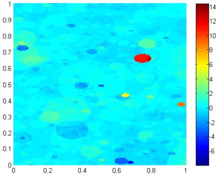
$$E = \mathrm{diag}(1, 1.2),\ \sigma = 0.1,\ \beta = 1.6 \in (1.5, 2.5)$$

Top : $\alpha = 2,\ H = 0.3 = H_1,\ H_2 = 0.25,$ Bottom : $\alpha = 1.8,\ H = 0.33 = H_1,\ H_2 = 0.28$

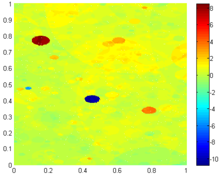
Illustration



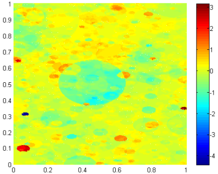
$\alpha = 1.1$



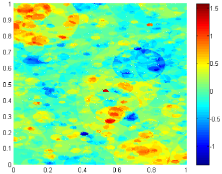
$\alpha = 1.3$



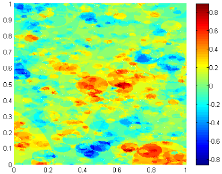
$\alpha = 1.5$



$\alpha = 1.7$



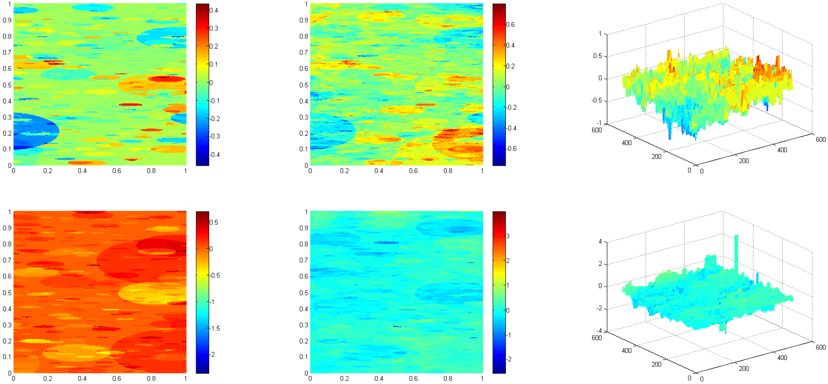
$\alpha = 1.9$



$\alpha = 2$

$E = \text{diag}(1, 1.2),\ \sigma = 0.1,\ \beta = 1.6 \in (1.5, 2.5)$

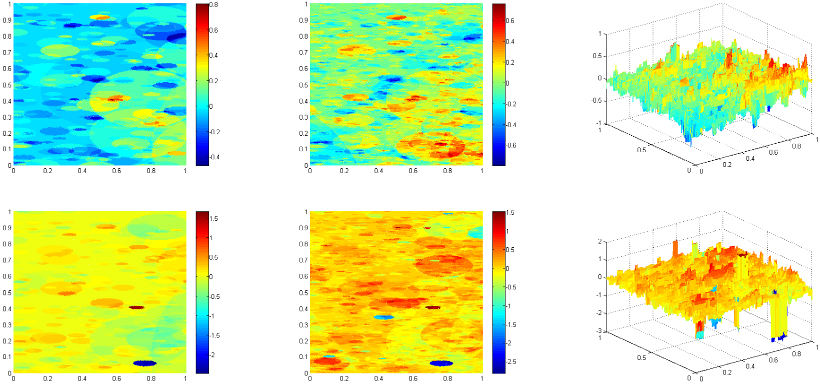
Illustration



$$E = \text{diag}(1, 1.8),\ \sigma = 0.1,\ \beta = 1.9 \in (1.8, 2.8)$$

Top : $\alpha = 2,\ H = 0.45 = H_1,\ H_2 = 0.25$, Bottom : $\alpha = 1.8,\ H = 0.5 = H_1,\ H_2 = 0.28$

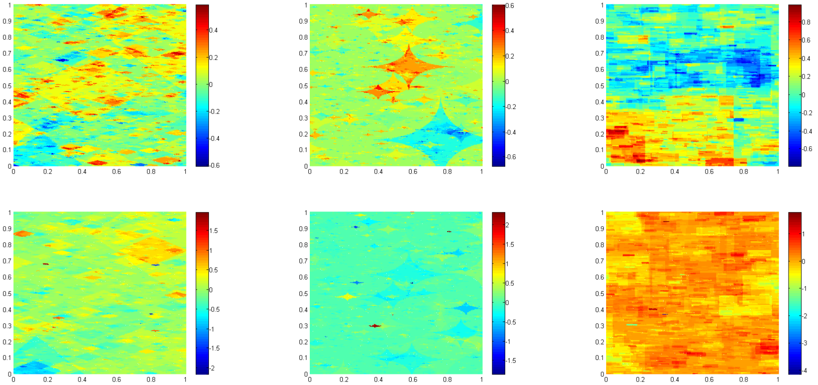
Illustration



$$E = \mathrm{diag}(1, 1.5),\ \sigma = 0.1,\ \beta = 1.75 \in (1.5, 2.5)$$

Top : $\alpha = 2,\ H = 0.375 = H_1,\ H_2 = 0.25$, Bottom : $\alpha = 1.8,\ H = 0.417 = H_1,\ H_2 = 0.28$

Illustration



B_1 $\qquad$ $B_{1/2}$ $\qquad$ B_∞

$$E = \text{diag}(1, 1.5),\ \sigma = 0.1,\ \beta = 1.75 \in (1.5, 2.5)$$
$$\alpha = 2 \text{ and } H = 0.375 \text{ (top)},\ \alpha = 1.8 \text{ and } H = 0.417 \text{ (bottom)}$$