Tail process and its role in limit theorems

Bojan Basrak, University of Zagreb

The Fields Institute
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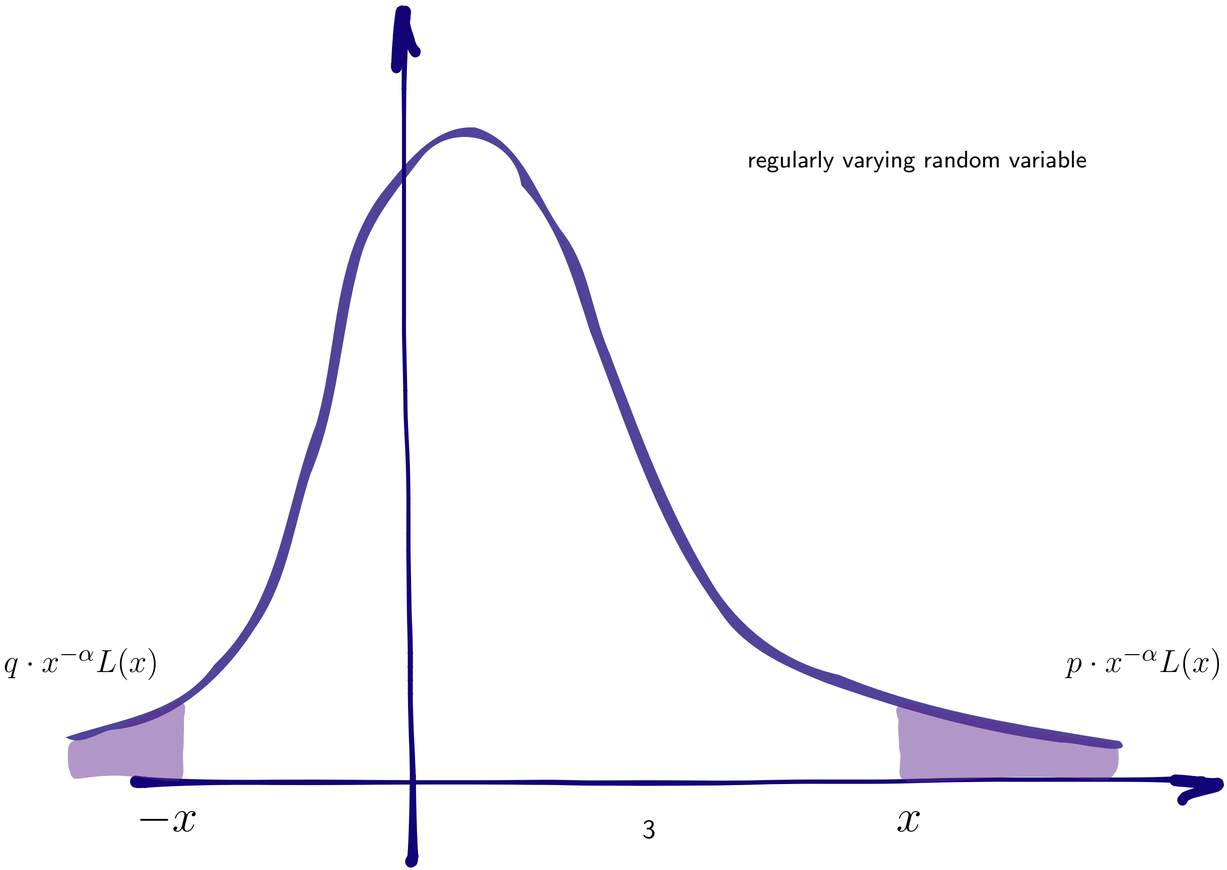
based on the joint work (in progress) with

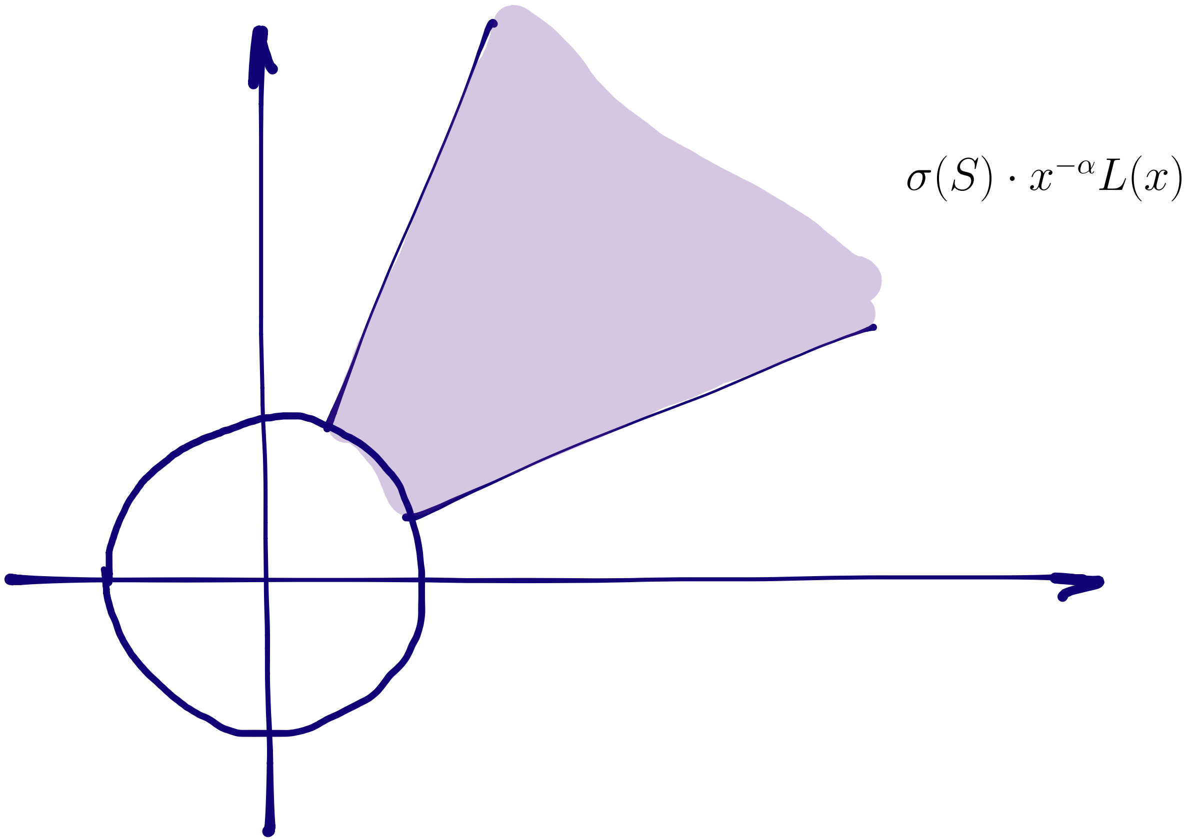
Philippe Soulier, Azra Tafro, Hrvoje Planinić

Stationary regularly varying sequences

Regular variation



Multivariate regular variation



Regularly varying process

- A stationary time series $(X_n)_n$ is said to be **regularly varying** if random vectors

$$(X_0, \ldots, X_k) \quad k \geq 0$$

are regularly varying for each k.

For a stationary regularly varying sequence there exists a **tail process**

$$(Y_t)_{t\in\mathbb{Z}}$$

such that

$$\left(\frac{X_t}{x}\right)_{t\in\mathbb{Z}}\Bigg|\,|X_0|>x \xrightarrow{d} (Y_t)_{t\in\mathbb{Z}}$$

and a **spectral tail process**

$$(\theta_t)_{t\in\mathbb{Z}}$$

independent of $|Y_0|$ such that

$$(Y_t)_t \stackrel{d}{=} |Y_0|(\theta_t)_t\,.$$

Moreover

$$|\theta_0|=1 \quad \text{and} \quad |Y_0|\sim \text{Pareto}(\alpha)\,.$$

There exists a sequence (a_n) such that

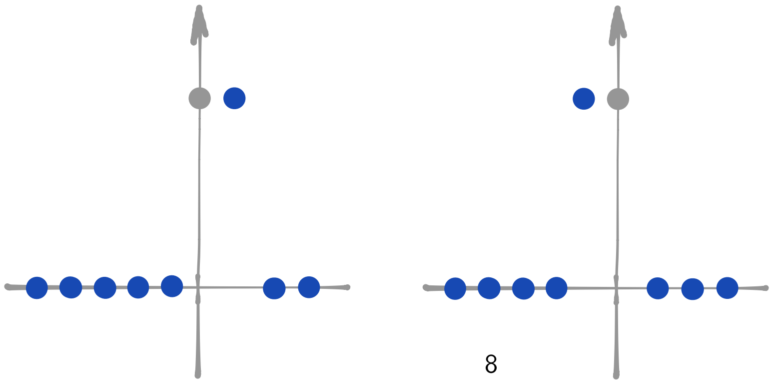
$$\left(\frac{X_t}{a_n}\right)_{t\in\mathbb{Z}} \Bigg| \, |X_0| > a_n \xrightarrow{d} (Y_t)_{t\in\mathbb{Z}}.$$

Examples (for simplicity, assume $\theta_0 = 1$)

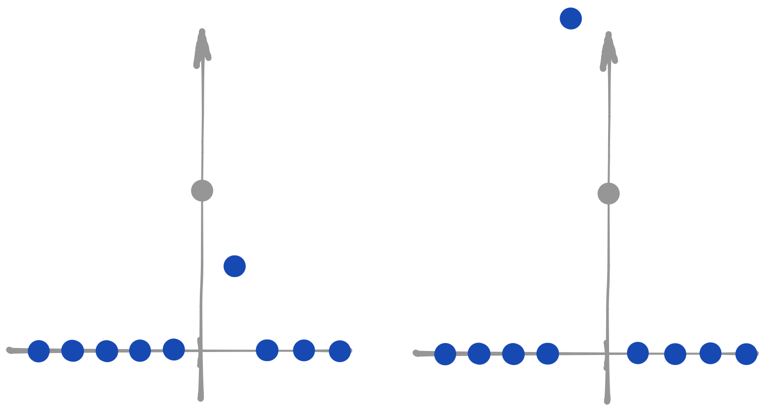
a) X_t iid RV(α), $\quad \theta_t = 0$, for $t \neq 0$.

b) $X_t = Z_t \vee Z_{t-1}$, Z_t iid RV(α),

$$\ldots, \theta_{-1}, \theta_0, \theta_1, \ldots \sim \begin{cases} \ldots, 0, 0, 1, 1, 0, \ldots & \text{w.p. } 1/2 \\ \ldots, 0, 1, 1, 0, 0, \ldots & \text{w.p. } 1/2 \end{cases}$$



c) $X_t = Z_t + \frac{1}{2} Z_{t-1}$, Z_t iid RV(α).



Independent observations

Regular variation assumption determines limiting behavior of

- point processes
- sums and random walks $S_n = X_1 + \cdots + X_n$
- maxima and other extremes $M_n = \max\{X_1, \ldots, X_n\}$
- records and record times

Complete convergence theorem 1

simple but powerful – cf. Leadbetter-Rootzén, Resnick

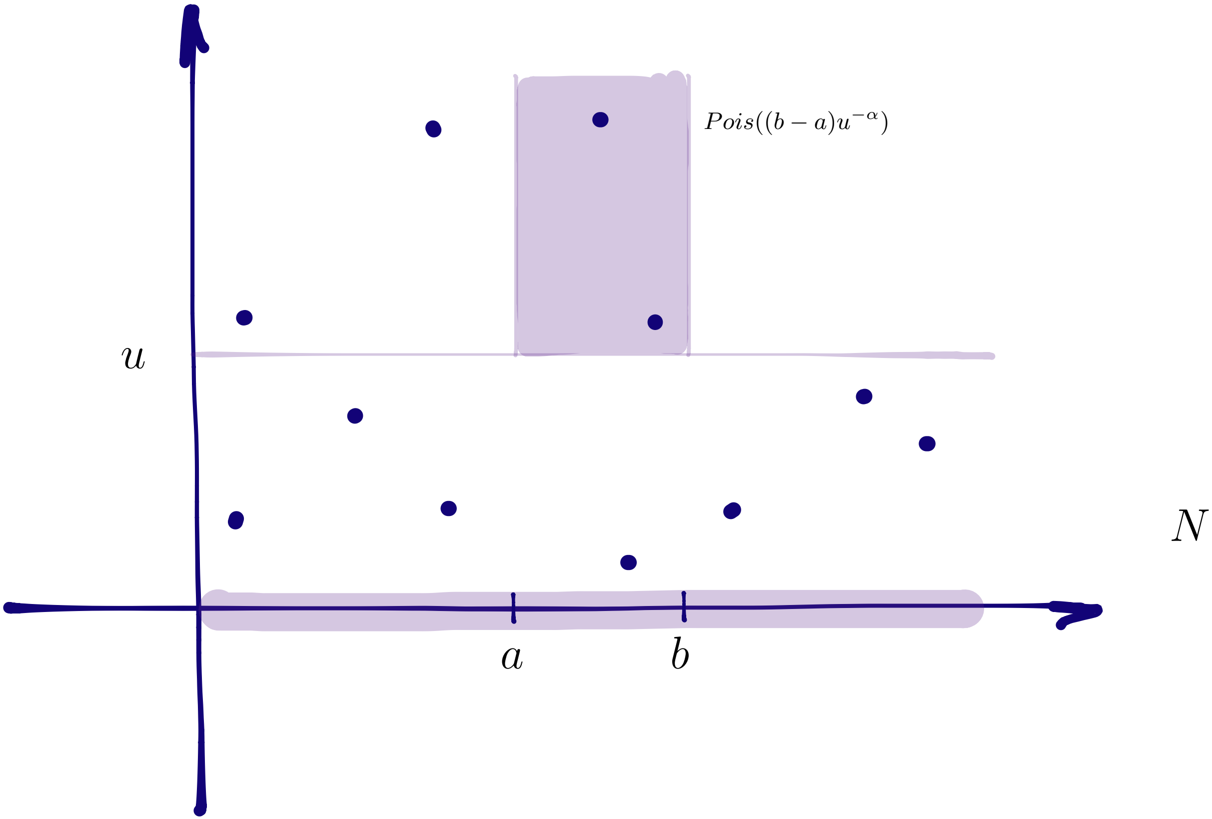
Theorem For iid $X_t \geq 0$, X_0 is reg. varying is if and only if

$$N_n = \sum_1^n \delta_{\frac{i}{n}, \frac{X_i}{a_n}} \xrightarrow{d} N = \sum_i \delta_{T_i, P_i} \, ,$$

where N is PRM(Leb$\times d(-x^{-\alpha})$).

So

$$P(M_n/a_n \leq u) = P(N_n([0,1] \times (u, \infty)) = 0) \to P(N([0,1] \times (u, \infty)) = 0) = e^{-u^{-\alpha}}$$



$Pois((b-a)u^{-\alpha})$
u
N
a
b

Extremes of dependent sequences cluster

Strongly mixing observations

Mori (1977) showed if

$$N_n \xrightarrow{d} N$$

then

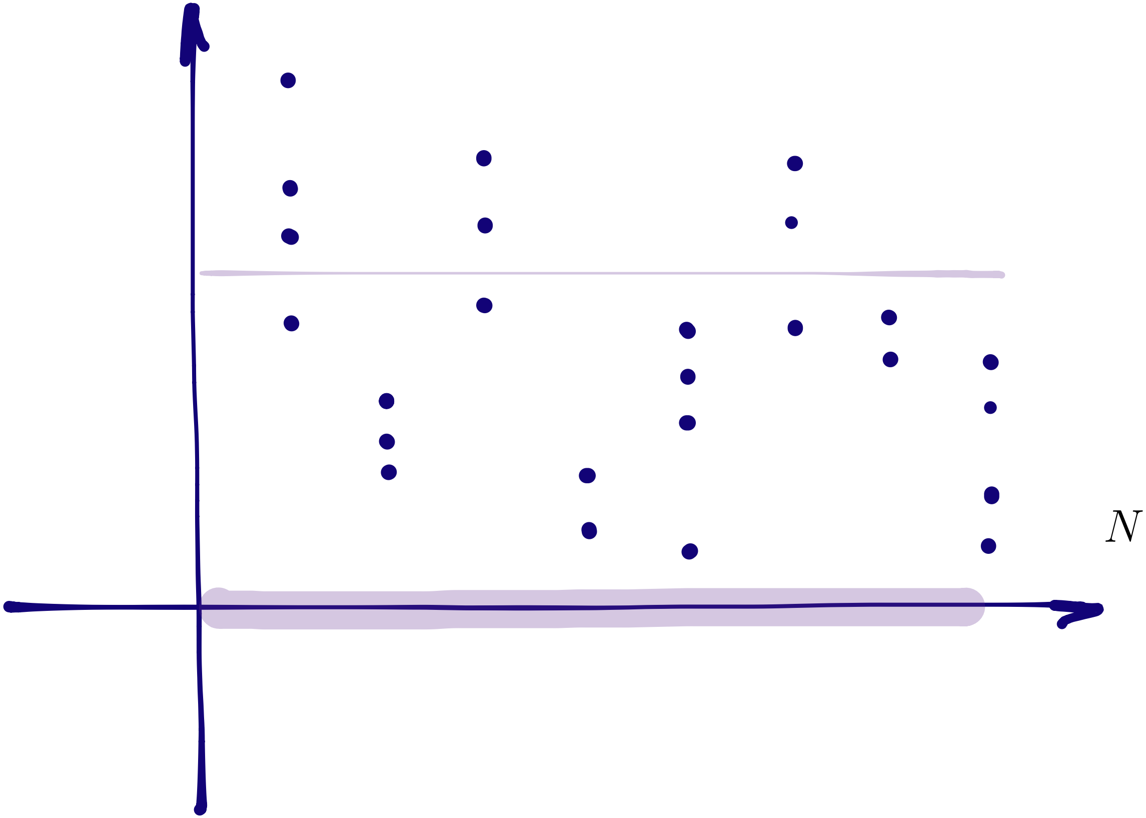
$$N = \sum_i \sum_j \delta_{T_i, P_i Q_{i,j}}$$

where

- ▷ $\sum_i \delta_{T_i,P_i}$ is a PRM($\vartheta \cdot \text{Leb} \times d(-x^{-\alpha})$) with $\vartheta \in (0,1]$
- ▷ $\sum_j \delta_{Q_{ij}}$ is an iid sequence of point processes in $[-1,1]$, independent of PRM above.

It was not really clear

- what is ϑ
- what is the distribution of $\sum_j \delta_{Q_{ij}}$
- what would be a sufficient condition for such a convergence



N

Anti-clustering condition

or finite mean cluster size condition

High level exceedances are not clustering for "too long", i.e for some $r_n \to \infty$ and $r_n/n \to 0$:

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left(\bigvee_{m \le |i| \le r_n} |X_i| > a_n u \,\middle|\, |X_0| > a_n u \right) = 0\,, \quad u > 0\,. \quad (1)$$

It implies

$$Y_m \xrightarrow{P} 0, \qquad \text{as } |m| \to \infty.$$

Main technical lemma

Denote $L_Y = \sup_{i\in\mathbb{Z}} |Y_i|$ and $M_n = \max |X_1|, \ldots, |X_n|$, then under the assumptions above

$$\left(\sum_{i=1}^{r_n} \delta_{X_i/M_{r_n}}, \frac{M_{r_n}}{a_n} \,\middle|\, M_{r_n} > a_n \right) \Rightarrow \left(\sum_i \delta_{Y_i/L_Y}, L_Y \,\middle|\, \sup_{j<0} |Y_j| \le 1 \right)$$

with two components on the right hand side being independent – B. & Tafro (2016).

Complete convergence theorem 2

building on Davis & Resnick, Davis & Hsing, Davis & Mikosch

tafro,b.(2016) /krizmanić, segers, b. (2012)

Theorem Under strong mixing and a.c., as $n \to \infty$,

$$N_n \xrightarrow{d} N = \sum_{i,j} \delta_{(T_i, P_i Q_{ij})} ,$$

where

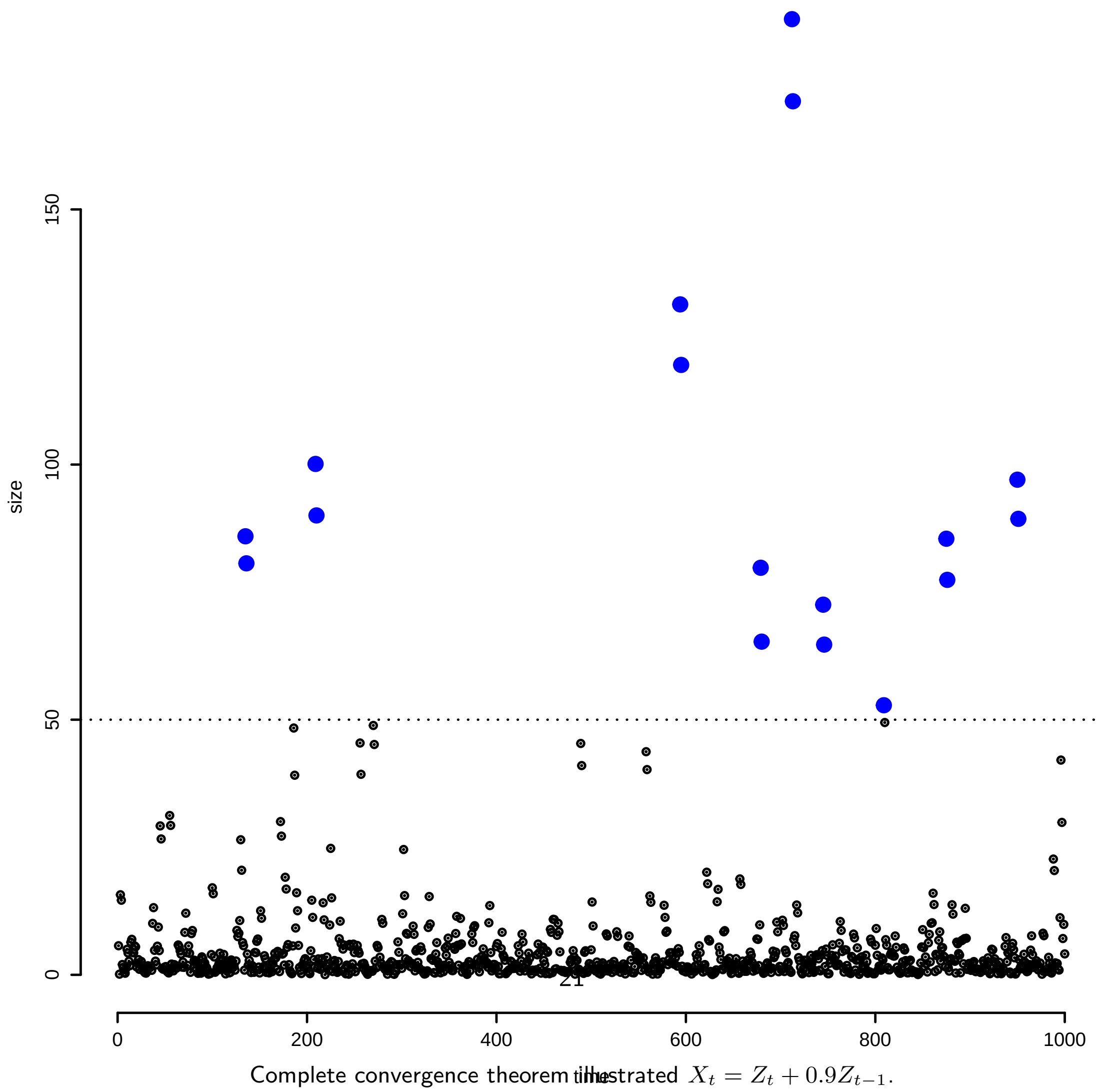
- $\sum_i \delta_{T_i, P_i}$ is a Poisson process on $[0,1] \times (0,\infty]$ with intensity ϑLeb $\times$ $d(-x^{-\alpha})$
- $(\sum_j \delta_{Q_{ij}})_i$ is an iid sequence of point processes independent of the process above

Here ϑ is **the extremal index** of the sequence $|X_t|$ with representation

$$\vartheta = P(\bigvee_{i \geq 1} |Y_i| \leq 1) = P(\bigvee_{i \leq -1} |Y_i| \leq 1) > 0.$$

While cluster shapes satisfy

$$\sum_j \delta_{Q_j} \stackrel{d}{=} \sum_i \delta_{Y_i/L_Y} \Big| \sup_{j<0} |Y_j| \leq 1$$



Complete convergence theorem illustrated $X_t = Z_t + 0.9Z_{t-1}$.

As in the iid case one can prove (functional) limit theorems for

- ▷ partial maxima $M_{\lfloor nt \rfloor}$
- ▷ partial sums $S_{\lfloor nt \rfloor}$ under additional conditions and unusual topologies (Avram & Taqqu, B. Krizmanić, Segers, or Jakubowski...)

However...

- in the limit there is a loss of information about the order
- one cannot find the limit of $S_{\lfloor nt \rfloor}$ even for some very simple models
- it is difficult to say much about records or record times

Space for ordered clusters

We introduce a new space

$$\tilde{l}_0 = l_0/\sim \qquad \text{where } l_0 = \{\boldsymbol{x} = (x_i)_{i\in\mathbb{Z}} : \lim_{|i|\to\infty} x_i = 0\}$$

and $\boldsymbol{x} \sim \boldsymbol{y}$ if $d(\boldsymbol{x}, \boldsymbol{y}) = 0$ with

$$d(\boldsymbol{x}, \boldsymbol{y}) = \inf_k \sup_j |x_j - y_{j+k}|.$$

Adapting d to $\tilde{l}_0$ produces a separable and complete metric space.

Technical lemma in $\tilde{l}_0$

large deviations result

Under a.c. assumption as $n \to \infty$

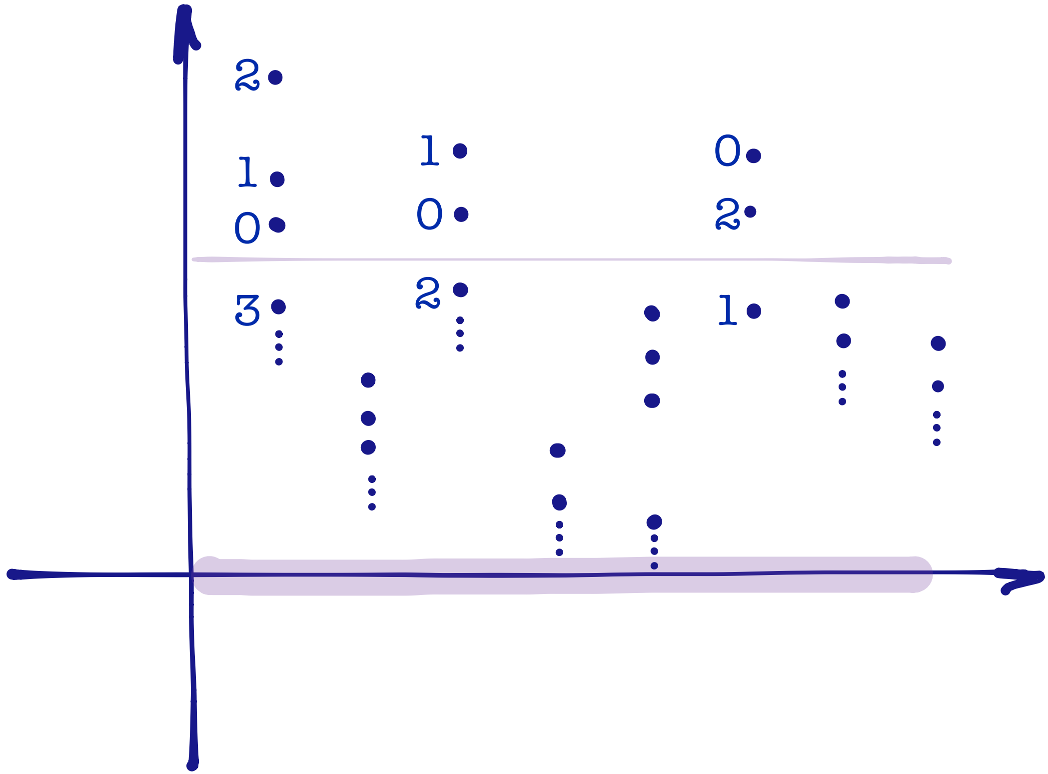
$$\left(\frac{X_1, \ldots, X_{r_n}}{a_n} \,\Big|\, M_{r_n} > a_n \right) \Rightarrow \left(Y_i,\ i \in \mathbb{Z} \,\Big|\, \sup_{j<0} |Y_j| \le 1 \right)$$

in $\tilde{l}_0 \setminus \{\mathbf{0}\}$. The space is not locally compact, so we need to use $w^{\#}$ topology (cf. Daley–Vere Jones). Related to Hult & Samorodnitsky (2010) and Mikosch & Wintenberger (2016) large deviations results.

As before, conditionally on $\sup_{j<0} |Y_j| \le 1$, random variable $L_Y = \sup_{i \in \mathbb{Z}} |Y_i|$ and random cluster $(Y_i / L_Y)_i$ are independent.

Complete convergence theorem 3

very technical but order is preserved



Partial sums converge again

Under assumptions above (for $\alpha \in (0,1)$ plus an additional one for $\alpha \in [1,2)$) we prove functional limit theorem for

$$\frac{S_{\lfloor nt \rfloor}}{a_n}, \quad t \geq 0,$$

but the limit is a "process" in a new space – $E[0,1]$, M_2. Assume for convenience that X_i's are symmetric if $\alpha \geq 1$.

Similarly one can prove the limiting theorem in a more standard space $(D[0,1], M_1)$ for

$$\sup_{s \leq t} \frac{S_{\lfloor ns \rfloor}}{a_n}, \quad t \geq 0.$$

Space $E[0, 1]$

Whitt 2002

Elements are triples

$$(x, T, \{I(t) : t \in T\}),$$

where $x \in D[0, 1]$, T is a countable subset of $[0, 1]$ with

$$Disc(x) \subseteq T,$$

and, for each $t \in T$, $I(t)$ is a closed bounded interval such that

$$x(t), x(t-) \in I(t) .$$

Moreover, for each $\varepsilon > 0$, there are at most finitely many times t with diameter of $I(t)$ greater than ε.

Functional limit theorem

in the space E

planinić, soulllier, b. (2016)

Theorem Under assumptions above for $\alpha \in (0, 2)$ as $n \to \infty$,

$$\frac{S_{\lfloor nt \rfloor}}{a_n} \xrightarrow{d} S^E \,,$$

for some $S^E = (S, T_S, \{I_S(t) : t \in T_S\})$ with S being an α–stable Lévy process.

Remark

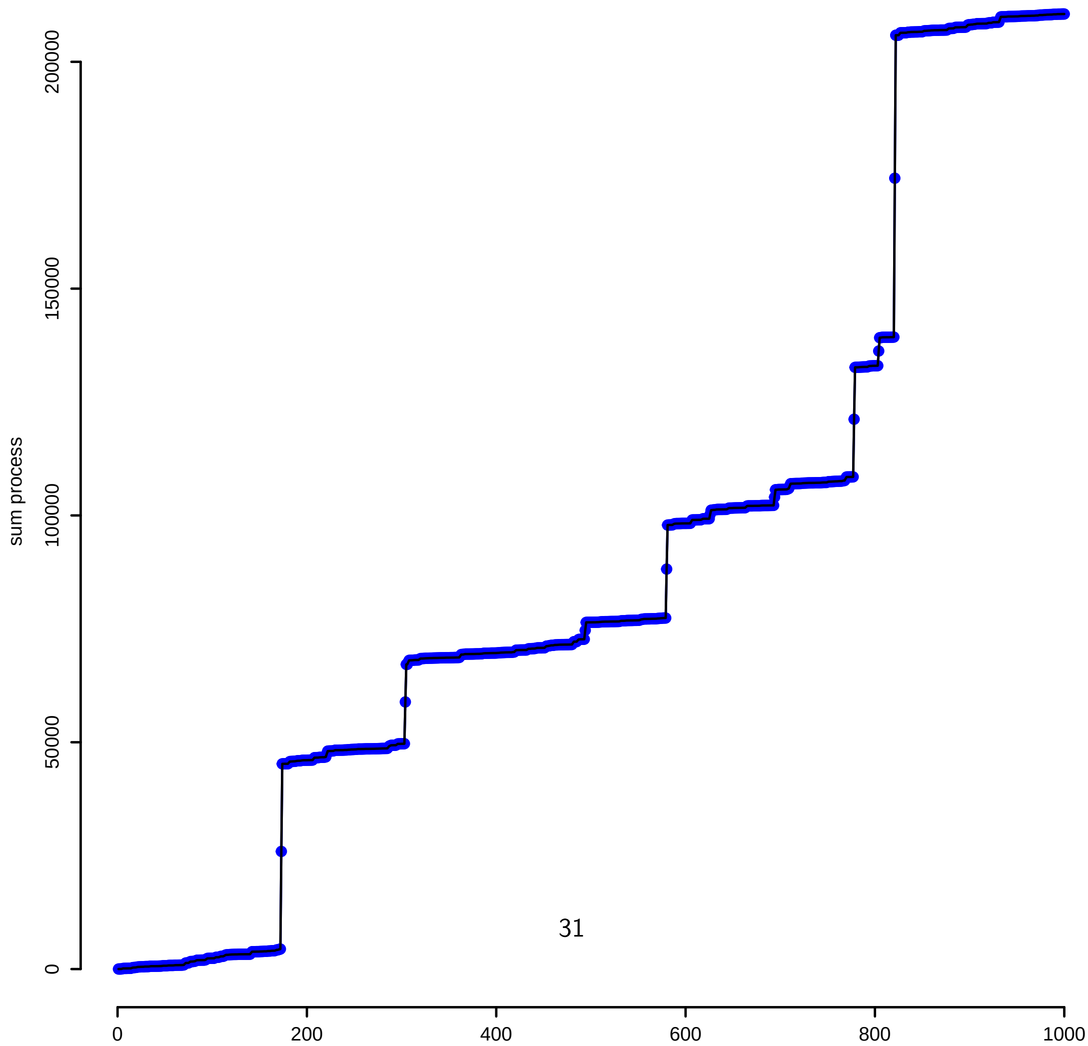
In the limit $S^E = (S, T_S, \{I_S(t) : t \in T_S\})$, countable set T_S includes all the discontinuities of the stable process S, and S can be trivial.

All three components of the process S^E can be expressed in the terms of the tail process.

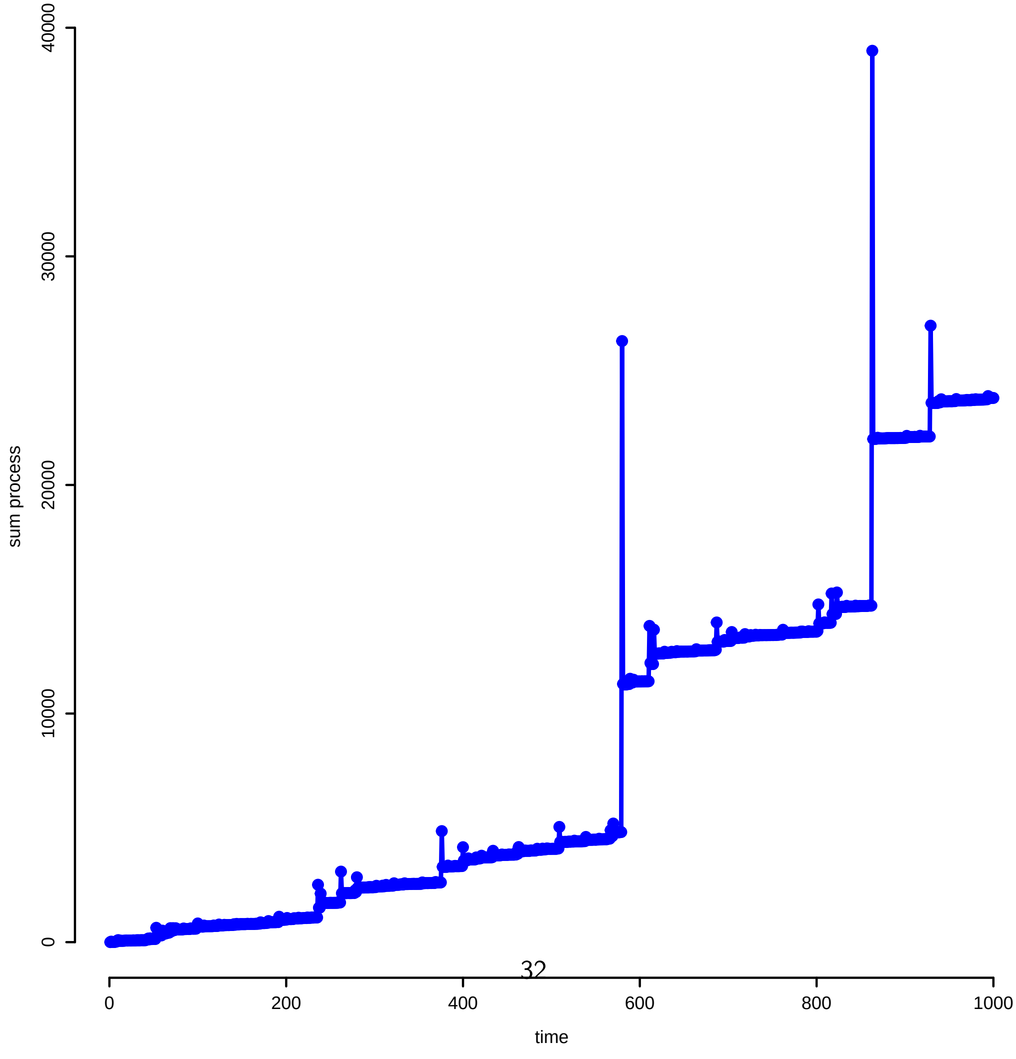
One can prove that in $D[0,1]$ with M_1 topology

$$\sup_{s \le t} \frac{S_{\lfloor ns \rfloor}}{a_n} \xrightarrow{d} \sup_{s \le t} S^E(s)\,.$$

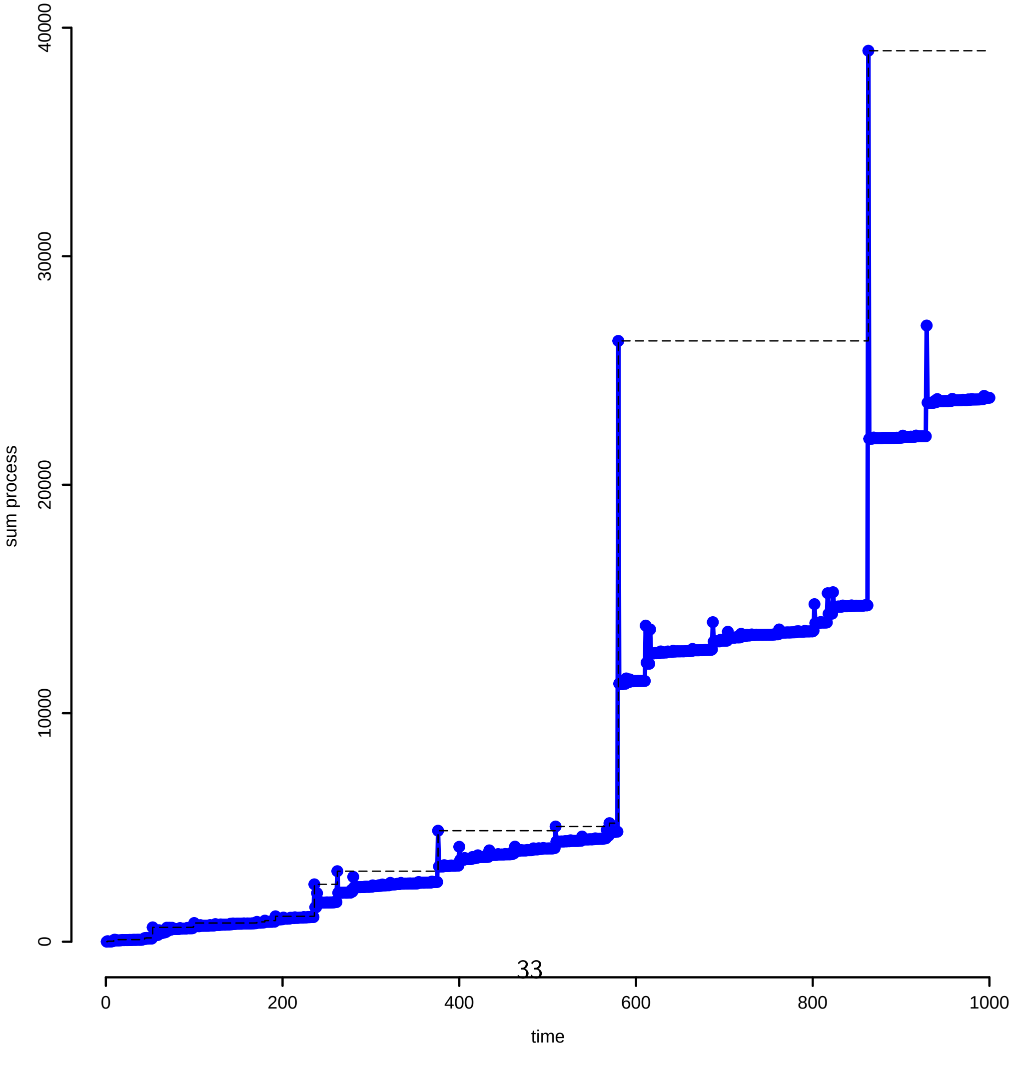
Partial sums for $X_t = Z_t + 0.9Z_{t-1}$.



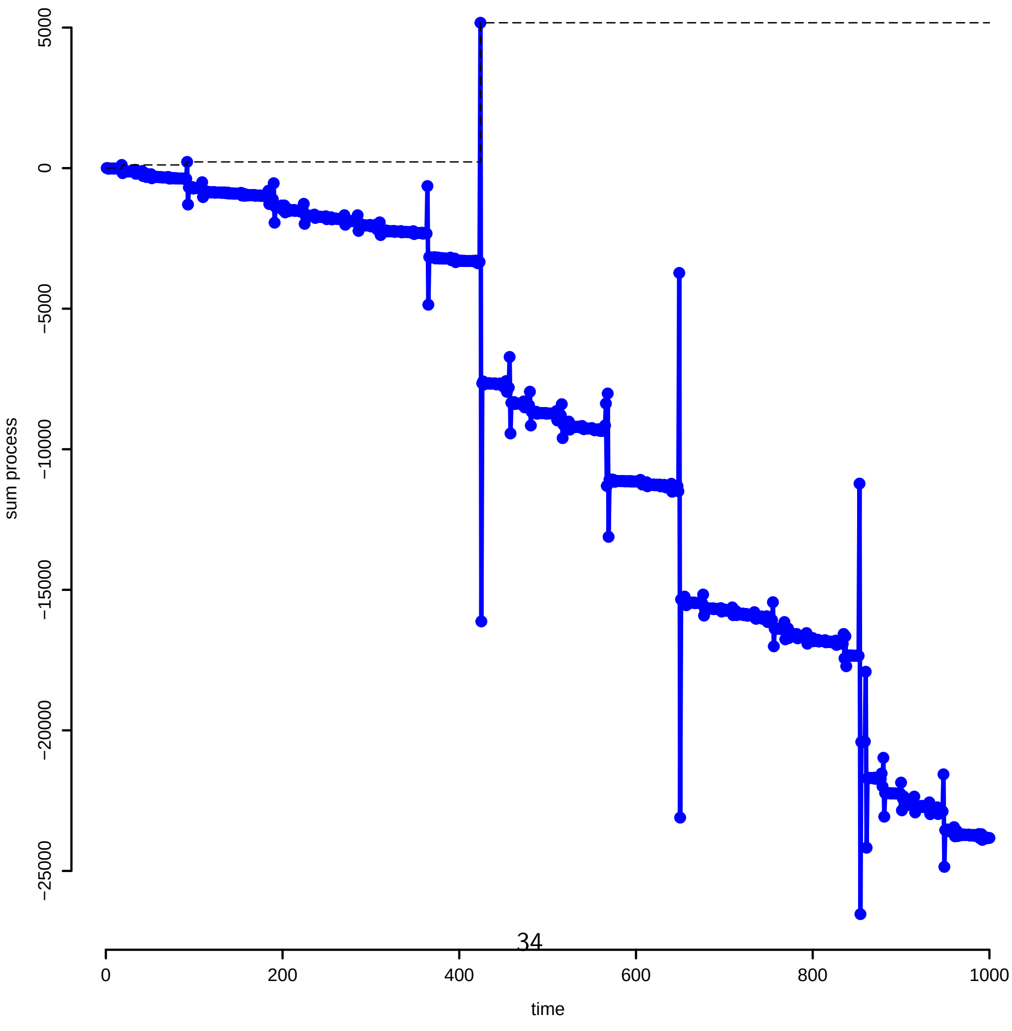
Partial sums for $X_t = Z_t - 0.7Z_{t-1}$.



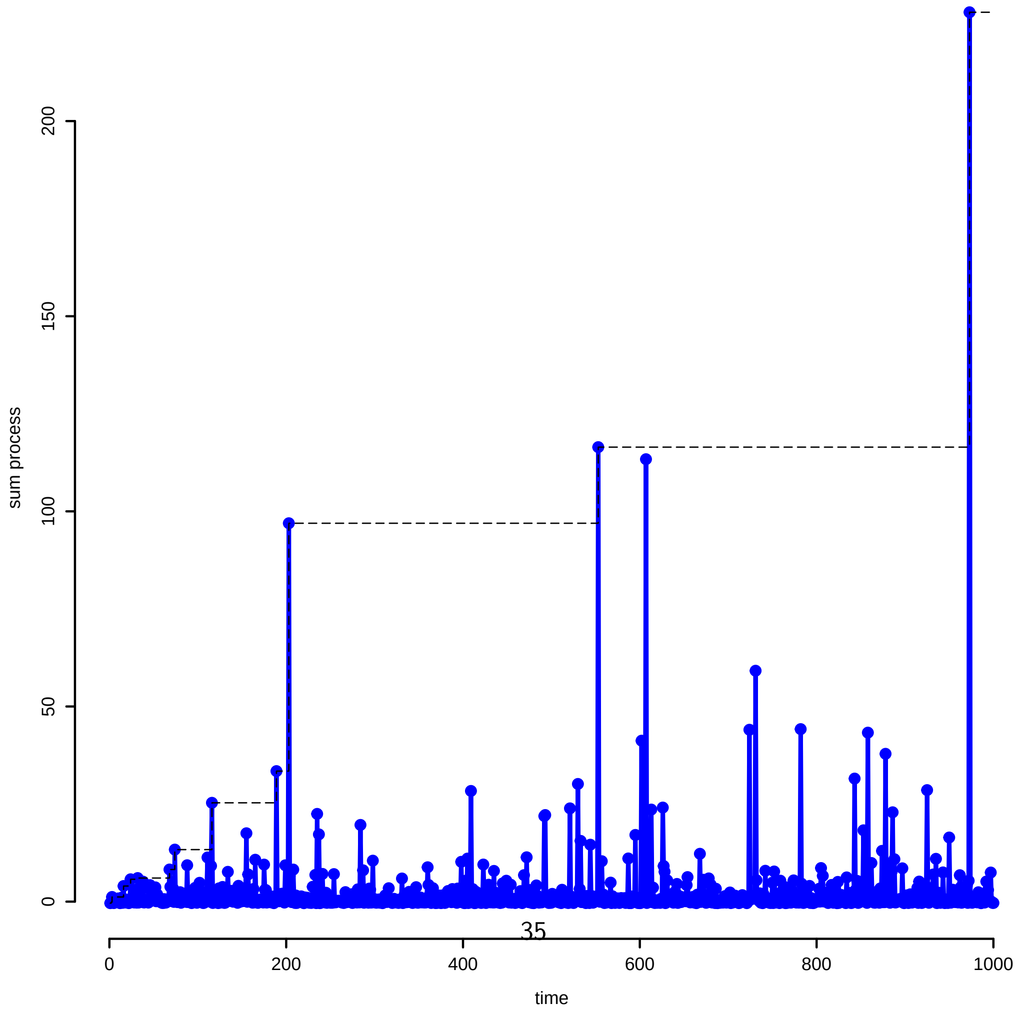
Running max of sums for $X_t = Z_t - 0.7Z_{t-1}$.



Sums of $X_t = Z_t - 2.5Z_{t-1} + Z_{t-2}$.



Sums of $X_t = Z_t - Z_{t-1}$.



Record times

It is well known (Resnick, 1987) that for any iid sequence from a continuous distribution, point process of record times converges, ie

$$R_n = \sum_{i=1}^{\infty} \delta_{i/n} \mathbb{I}_{\{X_i \text{ is a record}\}}$$

satisfies

$$R_n \xrightarrow{d} R = \sum_{i \in \mathbb{Z}} \delta_{R_i}$$

where R is a so called **scale invariant Poisson process** on $(0, \infty)$ with intensity dx/x .

Record times are much more difficult to handle when dependence is present, therefore we assume

▷ sequence (X_n), maybe after monotone transformation, is regularly varying, strong mixing and a.c. holds.

▷ with probability one all nonzero values of the tail process Y_i are mutually different .

The limit of record times

planinić, soulllier, b. (2016)

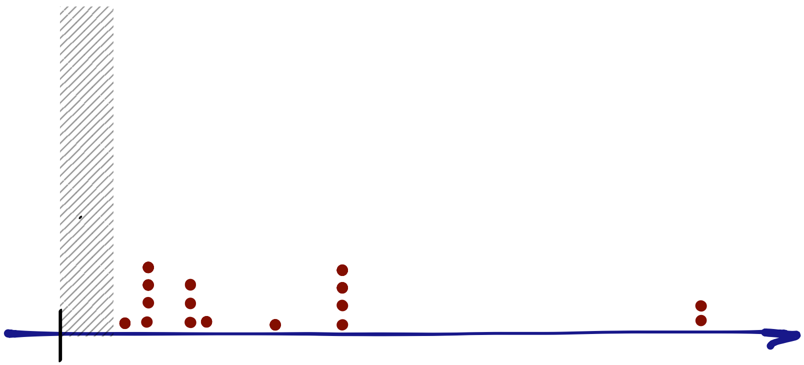
Theorem Under assumptions above as $n \to \infty$,

$$R_n \xrightarrow{d} R' = \sum_{i \in \mathbb{Z}} \delta_{R_i \kappa_i} ,$$

where

- ▷ $\sum_{i \in \mathbb{Z}} \delta_{R_i}$ is Poisson process on $(0, \infty)$ with intensity dx/x and
- ▷ $(\kappa_i)_{i \in \mathbb{Z}}$ is an iid sequence independent of it.

Records are broken in clusters.



Remark

The limit R' does not depend on ϑ directly.

Moreover, κ_i have the same distribution as

$$\sum_{j=-\infty}^{\infty} \mathbb{I}_{\{Q_j > \sup_{i<j} Q_i \vee e^{-W/\alpha}\}} \,,$$

where W is standard exponential and

$$(Q_j)_j \stackrel{d}{=} \left(\left(\frac{Y_i}{L_Y} \right)_i \,\middle|\, \sup_{j<0} |Y_j| \leq 1 \right)$$

Summary

A stationary regularly varying sequence (X_t)

- has a tail process (Y_t)
- the clusters of extremes can be described by (Y_t)
- point processes N_n have a limit characterized by (Y_t) with order preserved in the space $[0,1] \times \tilde{l}_0$.
- random walks with steps (X_t) have an "α–stable" limit for $\alpha \in (0,2)$ but in M_2 on $E[0,1]$ (càdlàg functions are ok only for some special cases).
- record times have a surprisingly simple compound Poisson structure in the limit.

Thanks