Chapter 16

Vector Calculus

16.7 Surface Integrals

Surface Integrals

The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length.

Suppose f is a function of three variables whose domain includes a surface S.

We will define the surface integral of f over S in such a way that, in the case where $f(x, y, z) = 1$, the value of the surface integral is equal to the surface area of S.

We start with parametric surfaces and then deal with the special case where S is the graph of a function of two variables.

Parametric Surfaces

Parametric Surfaces (1 of 6)

Suppose that a surface S has a vector equation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \qquad (u,v) \in D$$

We first assume that the parameter domain D is a rectangle and we divide it into subrectangles R_{ij} with dimensions Δu and Δv.

Then the surface S is divided into corresponding patches S_{ij} as in Figure 1.

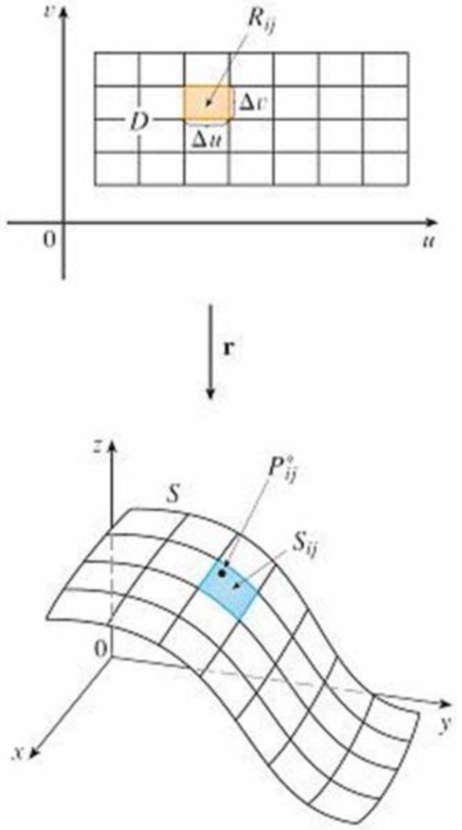


Figure 1

Parametric Surfaces (2 of 6)

We evaluate f at a point P_{ij}^* in each patch, multiply by the area ΔS_{ij} of the patch, and form the Riemann sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*) \Delta S_{ij}$$

Then we take the limit as the number of patches increases and define the **surface integral of *f* over the surface *S*** as

$$\textbf{1} \qquad \iint_S f(x, y, z)\, dS = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*) \Delta S_{ij}$$

Notice the analogy with the definition of a line integral and also the analogy with the definition of a double integral.

Parametric Surfaces (3 of 6)

To evaluate the surface integral in Equation 1 we approximate the patch area ΔS_{ij} by the area of an approximating parallelogram in the tangent plane.

In our discussion of surface area we made the approximation

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v| \; \Delta u \; \Delta v$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

are the tangent vectors at a corner of S_{ij}.

Parametric Surfaces (4 of 6)

If the components are continuous and $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonzero and nonparallel in the interior of *D*, it can be shown from Definition 1, even when *D* is not a rectangle, that

2 $$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

This should be compared with the formula for a line integral:

$$\int_C f(x, y, z)\, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt$$

Parametric Surfaces (5 of 6)

Observe also that

$$\iint_S 1\, dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA = A(S)$$

Formula 2 allows us to compute a surface integral by converting it into a double integral over the parameter domain D.

When using this formula, remember that $f(\mathbf{r}(u, v))$ is evaluated by writing $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ in the formula for $f(x, y, z)$.

Example 1

Compute the surface integral $\iint_S x^2 \, dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution:

We use the parametric representation

$$x = \sin\phi\cos\theta \qquad y = \sin\phi\sin\theta \qquad z = \cos\phi \qquad 0 \le \phi \le \pi \qquad 0 \le \theta \le 2\pi$$

That is,

$$\mathbf{r}(\phi, \theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\cos\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$$

We can compute that

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi$$

Example 1 – *Solution*

Therefore, by Formula 2,

$$\iint_S x^2\, dS = \iint_D (\sin\phi\cos\theta)^2 \left|\mathbf{r}_\phi \times \mathbf{r}_\theta\right| dA$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^2\phi\cos^2\theta\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \cos^2\theta\, d\theta \int_0^{\pi} \sin^3\phi\, d\phi$$

$$= \int_0^{2\pi} \tfrac{1}{2}(1+\cos 2\theta)\, d\theta \int_0^{\pi} (\sin\phi - \sin\phi\cos^2\phi)\, d\phi$$

$$= \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} \left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi}$$

$$= \frac{4\pi}{3}$$

Parametric Surfaces (6 of 6)

If a thin sheet (say, of aluminum foil) has the shape of a surface S and the density (mass per unit area) at the point (x, y, z) is $\rho(x, y, z)$, then the total **mass** of the sheet is

$$m = \iint_S \rho(x,y,z)\, dS$$

and the **center of mass** is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m}\iint_S x\rho(x,y,z)\, dS \qquad \bar{y} = \frac{1}{m}\iint_S y\rho(x,y,z)\, dS \qquad \bar{z} = \frac{1}{m}\iint_S z\rho(x,y,z)\, dS$$

Graphs of Functions

Graphs of Functions (1 of 3)

Any surface S with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations

$$x = x \qquad y = y \qquad z = g(x, y)$$

and so we have

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right)\mathbf{k}$$

Thus

$$\mathbf{3} \qquad \mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Graphs of Functions (2 of 3)

Therefore, in this case, Formula 2 becomes

4 $$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA$$

Similar formulas apply when it is more convenient to project *S* onto the *yz*-plane or *xz*-plane. For instance, if *S* is a surface with equation $y = h(x, z)$ and *D* is its projection onto the *xz*-plane, then

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,h(x,z),z)\sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\,dA$$

Example 2

Evaluate $\iint_S y \, dS$, where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$. (See Figure 2.)

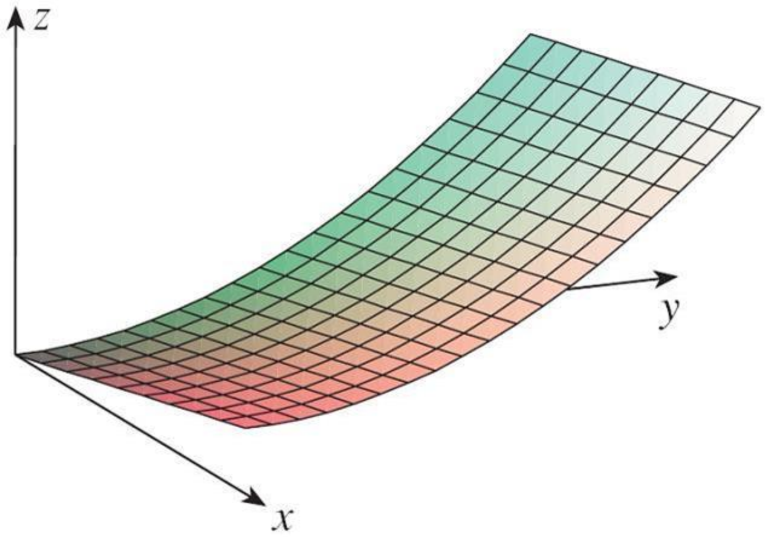


Figure 2

Example 2 – *Solution*

Since

$$\frac{\partial z}{\partial x} = 1 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 2y$$

Formula 4 gives

$$\begin{aligned}
\iint_S y\, dS &= \iint_D y\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\
&= \int_0^1 \int_0^2 y\sqrt{1 + 1 + 4y^2}\, dy\, dx \\
&= \int_0^1 dx \sqrt{2} \int_0^2 y\sqrt{1 + 2y^2}\, dy \\
&= \sqrt{2}\left(\tfrac{1}{4}\right)\tfrac{2}{3}(1 + 2y^2)^{\frac{3}{2}}\Big]_0^2 = \frac{13\sqrt{2}}{3}
\end{aligned}$$

Graphs of Functions (3 of 3)

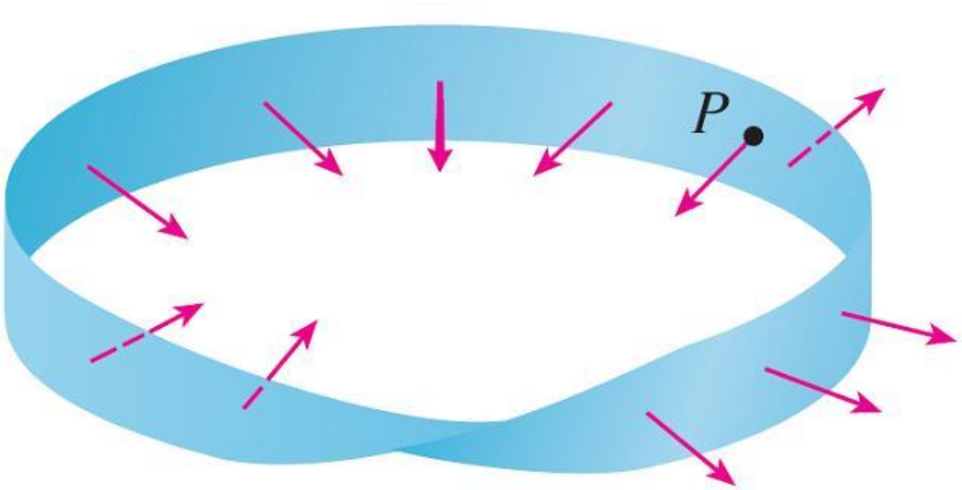
If S is a piecewise-smooth surface, that is, a finite union of smooth surfaces S_1, S_2, . . . , S_n that intersect only along their boundaries, then the surface integral of f over S is defined by

$$\iint_S f(x, y, z)\, dS = \iint_{S_1} f(x, y, z)\, dS + \cdots + \iint_{S_n} f(x, y, z)\, dS$$

Oriented Surfaces

Oriented Surfaces (1 of 9)

To define surface integrals of vector fields, we need to rule out nonorientable surfaces such as the Möbius strip shown in Figure 4. [It is named after the German geometer August Möbius (1790–1868).]

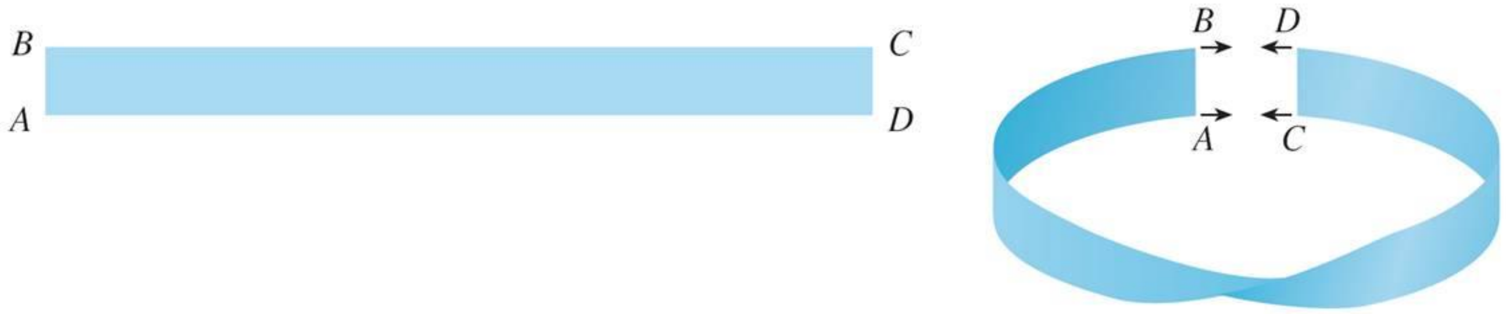


A Möbius strip

Figure 4

Oriented Surfaces (2 of 9)

You can construct one for yourself by taking a long rectangular strip of paper, giving it a half-twist, and taping the short edges together as in Figure 5.



Constructing a Möbius strip

Figure 5

Oriented Surfaces (3 of 9)

If an ant were to crawl along the Möbius strip starting at a point P, it would end up on the "other side" of the strip (that is, with its upper side pointing in the opposite direction).

Then, if the ant continued to crawl in the same direction, it would end up back at the same point P without ever having crossed an edge. (If you have constructed a Möbius strip, try drawing a pencil line down the middle.)

Therefore a Möbius strip really has only one side.

From now on we consider only orientable (two-sided) surfaces.

Oriented Surfaces (4 of 9)

We start with a surface S that has a tangent plane at every point (x, y, z) on S (except at any boundary point).

There are two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2 = -\mathbf{n}_1$ at (x, y, z). (See Figure 6.)

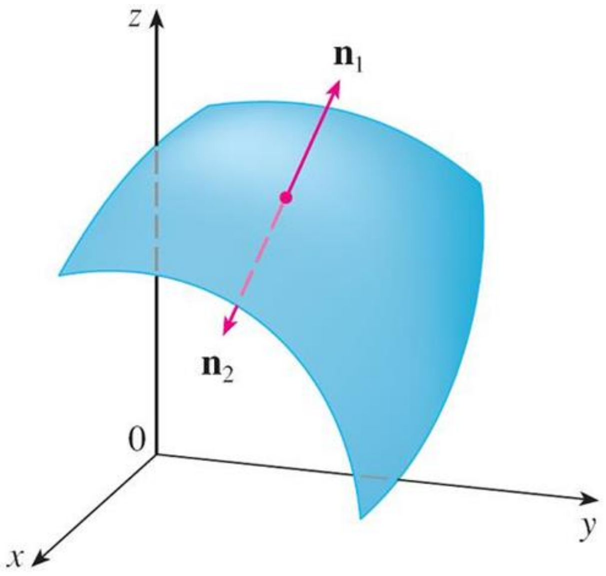
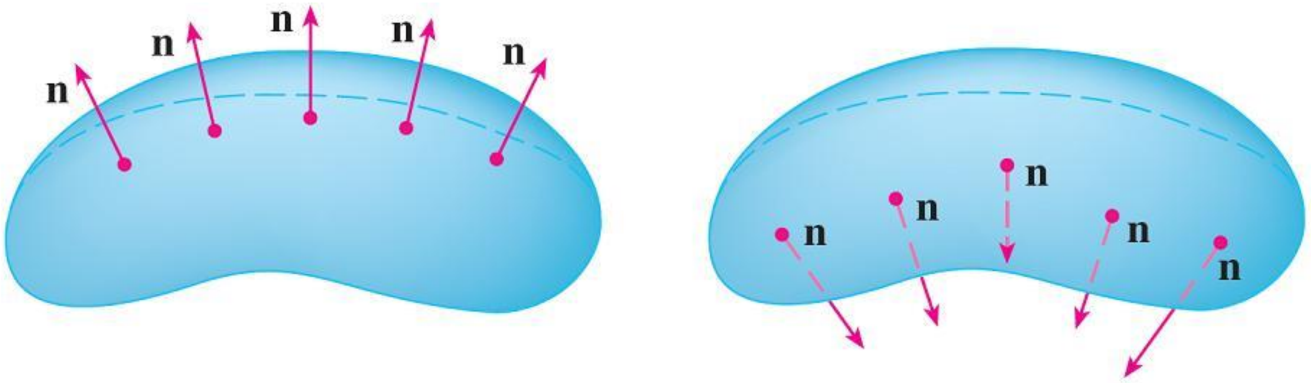


Figure 6

Oriented Surfaces

If it is possible to choose a unit normal vector **n** at every such point (x, y, z) so that **n** varies continuously over S, then S is called an **oriented surface** and the given choice of **n** provides S with an **orientation**.

There are two possible orientations for any orientable surface (see Figure 7).



The two orientations of an orientable surface

Figure 7

Oriented Surfaces (6 of 9)

For a surface $z = g(x, y)$ given as the graph of g, we use Equation 3 to associate with the surface a natural orientation given by the unit normal vector

$$\textbf{5} \qquad \mathbf{n} = \frac{-\dfrac{\partial g}{\partial x}\mathbf{i} - \dfrac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial g}{\partial x}\right)^2 + \left(\dfrac{\partial g}{\partial y}\right)^2}}$$

Since the **k**-component is positive, this gives the *upward* orientation of the surface.

Oriented Surfaces (7 of 9)

If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector

$$\mathbf{6} \qquad \mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

and the opposite orientation is given by $-\mathbf{n}$.

For instance, the parametric representation is

$$\mathbf{r}(\phi, \theta) = a \sin\phi \cos\theta\,\mathbf{i} + a \sin\phi \cos\theta\,\mathbf{j} + a \cos\phi\,\mathbf{k}$$

for the sphere $x^2 + y^2 + z^2 = a^2$.

Oriented Surfaces (8 of 9)

We know that

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2 \sin^2\phi \cos\theta\,\mathbf{i} + a^2 \sin^2\phi \sin\theta\,\mathbf{j} + a^2 \sin\phi \cos\phi\,\mathbf{k}$$

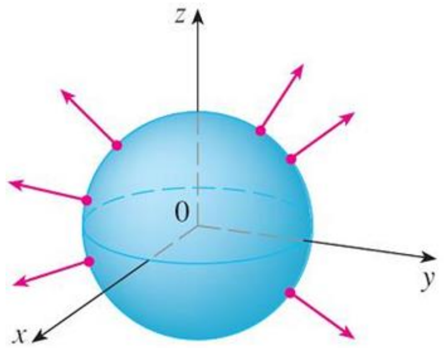
and

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin\phi$$

So the orientation induced by $\mathbf{r}(\phi, \ \theta)$ is defined by the unit normal vector

$$\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k} = \frac{1}{a}\mathbf{r}(\phi, \theta)$$

Observe that **n** points in the same direction as the position vector, that is, outward from the sphere (see Figure 8).

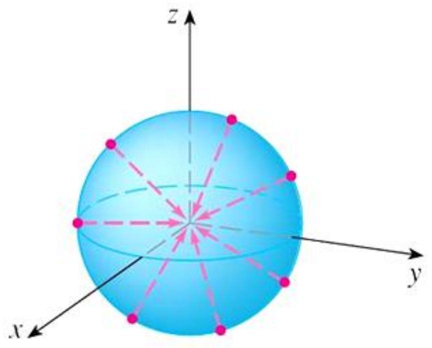


Positive orientation

Figure 8

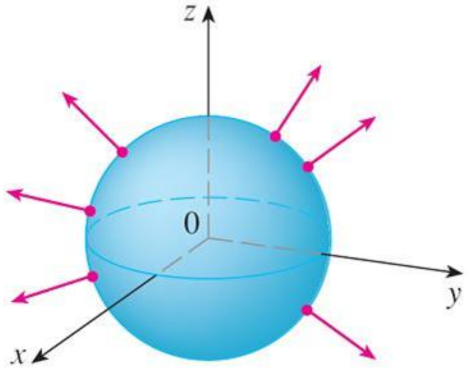
Oriented Surfaces

The opposite (inward) orientation would have been obtained (see Figure 9) if we had reversed the order of the parameters because $\mathbf{r}_\theta \times \mathbf{r}_\phi = -\mathbf{r}_\phi \times \mathbf{r}_\theta$.



Negative orientation

Figure 9



Positive orientation

Figure 8

For a **closed surface**, that is, a surface that is the boundary of a solid region *E*, the convention is that the **positive orientation** is the one for which the normal vectors point *outward* from *E*, and inward-pointing normals give the negative orientation (see Figures 8 and 9).

Surface Integrals of Vector Fields

Surface Integrals of Vector Fields (1 of 11)

Suppose that S is an oriented surface with unit normal vector **n**, and imagine a fluid with density $\rho(x, y, z)$ and velocity field $\mathbf{v}(x, y, z)$ flowing through S. (Think of S as an imaginary surface that doesn't impede the fluid flow, like a fishing net across a stream.)

Then the rate of flow (mass per unit time) per unit area is $\rho\mathbf{v}$.

Surface Integrals of Vector Fields (2 of 11)

If we divide S into small patches S_{ij}, as in Figure 10 (compare with Figure 1).

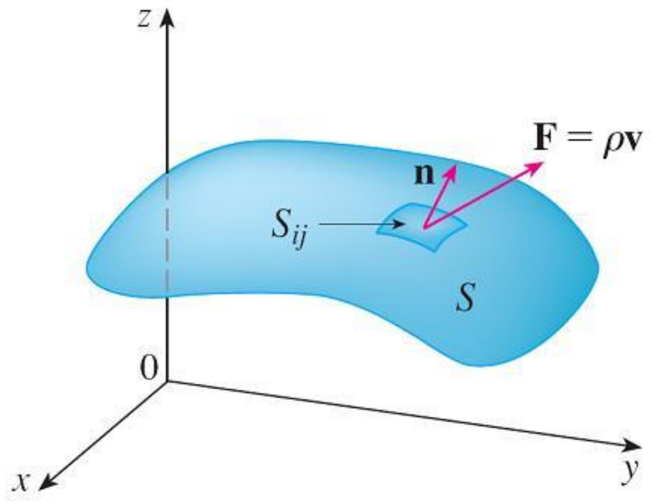


Figure 10

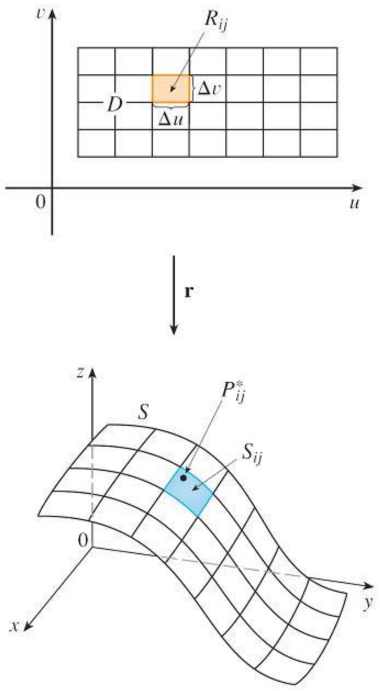


Figure 1

Surface Integrals of Vector Fields (3 of 11)

Then S_{ij} is nearly planar and so we can approximate the mass of fluid per unit time crossing S_{ij} in the direction of the normal **n** by the quantity

$$(\rho \mathbf{v} \cdot \mathbf{n}) A(S_{ij})$$

where ρ, **v**, and **n** are evaluated at some point on S_{ij}. (Recall that the component of the vector $\rho\mathbf{v}$ in the direction of the unit vector **n** is $\rho\mathbf{v} \cdot \mathbf{n}$.)

Surface Integrals of Vector Fields

By summing these quantities and taking the limit we get, according to Definition 1, the surface integral of the function $\rho\mathbf{v} \cdot \mathbf{n}$ over S:

7 $$\iint_S \rho\mathbf{v} \cdot \mathbf{n}\, dS = \iint_S \rho(x, y, z)\mathbf{v}(x, y, z) \cdot \mathbf{n}(x, y, z)\, dS$$

and this is interpreted physically as the rate of flow through S.

If we write $\mathbf{F} = \rho\mathbf{v}$, then $\mathbf{F}$ is also a vector field on $\mathbb{R}^3$ and the integral in Equation 7 becomes

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

Surface Integrals of Vector Fields (5 of 11)

A surface integral of this form occurs frequently in physics, even when **F** is not ρ**v**, and is called the *surface integral* (or *flux integral*) of **F** over *S*.

8 Definition If **F** is a continuous vector field defined on an oriented surface *S* with unit normal vector **n**, then the **surface integral of F over *S*** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

This integral is also called the **flux** of **F** across *S*.

In words, Definition 8 says that the surface integral of a vector field over *S* is equal to the surface integral of its normal component over *S*.

Surface Integrals of Vector Fields (6 of 11)

If S is given by a vector function $\mathbf{r}(u, v)$, then $\mathbf{n}$ is given by Equation 6, and from Definition 8 and Equation 2 we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, dS$$

$$= \iint_D \left[\mathbf{F}(\mathbf{r}(u,v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right] |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

where D is the parameter domain. Thus we have

9 $$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

Example 4

Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution:

As in Example 1, we use the parametric representation

$$\mathbf{r}(\phi,\theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k} \quad 0 \le \phi \le \pi \quad 0 \le \theta \le 2\pi$$

Then

$$F(\mathbf{r}(\phi,\theta)) = \cos\phi\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\theta\,\mathbf{k}$$

and,

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi\cos\theta\,\mathbf{i} + \sin^2\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\phi\,\mathbf{k}$$

Example 4 – *Solution*

Therefore $\mathbf{F}(\mathbf{r}(\phi,\theta)\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta) = \cos\phi\sin^2\phi\cos\theta + \sin^3\phi\ \sin^2\theta + \sin^2\phi\cos\phi\cos\theta$

and, by Formula 9, the flux is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta)\ dA$$

$$= \int_0^{2\pi}\int_0^{\pi}(2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta)\ d\phi\ d\theta$$

$$= 2\int_0^{\pi}\sin^2\phi\cos\phi\ d\phi\int_0^{2\pi}\cos\theta\ d\theta + \int_0^{\pi}\sin^3\phi\ d\phi\int_0^{2\pi}\sin^2\theta\ d\theta$$

$$= 0 + \int_0^{\pi}\sin^3\phi\ d\phi\int_0^{2\pi}\sin^2\theta\ d\theta \quad \left(\text{since} \int_0^{2\pi}\cos\theta d\theta = 0\right)$$

$$= \frac{4\pi}{3}$$

by the same calculation as in Example 1.

Surface Integrals of Vector Fields (7 of 11)

If, for instance, the vector field in Example 4 is a velocity field describing the flow of a fluid with density 1, then the answer, $\frac{4\pi}{3}$, represents the rate of flow through the unit sphere in units of mass per unit time.

In the case of a surface S given by a graph $z = g(x, y)$, we can think of x and y as parameters and use Equation 3 to write

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} \right)$$

Surface Integrals of Vector Fields (8 of 11)

Thus Formula 9 becomes

$$\textbf{10} \quad \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA$$

This formula assumes the upward orientation of *S*; for a downward orientation we multiply by −1.

Similar formulas can be worked out if *S* is given by $y = h(x, z)$ or $x = k(y, z)$.

Although we motivated the surface integral of a vector field using the example of fluid flow, this concept also arises in other physical situations.

Surface Integrals of Vector Fields (9 of 11)

For instance, if **E** is an electric field, then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the **electric flux** of **E** through the surface *S*. One of the important laws of electrostatics is **Gauss's Law**, which says that the net charge enclosed by a closed surface *S* is

$$\textbf{11} \qquad Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}$$

where ε_0 is a constant (called the permittivity of free space) that depends on the units used. (In the SI system, $\varepsilon_0 \approx 8.8542 \times 10^{-12} C^2/\mathrm{N} \cdot \mathrm{m}^2$.)

Surface Integrals of Vector Fields

Therefore, if the vector field **F** in Example 4 represents an electric field, we can conclude that the charge enclosed by S is $Q = \frac{4}{3}\pi\varepsilon_0$.

Another application of surface integrals occurs in the study of heat flow. Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$. Then the **heat flow** is defined as the vector field

$$\mathbf{F} = -K\nabla u$$

where K is an experimentally determined constant called the **conductivity** of the substance.

Surface Integrals of Vector Fields (11 of 11)

The rate of heat flow across the surface S in the body is then given by the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}$$