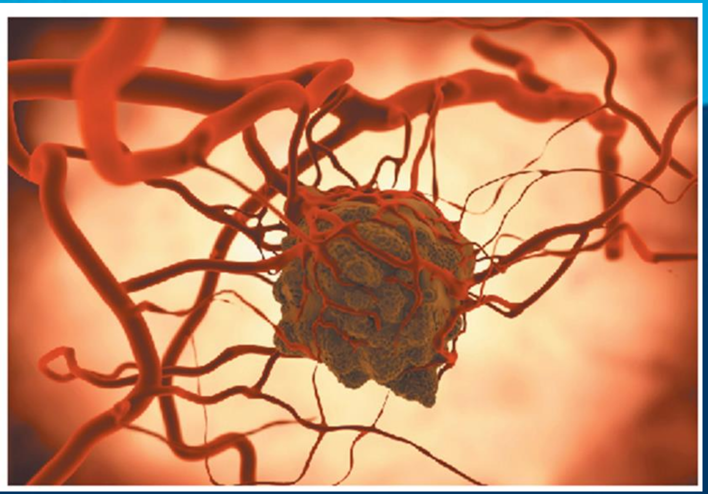


Chapter 15

Multiple Integrals

15.9 Change of Variables in Multiple Integrals

Change of Variables in Multiple Integrals (1 of 19)

In one-dimensional calculus we often use a change of variable (a substitution) to simplify an integral. By reversing the roles of *x* and *u*, we can write

1 $$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

where $x = g(u)$ and $a = g(c)$, $b = g(d)$. Another way of writing Formula 1 is as follows:

2 $$\int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du$$

Change of Variables in Multiple Integrals (2 of 19)

A change of variables can also be useful in double integrals. We have already seen one example of this: conversion to polar coordinates. The new variables r and θ are related to the old variables x and y by the equations

$$x = r\cos\theta \qquad y = r\sin\theta$$

and the change of variables formula can be written as

$$\iint_R f(x, y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane.

Change of Variables in Multiple Integrals (3 of 19)

More generally, we consider a change of variables that is given by a **transformation** T from the uv-plane to the xy-plane:

$$T(u, v) = (x, y)$$

where x and y are related to u and v by the equations

$$\mathbf{3} \qquad x = g(u, v) \qquad y = h(u, v)$$

or, as we sometimes write,

$$x = x(u, v) \qquad y = y(u, v)$$

Change of Variables in Multiple Integrals

We usually assume that T is a $\boldsymbol{C^1}$ **transformation**, which means that g and h have continuous first-order partial derivatives.

A transformation T is really just a function whose domain and range are both subsets of $\mathbb{R}^2$.

If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the **image** of the point (u_1, v_1).

If no two points have the same image, T is called **one-to-one**.

Change of Variables in Multiple Integrals (5 of 19)

Figure 1 shows the effect of a transformation *T* on a region *S* in the *uv*-plane.

T transforms *S* into a region *R* in the *xy*-plane called the **image of *S***, consisting of the images of all points in *S*.

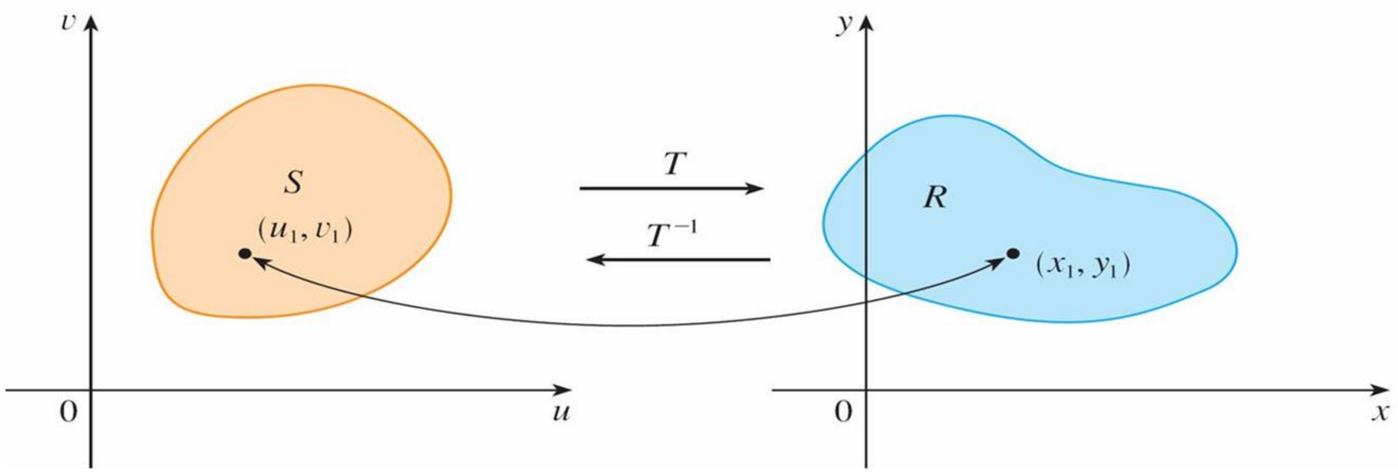


Figure 1

Change of Variables in Multiple Integrals

If T is a one-to-one transformation, then it has an **inverse transformation** T^{-1} from the xy-plane to the uv-plane and it may be possible to solve Equations 3 for u and v in terms of x and y:

$$u = G(x, y) \qquad v = H(x, y)$$

Example 1

A transformation is defined by the equations

$$x = u^2 - v^2 \qquad y = 2uv$$

Find the image of the square

$$S = \{(u, v) \mid 0 \le u \le 1, \ 0 \le v \le 1\}.$$

Solution:

The transformation maps the boundary of *S* into the boundary of the image.

So we begin by finding the images of the sides of *S*.

Example 1 – *Solution* (1 of 4)

The first side, S_1, is given by $v = 0$ $(0 \le u \le 1)$. (See Figure 2.)

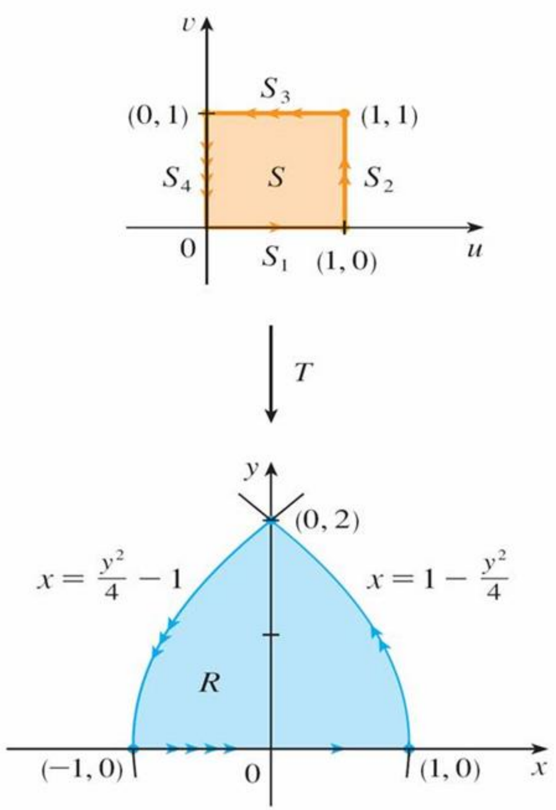


Figure 2

Example 1 – *Solution* (2 of 4)

From the given equations we have $x = u^2$, $y = 0$, and so $0 \leq x \leq 1$. Thus S_1 is mapped into the line segment from (0, 0) to (1, 0) in the *xy*-plane.

The second side, S_2, is $u = 1$ $(0 \leq v \leq 1)$ and, putting $u = 1$ in the given equations, we get

$$x = 1 - v^2 \qquad y = 2v$$

Example 1 – *Solution* (3 of 4)

Eliminating v, we obtain

$$\textbf{4} \qquad x = 1 - \frac{y^2}{4} \qquad 0 \le x \le 1$$

which is part of a parabola.

Similarly, S_3 is given by $v = 1$ $(0 \le u \le 1)$, whose image is the parabolic arc

$$\textbf{5} \qquad x = \frac{y^2}{4} - 1 \qquad -1 \le x \le 0$$

Example 1 – *Solution* (4 of 4)

Finally, S_4 is given by $u = 0$ $(0 \le v \le 1)$ whose image is $x = -v^2$, $y = 0$, that is, $-1 \le x \le 0$. (Notice that as we move around the square in the counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.)

The image of S is the region R (shown in Figure 2) bounded by the x-axis and the parabolas given by Equations 4 and 5.

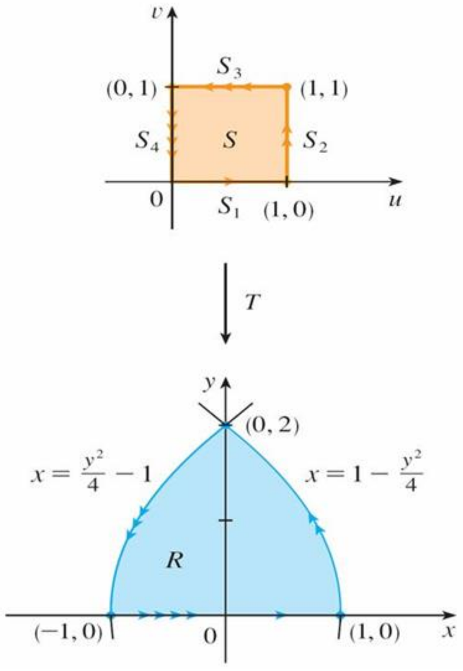


Figure 2

Change of Variables in Multiple Integrals (7 of 19)

Now let’s see how a change of variables affects a double integral. We start with a small rectangle S in the uv-plane whose lower left corner is the point (u_0, v_0) and whose dimensions are Δu and Δv. (See Figure 3.)

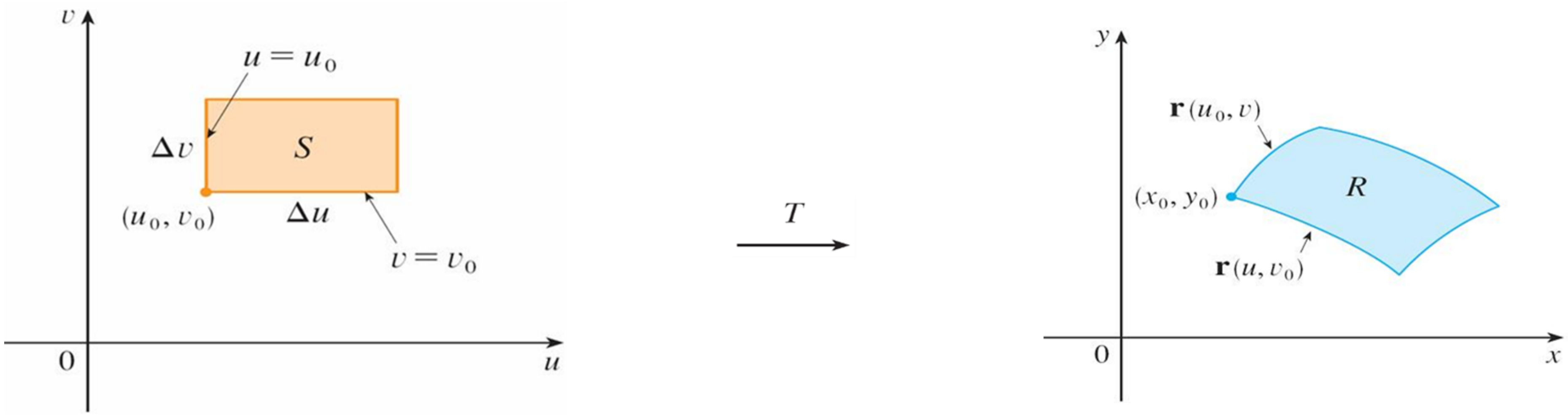


Figure 3

Change of Variables in Multiple Integrals

The image of S is a region R in the xy-plane, one of whose boundary points is $(x_0, y_0) = T(u_0, v_0)$.

The vector

$$\mathbf{r}(u, v) = g(u, v)\,\mathbf{i} + h(u, v)\,\mathbf{j}$$

is the position vector of the image of the point (u, v).

The equation of the lower side of S is $v = v_0$, whose image curve is given by the vector function $\mathbf{r}(u, v_0)$.

Change of Variables in Multiple Integrals

The tangent vector at (x_0, y_0) to this image curve is

$$\begin{aligned}\mathbf{r}_u &= g_u(u_0, v_0)\,\mathbf{i} + h_u(u_0, v_0)\,\mathbf{j} \\ &= \frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j}\end{aligned}$$

Similarly, the tangent vector at (x_0, y_0) to the image curve of the left side of S (namely, $u = u_0$) is

$$\begin{aligned}\mathbf{r}_v &= g_v(u_0, v_0)\,\mathbf{i} + h_v(u_0, v_0)\,\mathbf{j} \\ &= \frac{\partial x}{\partial v}\,\mathbf{i} + \frac{\partial y}{\partial v}\,\mathbf{j}\end{aligned}$$

Change of Variables in Multiple Integrals

We can approximate the image region $R = T(S)$ by a parallelogram determined by the secant vectors

$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u,\ v_0) - \mathbf{r}(u_0,\ v_0) \quad \mathbf{b} = \mathbf{r}(u_0,\ v_0 + \Delta v) - \mathbf{r}(u_0,\ v_0)$$

shown in Figure 4.

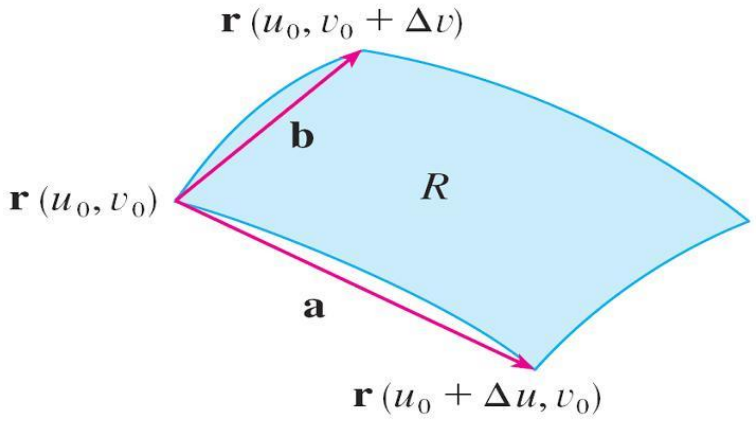


Figure 4

But
$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u,\ v_0) - \mathbf{r}(u_0,\ v_0)}{\Delta u}$$

and so
$$\mathbf{r}(u_0 + \Delta u,\ v_0) - \mathbf{r}(u_0,\ v_0) \approx \Delta u\ \mathbf{r}_u$$

Similarly
$$\mathbf{r}(u_0,\ v_0 + \Delta v) - \mathbf{r}(u_0,\ v_0) \approx \Delta v\ \mathbf{r}_v$$

This means that we can approximate R by a parallelogram determined by the vectors $\Delta u\ \mathbf{r}_u$ and $\Delta v\ \mathbf{r}_v$. (See Figure 5.)

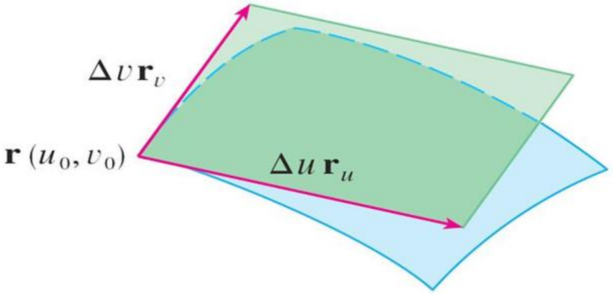


Figure 5

Change of Variables in Multiple Integrals

Therefore we can approximate the area of *R* by the area of this parallelogram, which is

$$\textbf{6} \qquad \left|(\Delta u\ \mathbf{r}_u)\times(\Delta v\ \mathbf{r}_v)\right| = \left|\mathbf{r}_u \times \mathbf{r}_v\right|\Delta u\ \Delta v$$

Computing the cross product, we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix}\mathbf{k} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}\mathbf{k}$$

Change of Variables in Multiple Integrals (13 of 19)

The determinant that arises in this calculation is called the *Jacobian* of the transformation and is given a special notation.

7 Definition The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

With this notation we can use Equation 6 to give an approximation to the area ΔA of R:

$$\textbf{8} \quad \Delta A \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u \, \Delta v$$

where the Jacobian is evaluated at (u_0, v_0).

Change of Variables in Multiple Integrals

Next we divide a region S in the uv-plane into rectangles S_{ij} and call their images in the xy-plane R_{ij}. (See Figure 6.)

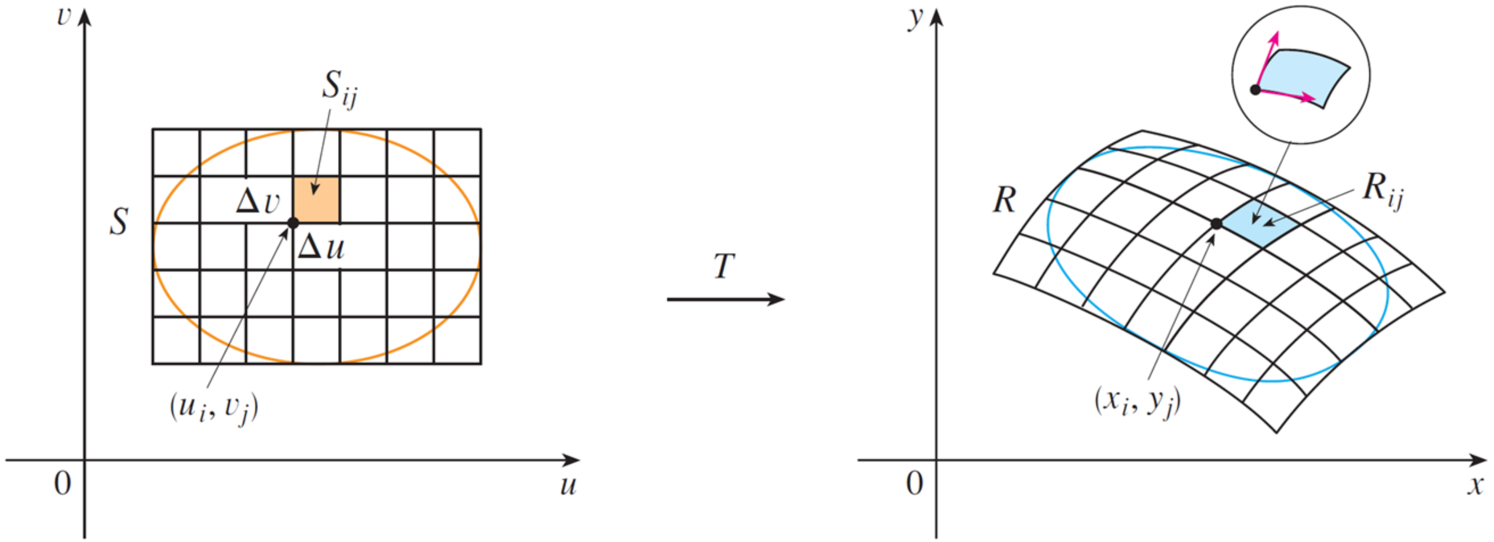


Figure 6

Change of Variables in Multiple Integrals

Applying the approximation (8) to each R_{ij}, we approximate the double integral of f over R as follows:

$$\iint_R f(x, y)\,dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_i)\,\Delta A$$

$$\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(g(u_i, v_j), h(u_i, v_i)\right) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u\, \Delta v$$

where the Jacobian is evaluated at (u_i, v_j). Notice that this double sum is a Riemann sum for the integral

$$\iint_S f\left(g(u, v), h(u, v)\right) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

Change of Variables in Multiple Integrals

The foregoing argument suggests that the following theorem is true.

9 Change of Variables in a Double Integral Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x, y)\,dA = \iint_S f\big(x(u, v), y(u, v)\big)\left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv$$

Theorem 9 says that we change from an integral in x and y to an integral in u and v by expressing x and y in terms of u and v and writing

$$dA = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv$$

Change of Variables in Multiple Integrals (17 of 19)

Notice the similarity between Theorem 9 and the one-dimensional formula in Equation 2.

Instead of the derivative $\frac{dx}{du}$, we have the absolute value of the Jacobian, that is, $\left|\frac{\partial(x, y)}{\partial(u, v)}\right|$.

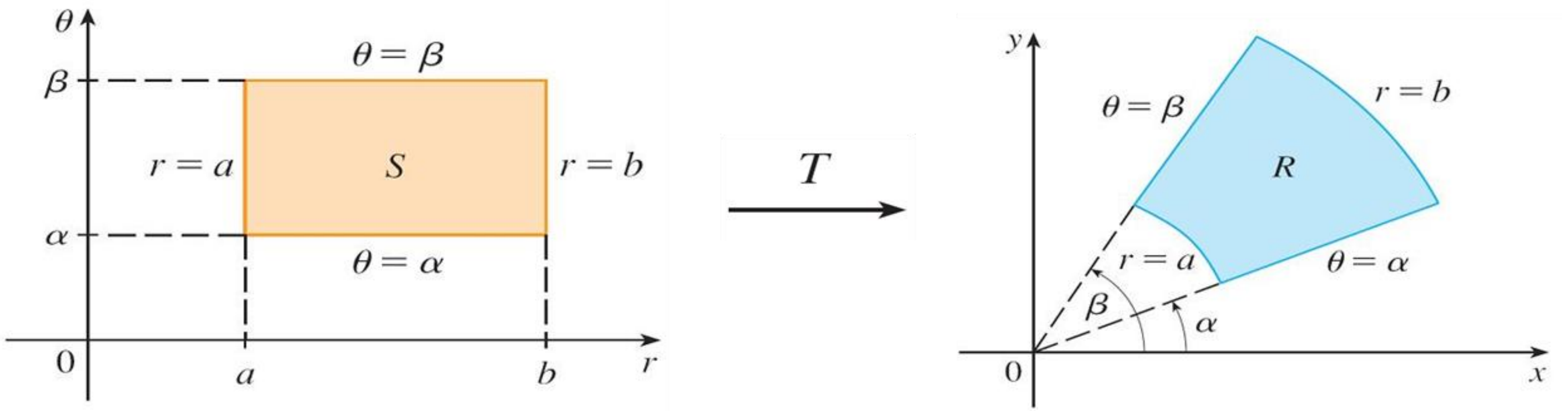
As a first illustration of Theorem 9, we show that the formula for integration in polar coordinates is just a special case.

Change of Variables in Multiple Integrals (18 of 19)

Here the transformation *T* from the *rθ*-plane to the *xy*-plane is given by

$$x = g(r,\ \theta) = r\cos\theta \quad y = h(r,\ \theta) = r\sin\theta$$

and the geometry of the transformation is shown in Figure 7. *T* maps an ordinary rectangle in the *rθ*-plane to a polar rectangle in the *xy*-plane.



The polar coordinate transformation

Figure 7

Change of Variables in Multiple Integrals

The Jacobian of *T* is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r > 0$$

Thus Theorem 9 gives

$$\begin{aligned} \iint_R f(x, y)\,dx\,dy &= \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| dr\,d\theta \\ &= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \end{aligned}$$

Triple Integrals

Triple Integrals (1 of 2)

There is a similar change of variables formula for triple integrals.

Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space by means of the equations

$$x = g(u, v, w) \quad y = h(u, v, w) \quad z = k(u, v, w)$$

Triple Integrals (2 of 2)

The **Jacobian** of T is the following 3×3 determinant:

$$\textbf{12} \qquad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Under hypotheses similar to those in Theorem 9, we have the following formula for triple integrals:

$$\textbf{13} \qquad \iiint_R f(x, y, z)\,dV = \iiint_S f\big(x(u, v, w), y(u, v, w), z(u, v, w)\big) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\,dv\,dw$$

Example 4

Use Formula 13 to derive the formula for triple integration in spherical coordinates.

Solution:

Here the change of variables is given by

$$x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta \quad z = \rho \cos\phi$$

We compute the Jacobian as follows:

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi \cos\theta & -\rho \sin\phi \sin\theta & \rho \cos\phi \cos\theta \\ \sin\phi \sin\theta & \rho \sin\phi \cos\theta & \rho \cos\phi \sin\theta \\ \cos\phi & 0 & -\rho \sin\phi \end{vmatrix}$$

Example 4 – *Solution* (1 of 2)

$$= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} - \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi\left(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta\right)$$

$$- \rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)$$

$$= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin\phi\sin^2\phi$$

$$= -\rho^2\sin\phi$$

Since $0 \le \phi \le \pi$, we have $\sin\phi \ge 0$.

Example 4 – *Solution* (2 of 2)

Therefore

$$\left|\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)}\right| = \left|-\rho^2 \sin\phi\right|$$

$$= \rho^2 \sin\phi$$

and Formula 13 gives

$$\iiint_R f(x, y, z)\, dV$$

$$= \iiint_S f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$