Classical Fields

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Contents

Chapter 1

Special Relativity

In physics, a "theory of relativity" is an assertion that the Universe is objective. Its goal is to provide an isomorphism between sets of measurements performed in two (different) inertial frames of reference via an automorphism of the underlying space-time. *Galilean* relativity is based on the concepts of a universal or "absolute" space and an "absolute" time, by which is meant that measurements of spatial distances and time intervals are observer (or frame) independent provided that the spatial distances are measured by a simultaneous measurement of the endpoints. Toward the end of the nineteenth century, however, Maxwell's formulation of electromagnetism, which was completed in 1865, had exposed certain fundamental inconsistencies between the new and extremely successful electromagnetic theory and the Galilean conception of space and time. Today we know that Galilean transformations cease to yield results that agree with experiment when the relative velocity of the two frames being compared is a significant fraction of the speed of light.

In 1887, A. Michelson and E. Morley were able to provide convincing evidence, by means of a very clever and now famous experiment named after them, that the speed of light is the same in all directions and that light does not require a medium in which to travel. At the time their experiment was performed such a medium was assumed to exist because electromagnetic waves were not considered to be different from other well-known mechanical waves (eg. sound) and all mechanical waves were known to require a medium in which to propagate. The putative medium in which light traveled was dubbed the **luminiferous aether** and was thought to pervade all of space. When wave propagation occurs in a medium, the frame that is at rest relative to it assumes a special place in the theory and the "speed" of the wave is its speed as measured in this frame. Thus, the speed of sound in air at STP is approximately $c = 343$ m/s in the frame of the air. An inertial observer moving relative to the air with a velocity $\vec{v}$ in the direction of the wave propagation or opposite it would observe that the speed of the wave is $c \mp v$, in accordance with the principles of Galilean relativity. The Michelson and Morley experiment was designed to measure the velocity of the earth relative to the luminiferous aether as it revolves around the sun during the course of a year. The results were null and the speed of light was found to be the same for propagation in all directions, indicating that the aether was absent. If no such medium exists then the wave speed could be *the same for all inertial observers*, in other words, a universal constant of nature. This agreed with Maxwell's theory of electromagnetism, by which electromagnetic waves propagate in a vacuum at a speed that depends only on the fundamental constants. Motivated by the Michelson-Morley result and by Maxwell's theory, A. Einstein recognized in 1905 that the failure of Galilean relativity at high relative velocities is a consequence of the breakdown of the concepts of "absolute" space and "absolute" time mentioned above. When they are abandoned and replaced by the experimental requirement that the speed of light is the same *in all inertial frames*, we arrive at a dramatically new conception of space and time and therefore of mechanics as well. This modification is known as Einstein's "special" theory of relativity, or simply **Special Relativity** and is the topic of this chapter.

We introduce Einstein's theory in this chapter. We will not dwell much on the questions and experiments that led up to it, neither shall we concern ourselves too much with the apparent paradoxes (there are many, all of them safely resolved). It is assumed that the reader has had some exposure to the topic, so we rather concentrate on a mathematical formulation of the theory and a framework that will be useful for the objectives of these notes.

1.1 The Principle of Covariance

It is a general principle that the laws of physics must be the same in all inertial frames. If this were not true, there would be no way to compare the measurements of one inertial observer with those of any other.

Mathematically, the fundamental laws of physics would be same in all inertial frames of reference if the equations describing them have the same form in all inertial frames, that is, if the set of transformations that relate one inertial frame to another (automorphisms of space-time) would, when applied to the two sides of any fundamental equation of physics, transform each side in precisely the same way as the other. This is the principle of **covariance** and equations that have this property are said to be **covariant**. To further elaborate on this idea, we recall that the transformations that relate two inertial frames will in turn determine the transformation properties of physical quantities such as velocity, acceleration, etc. If they leave a physical quantity the same in every inertial frame then that quantity is an **invariant** or a **scalar**. Other quantities may not remain invariant but they will transform in a prescribed way. Covariance requires that both sides of the fundamental equations must have the same transformation properties. Thus a scalar quantity can only be related to another scalar quantity, a vector to a vector and so on.

We will see below that Newton's laws are covariant under Galilean transformations but Maxwell's equations are not. This signals an incompatibility between mechanics and electromagnetism, and incompatibilities always indicate that modifications to one or both theories are required at a fundamental level. While it is possible that both theories are wrong, it is more fruitful at first to accept one as correct and modify the other so as to make *its* equations covariant under the transformations that are compatible with the first. Given the fundamental agreement between the predictions of electromagnetism and the experiment of Michaelson and Morley, Einstein chose the transformations that preserve the form of Maxwell's equations over the Galilean transformations of Newtonian mechanics. The result is a new formulation of classical mechanics that accounts for the fact that the speed of light is a finite and universal constant of nature. In the end, of course, only experiment can decide which theory is correct and, indeed, in the years that followed Einstein's 1905 paper, it has resoundingly confirmed his choice.

1.1.1 Galilean tranformations

We are familiar with Galilean relativity, which we may conveniently think of as two sets of transformations *viz.,* the "boosts"

$$\vec{r} \to \vec{r}' = \vec{r} - \vec{v}t, \quad t \to t' = t \tag{1.1.1}$$

(provided that the frames are coincident at $t = 0$) and spatial rotations

$$\vec{r} \to \vec{r}' = \widehat{R}\,\vec{r}, \quad t \to t' = t \tag{1.1.2}$$

where $\widehat{R}$ is a rotation matrix (see figure 1.1). The second of (1.1.1) expresses the absoluteness of time intervals, as $dt' = dt$ is the same for all inertial observers. To see that spatial intervals are also absolute one must remember that the measurement of a distance involves a *simultaneous* measurement of the endpoints and therefore one has

$$|d\vec{r}'|_{dt'=0} = |d\vec{r} - \vec{v}dt|_{dt'=dt=0} = |d\vec{r}|. \tag{1.1.3}$$

Consider a single particle within a collection of N particles with interactions between them. If we label the particles by integers, Newton's equations describing the evolution of a single particle, say particle n, may be written as,

$$m_n \frac{d^2\vec{r}_n}{dt^2} = \vec{F}_n^{ext} + \vec{F}_n^{int} = \vec{F}_n^{ext} + \sum_{m\neq n} \vec{F}_{m\to n}^{int}, \tag{1.1.4}$$

where $\vec{F}_{m\to n}^{int}$ represents the (internal) force that particle m exerts over particle n. Assume that the external forces are invariant under Galilean boosts,

$$\vec{F}_n^{\prime ext} = \vec{F}_n^{ext}, \tag{1.1.5}$$

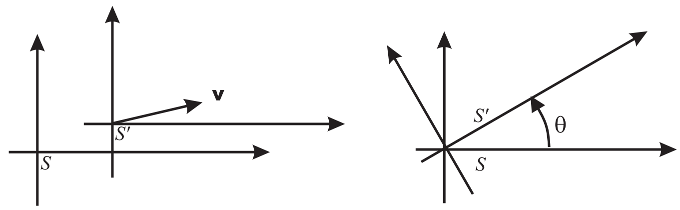

Figure 1.1: Boosts and rotations

and that the internal forces are derivable from a potential that depends only on the spatial distance between the particles, *i.e.*,

$$\vec{F}_n^{int} = -\vec{\nabla}_n \Phi_n^{int} = -\sum_{m \neq n} \vec{\nabla}_n \Phi_{nm}(|\vec{r}_n - \vec{r}_m|). \tag{1.1.6}$$

This is compatible with the third law (of action and reaction) and it also makes the internal forces invariant under Galilean boosts. To see that this is so, specialize to just one space dimension and write the transformations in the following form (we are making this more complicated than it really is so as to introduce methods that will be useful in more complicated situations)

$$\begin{bmatrix} dt' \\ dx' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -v & 1 \end{bmatrix} \begin{bmatrix} dt \\ dx \end{bmatrix} \tag{1.1.7}$$

and the inverse transformations as

$$\begin{bmatrix} dt \\ dx \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix} \begin{bmatrix} dt' \\ dx' \end{bmatrix}. \tag{1.1.8}$$

We can now read off

$$\frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} \tag{1.1.9}$$

and

$$\frac{\partial}{\partial x'} = \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}. \tag{1.1.10}$$

Therefore

$$\frac{\partial}{\partial x'_n}\Phi_{nm}(|x'_n - x'_m|) = \frac{\partial}{\partial x_n}\Phi_{nm}(|x_n - x_m|), \tag{1.1.11}$$

as claimed and the r.h.s. of Newton's equations are invariant. Moreover $dt' = dt$ and the transformation is linear so that the left hand side (l.h.s.) of Newton's equations is also invariant under these transformations. Therefore, subject to the conditions of (1.1.5) and (1.1.6), the equations of Newtonian dynamics are invariant under Galilean boosts.

1.1.2 Lorentz Transformations

In electrodynamics, on the other hand, in free space one typically ends up with the wave equation,

$$\Box_x \psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \vec{\nabla}^2 \psi = 0, \tag{1.1.12}$$

where c is the speed of light in the vacuumn, determined by two constants, *viz.,* the permitivity and the permeability of space, and ψ is the "wave function", which can be the electromagnetic scalar or vector potential. Now it is an experimental fact that the speed of light in a vacuum is universal, *i.e.,* the same for all inertial observers. However, then (1.1.12) is not invariant under Galilean transformations. Using the transformations in (1.1.7) and (1.1.8) we have

$$\frac{\partial^2}{\partial t'^2} = \left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right) \tag{1.1.13}$$

and

$$\vec{\nabla}'^2 = \vec{\nabla}^2. \tag{1.1.14}$$

Plugging this into the wave equation, we find

$$\frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \vec{\nabla}'^2 \to \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + \frac{2\vec{v}}{c^2}\cdot\vec{\nabla}\frac{\partial}{\partial t} + \frac{1}{c^2}(\vec{v}\cdot\vec{\nabla})(\vec{v}\cdot\vec{\nabla}), \tag{1.1.15}$$

but only the first two terms on the r.h.s. correspond to the wave-equation. Moreover, there is no known kinetic transformation of the wave-function that can return the wave equation to its original form,[1] so we must conclude that the electromagnetic wave-equation is not invariant under Galilean transformations. This signals an incompatibility between electromagnetism and Newtonian mechanics, therefore, by the principle of covariance, one or both of them must be modified. As we now know, Maxwell's theory was preferred over Newtonian mechanics, which leads us to ask: what are the transformations that keep Maxwell's equations covariant? Once we have answered this question we will be in a position to address the problem of constructing a theory of mechanics that is indeed covariant under them.

To answer the first question, assume that the transformations that relate two inertial frames continue to be linear (as the Galilean transformations are) and think of the wave-equation as made up of two distinct parts: the second order differential operator, "$\Box_x$", and

[1] Problem: Show that, on the contrary, the Schroedinger equation is invariant under Galilean transformations if they are supplemented with the following kinetic transformation of the wave-function:

$$\psi \to \psi' = e^{-\frac{i}{\hbar}(\vec{p}\cdot\vec{r} - Et)}\psi$$

where $\vec{p} = m\vec{v}$ and $E = m\vec{v}^2/2$. What does this mean?

the wave function, ψ, each transforming in its own way under the above transformations. For covariance, we will require "$\square_x$", to transform as a scalar (invariant). Let us work with Cartesian systems and consider some general transformations of the form

$$\begin{aligned} t &\rightarrow t' = t'(t, \vec{r}), \\ \vec{r} &\rightarrow \vec{r}' = \vec{r}'(t, \vec{r}). \end{aligned} \tag{1.1.16}$$

They must be

1. <u>one-to-one</u>: so that observers may be able to uniquely relate observations, and
2. <u>invertible</u>: so that the transformations can be made from any observer to the other – there is no preferred observer.

Our functions must therefore be bijective. As we have assumed that the transformations are linear, they will have the form

$$\begin{aligned} t' &= -\frac{1}{c^2}(L_{00}t + \sum_i L_{0i}x_i), \\ x_i' &= L_{i0}t + \sum_j L_{ij}x_j. \end{aligned} \tag{1.1.17}$$

The reason for this peculiar definition of the coefficents will become clear later. For now let us only note that the L's are some constants that we would like to evaluate. In matrix form the transformations could be written as

$$\begin{bmatrix} dt' \\ dx_i' \end{bmatrix} = \begin{bmatrix} -\frac{L_{00}}{c^2} & -\frac{L_{0j}}{c^2} \\ L_{i0} & L_{ij} \end{bmatrix} \begin{bmatrix} dt \\ dx_j \end{bmatrix}. \tag{1.1.18}$$

The matrix on the r.h.s. is really a 4×4 matrix and L_{ij} represents a 3×3 matrix of purely spatial transformations. It must be invertible because the transformation is required to be bijective. For example, if $L_{00} = -c^2$ and $L_{0i} = 0 = L_{i0}$, the resulting transformations are purely spatial, transforming $x_i \rightarrow x_i' = \sum_j L_{ij}x_j$ and leaving $t \rightarrow t' = t$ unchanged. Clearly, therefore, the wave-operator,

$$\square_x \rightarrow \square_x' = \partial_{t'}^2 - \vec{\nabla}'^2 = \partial_t^2 - \vec{\nabla}'^2, \tag{1.1.19}$$

is a scalar if and only if L_{ij} is a spatial rotation, because only then will $\vec{\nabla}'^2 = \vec{\nabla}^2$.

More interesting are the "boosts", which involve inertial observers with relative velocities. Now $L_{i0} \neq 0 \neq L_{0i}$. Consider relative velocities along the x direction and the transformation

$$\begin{bmatrix} dt' \\ dx_1' \\ dx_2' \\ dx_3' \end{bmatrix} = \begin{bmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} dt \\ dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}. \tag{1.1.20}$$

Notice that we have set $x'_2 = x_2$ and $x'_3 = x_3$. This is because we assumed that space is homogeneous and isotropic so that a boost in the x_1 direction has no effect on the orthogonal coordinates x_2 and x_3. We can consider then only the effective two dimensional matrix

$$\begin{bmatrix} dt' \\ dx' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} dt \\ dx \end{bmatrix} \tag{1.1.21}$$

(where $x_1 := x$). Thus we find the inverse transformation

$$\begin{bmatrix} dt \\ dx \end{bmatrix} = \frac{1}{\|\|} \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \begin{bmatrix} dt' \\ dx' \end{bmatrix}, \tag{1.1.22}$$

where $\|\|$ represents the determinant of the transformation, $\|\| = \alpha\delta - \beta\gamma$ and we have

$$\begin{aligned} \frac{\partial}{\partial t'} &= \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} = \frac{1}{\|\|}\left(+\delta\frac{\partial}{\partial t} - \gamma\frac{\partial}{\partial x}\right) \\ \frac{\partial}{\partial x'} &= \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} = \frac{1}{\|\|}\left(-\beta\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial x}\right), \end{aligned} \tag{1.1.23}$$

turning our wave-operator into

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \vec{\nabla}'^2 &= \frac{1}{\|\|^2}\left(\frac{1}{c^2}\left(+\delta\frac{\partial}{\partial t} - \gamma\frac{\partial}{\partial x}\right)^2 - \left(-\beta\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial x}\right)^2\right) \\ &= \frac{1}{\|\|^2}\left((\delta^2/c^2 - \beta^2)\frac{\partial^2}{\partial t^2} - (\alpha^2 - \gamma^2/c^2)\frac{\partial^2}{\partial x^2}\right. \\ &\qquad \left. -2(\alpha\beta - \gamma\delta/c^2)\frac{\partial^2}{\partial t\partial x}\right). \end{aligned} \tag{1.1.24}$$

If it is to remain form invariant, the right hand side above has to look the same in the frame S and we need to set

$$\begin{aligned} \frac{\delta^2}{c^2} - \beta^2 &= \frac{\|\|^2}{c^2}, \\ \alpha^2 - \frac{\gamma^2}{c^2} &= \|\|^2, \\ \alpha\beta - \frac{\gamma\delta}{c^2} &= 0. \end{aligned} \tag{1.1.25}$$

We have four unknowns and three constraints, so there is really just one parameter that determines all the unknowns. It is easy to find. Note that setting

$$\delta = \|\| \cosh\eta, \qquad \beta = \frac{\|\|}{c}\sinh\eta \tag{1.1.26}$$

solves the first of these equations, as

$$\alpha = ||| \cosh\omega, \qquad \gamma = c||| \sinh\omega \tag{1.1.27}$$

solves the second. The last equation is then a relationship between η and ω. It implies that

$$\sinh\eta\cosh\omega - \sinh\omega\cosh\omega = \sinh(\eta-\omega) = 0 \to \eta = \omega. \tag{1.1.28}$$

Our boost in the x direction therefore looks like

$$\begin{bmatrix} dt' \\ dx' \end{bmatrix} = ||| \begin{bmatrix} \cosh\eta & \frac{1}{c}\sinh\eta \\ c\sinh\eta & \cosh\eta \end{bmatrix} \begin{bmatrix} dt \\ dx \end{bmatrix}. \tag{1.1.29}$$

We notice that $|||$ is not determined. We will henceforth take it to be unity.

What is the meaning of the parameter η? Consider a test body having a velocity u as observed in the S frame. Its velocity as measured in the S' frame would be (the velocity does not transform as a vector)

$$u' = \frac{dx'}{dt'} = \frac{(\cosh\eta)dx + c(\sinh\eta)dt}{(\cosh\eta)dt + \frac{1}{c}(\sinh\eta)dx}. \tag{1.1.30}$$

Dividing by $(\cosh\eta)dt$ we find

$$u' = \frac{u + c\tanh\eta}{1 + \frac{u}{c}\tanh\eta}. \tag{1.1.31}$$

Now suppose that the body is at rest in the frame S. This would mean that $u = 0$. But, if S' moves with a velocity v relative to S, we can say that S should move with velocity $-v$ relative to S'. Therefore, because the test body is at rest in S, its velocity relative to S' should be $u' = -v$. Our formula gives

$$u' = -v = c\tanh\eta \to \tanh\eta = -\frac{v}{c}. \tag{1.1.32}$$

This in turn implies that

$$\cosh\eta = \frac{1}{\sqrt{1 - v^2/c^2}}, \qquad \sinh\eta = -\frac{v/c}{\sqrt{1 - v^2/c^2}}, \tag{1.1.33}$$

so we can write the transformations in a recognizable form

$$\begin{aligned} t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}, \\ x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \end{aligned}$$

$$\begin{aligned} y' &= y, \\ z' &= z. \end{aligned} \tag{1.1.34}$$

Notes:

- These are the Lorentz transformations of the special theory of relativity.[2] They reduce to Galilean transformations when $v/c \ll 1$.
- Because $\tanh\eta \in (-1, 1)$ it follows that the transformations are valid only for $v < c$. The velocity of light is the *limiting* velocity of material bodies and observers. There exists no transformation from the rest frame of light to the rest frame of a material body.
- In general the matrix $\hat{L}$ is made up of boosts *and* rotations. Rotations do not, in general, commute with boosts and two boosts can lead to an overall rotation.
- Lorentz transformations keep the interval

  $$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{1.1.35}$$

  invariant[3] *i.e.,* the same for all observers. The interval ds is known as the proper distance and ds/c is known as the proper time (it's not difficult to see that when $d\vec{r} = 0$, $ds/c = dt$ *i.e.,* it is the time measured on a clock that is stationary in the frame). Like the proper distance, the proper time is an invariant. The transformations that keep an interval like (1.1.35) invariant form the Lie group $SO(3, 1)$.

1.2 Elementary consequences of Lorentz transformations

Our transformations mix up space and time, so there is no way for it but to consider both time and space as part of a single entity: "space-time". This is a four dimensional manifold. A point in space-time is called an event and involves not just its spatial location but also the time at which the event occurred.

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[2]For the very curious: Lorentz transformations can be put in four categories:

– Proper orthochronous: $L_+^\uparrow$ with $\|\| = +1, L_{00}/c^2 \leq -1$

– Proper non-orthochronous: $L_+^\downarrow$ with $\|\| = +1, L_{00}/c^2 \geq +1$

– Improper orthochronous: $L_-^\uparrow$ with $\|\| = -1, L_{00}/c^2 \leq -1$

– Improper non-orthochronous: $L_-^\downarrow$ with $\|\| = -1, L_{00}/c^2 \geq +1$

What we have are therefore proper orthochronous transformations.

[3]Problem: Prove this by direct substitution.

1.2.1 Simultaneity

The single most important consequence of the Lorentz transformations is that the concept of "simultaneity" is no longer absolute. Consider two events that are spatially separated, but occur at the same time as measured in the frame of an observer, S. Thus $dx \neq 0$ but $dt = 0$. According to (1.1.34),

$$dt' = \frac{-v dx/c^2}{\sqrt{1 - v^2/c^2}} \neq 0 \tag{1.2.1}$$

Thus events that are regarded as simultaneous in one frame are not so regarded in another frame, which is moving relative to the first.[4]

1.2.2 Length Contraction

Another interesting consequence is that length measurements of objects that are moving relative to an observer are smaller than measurements performed in the frame in which the objects are at rest. (The rest frame of a body is called the "proper" frame of the body). To understand how this comes about, one must recognize that to correctly measure the spatial distance between two points, their positions must be ascertained simultaneously. Let S be the frame in which the body is at rest and let S' be an observer moving at velocity v relative to S. Since a measurement of the body's length involves a simultaneous measurement of its endpoints, we should have $dt' = 0$. By the Lorentz transformations, this means that $dt = v dx/c^2$ and therefore

$$dx' = \frac{dx - v dt}{\sqrt{1 - v^2/c^2}} = dx\sqrt{1 - v^2/c^2}. \tag{1.2.2}$$

But dx represents the length of the body as measured in its proper frame, so its length as measured by S' *i.e.,* dx', is "contracted" by a factor of $\sqrt{1 - v^2/c^2}$.

1.2.3 Time Dilation

Measurements of time intervals are also naturally observer dependent. Let S be the proper frame of a clock, which is moving relative to an observer S' with a velocity v. Being stationary in S, we might say that $dx = 0$ and $dt = ds/c$ represents the proper time intervals of the clock. The Lorentz transformation then tells us that time intervals read off by S' are related to proper time intervals according to

$$dt' = \frac{dt}{\sqrt{1 - v^2/c^2}} \tag{1.2.3}$$

[4] Problem: Show that "future" and "past" are absolute; *i.e.,* show that if event "1" occurs in the past of event "2" in any inertial frame then "1" will occur in the past of "2" in *every* reference frame.

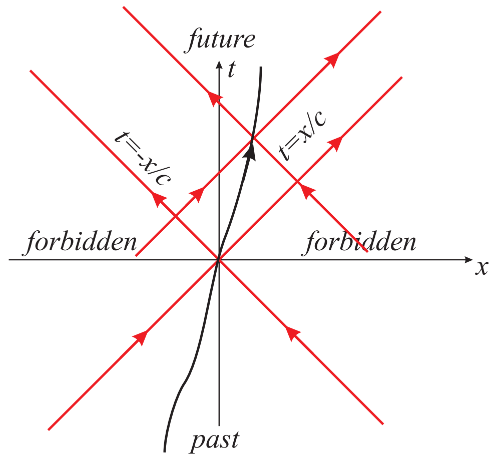

Figure 1.2: The light cone

This is known as "time dilation". Physically, this may be understood by noticing that while the time interval is measured in S **colocally** (at the same place), it is not so in S'. The clock in S appears to be "running slow" to the observer S'. This was in fact predicted in 1897 (long before Einstein's theory) by Louis Larmor who noticed the effect for electrons orbiting the nucleus of atoms.

It is often valuable to understand these phenomena in terms of world (space-time) diagrams. Thus, in figure 1.2 we show a two dimensional universe with the $y-$axis representing time from the point of view of some inertial observer, S. The red lines represent the path of light rays emanating from the origin in the upper half plane and terminating at the origin in the lower half plane. Consider a particle whose path, represented by the black curve, passes through the origin. (This can be arranged by resetting the origin of space and time). At no instant on this path may its slope, dt/dx, be less than or even equal to $1/c$, otherwise our particle would be traveling faster than or at the speed of light at that instant. Therefore the path lies wholly between the boundaries provided by the lines $t = \pm x/c$. This is the light cone. The region within the light cone and to the future is called the future light cone, the region within the light cone and in the past is called the past light cone and the regions on the left and right sides of the light cone are forever forbidden to the particle in the sense that it can *never* physically reach them. At any moment in time, the particle may only receive information from (and thus be influenced by) events within its own past light cone. Thus a fundamental role of Relativity is to restrict the domain of causal influence on any event. However, notice that as our particle travels along its world line, regions that were previously inaccessible begin to fall within its past light cone and become accessible as shown until, after an infinite time, its past

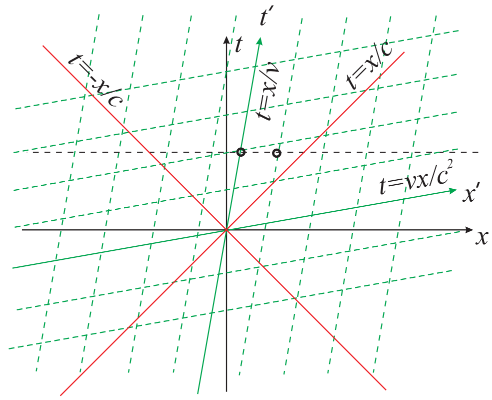

Figure 1.3: Two frames compared to each other

light cone encompasses the whole of the universe.

Figure 1.3 shows two inertial frames drawn in the same diagram. Let the (t, x) coordinate system represent an observer S and consider what the reference frame of an observer moving at velocity v relative to S might look like. The t' axis is the axis for which $x' = 0$, *i.e.,* in the (t, x) frame it is given by the straight line $t = x/v$ as shown in green. On the other hand, the x' axis is the one for which $t' = 0$, *i.e.,* it is given by $t = vx/c^2$, also shown in green in the figure. Consider two events that are spatially separated but occur simultaneously in S. These are represented by small circles on a horizontal ($t =$ const.) line. We see immediately that they do not fall on the same $t' =$ const. line. This graphically encapsulates the relativity of simultaneity. One can similarly visualize both length contraction and time dilation by projecting respectively on the x and t directions.[5]

1.2.4 Velocity Addition

Let us also recall the so-called law of "composition of velocities". Consider a particle whose velocity is being measured in two frames S and S'. Suppose that frame S' has a speed v in the positive $x-$direction relative to S then how do the particle velocities, as measured by S and S' relate, to one another? By definition, the velocity measured by S' will be

$$\begin{aligned} u'_x &= \frac{dx'}{dt'} = \frac{dx - vdt}{dt - vdx/c^2} = \frac{u_x - v}{1 - u_x v/c^2} \\ u'_y &= \frac{dy'}{dt'} = \frac{dy\sqrt{1 - v^2/c^2}}{dt - vdx/c^2} = \frac{u_y\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2} \end{aligned}$$

[5]Problem: Do this!

$$u'_z = \frac{dz'}{dt'} = \frac{dz\sqrt{1-v^2/c^2}}{dt - vdx/c^2} = \frac{u_z\sqrt{1-v^2/c^2}}{1-u_x v/c^2} \tag{1.2.4}$$

and they all reproduce the Galilean result when $c \to \infty$.

1.2.5 Relativistic (Velocity) Aberration

Finally, we can compare directions in space, as measured by two different observers. For example, if the particle is moving in the $x-y$ plane ($u_z = 0$), let us see how two observers may describe its direction of motion. According to observer S' the (tangent of the) angle made with the positive $x-$axis will be

$$\tan\theta' = \frac{u'_y}{u'_x} = \frac{u_y\sqrt{1-v^2/c^2}}{u_x - v} = \frac{u\sin\theta\sqrt{1-v^2/c^2}}{u\cos\theta - v} \tag{1.2.5}$$

where θ is measured in S and u is the particle speed as measured in S. Notice that it depends on the speed of the particle as well as the relative speed of the frames. If the "particle" were a photon, *i.e.,* in the case of light propagation, $u = c$ and

$$\tan\theta' = \frac{\sin\theta\sqrt{1-v^2/c^2}}{\cos\theta - v/c} \tag{1.2.6}$$

This is the formula for light **aberration**.[6],[7]

1.3 Tensors on the fly

One lesson that we learn is that we must work with the position vectors of *events* and these are "four-vectors", *i.e.,* vectors having one time and three space components. It is no longer useful or even correct to think of space and time as separate entities because the Lorentz transformations mix the two. Continuing with a Cartesian system, label the coordinates as follows:

$$x^\mu = (x^0, x^i): \quad \mu \in \{0,1,2,3\}, \quad x^0 = t, \quad x^i = x_i. \tag{1.3.1}$$

[6] Problem: Show that the formula can be simplified to

$$\tan\frac{\theta'}{2} = \tan\frac{\theta}{2}\sqrt{\frac{1-v/c}{1+v/c}}$$

[7] Problem: Show that for small angles, in the limit $v/c \to 0$ and up to first order in v/c, the aberration angle $\Delta\theta = \theta - \theta'$ is given by

$$\Delta\theta \approx \frac{v}{c}\sin\theta$$

Let us be particular about the position of the indices as superscripts, distinguishing between superscripts and subscripts (soon we will see that this is important) and consider a displacement, dx^μ, in frame S letting the corresponding displacement in frame S' be dx'^μ. By our transformations we know that

$$dx^\mu \to dx'^\mu = \sum_\nu L^\mu{}_\nu dx^\mu, \tag{1.3.2}$$

where $L^\mu{}_\nu$ is precisely the matrix we derived earlier for the special case of boosts in the x direction. In that case

$$\begin{aligned} L^0{}_0 &= -L_{00}/c^2 = \cosh\eta, \\ L^0{}_1 &= -L_{01}/c^2 = \sinh\eta/c, \quad L^0{}_i = 0 \ \ \forall \ \ i \in \{2,3\}, \\ L^1{}_0 &= L_{10} = c\sinh\eta, \quad L^i{}_0 = 0 \ \ \forall \ \ i \in \{2,3\}, \\ L^1{}_1 &= L_{11} = \cosh\eta, \quad L^i{}_j = \delta^i_j \ \ \forall \ \ i,j \in \{2,3\}, \end{aligned} \tag{1.3.3}$$

where

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1.3.4}$$

is the usual Kronecker $\hat\delta$ (unit matrix).

In space-time, we may set up a vector space V at each point, P, by defining a set of four vectors, $\{\widehat{u}_{(\mu)}\}$, called a **tetrad frame**, spanning V. Suppose we choose the basis vectors in such a way that $u_{(\mu)}$ points in the direction of increasing x^μ at P, then an arbitrary proper displacement in space-time can be expressed as $d\vec{s} = \sum_\mu dx^\mu \widehat{u}_{(\mu)}$. Since the displacement itself should not depend on the observer, but dx^μ transforms according to (1.3.2), it follows that under a Lorentz transformation

$$\widehat{u}_{(\mu)} \to \widehat{u}'_{(\mu)} = \sum_\alpha \widehat{u}_{(\alpha)} (L^{-1})^\alpha{}_\mu \tag{1.3.5}$$

A vector is any object of the form $\vec{A} = \sum_\mu A^\mu \widehat{u}_{(\mu)}$, with four "contravariant" components, A^μ, each of which transforms as dx^μ (so that $\vec{A}$ is also observer independent), *i.e.,*

$$A^\mu \to A'^\mu = \sum_\nu L^\mu{}_\nu A^\nu. \tag{1.3.6}$$

The set of all of all vectors forms the vector space, V, with addition defined by

$$\vec{A} + \vec{B} = \sum_\mu (A^\mu + B^\mu)\widehat{u}_{(\mu)} \tag{1.3.7}$$

and scalar multiplication by

$$a\vec{A} = \sum_\mu (aA^\mu)\widehat{u}_{(\mu)}, \tag{1.3.8}$$

where $a \in \mathbb{R}$. The additive identity is the zero vector ($A^\mu = 0 \ \forall \ \mu$), the additive inverse of $\vec{A}$ is the vector $-\vec{A} = \sum_\mu (-A^\mu)\widehat{u}_{(\mu)}$ and the number "1" is the identity of scalar multilication. All the axioms of a vector space are satisfied by these definitions.

It is both customary and useful to think of a vector in terms of its components, but it is somewhat inconvenient to explicitly write out the summation (Σ) every time we have a sum over components. We notice, however, that only repeated indices get summed over; therefore we will use Einstein's convention and drop the symbol Σ, but now with the understanding that repeated indices, occurring in pairs in which one member appears "up" (as a superscript) and the other "down" (as a subscript), automatically implies a sum. Thus, for example, we would write the above transformation of contravariant vectors as

$$A^\mu \to A'^\mu = L^\mu{}_\nu A^\nu. \tag{1.3.9}$$

Notice that the derivative operator does not transform as dx^μ, but according to the inverse transformation. In other words:

$$\frac{\partial}{\partial x^\mu} := \partial_\mu \to \frac{\partial}{\partial x'^\mu} := \partial'_\mu = \frac{\partial x^\alpha}{\partial x'^\mu}\partial_\alpha. \tag{1.3.10}$$

But since $\partial x'^\mu/\partial x^\alpha = L^\mu{}_\alpha$, and

$$\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial x^\beta} = \delta^\alpha{}_\beta = (L^{-1})^\alpha{}_\mu L^\mu{}_\beta, \tag{1.3.11}$$

it follows that

$$\partial'_\mu = \frac{\partial x^\alpha}{\partial x'^\mu}\partial_\alpha = (L^{-1})^\alpha{}_\mu \partial_\alpha, \tag{1.3.12}$$

which is the same as the transformation of the basis vectors according to (6.2.6). Moreover, if $\phi(x)$ is any scalar function, *i.e.,* $\phi(x) \to \phi'(x') = \phi(x)$, then

$$dx'^\mu \partial'_\mu \phi'(x') = (L^{-1})^\beta{}_\mu L^\mu{}_\alpha dx^\alpha \partial_\beta \phi(x) = \delta^\beta{}_\alpha dx^\alpha \partial_\beta \phi(x) = dx^\mu \partial_\mu \phi(x) \tag{1.3.13}$$

shows that $dx^\mu \partial_\mu \phi$ is invariant.

Given any vector space, V, one can consider the space of all linear maps from V to the real numbers, *i.e.,* maps of the form $\vec{\omega} : V \to \mathbb{R}$, *i.e.,* for any vector $\vec{A}$,

$$\vec{\omega}(\vec{A}) \ \in \ \mathbb{R}$$

and for any two vectors $\vec{A}$ and $\vec{B}$,

$$\vec{\omega}(a\vec{A} + b\vec{B}) = a\vec{\omega}(\vec{A}) + b\vec{\omega}(\vec{B})$$

where a and b are real numbers. One may now define the sum of two linear maps by

$$(a\vec{\omega} + b\vec{\eta})(\vec{A}) = a\vec{\omega}(\vec{A}) + b\vec{\eta}(\vec{A}) \tag{1.3.14}$$

then one can show that these maps themselves form a vector space of the same dimension as V, called the **dual** vector space, $^*V$. Given the tetrad $\{\widehat{u}_{(\mu)}\}$, spanning V, we could introduce a basis for the dual vector space, $\{\widehat{\theta}^{(\mu)}\}$, by requiring that

$$\widehat{\theta}^{(\nu)}(\widehat{u}_{(\mu)}) = \delta^\nu_\mu \tag{1.3.15}$$

For the definition above to remain invariant, it must hold that, under a Lorentz transformation,

$$\widehat{\theta}^{(\mu)} \to \widehat{\theta}'^{(\mu)} = L^\mu{}_\alpha \widehat{\theta}^{(\alpha)}. \tag{1.3.16}$$

Any member of the dual vector space, $\vec{\omega}$ can now be expressed as $\vec{\omega} = \omega_\mu \widehat{\theta}^{(\mu)}$. ω_μ are called the "covariant" components of $\vec{\omega}$. They will transform as

$$\omega_\mu \to \omega'_\mu = \omega_\alpha (L^{-1})^\alpha{}_\mu, \tag{1.3.17}$$

so that, given any vector, $\vec{A}$, and any dual vector, $\vec{\omega}$, one forms a scalar

$$\vec{\omega}(\vec{A}) = \omega_\mu A^\mu. \tag{1.3.18}$$

This is the four dimensional dot product, the analogue of the three dimensional dot product we are familiar with. The simplest example of a dual vector is the gradient of a scalar function:

$$\nabla\phi(x) = \widehat{\theta}^{(\mu)} \partial_\mu \phi(x), \tag{1.3.19}$$

as we saw earlier. Notice that

$$\nabla\phi(d\vec{s}) = \partial_\mu \phi dx^\mu = d\phi \tag{1.3.20}$$

returns the change in the scalar function due to an infinitesimal displacement.

The form of (1.3.18) suggests that we could equivalently think of vectors as linear maps on dual vector spaces, *i.e.,*

$$\vec{A}(\vec{\omega}) \ \in \ \mathbb{R}. \tag{1.3.21}$$

Then, if we require

$$\widehat{u}_{(\mu)}(\widehat{\theta}^{(\nu)}) = \delta^\nu_\mu, \tag{1.3.22}$$

it follows that

$$\vec{A}(\vec{\omega}) = \vec{\omega}(\vec{A}) = \omega_\mu A^\mu, \tag{1.3.23}$$

that is, the dual of the dual vector space is the vector space itself.

The concept of vectors is generalized to tensors by simply defining a rank $(0, n)$ tensor to be a multilinear map[8] from an ordered collection of vectors to $\mathbb{R}$, *i.e.,*

$$\mathbb{T} : V \otimes V \ldots \otimes V \ (n \text{ times}) \to \mathbb{R}. \tag{1.3.24}$$

[8] A multilinear map acts linearly on all its arguments.

A basis for $\mathbb{T}$ will be $\widehat{\theta}^{(\mu_1)} \otimes \widehat{\theta}^{(\mu_2)} \ldots \otimes \widehat{\theta}^{(\mu_n)}$ and we could express $\mathbb{T}$ as

$$\mathbb{T} = T_{\mu_1\mu_2\ldots\mu_n}\widehat{\theta}^{(\mu_1)} \otimes \widehat{\theta}^{(\mu_2)} \ldots \otimes \widehat{\theta}^{(\mu_n)}. \tag{1.3.25}$$

Its covariant components will transform as n copies of a dual vector,

$$T'_{\mu\nu\lambda\ldots} = T_{\alpha\beta\gamma\ldots}(L^{-1})^{\alpha}{}_{\mu}(L^{-1})^{\beta}{}_{\nu}(L^{-1})^{\gamma}{}_{\lambda}\ldots. \tag{1.3.26}$$

Similarly, we could define a rank $(m, 0)$ tensor to be a multilinear map from an ordered collection of dual vectors to $\mathbb{R}$, *i.e.*,

$$\mathbb{T} : {}^*V \otimes {}^*V \ldots \otimes {}^*V \ (m \text{ times}) \to \mathbb{R}. \tag{1.3.27}$$

By the same reasoning, a basis for $\mathbb{T}$ will be $\widehat{u}_{(\mu_1)} \otimes \widehat{u}_{(\mu_2)} \ldots \otimes \widehat{u}_{(\mu_m)}$ and we could express $\mathbb{T}$ as

$$\mathbb{T} = T^{\mu_1\mu_2\ldots\mu_m}\widehat{u}_{(\mu_1)} \otimes \widehat{u}_{(\mu_2)} \ldots \otimes \widehat{u}_{(\mu_m)} \tag{1.3.28}$$

so that its contravariant components will transform as m copies of a vector,

$$T^{\mu\nu\lambda\ldots} = L^{\mu}{}_{\alpha}L^{\nu}{}_{\beta}L^{\lambda}{}_{\gamma}T^{\alpha\beta\gamma\ldots} \tag{1.3.29}$$

More generally, we define "mixed" tensors as multilinear maps from an ordered collection of vectors *and* dual vectors, $\mathbb{T} : {}^*V \otimes {}^*V \ldots \otimes {}^*V \ (m \text{ times}) \otimes V \otimes V \ldots \otimes V \ (n \text{ times}) \to \mathbb{R}$ and express it as

$$\mathbb{T} = T^{\mu\nu\ldots}{}_{\lambda\kappa\ldots}\widehat{u}_{(\mu)} \otimes \widehat{u}_{(\nu)} \ldots \otimes \widehat{\theta}^{(\lambda)} \otimes \widehat{\theta}^{(\kappa)} \ldots, \tag{1.3.30}$$

with V and ${}^*V$ appearing in any order in the product (the above is simply one example). In this case, the tensor is said to have rank (m, n). Thus scalars, vectors and dual vectors are special cases of tensors: scalars are tensors of rank $(0, 0)$, vectors are tensors of rank $(1, 0)$ and dual vectors are tensors of rank $(0, 1)$. Just as we think of vectors and dual vectors in terms of their components, we will also think of tensors in terms of their components. Thus we will speak of contravariant, covariant and mixed tensors according to their components.

There is a one to one relationship between the covariant and contravariant tensors: for every covariant tensor we can find a contravariant tensor and vice-versa. To see how this comes about, let us rewrite the proper distance (1.1.35) in a slightly different way. Let η be the rank $(0, 2)$ tensor

$$\eta = \eta_{\mu\nu}\theta^{(\mu)} \otimes \theta^{(\nu)}. \tag{1.3.31}$$

and $d\vec{s} = dx^{\alpha}\widehat{u}_{(\alpha)}$ be an infinitesimal displacement vector, then

$$ds^2 = -\eta(d\vec{s} \otimes d\vec{s}) = -\eta_{\mu\nu}dx^{\mu}dx^{\nu}. \tag{1.3.32}$$

The tensor η is called the **Minkowski metric**. According to (1.1.35), $\eta_{\mu\nu}$ is the matrix

$$\widehat{\eta} = \eta_{\mu\nu} = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.3.33}$$

or, simply, "diag$(-c^2, 1, 1, 1)$". It is a covariant tensor of rank two. To double check its transformation properties note that, given that ds^2 is invariant, we must have

$$-\, ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \to \eta'_{\alpha\beta} dx'^\alpha dx'^\beta = \eta'_{\alpha\beta} L^\alpha{}_\mu L^\beta{}_\nu dx^\mu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{1.3.34}$$

which implies that

$$\eta_{\mu\nu} = L^\alpha{}_\mu L^\beta{}_\mu \eta'_{\alpha\beta}, \tag{1.3.35}$$

or, by taking inverses,

$$\eta'_{\alpha\beta} = (L^{-1})^\mu{}_\alpha (L^{-1})^\nu{}_\beta \eta_{\mu\nu}. \tag{1.3.36}$$

However, $\eta_{\mu\nu}$ is required to be an invariant tensor in Special Relativity, $\eta'_{\mu\nu} \equiv \eta_{\mu\nu}$, and this can be used in conjunction with (1.3.35) to derive expressions for the matrices $\hat{L}$. It is an alternative way of deriving the Lorentz transformations through the generators of the transformation (see Appendix A). Now the metric is invertible ($\|\hat{\eta}\| \neq 0$), with inverse

$$\widehat{\eta}^{-1} \stackrel{\text{def}}{=} \eta^{\mu\nu} = \begin{bmatrix} -\frac{1}{c^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.3.37}$$

(It may be easily shown that the inverse metric $\eta^{\mu\nu}$ transforms as

$$\eta'^{\alpha\beta} = L^\alpha{}_\mu L^\beta{}_\nu \eta^{\mu\nu} = \eta^{\alpha\beta}, \tag{1.3.38}$$

i.e., according the rule for a contravariant tensor of rank two.)

But, according to (1.3.18), scalars also result from the action of a dual vector on a vector. Thus we expect that $\eta_{\mu\nu} dx^\nu$ should transform as a *covariant* vector. In general, consider a contravariant vector A^μ and construct the quantity

$$A_\mu = \eta_{\mu\nu} A^\nu. \tag{1.3.39}$$

How does it transform? We see that

$$A_\mu \to A'_\mu = \eta_{\mu\nu} L^\nu{}_\alpha A^\alpha = \eta_{\mu\nu} L^\nu{}_\alpha \eta^{\alpha\gamma} \eta_{\gamma\lambda} A^\lambda = (\eta_{\mu\nu} L^\nu{}_\alpha \eta^{\alpha\gamma})(\eta_{\gamma\lambda} A^\lambda), \tag{1.3.40}$$

where we have used $\eta^{\alpha\gamma}\eta_{\gamma\lambda} = \delta^{\alpha}{}_{\lambda}$. But notice that (1.3.35) implies the identity

$$\begin{aligned}
\eta_{\alpha\beta} &= \eta_{\mu\nu} L^{\mu}{}_{\alpha} L^{\nu}{}_{\beta} \to \eta^{\gamma\alpha}\eta_{\alpha\beta} = \eta_{\mu\nu}\eta^{\gamma\alpha} L^{\mu}{}_{\alpha} L^{\nu}{}_{\beta} \\
\to \delta^{\gamma}{}_{\beta} &= (\eta_{\nu\mu} L^{\mu}{}_{\alpha}\eta^{\alpha\gamma}) L^{\nu}{}_{\beta} = (L^{-1})^{\gamma}{}_{\nu} L^{\nu}{}_{\beta} \\
\to (L^{-1})^{\gamma}{}_{\nu} &= \eta_{\nu\mu} L^{\mu}{}_{\alpha}\eta^{\alpha\gamma}. \qquad (1.3.41)
\end{aligned}$$

Therefore (1.3.40) reads

$$A'_{\mu} = A_{\gamma}(L^{-1})^{\gamma}{}_{\mu}, \qquad (1.3.42)$$

which is the transformation of a covariant vector. The Minkowski metric therefore maps contravariant vectors to covariant vectors. In the same way it maps contravariant tensors to covariant tensors:

$$T_{\alpha_1,\alpha_2,\ldots\alpha_n} = \eta_{\alpha_1\beta_1}\eta_{\alpha_2\beta_2}\ldots\eta_{\alpha_n\beta_n} T^{\beta_1,\beta_2,\ldots\beta_n}. \qquad (1.3.43)$$

Likewise, the inverse metric $\eta^{\mu\nu}$ maps covariant vectors to contravariant vectors, *i.e.*, the quantity A^{μ} defined by

$$A^{\mu} = \eta^{\mu\nu} A_{\nu}, \qquad (1.3.44)$$

transforms as a contravariant vector.[9] Therefore, it maps covariant tensors to contravariant tensors:

$$T^{\alpha_1,\alpha_2,\ldots\alpha_n} = \eta^{\alpha_1\beta_1}\eta^{\alpha_2\beta_2}\ldots\eta^{\alpha_n\beta_n} T_{\beta_1,\beta_2,\ldots\beta_n}. \qquad (1.3.45)$$

This relationship between covariant tensors and contravariant tensors is why we originally defined the boosts as in (1.1.18). Thus, $L^{\mu}{}_{\nu} = \eta^{\mu\alpha} L_{\alpha\nu}$ which gives

$$\begin{aligned}
L^{0}{}_{0} &= \eta^{00} L_{00} = -L_{00}/c^2, \\
L^{0}{}_{i} &= \eta^{00} L_{0i} = -L_{0i}/c^2, \\
L^{i}{}_{0} &= \eta^{ij} L_{j0} = L_{i0}, \\
L^{i}{}_{j} &= \eta^{ik} L_{kj} = L_{ij}. \qquad (1.3.46)
\end{aligned}$$

Moreover, there is a natural way to define the (invariant) magnitude of a four-vector, A^{μ}. It is simply

$$\vec{A}^2 = A^{\mu} A_{\mu} = \eta_{\mu\nu} A^{\mu} A^{\nu} = \eta^{\mu\nu} A_{\mu} A_{\nu}, \qquad (1.3.47)$$

[9]Problem: Show this.

which is the equivalent of the familiar way of defining the magnitude of an ordinary three-vector.[10] For example, the familiar operator $\Box_x$ can be written as

$$\Box_x = -\eta^{\mu\nu}\partial_\mu\partial_\nu = -\partial^2, \tag{1.3.48}$$

in which form it is manifestly a scalar. More generally, the action of the metric on any two vectors will result in a scalar,

$$\eta(\vec{A}\otimes\vec{B}) = \eta_{\mu\nu}A^\mu B^\nu \tag{1.3.49}$$

called the **inner product**, or **dot product**, $\vec{A}\cdot\vec{B}$, of the vectors $\vec{A}$ and $\vec{B}$.

Finally, just as in three dimensions, the four dimensional permutation symbol, $[\mu\nu\alpha\beta]$,

$$[\mu\nu\alpha\beta] = \begin{cases} +1 & \text{if } \mu\nu\alpha\beta \text{ is an even permutation of 0123,} \\ 0 & \text{if } \mu\nu\alpha\beta \text{ is not permutation of 0123, and} \\ -1 & \text{if } \mu\nu\alpha\beta \text{ is an odd permutation of 0123,} \end{cases} \tag{1.3.50}$$

transforms as a tensor called the Levi-Civita tensor, $\epsilon_{\mu\nu\alpha\beta}$. (Recall that a permutation of any set, say 0123, is an exchange of two of the members of the set. An even permutation is the ordering one gets after an even number of exchanges. For example, 1032 is an even permutation of 0123. An odd permutation is the ordering obtained after an odd number of exchanges, for example 1302 is an odd permutation of 0123.)[11]

We see once again that the basic difference between Newtonian space and Lorentzian space-time is that, in the case of the former, space and time do not mix and both are absolute. In Newtonian mechanics there is no concept of an invariant distance between events and it is sufficient to consider only spatial distances. Because spatial distances are absolute, Pythagoras' theorem ensures that the metric is just the Kronecker δ (with three positive eigenvalues), so there is no need to distinguish between covariant and contravariant indices. In the case of a Lorentzian space-time an observer's measurements of space and time are not independent, neither is each absolute by itself and so one is forced to consider the "distance" between *events* in space-time. The metric, $\eta_{\mu\nu}$, for space-time has signature $(-1,3)$ *i.e.,* it has one negative eigenvalue and three positive eigenvalues.

For an arbitrary boost specified by a velocity $\vec{v} = (v_1, v_2, v_3) = (v^1, v^2, v^3)$, we find the following Lorentz transformations:

$$L^0{}_0 \;=\; \frac{1}{\sqrt{1-\vec{v}^2/c^2}} \stackrel{\text{def}}{=} \gamma,$$

[10] When $A^2 < 0$ the vector points within the light cone and is said to be "time-like". When $A^2 > 0$ it points outside the light cone and is called "space-like" and when $A^2 = 0$ the vector A is "light-like" or "null", pointing along the light cone.

[11] Problem: Show that the four dimensional Levi-Civita symbol is a rank $(0,4)$ tensor by verifying its transformation properties.

$$
\begin{aligned}
L^i{}_0 &= -\gamma v^i, \\
L^0{}_i &= -\frac{\gamma v_i}{c^2}, \\
L^i{}_j &= \delta^i{}_j + (\gamma - 1)\frac{v^i v_j}{\vec{v}^2}.
\end{aligned} \tag{1.3.51}
$$

These are most easily derived using (1.3.35) and the fact that $\eta'_{\mu\nu} = \eta_{\mu\nu}$. A more compact way to write (1.3.51) is

$$
\begin{aligned}
t' &= \gamma \left[t - \frac{(\vec{v} \cdot \vec{r})}{c^2} \right] \\
\vec{r}' &= \vec{r} - \gamma \vec{v} t + (\gamma - 1)\frac{\vec{v}}{v^2}(\vec{v} \cdot \vec{r})
\end{aligned} \tag{1.3.52}
$$

for a general $\vec{v}$.

Spatial volume elements are not invariant under Lorentz transformations. We can make a rough argument for this as follows: suppose that the volume measured by the proper observer is dV then the observer moving relative to this observer with a velocity $\vec{v}$ will observe the length dimension in the direction of motion contracted according to (1.2.2) and all perpendicular length dimensions will remain unchanged, so we expect $dV' = dV/\gamma$. A more precise treatment follows by mimicking the argument for length contraction. Consider the transformation form $(t, x) \to (t'x')$

$$
dt' = L^0{}_0 dt + L^0{}_j dx^j, \quad dx'^i = L^i{}_0 dt + L^i{}_j dx^j \tag{1.3.53}
$$

with $dt' = 0$ because length measurements must be made subject to a simultaneous measurement of the endpoints in every frame. Therefore $dt = -L^0{}_j dx^j/\gamma$ and

$$
dx'^i = \left(-\frac{1}{\gamma} L^i{}_0 L^0{}_j + L^i{}_j \right) dx^j = \left(\delta^i_j + \frac{(1-\gamma)}{\gamma} \frac{v^i v_j}{v^2} \right) dx^j, \tag{1.3.54}
$$

so taking the Jacobian of the transformation gives

$$
d^3\vec{r} \to d^3\vec{r}' = d^3\vec{r} \left| \frac{\partial x'^i}{\partial x^j} \right| = d^3\vec{r}/\gamma. \tag{1.3.55}
$$

The four dimensional volume element, d^4x, is invariant for proper Lorentz transformations.

A consequence of the Lorentz transformation of volume is that the three dimensional δ function, $\delta^{(3)}(\vec{r} - \vec{r}_0)$, which is defined according to

$$
\int d^3\vec{r}\delta^{(3)}(\vec{r} - \vec{r}_0) = 1 \tag{1.3.56}
$$

cannot be invariant either. If we require the defining integral to remain invariant then

$$\delta^{(3)}(\vec{r}-\vec{r}_0) \to \delta'^{(3)}(\vec{r}'-\vec{r}'_0) = \gamma\delta^{(3)}(\vec{r}-\vec{r}_0). \tag{1.3.57}$$

The four dimensional delta function, $\delta^{(4)}(x-x_0)$, will, however, be invariant.

1.4 Waves and the Relativistic Doppler Effect

Maxwell's equations for the electromagnetic field, A_μ, in Lorentz gauge read

$$\Box_x A_\mu = j_\mu. \tag{1.4.1}$$

In the absence of sources, this is just the wave equation with c being the speed of propagation; in one spatial dimension

$$\left[\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right] A_\mu(t,x) = 0 \tag{1.4.2}$$

and a typical solution will look like a linear combination of plane waves of varying amplitudes and frequencies,

$$A_\mu^{(k)}(t,x) = A_\mu^{(0)}(k,\omega)e^{i(kx-\omega t)}, \tag{1.4.3}$$

subject to $k^2 - \omega^2/c^2 = 0$. Because A_μ transforms as a vector, the exponent must transform as a scalar, *i.e.,* $kx - \omega t$ must be an invariant. This is only possible if $k_\mu = (-\omega, k)$ transforms as a covariant vector,

$$\omega' = \gamma(\omega - vk), \quad k' = \gamma(k - v\omega/c^2) \tag{1.4.4}$$

i.e., $k^\mu = (\omega/c^2, k)$ transforms as $x^\mu = (t, x)$. In particular, the first relation tells us that

$$f' = f\sqrt{\frac{1-v/c}{1+v/c}}, \tag{1.4.5}$$

which is the expression for the Doppler shifting of light in the frame of an observer moving with a velocity v, taken as positive when the observer is traveling in the direction of the propagating light wave. Thus an observer moving "away from" the source sees a **red-shifting** of the light, *i.e.,* a shifting toward lower frequencies, and an observer moving toward the source sees a **blue-shifting**, *i.e.,* a shifting toward higher frequencies. If the observer's speed is small compared to the speed of light, the linear approximation of (1.4.5) gives

$$f' \approx \left(1 - \frac{v}{c}\right) f, \tag{1.4.6}$$

which should be compared with the Doppler shifting for ordinary mechanical waves that propagate in a medium. We get the observed wavelengths either directly, by requiring $\lambda f = c = \lambda' f'$, or by using the second relation in (1.4.4),

$$\lambda' = \lambda\sqrt{\frac{1+v/c}{1-v/c}}. \tag{1.4.7}$$

The "redshift" factor is defined as

$$z = \frac{\lambda' - \lambda}{\lambda} = \sqrt{\frac{1+v/c}{1-v/c}} - 1. \tag{1.4.8}$$

and $z \approx \frac{v}{c}$ when $v \ll c$. Because all inertial observers are equivalent in special relativity there is no separate effect for "moving sources" as there is in the case of mechanical waves. Yet, one may wonder why there is an effect at all, considering that light requires no medium in which to travel and its speed in all reference frames is the same. The Doppler effect for light originates in time dilation.

1.5 Dynamics in Special Relativity

The relativistic point particle extremizes its "proper time" (this can be thought of as a generalization of Fermat's principle, which was originally enunciated for the motion of Newton's "light corpuscles"),

$$\mathcal{S}_p = -mc^2 \int d\tau = -mc\int_1^2 \sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu} = -mc^2\int_1^2 dt\sqrt{1-\frac{\vec{v}^2}{c^2}} \tag{1.5.1}$$

where $d\tau = ds/c = \frac{1}{c}\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu}$ is the proper time and the constant "mc^2" is chosen so that $\mathcal{S}_\mathrm{p}$ has the dimension of action (or angular momentum: J·s). One sees quite easily that this action principle reduces to Hamilton's principle (with zero potential energy, $V = 0$) when the velocity of the particle relative to the observer is small compared with the velocity light, for then

$$\sqrt{1-\frac{\vec{v}^2}{c^2}} \approx 1 - \frac{1}{2}\frac{\vec{v}^2}{c^2} \tag{1.5.2}$$

which, when inserted into (1.5.1), gives

$$\mathcal{S}_p \approx \int_1^2 dt\left[\frac{1}{2}m\vec{v}^2 - mc^2\right]. \tag{1.5.3}$$

The second term is, of course, just a constant (later to be identified with the rest mass energy of the particle) and can be dropped without affecting either the equations of motion or the conservation laws. The first term is the non-relativistic kinetic energy of the particle and the action is therefore just that of a free non-relativistic point particle.

The momentum conjugate to x^i is

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = \frac{mv_i}{\sqrt{1-\vec{v}^2/c^2}} = \gamma m v_i, \tag{1.5.4}$$

where $\mathcal{L}$ is the Lagrangian of (1.5.1). The momentum of (1.5.4) reduces to its non-relativistic version, $p_i = mv_i$, when $|\vec{v}| << c$. Euler's equations give

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m\vec{v}) = \frac{d}{dt}\frac{m\vec{v}}{\sqrt{1-\vec{v}^2/c^2}} = 0, \tag{1.5.5}$$

which are the relativistic equations of motion of the particle. The Lagrangian does not depend explicitly on time, so we expect that the Hamiltonian is the total energy and is conserved,

$$E = \mathcal{H} = p_i\dot{x}^i - \mathcal{L} = \frac{m\vec{v}^2}{\sqrt{1-\vec{v}^2/c^2}} + mc^2\sqrt{1-\vec{v}^2/c^2} = \frac{mc^2}{\sqrt{1-\vec{v}^2/c^2}} \tag{1.5.6}$$

The quantity

$$m_R = \frac{m}{\sqrt{1-\vec{v}^2/c^2}} \tag{1.5.7}$$

is generally called the "relativistic mass" or simply "mass" of the particle, whereas the parameter m we used initially is called the "rest mass" of the particle and can be thought of as its mass when measured in its proper frame ($\vec{v} = 0$). We have just obtained the famous Einstein relation,

$$E = m_R c^2. \tag{1.5.8}$$

Notice that the energy of the particle is not zero in the rest frame. In this frame the particle possesses an energy, $E = mc^2$, which is exclusively associated with its proper (rest) mass. Furthermore, expanding E in powers of $\vec{v}$ we find

$$E = mc^2 + \frac{1}{2}m\vec{v}^2 + \frac{3}{8}m\frac{\vec{v}^4}{c^2} + \ldots \tag{1.5.9}$$

The second term is the Newtonian kinetic energy and the higher order terms are all corrections to the Newtonian expression. The kinetic energy, K, in special relativity is defined via the relation

$$E = K + mc^2, \tag{1.5.10}$$

so it is what remains after its rest mass energy is subtracted from its total energy.

The Hamiltonian is obtained by a Legendre transformation of the Lagrangian and is expressed in terms of the momenta and coordinates but not the velocities. This is easily accomplished by noting that (1.5.4) gives

$$\vec{p}^2 = \frac{m^2\vec{v}^2}{1-\vec{v}^2/c^2} \to \frac{\vec{v}}{c} = \frac{\vec{p}}{\sqrt{\vec{p}^2+m^2c^2}} \tag{1.5.11}$$

Thus we get

$$1 - \frac{\vec{v}^2}{c^2} = \frac{m^2c^2}{\vec{p}^2+m^2c^2}, \tag{1.5.12}$$

which, when inserted into (1.5.6), gives another well known result,

$$\mathcal{H} = E = \sqrt{\vec{p}^2c^2+m^2c^4}. \tag{1.5.13}$$

Again we recover the rest mass energy, $E = mc^2$ when we set $\vec{p} = 0$.

Let us note that the momentum

$$p_i = m\gamma v_i = m\frac{dt}{d\tau}\frac{dx_i}{dt} = m\frac{dx_i}{d\tau} \tag{1.5.14}$$

is quite manifestly the spatial component of the four-vector

$$p_\mu = m\frac{dx_\mu}{d\tau} \equiv mU_\mu, \tag{1.5.15}$$

where $U^\mu = dx^\mu/d\tau$ is the "four velocity" of the particle and transforms as a vector.[12] The quantity p_μ is called its "four momentum". Its time component is

$$p_0 = -mc^2\frac{dt}{d\tau} = -\frac{mc^2}{\sqrt{1-\vec{v}^2/c^2}} = -E \tag{1.5.16}$$

so the spatial momentum and the energy are components of one four-vector momentum,

$$p^\mu = m\frac{dx^\mu}{d\tau}, \quad p^0 = \frac{E}{c^2}, \quad p^i = \frac{mv^i}{\sqrt{1-\vec{v}^2/c^2}} \tag{1.5.17}$$

Formula (1.5.13) for the energy is now seen to result from a purely kinematic relation, because

$$p^2 = \eta_{\mu\nu}p^\mu p^\mu = m^2\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -m^2\left[\frac{ds}{d\tau}\right]^2 = -m^2c^2 \tag{1.5.18}$$

[12] Problem: Convince yourself that $p^\mu = mdx^\mu/d\tau$ is indeed a four-vector under Lorentz transformations. Remember that the proper time, τ, is a scalar. Notice that the four momentum can be written as

$$p^\mu = mU^\mu = m\frac{dx^\mu}{d\tau} = m\frac{dt}{d\tau}\frac{dx^\mu}{dt} = m\gamma v^\mu$$

where $v^\mu = dx^\mu/dt = v^\mu = (1, \vec{v})$. v^μ does **not** transform as a vector. Why?

(the kinematic relation being, of course $U_\mu U^\mu = -c^2$, *remember it*). Therefore, expanding the l.h.s.,

$$p^2 = -\frac{E^2}{c^2} + \vec{p}^2 = -m^2c^2, \;\Rightarrow\; E^2 = \vec{p}^2c^2 + m^2c^4 \tag{1.5.19}$$

Interestingly, taking the square root allows for both positive and negative energies but we have chosen the positive sign, thereby excluding negative energy free particles by *fiat*.[13]

Euler's equations as given in (1.5.5) are not in a manifestly covariant form. They can, however, be put in such a form if we multiply by γ, expressing them as

$$\gamma\frac{d\vec{p}}{dt} = \frac{dt}{d\tau}\frac{d\vec{p}}{dt} = \frac{d\vec{p}}{d\tau} = 0 \tag{1.5.20}$$

This is the equation of motion for a *free* particle, so the r.h.s. is zero. The l.h.s. transforms as the spatial components of a four-vector and we need not worry about the transformation properties of the r.h.s., since it vanishes. In the presence of an external force the r.h.s. should not vanish and the principle of covariance requires that both sides of the equations of motion should transform in the same way under Lorentz transformations. Let us tentatively write a covariant equation of motion as

$$\frac{dp^\mu}{d\tau} = f^\mu \tag{1.5.21}$$

where f^μ is a four-vector. It must be interpreted as the relativistic equivalent of Newton's force. If m is constant then

$$f^\mu = m\frac{dU^\mu}{d\tau}$$

and, because $U^2 = -c^2$, the "four force" must satisfy one constraint, *i.e.,*

$$f \cdot U = f^\mu U_\mu = 0, \tag{1.5.22}$$

which means that not all its components are independent. But what is the connection between f^μ and the familiar concept of the Newtonian force, which we will call $\vec{F}_N$? To find it consider the proper frame, $\overline{S}$, of the particle (quantities in this instantaneous rest frame will be represented by an over-bar). In this frame $\tau = \bar{t}$, $\overline{p}^0 = m$ and $\overline{p}^i = 0$. It follows that the time component of the l.h.s of (1.5.21) is zero (assuming m is constant) and therefore so is $\overline{f}^0$. The spatial part of the force equation then reads

$$\frac{d\overline{p}^i}{d\bar{t}} = m\overline{a}^i = \overline{f}^i, \tag{1.5.23}$$

[13] While this works well in the classical theory, it fails to be consistent in the quantum theory. When quantizing classical, Lorentz invariant theories, both positive and negative energies must be included on an equal footing.

where $\overline{a}^i$ is the particle's acceleration relative to $\overline{S}$, generally referred to as its **proper acceleration**. Naturally, we identify F_N^i with $\overline{f}^i$ or, equivalently, with $m\overline{a}^i$. We will discuss the connection between the proper acceleration, $\overline{a}^i$, and the acceleration as measured in S,

$$a^i = \frac{d^2x^i}{dt^2}$$

in a later section. For the present take

$$\overline{f}^\mu = (0, \vec{F}_N). \tag{1.5.24}$$

To determine f^μ in an arbitrary frame we only need to perform a boost because f^μ is a genuine four-vector. Therefore, in a frame in which the instantaneous velocity of the particle is $\vec{v}$, we find in particular that

$$f^0 = \gamma\frac{\vec{v}\cdot\vec{F}_N}{c^2} \tag{1.5.25}$$

i.e.,

$$\frac{dE}{dt} = \vec{v}\cdot\vec{F}_N \tag{1.5.26}$$

The equation says that the rate of energy gain (loss) of the particle is simply the power transferred to (or from) the system by the external Newtonian forces. The same boost also gives the spatial components of the relativistic force in an arbitrary frame as

$$\vec{f} = \vec{F}_N + (\gamma-1)\frac{\vec{v}}{v^2}(\vec{v}\cdot\vec{F}_N) \tag{1.5.27}$$

and we notice that the component of $\vec{f}$ perpendicular to the velocity is equal to the corresponding component of the Newtonian force, $\vec{f}_\perp = \vec{F}_{N\perp}$. However the component of the force in the direction of motion is enhanced over the same component of the Newtonian force by the factor of γ, *i.e.*, $\vec{f}_\parallel = \gamma\vec{F}_{N\parallel}$. Our expression also has the non-relativistic limit $(\gamma \approx 1)$ $\vec{f} \approx \vec{F}_N$, as it should.

We have given two forms of the action for the massive point particle in (1.5.1) although we have concentrated so far on the last of these. The first form,

$$\mathcal{S}_p = -mc\int_1^2 \sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu}, \tag{1.5.28}$$

is actually quite interesting. If λ is any parameter describing the particle trajectories then we could write this as

$$\mathcal{S}_p = -mc\int_1^2 d\lambda\sqrt{-\eta_{\mu\nu}U^\mu_{(\lambda)}U^\nu_{(\lambda)}} \tag{1.5.29}$$

where

$$U^\mu_{(\lambda)} = \frac{dx^\mu(\lambda)}{d\lambda} \tag{1.5.30}$$

is tangent to the trajectories $x^\mu(\lambda)$ and λ is an arbitrary parameter. The action is therefore reparameterization invariant and all that we have said earlier corresponds to a particular choice of λ $(= t)$. This is like fixing a "gauge", to borrow a term from electrodynamics.[14],[15]

The relativistic Hamilton-Jacobi equation is obtained by replacing the momenta, p_μ, by $\partial S/\partial x^\mu$ in (1.5.19),

$$\eta^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu}\right)\left(\frac{\partial S}{\partial x^\nu}\right) = -m^2c^2 \tag{1.5.31}$$

We could define $S = S' - mc^2 t$ and write the equation in terms of S',

$$\frac{1}{2m}(\vec{\nabla}S')^2 - \left(\frac{\partial S'}{\partial t}\right) - \frac{1}{2mc^2}\left(\frac{\partial S'}{\partial t}\right)^2 = 0 \tag{1.5.32}$$

In this form the limit $c \to \infty$ compares directly with the expected Hamiltonian-Jacobi equation for the free non-relativistic particle.

1.6 Conservation Laws

We will now consider a system of relativistic particles and define the total particle momentum as

$$p^\mu = \sum_n p^\mu_n = \sum_n m_n \frac{dx^\mu_n}{d\tau_n} = \sum_n m_n \gamma_n v^\mu_n \tag{1.6.1}$$

[14] Problem: Starting from (1.5.28), treat all the coordinates of an event, x^μ, on the same footing (instead of singling out one of them – time – as a parameter) and define

$$p^{(\lambda)}_\mu = \frac{\partial \mathcal{L}}{\partial U^\mu_{(\lambda)}}.$$

Show that

$$\mathbb{H} = p^{(\lambda)}_\mu U^\mu_{(\lambda)} - \mathcal{L} = 0.$$

This is a consequence of reparameterization invariance.

[15] Problem: The square-root Lagrangian in (1.5.28) is inconvenient for a host of applications such as, for example, the quantization of a collection of free particle or working out the statistical mechanics of free, relativistic particles. A quadratic form, similar to the non-relativistic one, is preferable. This can be achieved by introducing an auxiliary function, χ, together with the action,

$$S = -\int d\lambda \left[\chi \eta_{\mu\nu} U^\mu_{(\lambda)} U^\nu_{(\lambda)} + \chi^{-1} m^2\right]$$

and treating x^μ and χ as independent functions with respect to which the action is to be extremized. Show that one obtains the expected equations of motion.

The rate at which each particle's four-momentum changes will depend on the net force acting upon it, according to

$$\frac{dp_n^\mu}{d\tau_n} = f_n^\mu, \tag{1.6.2}$$

but this is an inconvenient form of the equation of motion, particularly when dealing with many particles, because it describes the rate of change with respect to the proper time of the particle, which itself depends on its motion. Making use of the fact that $\gamma_n = dt/d\tau_n$, let's rewrite this equation in the form

$$\frac{dp_n^\mu}{dt} = \gamma_n^{-1} f_n^\mu \tag{1.6.3}$$

and therefore also

$$\frac{dp^\mu}{dt} = \sum_n \frac{dp_n^\mu}{dt} = \sum_n \gamma_n^{-1} f_n^\mu. \tag{1.6.4}$$

Concentrate, for the moment, on the spatial components only and, as before, let $\vec{f}_n$ be made up of two parts, *viz.,* (i) an "external" force, $\vec{f}_n^{\text{ext}}$, acting on the particle and (ii) an "internal" force, $\vec{f}_n^{\text{int}}$ acting on it due to its interactions with all the other particles within the system. Then

$$\vec{f}_n^{\text{int}} = \sum_{m\neq n} \vec{f}_{m\to n} \tag{1.6.5}$$

and it follows that

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\sum_n \vec{p}_n = \sum_{n,m\neq n} \gamma_n^{-1} \vec{f}_{m\to n} + \sum_n \gamma_n^{-1} \vec{f}_n^{\text{ext}} \tag{1.6.6}$$

For low particle velocities $\gamma_n \approx 1$ for all n and $\vec{f}_n \approx \vec{F}_n^N$, where $\vec{F}^N$ is the Newtonian force. In this limit, conservation of particle momentum in the absence of external forces follows by Newton's third law.

What about the internal forces in a fully relativistic scenario? In principle, no effect experienced at any world point x can have originated at a world point x' outside its past light cone, *i.e.,* at a time earlier than $t-|\vec{r}-\vec{r}'|/c$[16] because the speed of light is assumed to be the maximal speed at which information or influence can travel. Instantaneous particle interactions, in particular forces that depend only on the spatial distances between the particles (which are typically employed in Newtonian physics), cannot have the desired Lorentz transformation properties and therefore are impossible in the context of special relativity. This, of course, becomes relevant only for very high particle velocities because the change in position of one particle can influence another particle only after the information has had the time to propagate through the distance that separates the particles,

[16] This is called **retardation**.

which itself changes appreciably during this time if the relative velocities are high, making the interactions of particles depend in a complicated way on their motions. This is already evident from the expression $\sum_{n,m\neq n}\gamma_n^{-1}\vec{f}_{m\to n}$ describing the effects of the internal forces. One gets around this difficulty by imagining that the particles are immersed in a set of dynamical **fields** whose disturbances propagate in space-time with a speed not exceeding the speed of light.

A "field" should be thought of as a potential (or set of potentials) associated with each point of space-time. Disturbances in these potentials transfer energy and momentum from one event to another. Every field is realized by a function (or set of functions) with definite Lorentz transformation properties and one can have many kinds of fields, eg. scalar fields, vector fields, etc., depending on how the field transforms. Particle-particle interactions are then described in terms of **local interactions** of the particles with the fields, which involve an exchange of energy and momentum between the two at the world point of the particle. This exchange causes disturbances in the fields, which then propagate through space-time and, at a later time, exchange energy and momentum (locally) with other particles in the system. In this way fields act as mediators of inter-particle forces. A familiar example of this would be electrically charged particles in an electromagnetic field. The electromagnetic field is responsible for energy and momentum transfer between the particles via their electromagnetic interaction.

This picture is only consistent if the field that is responsible for mediating the interaction carries energy and momentum in its own right. We then define the total four momentum of the system by

$$P^\mu = p^\mu + \pi_{\rm f}^\mu \tag{1.6.7}$$

where $\pi_{\rm f}^\mu$ represents the field momentum and assert the following:

- In the absence of external forces the total four momentum of the system, which consists of the momentum of the particles and the field, is conserved.

This is a natural generalization of the non-relativistic statement about momentum conservation and, in fact, follows from Noether's theorem by space-time translation invariance, as we will see later. It is worth understanding why any generalization of the conservation law for momentum *must* involve the entire four-vector momentum if it is to be a covariant statement. It can be argued as follows: let ΔE and $\Delta\vec{P}$ represent the change in energy and momentum respectively of the system in some inertial frame S. In some other frame, S', we represent these quantities by $\Delta E'$ and $\Delta\vec{P}'$ respectively. Being components of a four vector, they are connected by the Lorentz transformation,

$$\begin{aligned}\Delta P'^i &= \frac{\gamma v^i}{c^2}\Delta E + \left(\delta^i_j + (\gamma-1)\frac{v^i v_j}{v^2}\right)\Delta P^j\\ \Delta E' &= \gamma\left(\Delta E + \vec{v}\cdot\Delta\vec{P}\right).\end{aligned} \tag{1.6.8}$$

It follows that if $\Delta\vec{P}$ vanishes (the spatial momentum is conserved) in S then it will vanish in S' if and only if ΔE vanishes as well and if ΔE vanishes (the total energy is conserved) in S then it will vanish in S' if and only if $\Delta\vec{P}$ also vanishes. Therefore energy and momentum conservation go hand in hand in Einstein's theory of relativity and one cannot be had without the other. This is a most remarkable fact. In Newtonian mechanics the two conservation laws are distinct: momentum conservation requires the absence external forces and a sufficient condition for the conservation of energy is that *all* forces acting on the system are conservative. No such condition appears in the relativistic version of the conservation law, which must therefore *always* hold provided that the momentum of the particles and fields are consistently taken into account.

Note that neither the total field momentum nor the total particle momentum is separately conserved since momentum may be exchanged between the two. This implies that the interaction forces between the particles do **not** satisfy Newton's third law *i.e.,* action is **not** equal and opposite to reaction.

It is often convenient to define the **center of momentum** frame in complete analogy with the non-relativistic case by setting the spatial components of the total momentum in that frame to zero, *i.e.,*

$$P^\mu_{\text{cm}} = \left(\frac{E_{\text{cm}}}{c^2}, \vec{0}\right) = (M_{\text{cm}}, \vec{0}), \tag{1.6.9}$$

which also defines the total rest mass, M_{cm}, of the system. The rest mass energy, $M_{\text{cm}}c^2$, contains all the rest energies of the particles that make up the system. It *also* contains their kinetic energy relative to the center of mass as well as the energies of their interactions with one another and of the fields involved in these interactions. In other words, the rest mass of the system contains the entire internal energy of the system and *it is conserved*. We know of four kinds of elementary fields, each with its characteristic interactions. From weakest to strongest they are the gravitational field, the fields associated with the weak interaction, the electromagnetic field and the fields associated with the strong interaction or chromodynamics. The gravitational field is associated with space-time itself and its description is unique. All the other fields are special cases of a single family of theories called "gauge theories". These will be discussed in the following chapters. The momentum in a generic frame, S, can be obtained by a Lorentz transformation and will involve only M_{cm} and the velocity of the center of mass relative to the Laboratory, $\vec{v}_{\text{cm}}$,

$$P^0 = M_{\text{cm}}\gamma_{\text{cm}}, \quad \vec{P} = M_{\text{cm}}\gamma\vec{v}_{\text{cm}}. \tag{1.6.10}$$

Thus in every physical system consisting of interacting particles, the center of mass will behave as a single particle with an effective mass (equal to M_{cm}).

1.7 Relativistic Collisions

The preceding discussion leads directly to the topic of collisions between relativistic particles. In this section we will briefly examine such collisions, assuming that whatever fields are present diminish rapidly enough to zero that the field contribution to the total momentum can be ignored when the particles are sufficiently far apart. We will then consider "free" incoming particles and "free" outgoing particles as we did earlier and conserve momentum according to $\sum_n p^\mu_{ni} = \sum_n p^\mu_{nf}$ as before, but this time taking care with the relativistic factors.

First consider a collision in which two incoming bodies with momenta

$$p_{1i} = \left(m_1 + \frac{K_1}{c^2}, \vec{p}_1\right), \quad p_{2i} = \left(m_2 + \frac{K_2}{c^2}, \vec{p}_2\right) \tag{1.7.1}$$

stick together to form a body of mass m_f, with momentum

$$p_f = \left(m_f + \frac{K_f}{c^2}, \vec{p}_f\right), \tag{1.7.2}$$

where we used the definition of the Kinetic energy, $E = mc^2 + K$. Conservation of momentum means that

$$\begin{aligned} m_1 + m_2 + \frac{K_1}{c^2} + \frac{K_2}{c^2} &= m_f + \frac{K_f}{c^2} \\ \vec{p}_1 + \vec{p}_2 &= \vec{p}_f, \end{aligned} \tag{1.7.3}$$

which three equations (the collision is planar) are sufficient to determine m_f and $\vec{p}_f$ in terms of the initial data. Such a collision is best viewed in the center of momentum frame in which $\vec{p}_f = 0$ and $m_f = M_{\text{cm}}$. Then, in this frame $p^\mu_{\text{cm}} = (M_{\text{cm}}, \vec{0})$

$$m_1 + m_2 + \frac{K'_1}{c^2} + \frac{K'_2}{c^2} = M_{\text{cm}}, \quad \vec{p}'_1 = -\vec{p}'_2 \tag{1.7.4}$$

where we used primes to denote quantities measured in the center of momentum frame or system (c.m.s.). The first equation gives the effective mass, *i.e.,* the mass-energy in the center of momentum frame. The second simply defines the center of momentum frame. If the velocity of the center of momentum frame as measured in the the Laboratory frame is v_{cm}, then

$$\vec{v}_{\text{cm}} = \frac{\vec{p}_{\text{cm}} c}{\sqrt{\vec{p}^2_{\text{cm}} + M^2_{\text{cm}} c^2}} = \frac{(\vec{p}_1 + \vec{p}_2)c}{\sqrt{(\vec{p}_1 + \vec{p}_2)^2 + M^2_{\text{cm}} c^2}} \tag{1.7.5}$$

and an appropriate boost recovers the solutions in that frame.

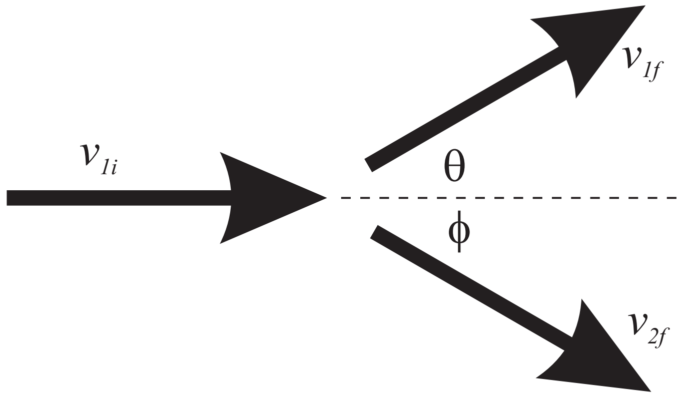

Figure 1.4: Two dimensional collision

Consider a collision in which two incident particles, "1" and "2", give rise to two outgoing particles, "3" and "4" (we label the particles differently because, in relativistic collisions, particle physics processes *may* cause one set of incoming particles to be transformed into a wholly different set of outgoing particles and we wish to allow for this possibility) and suppose particle "2" is at rest in the Laboratory frame. Let the $x-$ axis lie along the motion of "1" and let θ and ϕ be the angles made by "3" and "4" respectively with the $x-$axis, as in figure 1.4. Conserving the four momentum gives

$$m_1 + m_2 + \frac{K_1}{c^2} = m_3 + m_4 + \frac{K_3}{c^2} + \frac{K_4}{c^2}$$

$$p_1 = p_3 \cos\theta + p_4 \cos\phi$$

$$p_3 \sin\theta - p_4 \sin\phi = 0 \tag{1.7.6}$$

Note that in general

$$p^2c^2 + m^2c^4 = E^2 = (mc^2 + K)^2 \Rightarrow p^2 = 2mK + \frac{K^2}{c^2} \tag{1.7.7}$$

so these equations are to be solved for p_3, p_4 and one of the angles. As before, the other angle must be specified. Our strategy will be similar to the one we followed for non-relativistic collisions. Multiply the third equation in (1.7.6) by $\cos\phi$ and use the second to find

$$p_3 \sin\theta \cos\phi = p_4 \sin\phi \cos\phi = p_1 \sin\phi - p_3 \cos\theta \sin\phi. \tag{1.7.8}$$

This gives

$$p_3 \sin(\theta + \phi) = p_1 \sin\phi \Rightarrow p_3 = \frac{p_1 \sin\phi}{\sin(\theta + \phi)} \tag{1.7.9}$$

and inserting the result into the second equation of (1.7.6)

$$p_4 = \frac{p_1 \sin\theta}{\sin(\theta+\phi)}. \tag{1.7.10}$$

Finally, to determine θ, we must insert the above two formulae into the energy equation, but the energy equation is much more complicated than its non-relativistic counterpart! A welcome algebraic simplification occurs if the outgoing two particles fly off symmetrically, *i.e.,* with $\theta = \phi$ in the laboratory frame. If, moreover, the incident and outgoing particles have the same mass, m, then

$$p_3 = \frac{p_1}{2\cos\theta} = p_4 \tag{1.7.11}$$

and

$$E_1 + mc^2 = 2mc^2 + 2K, \quad K = \sqrt{\frac{p_1^2 c^2}{4\cos^2\theta} + m^2c^4} \tag{1.7.12}$$

yields, after a little bit of algebra,

$$\cos^2\theta = \frac{E_1^2 - m^2c^4}{c^2[(E_1 - mc^2)^2 - 4m^2c^4]}. \tag{1.7.13}$$

It is interesting to notice that in the extreme relativistic case, *i.e.,* when $E_1 \gg mc^2$, $\cos\theta \to 1$ and the separation angle approaches zero whereas, in the limit that the rest mass energy is much greater than the incident kinetic energy, $\cos\theta \to 0$ and the separation angle approaches $\pi/2$ radians. This, as we know, is the non-relativistic case.

The view from the center of momentum frame is different. Now the two particles scatter as shown in figure 1.5 because both the initial and final total spatial momentum must vanish,

$$\begin{aligned} &E_1' + E_2' = E_3' + E_4' = M_{\rm cm}c^2 \\ &\vec{p}_1' + \vec{p}_2' = 0 = \vec{p}_3' + \vec{p}_4' \Rightarrow \vec{p}_1' = -\vec{p}_2', \quad \vec{p}_3' = -\vec{p}_4'. \end{aligned} \tag{1.7.14}$$

Let the initial momenta lie along the $x-$axis and let the masses (initial and final) all be the same, say m as before. Then, because of the second equation above, $E_1' = E_2' = E_i'$ and $E_3' = E_4' = E_f'$ and because of energy conservation $2E_i' = 2E_f' = M_{\rm cm}c^2$. Therefore all momenta have the same magnitude. Since the final velocities are anti-parallel, there is only one final angle, ξ, between the outgoing particles and the $x-$axis, but there is not enough information to determine it. However, we can relate ξ to the angle θ in the Laboratory frame discussed earlier by performing a Lorentz transformation.[17]

[17] Problem: Determine the relationship between the angle of scattering, ξ, in the center of momentum frame and the angle θ in the Laboratory frame, assuming that particle "2" is initially at rest in this frame and that particles "3" and "4" leave the collision center symmetrically, as discussed.

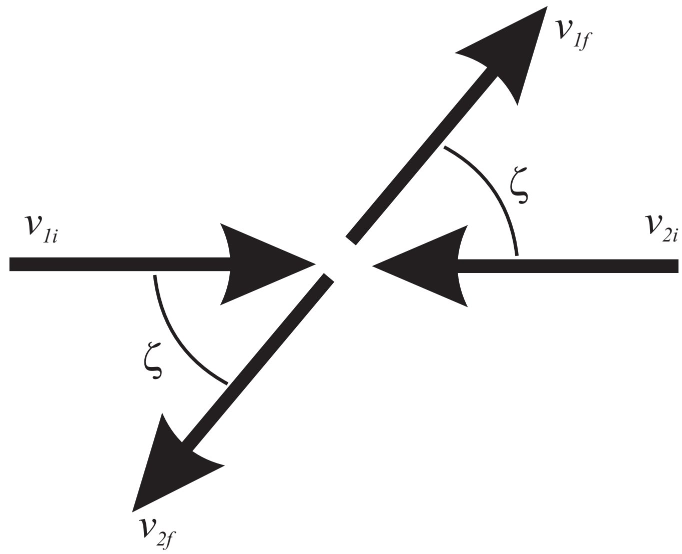

Figure 1.5: Two dimensional collision from the center of momentum frame

In general, a good amount of information about any collision in any frame is obtained directly from the Lorentz invariants. For example, if, in the Laboratory frame, our incoming particles had momenta p_{1i} and p_{2i} then, because p^2 is a Lorentz invariant, $p^2 = p^2_{\text{cm}}$ or

$$(p_{1i} + p_{2i})^2 = p^2_{1i} + p^2_{2i} + 2p_{1i} \cdot p_{2i} = p^2_{\text{cm}} \tag{1.7.15}$$

and therefore

$$(m_1^2 + m_2^2)c^4 + 2(E_{1i}E_{2i} - c^2\vec{p}_{1i} \cdot \vec{p}_{2i}) = M^2_{\text{cm}}c^4. \tag{1.7.16}$$

Using $E_i = m_i c^2 + K_i$ in each case, we arrive at

$$(m_1 + m_2)^2c^4 + 2(m_1c^2K_{2i} + m_2c^2K_{1i} + K_{1i}K_{2i} - c^2\vec{p}_{1i} \cdot \vec{p}_{2i}) = M^2_{\text{cm}}c^4, \tag{1.7.17}$$

so if particle "2" (say) is initially at rest in the Laboratory frame, then $\vec{p}_{2i} = 0$, $E_{2i} = m_2c^2$, $E_{1i} = m_1c^2 + K_{1i}$ where K_{1i} is the initial kinetic energy of particle "1" and

$$M^2_{\text{cm}}c^4 = (m_1 + m_2)^2c^4 + 2m_2c^2K_{1i}. \tag{1.7.18}$$

Thus the available center of momentum energy increases as the *square root* of the incident kinetic energy. Equation (1.7.16) holds whenever two particles collide whether or not the collision is inelastic and no matter how many the end products of the collision.

1.8 Accelerated Observers

A question of interest is how to relate an accelerated frame to an inertial one in the context of the special theory of relativity. Naturally this cannot be done directly because the Lorentz transformations only relate inertial frames. However, it can be done by considering a special one parameter family of inertial frames each of which is at rest relative to and coincident with the accelerated frame at *one particular* instant of time. Geometrically, this is equivalent to replacing the accelerated observer's curved world line in Minkowski space by a set of infinitesimal straight line segments along her world line. Each of the infinitisimal segments corresponds to an inertial frame over an infinitesimal path length. In this section we consider this problem in general and then specialize to one particular case: the "Rindler observer". Rindler observers, named after Wolfgang Rindler who first considered this problem, undergo a constant proper acceleration (recall that the proper acceleration is the acceleration of the detector w.r.t. a frame that is instantaneously at rest relative to it).

First we analyze the problem in two dimensions. Let S be an inertial frame and let $\widetilde{S}$ be the frame of the Rindler observer. $\widetilde{S}$ is not an inertial frame and cannot be directly connected to S within the context of the special theory, so introduce a one parameter family of inertial frames, $\{\overline{S}(s)\}$, each of which is instantaneously at rest relative to $\widetilde{S}$ and coincides with it at proper time s/c. If $\widetilde{S}$ possesses an acceleration, $\alpha(s)$, at s/c relative to the frame $\overline{S}(s)$ then $\alpha(s)$ is the **proper acceleration** of the Rindler observer,

$$\alpha(s) = \frac{d^2\overline{x}}{d\overline{t}^2}. \tag{1.8.1}$$

To begin with, we'll let $\alpha(s)$ be arbitrary. Now S and every member of the family $\overline{S}(s)$ are inertial frames and therefore they are related by Lorentz transformations. For a fixed s, we have

$$\begin{aligned} \overline{t} &= \gamma(t - vx/c^2) \\ \overline{x} &= \gamma(x - vt). \end{aligned} \tag{1.8.2}$$

where $v = v(s)$ is the velocity of frame $\overline{S}(s)$ relative to S. Defining the velocity $u = dx/dt$, we find

$$\overline{u} = \frac{d\overline{x}}{d\overline{t}} = \frac{(u - v)}{1 - uv/c^2} \tag{1.8.3}$$

and therefore (remember that we must keep $v(s)$ fixed because it represents the velocity of $\overline{S}(s)$ relative to S and $\overline{S}(s)$ is inertial)

$$d\overline{u} = \frac{du}{1 - uv/c^2} - \frac{(u - v)(-v/c^2 du)}{(1 - uv/c^2)^2} = \frac{du}{\gamma^2(1 - uv/c^2)^2}. \tag{1.8.4}$$

This gives

$$\overline{a} = \frac{d\overline{u}}{d\overline{t}} = \frac{a}{\gamma^3(1 - uv/c^2)^3} \tag{1.8.5}$$

Now, because the frame $\overline{S}(s)$ is instantaneously at rest relative to the the Rindler observer $\widetilde{S}$ at proper time s/c, it follows that $\overline{u}(s) = 0$ and $\overline{a}(s) = \alpha(s)$. Therefore $u = v(s)$ and[18,19]

$$\frac{a}{\left(1 - \frac{u^2}{c^2}\right)^{\frac{3}{2}}} = \overline{a} = \alpha(s). \tag{1.8.6}$$

But

$$a = \frac{du}{ds}\frac{ds}{dt} = \frac{c}{\gamma}\frac{du}{ds} \tag{1.8.7}$$

Therefore, integrating we find that

$$\int_{u_0}^{u} \frac{du}{\left(1 - \frac{u^2}{c^2}\right)} = c\tanh^{-1}(\frac{u}{c})\Big|_{u_0}^{u} = \frac{1}{c}\int_0^s ds\alpha(s) \tag{1.8.8}$$

where $u_0 = u(0)$ and $u = u(s)$, so

$$u = c\tanh\left(\frac{1}{c^2}\int_0^s ds\alpha(s) + \tanh^{-1}\frac{u_0}{c}\right) = c\tanh\eta \tag{1.8.9}$$

where we've called the argument of the hyperolic tangent on the right η. Again

$$u = \frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = \frac{c}{\gamma}\frac{dx}{ds} = \frac{c}{\cosh\eta}\frac{dx}{ds} \Rightarrow \frac{dx}{ds} = \sinh\eta \tag{1.8.10}$$

therefore

$$\begin{aligned} x - x_0 &= \int_0^s ds\sinh\eta \\ t - t_0 &= \frac{1}{c}\int_0^s ds\cosh\eta \end{aligned} \tag{1.8.11}$$

[18] Problem: Consider the same problem in four dimensions. We have seen that the relationship between the longitudinal component (*i.e.,* in the direction of the motion) of the acceleration in S and the corresponding component of the proper acceleration, is $\overline{a}_\| = \gamma^3 a_\|$. What is the relationship between the *transverse* component of the acceleration in S and the transverse component of the proper acceleration? Show that $\overline{a}_\perp = \gamma^2 a_\perp$.

[19] Problem: Using the result of the last problem together with equation (1.5.27), show that

$$\frac{d\vec{p}}{dt} = \gamma^3 m\vec{a}_\| + \gamma m\vec{a}_\perp$$

Then obtain this result directly by differentiating $\vec{p} = m\gamma\vec{v}$ w.r.t. t.

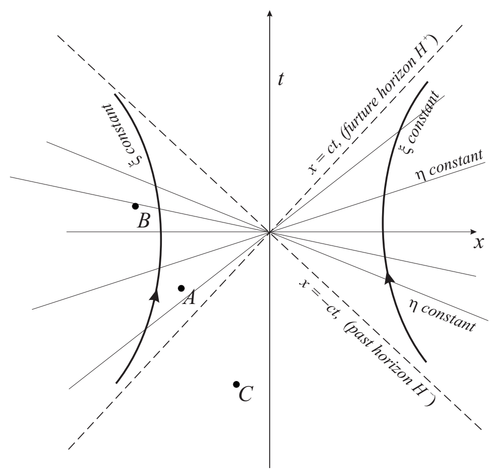

Figure 1.6: Trajectory of the Rindler observer

Without loss of generality choose $u_0 = 0$. Further, specialize to the case of a constant proper acceleration and let $\alpha(s) = a$, where $a =$ const., then $\eta = as/c^2$ and

$$x - x_0 = \frac{c^2}{a}\left[\cosh\frac{as}{c^2} - 1\right], \quad t - t_0 = \frac{c}{a}\sinh\frac{as}{c^2}. \tag{1.8.12}$$

This gives the trajectory of the accelerated observer as viewed by the inertial observer, S. Convenient initial conditions would be $x_0 = c^2/a$ at $t_0 = 0$ and we find that the trajectory may be expressed in the form

$$x^2 - c^2t^2 = \frac{c^4}{a^2}. \tag{1.8.13}$$

It is a hyperbola, shown in figure 1.6. Notice that the path of the accelerated observer may never cross the lines $x = \pm ct$. These lines represent "horizons" (past and future) that mark the boundaries of that portion of Minkowski space that is accessible to the accelerating observer. Not all of Minkowski space will be accessible to her, as is evident from the diagram in figure 1.6, where one sees that she can *never* receive information from events A and B and, while she will receive information from C, she will be unable to ascribe a time to it! The lines $x = \pm ct$ are called "Rindler horizons" because they apply only to accelerating observers. They divide Minkowski space into four causal wedges, called "Rindler wedges", defined by $x > c|t|$ (right), $x < c|t|$ (left), $ct > |x|$ (future) and $ct < |x|$ (past).

Let us consider one of these wedges. Define the coordinates

$$\xi = \frac{c^2}{2a}\ln\left[\frac{a^2}{c^4}(x^2 - c^2t^2)\right], \quad \widetilde{\eta} = \frac{c}{a}\tanh^{-1}\frac{ct}{x} \tag{1.8.14}$$

then, because a is an acceleration it follows that ξ has dimension of length and $\widetilde{\eta}$ has dimension of time. Both ξ and η range over the entire real line even though these coordinates do not cover all of Minkowski space. The inverse transformations are

$$x = \frac{c^2}{a} e^{a\xi/c^2} \cosh \frac{a\widetilde{\eta}}{c}, \quad t = \frac{c}{a} e^{a\xi/c^2} \sinh \frac{a\widetilde{\eta}}{c} \tag{1.8.15}$$

If $a > 0$, the new coordinates $(\widetilde{\eta}, \xi)$ cover only the right wedge in the (t, x) plane, *i.e.*, $x > c|t|$. They are called "Rindler" coordinates and define the "Rindler frame". In this frame, lines of constant ξ are hyperbolæ in the Minkowski frame and represent curves of constant proper acceleration equal to $ae^{-a\xi/c^2}$. The hyperbola $\xi = 0$ describes the trajectory of our particular accelerating observer and lines of constant $\widetilde{\eta}$ (time) are straight lines through the origin as shown in figure 1.6. The metric is

$$ds^2 = e^{2a\xi}(c^2 d\widetilde{\eta}^2 - d\xi^2), \quad \widetilde{\eta} \in (-\infty, \infty), \quad \xi \in (-\infty, \infty) \tag{1.8.16}$$

and the horizons are located at $\xi \to -\infty$. Another coordinatization that is often used is obtained by defining $y = c^2 e^{a\xi/c^2}/a$ (or, in terms of the original Minkowski coordinates: $y = \sqrt{x^2 - c^2t^2}$) then

$$ds^2 = \frac{a^2}{c^2} y^2 d\widetilde{\eta}^2 - dy^2, \quad \widetilde{\eta} \in (-\infty, \infty), \quad y \in (0, \infty) \tag{1.8.17}$$

gives another parametrization of Rindler space. In these coordinates, the Rindler observer is located on the (vertical) line $y = c^2/a$ and the horizons are located at $y = 0$. Notice that in both coordinatizations the horizons get defined by setting the time-time component of the metric to zero. This is a generic feature of time independent metrics.

But what exactly is a Rindler horizon and why does the coordinate system break down there? Notice that the definition of y is quite independent of a, but the definition of $\widetilde{\eta}$ depends on it therefore the proper time intervals of the observer will scale with her acceleration although proper distance does not. Thus consider Rindler observers with different proper accelerations living on vertical lines given by $y' = c^2/a'$. Notice that the greater the proper acceleration the smaller the value of y and, vice versa, the smaller the proper acceleration the greater the value of y. Now it should be clear that the proper distance between our observer and some other observer with $y = c^2/a$ will be fixed at $c^2|a' - a|/(aa')$. Think of this distance as the length of a rod connecting the two observers. If $a' > a$ then the observer with a' lies on the trailing end of the rod and, vice-versa, if $a' < a$ then the observer with a' is on the leading edge. Whereas in Galilean physics the two ends of a rod must have equal acceleration to keep the same length, in special relativity *the trailing end must accelerate a little bit faster to keep up!* This is because of length contraction: as the speed increases along the rod's length, its length also shrinks a little and the trailing end has to increase its velocity a little bit more in the same time interval to account for the shrinkage. Therefore observers on the "trailing end" *i.e.,* toward the horizon must accelerate more to "keep up". The horizon marks the stage at which the observer would need an *infinite* acceleration to keep up with the others.

Chapter 2

Scalar and Vector Fields

In theoretical mechanics, we developed a fairly sophisticated formalism to work with a system of point like particles. Such a system is described by a finite number of degrees of freedom, f, and its configuration space can be fully specified by a collection of variables, $\{q^1, \ldots, q^f\}$. The time evolution of the system is recovered by determining the "trajectories", $q^i(t)$, of these degrees of freedom and the equations that govern these trajectories are obtained from Hamilton's principle. The state of the system at any time is specified by giving the values of $q^i(t)$ and their velocities $\dot{q}^i(t)$ or momenta $p_i(t)$.

Then, when we started talking about fluid dynamics, we argued that there are situations in which it is more fruitful to think of the system of particles as "continuous". The system of particles is characterized by its Knudsen number, K, and if $K \ll 1$ then it may be deemed continuous. The natural way to think about such systems is via the concept of a "field", which is essentially a function or set of functions over space and time. For example, the description of fluids requires several "fields", *viz.,* the velocity vector field, $\vec{v}(\vec{r}, t)$, which represents the average velocity of a fluid molecule at any given point $\vec{r}$ in space at time t, the mass density field, $\rho(\vec{r}, t)$, and the pressure field, $p(\vec{r}, t)$.

The fundamental difference between a "field" and a system of point particles is that a field has an uncountably infinite number of degrees of freedom, a countable number at each event in space-time, yet all these degrees of freedom are captured by a finite set of functions. These functions are the fields that will be our concern in the rest of these notes. While fluid dynamics provides an example of a set of "material" fields, we are familiar with non-material fields as well, eg., the electromagnetic field. When examining the conservation laws for a relativistic system of particles, we argued for the need for such non-material fields: they act as intermediaries, allowing us to describe particle-particle interactions as local interactions (of the particles with the fields), which in their turn generate disturbances that propagate information about these interactions to other particles also interacting locally with the field.

There are also the field theories that describe elementary particles. In non-relativistic wave mechanics, a single particle is described by its Schroedinger wave function, $\psi(\vec{r}, t)$. Owing to the Born interpretation, which asserts that it is an amplitude for determining the probability of finding the particle in a given volume at any instant of time, the wave function itself cannot be regarded as describing a field. However, the Born interpretation fails when quantizing relativistic particles for two reasons: (a) the Born probability density is not positive definite if the wave equation is second order in time derivatives and (b) negative energy states arise generically in relativistic quantum mechanics (they are in fact required for completeness) and the energy spectrum is unbounded from below. The presence of these negative energy states implies that quantum mechanical transitions to an infinite set of lower energy states could occur, allowing, in principle, for the extraction of an unlimited amount of energy from a single particle. Moreover, Einstein's relation, $E = mc^2$, and the uncertainty principle, $\Delta E \Delta t \gtrsim \hbar$, together indicate that the number of particles within a system cannot be conserved because an arbitrary number of particles may be spontaneously produced out of the vacuum, provided that the resulting mass-energy exists for a short enough interval of time. Thus the *relativistic* wave function and wave equation, obtained via the usual Dirac procedure, cannot be understood within the Born framework and must be reinterpreted. We interpret the wave function as describing a *classical field* and the wave equation as the equation of motion for this field. In this view, the superposition principle is no longer required because the wave function, or what is now the classical field, is not endowed with the Born interpretation. Therefore the field equation need not be linear and non-linearities in the form of terms involving higher powers of the field may be added to the equation of motion. In this book we will discuss the classical dynamics of fields for which the equation of motion is derivable by Hamilton's principle from an action functional. [1]

Symmetries (such as Lorentz invariance) place strict restrictions on the possible self-consistent degrees of freedom of a physical field, but many possibilities still exist. Therefore, a theory of fields will not be a single theory but a general *framework*, containing tools for the description of experiments in which the concept of a field is useful. In this chapter, we will attempt to set up such a framework and we will do this in a way that mirrors our treatment of particles so that the powerful tools of mechanics can also be applied to great effect. However, because any field will have a finite number of degrees of freedom at *every point*, the Lagrangian, $\mathcal{L}$, of a field theory will be a functional, *i.e.,* an integral over space of some function (the Lagrange *density* function, $\mathfrak{L}$) of the fields and their derivatives.

The physical interpretation of the degrees of freedom being described by a classical field theory will vary from theory to theory. As mentioned, familiar examples are the vector

[1] Quantization of all classical fields occurs by following the standard Dirac procedure on the classical phase space of the field; the field and its conjugate momentum become quantum operators acting on a Hilbert space of states of particle number. In the quantum theory, the non-linearities get interpreted as interactions between the particles.

field representing the velocity and two scalar fields representing the density and pressure in fluid dynamics, and the four vector potential of electrodynamics. Fields representing other degrees of freedom may be constructed as well, as seen by the following example. An infinitely long elastic rod, laid along the $x-$axis, is able to sustain oscillatory displacements of its particles in a direction parallel to the rod. We "discretize" the rod by imagining that it is made up of a very large number of point like particles that are spaced a distance "a" apart and connected by massless springs (which simulate the interaction between neighboring atoms in the rod). Consider the nth particle and denote it's displacement from equilibrium by η_n. Its kinetic energy is

$$T_n = \frac{1}{2} m_n \dot{\eta}_n^2 \tag{2.0.1}$$

giving, for all the particles together, the total kinetic energy

$$T = \frac{1}{2} \sum_n m_n \dot{\eta}_n^2 \tag{2.0.2}$$

The total potential energy could be be written as the sum of the potential energies of the springs which are stretched (or compressed) according to the displacements of neighboring particles:

$$V = \frac{1}{2} \sum_n k_n (\eta_{n+1} - \eta_n)^2 \tag{2.0.3}$$

The Lagrangian for the entire system follows from Hamilton's prescription:

$$\mathcal{L} = T - V = \frac{1}{2} \sum_n m_n \left[\dot{\eta}_n^2 - \omega_n^2 (\eta_{n+1} - \eta_n)^2 \right] \tag{2.0.4}$$

which will be convenient to write in the following form

$$\mathcal{L} = \frac{1}{2} \sum_n a \left(\frac{m_n}{a} \right) \left[\dot{\eta}_n^2 - (a\omega_n)^2 \left(\frac{\eta_{n+1} - \eta_n}{a} \right)^2 \right] \tag{2.0.5}$$

Now consider taking the limit as $a \to 0$ of the above. The quantity m_n/a can be interpreted as the mass per unit length or the linear mass density of the rod, which we take to be some constant, μ (independent of position). It is clear that in the same limit,

$$\lim_{a\to 0} \frac{\eta_{n+1} - \eta_n}{a} = \lim_{a\to 0} \frac{\eta(t, x+a) - \eta(t, x)}{a} = \frac{\partial}{\partial x} \eta(t, x) \tag{2.0.6}$$

and that

$$\dot{\eta}_n \to \frac{\partial}{\partial t} \eta(t, x), \tag{2.0.7}$$

but what about $a\omega_n^2$? A little thought will show that this is related to the Young's modulus, Y, of the rod by

$$am_n\omega_n^2 = \frac{Y}{2} \tag{2.0.8}$$

which we'll take to be independent of position as well. Finally, we let

$$\sum_n a\,(\ldots)_n \;\;\to\;\; \int dx\,(\ldots)\,(t,x) \tag{2.0.9}$$

and thus, in the continuum limit, the Lagrangian functional can be written as

$$\mathcal{L}(\dot\eta,\eta',x) = \int_{-\infty}^{\infty} dx\left[\frac{\mu}{2}\left(\frac{\partial\eta(t,x)}{\partial t}\right)^2 - \frac{Y}{2}\left(\frac{\partial\eta(t,x)}{\partial x}\right)^2\right] \tag{2.0.10}$$

The quantity within square brackets is a density function (in this case a *linear* density) called the Lagrange density, $\mathfrak{L}$, and

$$\mathcal{S} = \int dt \int_{-\infty}^{\infty} dx\;\mathfrak{L}(\dot\eta(t,x),\eta'(t,x),x,t). \tag{2.0.11}$$

is the action. $\eta(t,x)$ is a "field", which may be thought of as the field of displacements from equilibrium of the rod's constituents, the elementary excitations of the rod, or "phonons" in one dimension.

2.1 Hamilton's Principle

We will take a general field theory to be one that is described by a Lagrangian *functional*,

$$\mathcal{L}[\phi^A,\partial_\mu\phi^A,t] = \int d^3\vec r\;\mathfrak{L}(\phi^A(t,\vec r),\partial_\mu\phi^A(t,\vec r),t,\vec r) \tag{2.1.1}$$

where $\phi^A(t,\vec r)$ denotes a field, which, we assume, exhibits definite space-time transformation properties and carries indices both related and possibly unrelated to space-time, all of which we collectively denote by A. If the field theory is Lorentz invariant then the action

$$\mathcal{S} = \int dt\;\mathcal{L}[\phi^A,\partial_\mu\phi^A,t] = \int d^4x\;\mathfrak{L}(\phi^A(x),\partial_\mu\phi^A(x),x) \tag{2.1.2}$$

must be a Lorentz scalar. Moreover, because actions must always have the mechanical dimension ml^2/t, it follows that the Lagrange functional, $\mathcal{L}$, will have the dimension of energy and the Lagrange density will have the dimension of energy density, $[\mathfrak{L}] \sim m/lt^2$.

The fields may transform under any finite dimensional representation of the Lorentz group, *i.e.,* as scalars (eg. the Higgs field), as vectors (eg. the electromagnetic field and other gauge fields of the standard model), as spinors (eg. Dirac/Weyl/Majorana fields for fermions) or as tensors (eg. gravitational field). In the case of a vector or tensor field, the index "A" would be one or more space-time indices. For example, for the electromagnetic field, A_μ, the superscript "A" represents the covariant space-time index μ and for the gravitational field, $g_{\mu\nu}$, "A" would represent the pair of covariant space-time indices $(\mu\nu)$. Likewise "A" could be a spinor index if ϕ^A is a spinor field, or it could even represent an internal index or a combination of space-time indices and internal indices if the field transforms according to some representation of an internal group in addition to the Lorentz group as, for example, "A" is the pair $\binom{a}{\mu}$ if ϕ^A represents the non-abelian gauge field A^a_μ, which we will describe in some detail shortly. We will assume that the equations of motion are to be derived from the action (2.1.2) using Hamilton's variational principle.

Following the standard arguments for point particles, we realize at the onset that field variations must also come in two types: (a) variations in the functional form of $\phi^A(x)$,

$$\phi^A(x) \to \phi'^A(x) = \phi^A(x) + \delta_0\phi^A(x) \tag{2.1.3}$$

and (b) variations that arise because of a change in the arguments,

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) \tag{2.1.4}$$

where ϵ^μ represents an infinitesimal change in x^μ. This induces a change in the field according to

$$\begin{aligned} \phi^A(x) &\to \phi^A(x') = \phi^A(x+\epsilon) = \phi^A(x) + \epsilon\cdot\partial\phi^A(x), \\ \delta_1\phi^A(x) &= \phi^A(x') - \phi^A(x) = \epsilon\cdot\partial\phi^A \end{aligned} \tag{2.1.5}$$

In general, a change is made up of both components, *i.e.*,

$$\begin{aligned} \phi^A(x) &\to \phi'^A(x') = \phi'^A(x+\epsilon) = \phi(x) + \epsilon\cdot\partial\phi^A + \delta_0\phi^A \\ \delta\phi^A(x) &= \phi'^A(x') - \phi^A(x) = \delta_0\phi^A(x) + \delta_1\phi^A(x) \end{aligned} \tag{2.1.6}$$

up to first order in the variations, of course. To apply Hamilton's principle to the action we first consider only functional variations that vanish at the boundary (usually taken to be at infinity). Then, using Einstein's summation convention for all indices,

$$\begin{aligned} \delta_0 S &= \int d^4x\,\delta\mathfrak{L} = \int d^4x \left[\frac{\partial\mathfrak{L}}{\partial\phi^A}\delta_0\phi^A + \frac{\partial\mathfrak{L}}{\partial(\partial_\mu\phi^A)}\delta_0\partial_\mu\phi^A\right] \\ &= \int d^4x \left[\frac{\partial\mathfrak{L}}{\partial\phi^A}\delta_0\phi^A + \partial_\mu\left(\frac{\partial\mathfrak{L}}{\partial(\partial_\mu\phi^A)}\delta_0\phi^A\right) - \partial_\mu\left(\frac{\partial\mathfrak{L}}{\partial(\partial_\mu\phi^A)}\right)\delta_0\phi^A\right] \end{aligned}$$

$$= \int d^4x \left[\frac{\partial \mathfrak{L}}{\partial \phi^A} - \partial_\mu \left(\frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi^A)} \right) \right] \delta_0 \phi^A \tag{2.1.7}$$

where we have interchanged ∂_μ and δ_0, because $[\partial_\mu, \delta_0]\phi^A = 0$, and used the fact that the integral of the total derivative is a surface term that vanishes exactly,

$$\int_{\mathcal{M}} d^4x \, \partial_\mu \left(\frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi^A)} \delta_0 \phi^A \right) = \int_{\partial \mathcal{M}} d\sigma_\mu \left(\frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi^A)} \delta_0 \phi^A \right) \equiv 0 \tag{2.1.8}$$

by our condition that the variation $\delta_0\phi$ vanishes there. But, as $\delta_0\phi$ is otherwise arbitrary, and the action is stationary ($\delta S = 0$), we necessarily arrive at Euler's equations

$$\frac{\partial \mathfrak{L}}{\partial \phi^A} - \partial_\mu \left(\frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi^A)} \right) = 0 \tag{2.1.9}$$

for a field theory governed by the Lagrange density $\mathfrak{L}$. Of course, Lagrange multipliers and the Euler-Lagrange equations may be used when constraints are involved. The left hand side of (2.1.9) is called the **Euler derivative** of $\mathfrak{L}$.

For example, if we apply these equations to the action we wrote down earlier for the elastic rod. The field is $\eta(t, x)$ and the Lagrange density does not depend on $\eta(t, x)$ but only on its derivatives. Therefore we have

$$\partial_t \left(\frac{\partial \mathfrak{L}}{\partial \dot{\eta}} \right) + \partial_x \left(\frac{\partial \mathfrak{L}}{\partial \eta'} \right) = 0 = \mu \frac{\partial^2 \eta}{\partial t^2} - Y \frac{\partial^2 \eta}{\partial x^2} \tag{2.1.10}$$

which is a wave equation for perturbations that travel at the speed of sound, $v_s = \sqrt{\frac{Y}{\mu}}$, in the rod.

2.2 Noether's Theorems

A symmetry of the action is a set of transformations of the fields and (or) the coordinates that leave the action invariant. Symmetries that arise because of the way in which the action is formulated (eg. reparametrization invariance of the free relativistic particle action) are called **non-dynamical** and symmetries that result from a specific feature or property of the matter or its evolution (eg. gauge invariance or general coordinate invariance) are called **dynamical**.

Noether's theorems explore the consequences of dynamical symmetries. The approach follows the general lines of reasoning that were introduced when the number of degrees of freedom was finite (*i.e.,* for point particles), but they are considerably more powerful for fields. Let the fields $\phi^A(x)$ and coordinates x^μ undergo the following variations

$$\delta\phi^A = G^A_a \delta\omega^a + T^{A\mu}_a \partial_\mu \delta\omega^a + \ldots, \quad \delta x^\mu = \epsilon^\mu = G^\mu_a \delta\omega^a + \ldots \tag{2.2.1}$$

under a set of symmetry transformations with differentiable parameters $\delta\omega^a(x)$. One can, of course, conceive of fields whose transformations involve even higher derivatives of $\delta\omega^a(x)$, but we will not have occasion to do so in these notes so we restrict our attention to cases for which only $G_a^A(x)$, $G_a^\mu(x)$ and $T_a^{A\mu}(x)$ are non-vanishing. The transformations will lead to a variation of the action according to

$$\delta S = \int [(\delta d^4x)\mathfrak{L} + d^4x\delta\mathfrak{L}]. \tag{2.2.2}$$

We evaluate each term above separately. The first results exclusively from the change in coordinates

$$x'^\mu = x^\mu + \epsilon^\mu(x) \quad \rightarrow \quad \frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu\epsilon^\mu(x) \tag{2.2.3}$$

so the change in measure, δd^4x is determined via the Jacobian

$$d^4x \rightarrow d^4x' = d^4x \det\left|\frac{\partial x'^\mu}{\partial x^\nu}\right| = d^4x \det(\delta^\mu_\nu + \partial_\nu\epsilon^\mu) = d^4x \det\widehat{J} \tag{2.2.4}$$

where the "hat" is used to denote a matrix when its component indices are suppressed, *i.e.,* the components of $\widehat{J} = \widehat{1+\partial\epsilon}$ are $J^\mu{}_\nu = \delta^\mu_\nu + \partial_\nu\epsilon^\mu$. But the determinant of a matrix $\widehat{J}$ is related to its trace according to[2]

$$\ln\det\widehat{J} = \operatorname{tr}\ln\widehat{J} \tag{2.2.5}$$

so that

$$\det\widehat{J} = e^{\operatorname{tr}\ln\widehat{J}} = e^{\operatorname{tr}\ln(\widehat{1+\partial\epsilon})} \approx e^{\operatorname{tr}\widehat{\partial\epsilon}} \approx 1 + \partial_\mu\epsilon^\mu \tag{2.2.6}$$

and it follows that

$$\delta d^4x = d^4x' - d^4x = d^4x(\partial\cdot\epsilon) \tag{2.2.7}$$

and therefore

$$\begin{aligned}\delta S &= \int d^4x[(\partial\cdot\epsilon)\mathfrak{L} + \delta\mathfrak{L}] = \int d^4x[(\partial\cdot\epsilon)\mathfrak{L} + \epsilon\cdot\partial\mathfrak{L} + \delta_0\mathfrak{L}] \\ &= \int d^4x\left[\partial_\mu(\mathfrak{L}\epsilon^\mu) + \frac{\partial\mathfrak{L}}{\partial\phi^A}\delta_0\phi^A + \frac{\partial\mathfrak{L}}{\partial(\partial_\mu\phi^A)}\delta_0\partial_\mu\phi^A\right]\end{aligned} \tag{2.2.8}$$

Interchanging δ_0 and ∂_μ ($[\delta_0, \partial_\mu]\phi = 0$), we find

$$\delta S = \int d^4x\left[\partial_\mu(\mathfrak{L}\epsilon^\mu) + \frac{\partial\mathfrak{L}}{\partial\phi^A}\delta_0\phi^A + \partial_\mu\left(\frac{\partial\mathfrak{L}}{\partial(\partial_\mu\phi^A)}\delta_0\phi^A\right) - \partial_\mu\left(\frac{\partial\mathfrak{L}}{\partial(\partial_\mu\phi^A)}\right)\delta_0\phi^A\right] \tag{2.2.9}$$

[2] This is easy to prove: let λ_n be the eigenvalues of $\hat{J}$, so

$$\det\widehat{J} = \prod_n \lambda_n \rightarrow \ln\det\widehat{J} = \ln(\prod_n \lambda_n) = \sum_n \ln\lambda_n = \operatorname{tr}\ln\widehat{J}$$

which simplifies to

$$\delta S = \int d^4x \left\{ \mathcal{E}_A \delta_0 \phi^A + \partial_\mu \left[\mathfrak{L} \epsilon^\mu + \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi^A)} \delta_0 \phi^A \right] \right\} \tag{2.2.10}$$

where $\mathcal{E}_A$ are the Euler derivatives of $\mathfrak{L}$ as given by the left hand side of (2.1.9). Exchange the functional variation of $\phi^A(x)$ for a total variation and write the variation of the action as

$$\delta S = \int d^4x \left\{ \mathcal{E}_A (\delta\phi^A - \epsilon \cdot \partial\phi^A) + \partial_\mu \left[\left(\mathfrak{L} \delta^\mu_\nu - \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi^A)} \partial_\nu \phi^A \right) \epsilon^\nu + \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi^A)} \delta\phi^A \right] \right\} \tag{2.2.11}$$

Using (2.2.1), the first term above may be rewitten as

$$\begin{aligned} & \int d^4x \; \mathcal{E}_A \left\{ (G^A_a - G^\mu_a \partial_\mu \phi^A) \delta\omega^a + T^{A\mu}_a \partial_\mu \delta\omega^a \right\} \\ = \quad & \int d^4x \; \left\{ [\mathcal{E}_A (G^A_a - G^\mu_a \partial_\mu \phi^A) - \partial_\mu (\mathcal{E}_A T^{A\mu}_a)] \delta\omega^a + \partial_\mu (\mathcal{E}_A T^{A\mu}_a \delta\omega^a) \right\} \end{aligned} \tag{2.2.12}$$

after an integration by parts. Therefore

$$\begin{aligned} \delta S \; = \; & \int d^4x \; [\mathcal{E}_A (G^A_a - G^\mu_a \partial_\mu \phi^A) - \partial_\mu (\mathcal{E}_A T^{A\mu}_a)] \delta\omega^a \\ & + \int d^4x \; \partial_\mu \left[\left(\mathfrak{L} \delta^\mu_\nu - \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi^A)} \partial_\nu \phi^A \right) \epsilon^\nu + \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi^A)} \delta\phi^A + \mathcal{E}_A T^{A\mu}_a \delta\omega^a \right] \end{aligned} \tag{2.2.13}$$

Because these are symmetry transformations, $\delta S = 0$. This is only possible if the current

$$j^\mu = \left[\left(\mathfrak{L} \delta^\mu_\nu - \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi^A)} \partial_\nu \phi^A \right) G^\nu_a + \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi^A)} G^A_a + \mathcal{E}_A T^{A\mu}_a \right] \delta\omega^a + \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi^A)} T^{A\nu}_a \partial_\nu \delta\omega^a \tag{2.2.14}$$

(called the **Noether current**) is conserved and, because the parameters are arbitrary, the identities

$$\mathcal{E}_A (G^A_a - G^\mu_a \partial_\mu \phi^A) - \partial_\mu (\mathcal{E}_A T^{A\mu}_a) = 0 \tag{2.2.15}$$

(called the **Noether identities**) are satisfied. Both must hold true regardless of whether or not the equations of motion are satisfied. Thus we have two (Noether) theorems:

- *To every differentiable symmetry of an action there exists a conserved current, an example of which is given in* (2.2.14), and
- *To every differentiable symmetry of an action there exists a set of differential identities, an example of which is given in* (2.2.15).

The existence of a conserved Noether current implies the existence of a conserved **Noether charge**; we see this by integrating the conservation equation for j^μ over a volume, V,

$$\frac{d}{dt}\int_V d^3\vec{r}\; j^0 = -\int_V d^3\vec{r}\;\vec{\nabla}\cdot\vec{j} = -\int_S dS\;\widehat{n}\cdot\vec{j} \tag{2.2.16}$$

by Gauss' theorem, where S is the surface bounding the volume V. If the fields and their first derivatives vanish on S, or, if S is taken to infinity and they decrease rapidly enough as infinity is approached, then the right hand side is zero and we find that the charge

$$Q = \int_V d^3\vec{r}\; j^0 \tag{2.2.17}$$

is conserved,

$$\frac{dQ}{dt} = 0. \tag{2.2.18}$$

The Noether current can be modified by the addition of any divergence free set of vectors Θ^μ constructed out of the fields and their derivatives, so it is not unique and, as a consequence, the Noether charge is also not unique. The differential identities of Noether's second theorem are trivial "on-shell" *i.e.,* on classical solutions of the field equations ($\mathcal{E}_A = 0$), so Euler's equations can be viewed as a particular solution of the Noether identities. What is important is that the identities are required to hold "off-shell" as well.

2.3 Lorentz Transformations and Translations

As all actions are required to be Lorentz scalars, one of the most basic applications of Noether's first theorem is to global Lorentz transformations and space-time translations (the group of transformations ISO(3,1)). These we take to be of the form $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$, where

$$\delta x^\mu = \epsilon^\mu = \delta\omega^\mu{}_\nu x^\nu + \delta a^\mu. \tag{2.3.1}$$

The first component of the transformations are the Lorentz transformations[3] and the second component are constant translations. Suppose that under these transformations

$$\delta\phi^A = G^A_{\alpha\beta}\delta\omega^{\alpha\beta} \tag{2.3.2}$$

[3]Problem: Consider an infinitesimal Lorentz transformation

$$x^\mu \to x'^\mu = (\delta^\mu{}_\nu + \delta\omega^\mu{}_\nu)x^\nu$$

and expand $x'^2 = x'^\mu x'_\nu$ to first order in $\delta\omega^\mu{}_\nu$. Because x^2 is a scalar we must have $x'^2 = x^2$. Show that this implies that $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

(since fields are Lorentz scalars, vectors or tensors, they must be invariant under constant translations), then if we define

$$\Theta^\mu{}_\nu = \mathfrak{L}\delta^\mu{}_\nu - \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi^A)}\partial_\nu \phi^A \tag{2.3.3}$$

and also

$$S^\mu{}_{\alpha\beta} = \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi^A)} G^A_{\alpha\beta}, \tag{2.3.4}$$

the conserved Noether current in (2.2.14) can be expressed as

$$j^\mu = \Theta^\mu{}_\nu \delta a^\nu + \left(\frac{1}{2}\left[\Theta^\mu{}_\alpha x_\beta - \Theta^\mu{}_\beta x_\alpha\right] + S^\mu{}_{\alpha\beta}\right)\delta\omega^{\alpha\beta} \tag{2.3.5}$$

where we have antisymmetrized the second term because $\delta\omega^{\alpha\beta}$ is antisymmetric; $\widehat{S}$ is already antisymmetric in the pair (α, β), by definition. Evidently, by considering pure translations ($\delta\widehat{\omega} = 0$) it follows that $\widehat{\Theta}$ is conserved. Likewise, the antisymmetric tensor

$$M^\mu{}_{\alpha\beta} = \frac{1}{2}(\Theta^\mu{}_\alpha x_\beta - \Theta^\mu{}_\beta x_\alpha) + S^\mu{}_{\alpha\beta}, \tag{2.3.6}$$

is also a conserved current by invariance under pure boosts and rotations ($\delta a^\mu = 0$).

We have obtained two important conserved quantities. The (mixed) second rank tensor, $\Theta^\mu{}_\nu$ is called the **canonical energy momentum** or **canonical stress energy** tensor of the field.[4] The rank three tensor $M^\mu{}_{\alpha\beta}$ is called the **total angular momentum** tensor density of the field and it is made up of two parts, *viz.,* the **orbital angular momentum** tensor density,

$$L^\mu{}_{\alpha\beta} = \frac{1}{2}(\Theta^\mu{}_\alpha x_\beta - \Theta^\mu{}_\beta x_\alpha) \tag{2.3.7}$$

and the **intrinsic angular momentum** tensor density, $S^\mu{}_{\alpha\beta}$. From the conservation of $M^\mu{}_{\alpha\beta}$ follows

$$\partial_\mu S^\mu{}_{\alpha\beta} = \frac{1}{2}(\Theta_{\alpha\beta} - \Theta_{\beta\alpha}) \tag{2.3.8}$$

and so $S^\mu{}_{\alpha\beta}$ is separately conserved if $\widehat{\Theta}$ is a symmetric tensor. Otherwise the intrinsic angular momentum of the field is not separately conserved and may be exchanged in favor of orbital angular momentum and vice-versa. The extent to which this occurs is measured by the antisymmetric part of the canonical energy momentum tensor.

[4]Problem: Compute the stress-energy tensor of the field $\eta(t, x)$, representing the excitations of an elastic rod.

We may define a 4-vector that is identified with the energy and momentum of the field,[5] when $\Theta^{\mu\nu}$ is a symmetric tensor,

$$\mathrm{P}^{\mu} = \int d^3\vec{r}\ \Theta^{0\mu} \tag{2.3.9}$$

($\wp^{\mu} = \Theta^{0\mu}$ is the momentum density of the field). Conservation of the Noether charge means that the four quantities associated with space-time translations, P^{μ}, are conserved,

$$\frac{d\mathrm{P}^{\mu}}{dt} = 0, \tag{2.3.10}$$

if the fields fall off rapidly enough at infinity or if they (together with their derivatives) vanish at the boundaries. Likewise, we define

$$\mathrm{M}_{\alpha\beta} = \int d^3\vec{r}\ M^0{}_{\alpha\beta} \tag{2.3.11}$$

whose spatial components are associated with the total angular momentum of the field. Again, conservation of charge means that the six quantities associated with boosts and rotations, $M^0{}_{\alpha\beta}$, are conserved,

$$\frac{d}{dt}\mathrm{M}_{\alpha\beta} = 0, \tag{2.3.12}$$

under the same conditions.

It turns out that $\Theta^{\mu\nu}$ is not always symmetric in its indices. If it is not symmetric it would not make a lot of sense to define the momentum four vector as above. However, we can exploit the fact that it is also not uniquely defined as a conserved quantity because we could add to it *any* divergence free, second rank tensor. In particular, the divergence, $\partial_\lambda k^{\lambda\mu\nu} = \Delta^{\mu\nu}$, of a third rank tensor, $k^{\lambda\mu\nu}$, which is antisymmetric in the two indices (λ, μ) would satisfy this condition because partial derivatives commute whereas $\widehat{k}$ is antisymmetric by construction,

$$\partial_\mu \Delta^{\mu\nu} = \partial_\mu \partial_\lambda k^{\lambda\mu\nu} = 0. \tag{2.3.13}$$

Therefore, if

$$t^{\mu\nu} = \Theta^{\mu\nu} + \Delta^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda k^{\lambda\mu\nu} \tag{2.3.14}$$

then $t^{\mu\nu}$ is still conserved.

Suppose that by doing this we could find a tensor, $t^{\mu\nu}$, that is symmetric and conserved. Then, notice that

$$\int d^3\vec{r}\ t^{0\mu} = \overline{\mathrm{P}}_{\mu} = \int d^3\vec{r}\ \Theta^{0\mu} + \int d^3\vec{r}\ \partial_i k^{i0\mu} = \int d^3\vec{r}\ \Theta^{0\mu} + \oint_{\Sigma} d^2\Sigma\ \widehat{n}_i\ k^{i0\mu}, \tag{2.3.15}$$

[5] Problem: Show that P^{μ} transforms as a 4-vector

where Σ is a two dimensional boundary of the spatial region over which we are integrating and $\widehat{n}^i$ is its unit normal. If the $k^{i0\mu}$ vanishes on this boundary or falls off sufficiently fast at infinity to make the second integral vanish then there is no difference between the *total* momentum defined using $\Theta^{0\mu}$ and the one defined using $t^{0\mu}$; the momentum 4-vector could be defined in terms of either energy momentum tensor. However, the energy momentum densities would differ.

Exploiting (2.3.8), one can construct a candidate for $k^{\lambda\mu\nu}$ from the intrinsic angular momentum tensor density as[6]

$$k^{\lambda\mu\nu} = -\left(S^{\lambda\mu\nu} + S^{\mu\nu\lambda} + S^{\nu\mu\lambda}\right), \tag{2.3.16}$$

and the resulting tensor $t^{\mu\nu} = \Theta^{\mu\nu} + \Delta^{\mu\nu}$ will be both symmetric and conserved.[7] Modified in this way, the tensor, $t^{\mu\nu}$, is called the **Belinfante-Rosenfeld** stress-energy tensor. Given $t^{\mu\nu}$, define the modified orbital angular momentum tensor density

$$\widetilde{L}^{\mu}{}_{\alpha\beta} = \frac{1}{2}(t^{\mu}{}_{\alpha}x_{\beta} - t^{\mu}{}_{\beta}x_{\alpha}). \tag{2.3.17}$$

and the modified spin angular momentum tensor density

$$\widetilde{S}^{\mu}{}_{\alpha\beta} = S^{\mu}{}_{\alpha\beta} - \frac{1}{2}(\Delta^{\mu}{}_{\alpha}x_{\beta} - \Delta^{\mu}{}_{\beta}x_{\alpha}), \tag{2.3.18}$$

so that the conserved total angular momentum tensor density can be written in terms of these modified tensor densities,

$$M^{\mu}{}_{\alpha\beta} = \widetilde{L}^{\mu}{}_{\alpha\beta} + \widetilde{S}^{\mu}{}_{\alpha\beta}. \tag{2.3.19}$$

It is straightforward that $\widetilde{L}^{\mu}{}_{\alpha\beta}$ is separately conserved ($\partial_\mu \widetilde{L}^{\mu}{}_{\alpha\beta} = 0$) because $t^{\mu\alpha}$ is conserved and symmetric. Therefore $\widetilde{S}^{\mu}{}_{\alpha\beta}$ is also separately conserved ($\partial_\mu \widetilde{S}^{\mu}{}_{\alpha\beta} = 0$). Moreover, we notice that $\widetilde{S}^{\mu}{}_{\alpha\beta}$ is itself a divergence,

$$\widetilde{S}^{\mu\alpha\beta} = \partial_\lambda Z^{\lambda\mu\alpha\beta} \tag{2.3.20}$$

where $Z^{\lambda\mu\{\alpha\beta\}} = 0 = Z^{\{\lambda\mu\}\alpha\beta}$.[8]

We can now define the modified orbital angular momentum tensor,

$$\widetilde{\mathrm{L}}_{\mu\nu} = \int d^3\vec{r}\ \widetilde{L}^{0}{}_{\mu\nu} = \int d^3\vec{r}\ (x_\mu \wp_\nu - x_\nu \wp_\mu), \tag{2.3.21}$$

[6] Problem: Show that $k^{\lambda\mu\nu}$ is antisymmetric in $(\lambda\mu)$: $k^{\{\lambda\mu\}\nu} = k^{\lambda\mu\nu} + k^{\mu\lambda\nu} = 0$.
[7] Problem: Show that $t^{\mu\nu}$ is a symmetric tensor: $t^{[\mu\nu]} = t^{\mu\nu} - t^{\nu\mu} = 0$
[8] Problem: Determine $Z^{\lambda\mu\alpha\beta}$ and prove that it satisfies the given symmetry properties.

where $\wp_\mu = t_{0\mu}$ is the momentum density. Then the orbital angular momentum is conserved

$$\frac{d}{dt}\widetilde{\mathrm{L}}_{\mu\nu} = 0, \tag{2.3.22}$$

if the fields fall off fast enough at infinity or vanish on the boundary. Equation (2.3.21) is reminiscent of the way in which ordinary orbital angular momentum is defined in point particle mechanics. Likewise, defining

$$\widetilde{\mathrm{S}}_{\mu\nu} = \int d^3\vec{r}\ \widetilde{S}^0{}_{\mu\nu} \tag{2.3.23}$$

we find that

$$\frac{d}{dt}\widetilde{\mathrm{S}}_{\mu\nu} = 0, \tag{2.3.24}$$

provided the same conditions hold. We will now apply these concepts to some field theories of elementary particles.

2.4 The Klein-Gordon Equation

The earliest attempt at a relativistic quantum mechanics was made by Oskar Klein and Walter Gordon, who proposed a relativistic version of the Schroedinger equation in order to describe relativistic electrons. They went about determining the wave equation by applying the quantization procedure used in non-relativistic quantum mechanics, $p_i \to -i\hbar\partial_i$, $\mathcal{H} = E \to i\hbar\partial_t$ directly to the relativistic constraint, $p^2 + m^2c^2 = 0$. The resulting operator was taken to annihilate the wave function and they proposed[9]

$$\left(\Box_x + \frac{m^2c^2}{\hbar^2}\right)\phi(x) = 0 \tag{2.4.1}$$

where $\Box_x$ is the four dimensional Laplacian. The equation is manifestly Lorentz invariant when $\phi(x)$ is taken to be a scalar function, *i.e.*, $\phi(x) \to \phi'(x') = \phi(x)$ (note that this means that $\delta_0\phi = -\delta_1\phi = -\epsilon\cdot\partial\phi$). The wave function is complex, and if we multiply this equation from the left by its complex conjugate, $\phi^*(x)$, we find

$$\phi^*\Box_x\phi - \phi\Box_x\phi^* = 0 \ \Rightarrow\ \partial_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) \stackrel{\text{def}}{=} \partial_\mu(\phi^*\overleftrightarrow{\partial^\mu}\phi) = 0, \tag{2.4.2}$$

which has the form of a continuity equation with $j^\mu = \alpha\phi^*\overleftrightarrow{\partial^\mu}\phi$ serving as a four vector current density (α is any constant). Separating the space and time derivatives in the last equation above,

$$\partial_\mu j^\mu = \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \tag{2.4.3}$$

[9] Problem: Starting from the relation $p^2 = -m^2c^2$ and the replacement rule $p_\mu \to -i\hbar\partial_\mu$, of ordinary non-relativistic quantum mechanics, derive the Klein-Gordon equation.

where $\rho = j^0$ should represent the Born probability density associated with the wave function.

As we will see shortly, the mechanical dimension of the wave function must be $[\phi] = \sqrt{ml}/t$. If ρ is to represent a probability density, $[\rho] = l^{-3}$, then $[\alpha] = t/ml^2$ and we take take $\alpha = -i/\hbar$,

$$\rho = \frac{i}{\hbar c^2}\phi^* \overleftrightarrow{\partial_t}\phi, \quad \vec{j} = -\frac{i}{\hbar}\phi^* \overleftrightarrow{\nabla}\phi, \tag{2.4.4}$$

so as to recover the correct non-relativistic limit for the probability current. Recalling from ordinary quantum mechanics that the inner product must be commensurate with the probability density, j^0, we define it according to (2.4.4),

$$\langle \phi, \phi' \rangle = \frac{i}{\hbar c^2}\int d^3\vec{r}\ \left(\phi^*(x)\overleftrightarrow{\partial_t}\phi'(x)\right), \tag{2.4.5}$$

but there are two problems with this interpretation, both related to the fact that the equation is second order in time:

- The probability density ρ is not positive definite, a feature that is necessary for the Born interpretation. This reflects the fact that the equation is second order in the time derivative. (It also means that we must specify both ϕ and $\dot{\phi}$ everywhere at some initial time, say $t = 0$.)

- There are positive and negative energy solutions and both are required for completeness, but negative energy states can lead to an energy spectrum that is not bounded from below. In a classical theory, this is not a problem because there is a mass gap, *i.e.,* an energy region, between $-mc^2$ and $+mc^2$, that is forbidden. As energy can only change continuously in a classical theory, if we simply *assume* that the universe began with all particles in states with $E > +mc^2$ then it will be impossible for particles to transition to negative energy states. Even in the quantum theory, it is not a problem so long as the scalar particles are free. However, if the particle interacts with, say, an electromagnetic field, this can cause problems. In quantum mechanics, changes in observables (in particular energy) can be discontinuous, so even if the universe began with all particles in positive energy states, transitions from positive to negative energy states can occur. In this way we could, in principle, indefinitely extract energy from a system and use the extracted energy to do work, but observations suggest that is impossible.

These problems are circumvented if we think of $\phi(x)$ as a classical scalar *field* described by the Klein-Gordon equation, instead of as the wave function of a relativistic particle. On the classical level, we avoid the appearance of negative energies as the energy and momentum densities of the field are given by the energy momentum tensor and the energy density of the *field* is not negative. When the field is quantized multi-particle states appear. As we mentioned in the introduction to this chapter, this is to be expected in a relativistic theory because the Einstein relation $E = mc^2$ together with the uncertainty principle, $\Delta E \Delta t \gtrsim \hbar$, permits multi-particle states to be created out of the vacuum, so long as they exist for short enough times. Thus particle quantum mechanics gets replaced by multi-particle relativistic *quantum field* theory.

2.4.1 Mode Expansion

A general solution to the free Klein-Gordon equation with no boundary surfaces can be given as

$$\phi(x) = \int d^4k \; a(k) e^{ik\cdot x} \delta(k^2 + m^2c^2/\hbar^2), \tag{2.4.6}$$

where $k_\mu = (-\omega, \vec{k}) = p_\mu/\hbar$. Expanding, absorbing constants into $a(k)$ and separating the space and time components,

$$\phi(t, \vec{r}) = \int_{-\infty}^{\infty} d\omega \int \frac{d^3\vec{k}}{2\omega_k} \; a(\omega, \vec{k}) e^{i(\vec{k}\cdot x - \omega t)} \left[\delta(\omega - \omega_k) + \delta(\omega + \omega_k)\right] \tag{2.4.7}$$

where $\omega_k = +\sqrt{\vec{k}^2 c^2 + m^2c^4/\hbar^2}$. The second $\delta-$function in brackets shows that including both positive and negative energy states is necessary for completeness. Integrating,

$$\phi(t, \vec{r}) = \int \frac{d^3\vec{k}}{2\omega_k} \left[a(\omega_k, \vec{k}) e^{i(\vec{k}\cdot x - \omega t)} + a(-\omega_k, \vec{k}) e^{i(\vec{k}\cdot x + \omega t)}\right] \tag{2.4.8}$$

and making the change of variables $\vec{k} \to -\vec{k}$ in the second integral,

$$\phi(t, \vec{r}) = \int \frac{d^3\vec{k}}{2\omega_k} \left[a(\omega_k, \vec{k}) e^{i(\vec{k}\cdot x - \omega t)} - a(-\omega_k, -\vec{k}) e^{-i(\vec{k}\cdot x - \omega t)}\right] \tag{2.4.9}$$

or, calling $-a(-E_k, -\vec{k}) = b^*(E_k, \vec{k})$, we have

$$\phi(x) = \int \frac{d^3\vec{k}}{2\omega_k} \left[a(k) e^{ik\cdot x} + b^*(k) e^{-ik\cdot x}\right] \tag{2.4.10}$$

where it is now understood that $k_\mu = (-\omega_k, \vec{k})$. If the field is real then $b(k) = a(k)$.

Let $u_k^{(\pm)}(x) = e^{\pm ik\cdot x}$. As plane waves, $u_k^{(\pm)}(x)$ are not normalizable in infinite space, but they form a complete, orthogonal basis. Applying (2.4.5), we find

$$\langle u_{k'}^{(+)}, u_k^{(+)} \rangle = \frac{(2\pi)^3 (2\omega_k)}{\hbar c^2} \delta(\vec{k} - \vec{k}')$$

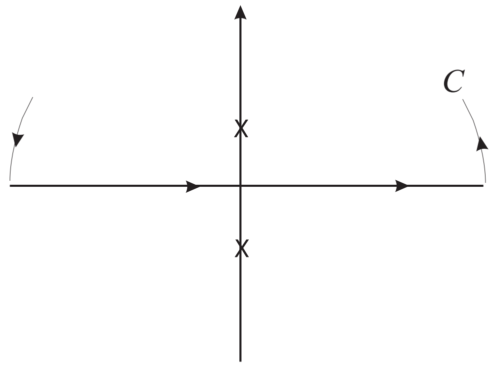

Figure 2.1: Contour for the static Green's function.

$$\langle u_{k'}^{(-)}, u_k^{(-)} \rangle = -\frac{(2\pi)^3(2\omega_k)}{\hbar c^2}\delta(\vec{k}-\vec{k}')$$
$$\langle u_{k'}^{(-)}, u_k^{(+)} \rangle = \langle u_{k'}^{(+)}, u_k^{(-)} \rangle = 0. \tag{2.4.11}$$

The energy and momentum carried by the mode functions, $u_k^{(\pm)}(x)$, are respectively

$$\widehat{H} u_k^{(\pm)}(x) = i\hbar\frac{\partial}{\partial t} u_k^{(\pm)}(x) = \pm E_k u_k^{(\pm)}(x) \tag{2.4.12}$$

and

$$\widehat{p}\ u_k^{(\pm)}(x) = -i\hbar\vec{\nabla} u_k^{(\pm)}(x) = \pm\vec{p} u_k^{(\pm)}(x), \tag{2.4.13}$$

where $E_k = \hbar\omega_k$ and $\vec{p} = \hbar\vec{k}$, as expected.

2.4.2 Green's Functions

If $j(x)$ is a source for the field so that ϕ satisfies the inhomogeneous wave equation,

$$\left(\Box_x + \frac{m^2c^2}{\hbar^2}\right)\phi(x) = j(x) \tag{2.4.14}$$

then $\phi(x)$ is readily determined by quadratures via the Green's function approach.

Static Green's Function

Let us first consider the static case, for which $j = j(\vec{r})$ and $\phi = \phi(\vec{r})$, then

$$\left(-\vec{\nabla}^2 + \frac{m^2c^2}{\hbar^2}\right)\phi(\vec{r}) = j(\vec{r}) \tag{2.4.15}$$

and $\phi(\vec{r})$ can be determined from

$$\phi(\vec{r}) = \int d^3\vec{r}' G(\vec{r},\vec{r}')j(\vec{r}') \tag{2.4.16}$$

where

$$\left(-\vec{\nabla}^2 + \frac{m^2c^2}{\hbar^2}\right) G(\vec{r},\vec{r}') = \delta^3(\vec{r}-\vec{r}') \tag{2.4.17}$$

is the Green's function associated with the operator, $-\vec{\nabla}^2 + m^2c^2/\hbar^2$. Assuming no bounding surfaces (for simplicity) and noting that translation invariance of the Laplacian requires $G(\vec{r},\vec{r}') = G(\vec{r}-\vec{r}')$, we Fourier expand,

$$G(\vec{r},\vec{r}') = \int \frac{d^3\vec{k}}{(2\pi\hbar)^3}\, G(\vec{k})e^{\frac{i}{\hbar}\vec{k}\cdot(\vec{r}-\vec{r}')} \tag{2.4.18}$$

It follows that $G(\vec{k}) = \hbar^2(\vec{k}^2 + m^2c^2)^{-1}$ and therefore,

$$G(\vec{r},\vec{r}') = \frac{1}{(2\pi)^3\hbar}\int d^3\vec{k}\, \frac{e^{\frac{i}{\hbar}\vec{k}\cdot(\vec{r}-\vec{r}')}}{\vec{k}^2 + m^2c^2} \tag{2.4.19}$$

Performing the angular integrations,

$$G(\vec{r},\vec{r}') = -\frac{i}{(2\pi)^2\Delta r}\int_{-\infty}^{\infty} kdk\, \left(\frac{e^{\frac{i}{\hbar}k\Delta r}}{\vec{k}^2 + m^2c^2}\right) \tag{2.4.20}$$

where $\Delta r = |\vec{r}-\vec{r}'|$ and $k = |\vec{k}|$. The integral may be performed in the complex plane (see figure 2.1) by closing the contour in the upper half plane to ensure that the semi-circle at infinity does not contribute (Jordan's lemma). The contour encloses only the simple pole at $k = +imc$ and therefore

$$G(\vec{r}-\vec{r}') = \frac{1}{4\pi}\frac{e^{-\frac{mc}{\hbar}\Delta r}}{\Delta r}. \tag{2.4.21}$$

Thus the "Yukawa potential" due to a point source at the origin, $j(\vec{r}) = j_0\delta(\vec{r})$, from (2.4.16), is

$$\phi(r) = \frac{j_0}{4\pi r}\, e^{-\frac{mc}{\hbar}r}. \tag{2.4.22}$$

As $m \to 0$, it approaches the expected potential due to a massless field.

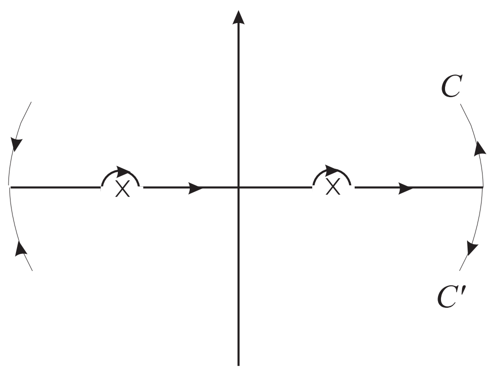

Figure 2.2: Contour for the "retarded" Green's function.

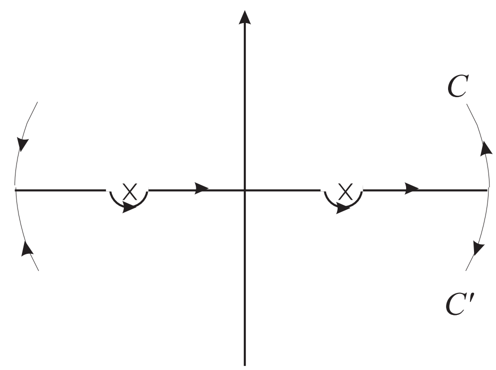

Figure 2.3: Contour for the "advanced" Green's function.

Causal Green's Function

If $j(x)$ is time dependent, then $\phi(x)$ can be determined from

$$\phi(x) = c\int d^4x' G(x,x')j(x') \tag{2.4.23}$$

where

$$\left(\Box_x + \frac{m^2c^2}{\hbar^2}\right) G(x,x') = \frac{1}{c}\delta^4(x-x') \tag{2.4.24}$$

is the Green's function associated with the Klein-Gordon operator, $\Box_x + m^2c^2/\hbar^2$. As before, assuming no bounding surfaces and noting that translation invariance of the Laplacian requires $G(x,x') = G(x-x')$, we Fourier expand,

$$G(x,x') = \int \frac{cd^4k}{(2\pi\hbar)^4}\, G(k)e^{\frac{i}{\hbar}k\cdot(x-x')} \tag{2.4.25}$$

It follows that $G(k) = \hbar^2(k^2+m^2c^2)^{-1}$ and therefore,

$$\begin{aligned} G(x,x') &= \int \frac{d^3\vec{k}}{(2\pi)^3\hbar}\, e^{\frac{i}{\hbar}\vec{k}\cdot(\vec{r}-\vec{r}')} \int_{-\infty}^{\infty} \frac{cdE}{2\pi\hbar} \frac{e^{-\frac{i}{\hbar}E(t-t')}}{-E^2+\vec{k}^2c^2+m^2c^4} \\ &= -\int \frac{d^3\vec{k}}{(2\pi)^3\hbar}\, e^{\frac{i}{\hbar}\vec{k}\cdot\Delta\vec{r}} \int_{-\infty}^{\infty} \frac{cdE}{2\pi\hbar} \frac{e^{-\frac{i}{\hbar}E\Delta t}}{(E-E_k)(E+E_k)} \end{aligned} \tag{2.4.26}$$

where $\Delta t = t-t'$ and $\Delta\vec{r} = \vec{r}-\vec{r}'$ and we have assumed spherical symmetry for simplicity. The integral has two simple poles, located at $E = \pm E_k$, so we regulate it by evaluating the energy integral by contour integration in the complex E plane (see figures 2.2 and

2.3). How one chooses to go around the poles will determine (or, is determined by) the boundary conditions on the field and the particular Green's function obtained.

To ensure that the integrals over the semi-circles at infinity do not contribute to the integral, we must close the contour in the lower half plane when $\Delta t > 0$ and in the upper half plane when $\Delta t < 0$, according to Jordan's lemma. Next, we must determine which of the poles to include in the contour, *i.e.,* how to go around the poles. Suppose we insist that any effect of changes in the source at t' may be felt at a distant point only in the future of t' and no effect may be felt at a distant point in the past of t'. This is tantamount to requiring that $G(x, x')$ vanishes when $\Delta t < 0$, so none of the poles should be contained within the contour that closes in the upper half plane. We are left with the **causal** or **retarded** Green's function, $G_R(x, x')$, (which should be familiar from electrodynamics) shown in figure 2.2. Because both poles are included within the contour of integration, both positive and negative frequencies are propagated forwards in time. Figure 2.3, on the contrary represents precisely the opposite situation: here the effects of a change in the source are felt at a distant point only in the past of the change. It leads to the **acausal** or **advanced** Green's function, $G_A(x, x')$ for which positive and negative frequencies are propagated backwards in time. In a classical theory we require causality, so choose the causal Green's function,

$$G_R(x, x') = -\frac{c\Theta(\Delta t)}{(2\pi)^3\hbar^2} \int \frac{d^3\vec{k}}{E_k}\, e^{\frac{i}{\hbar}\vec{k}\cdot\Delta\vec{r}} \sin\left(\frac{E_k \Delta t}{\hbar}\right) \tag{2.4.27}$$

where $\Theta(\Delta t)$ is the Heaviside function, guaranteeing that $G_R(x - x')$ vanishes if $t < t'$. Assuming spherical symmetry, for convenience, and integrating over the angular coordinates, the $k-$ integrals can be written as

$$G_R(x, x') = -\frac{c\Theta(\Delta t)}{2\pi^2\hbar\Delta r} \int_{-\infty}^{\infty} \frac{kdk}{E_k}\, e^{\frac{i}{\hbar}k\Delta r} \sin\left(\frac{E_k \Delta t}{\hbar}\right) \tag{2.4.28}$$

where we used $k = |\vec{k}|$ and $\Delta r = |\Delta\vec{r}|$. The integral may be performed in the complex plane over a contour that closes in the upper half plane (Jordan's lemma) and avoids the cut due to the branch point on the positive imaginary axis, at $k = +imc$, as shown in figure 2.4; one finds

$$G_R(x, x') = \frac{\Theta(\Delta t)}{4\pi}\left[\frac{1}{\Delta r}\delta(\Delta r - c\Delta t) - \frac{mc}{\hbar}\frac{J_1(\frac{mc}{\hbar}\sqrt{c^2\Delta t^2 - \Delta r^2})}{\sqrt{c^2\Delta t^2 - \Delta r^2}}\right], \tag{2.4.29}$$

provided that $\Delta s^2 = c^2\Delta t^2 - \Delta r^2 > 0$, and zero otherwise. Above, J_1 is the Bessel function of the first kind and the Heaviside function ensures that the event x' preceeds the event x ($t > t'$). If the Klein-Gordon field is massless then the retarded Green's function is just the familiar one from the theory of electromagnetism.

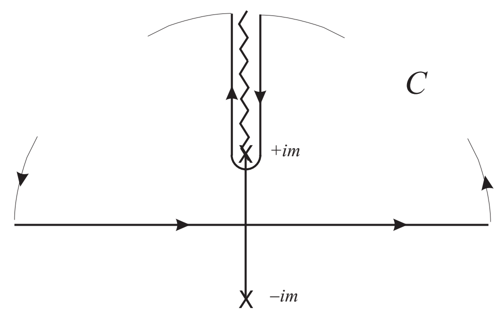

Figure 2.4: Contour for evaluating $G_R(x, x')$ in (2.4.28).

2.5 Scalar Fields

Let us now turn the Klein-Gordon equation into a Lagrangian field theory. We begin with the "real" scalar field.

2.5.1 Action and Symmetries

The "free, linear field" is described by the Klein-Gordon equation, which is derivable by Hamilton's principle from the action

$$\begin{aligned} S &= -\frac{1}{2}\int d^4x \left[\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{m^2c^2}{\hbar^2}\phi^2\right] \\ &= \frac{1}{2}\int d^4x \left[\frac{1}{c^2}\dot{\phi}^2(t,\vec{r}) - \vec{\nabla}\phi(t,\vec{r})\cdot\vec{\nabla}\phi(t,\vec{r}) - \frac{m^2c^2}{\hbar^2}\phi^2(t,\vec{r})\right], \end{aligned} \tag{2.5.1}$$

where m is associated with the mass of the field, c is the speed of light and $\hbar$ is Planck's constant. S will have the mechanical dimension of "action", *i.e.,* $[S] = ml^2/t$, if $[\phi] = \sqrt{ml}/t$.

The Lagrangian can be thought of as having the traditional form $T - V$, if the derivative terms can be associated with the field momentum and the non-derivative term with a field "potential". One can imagine generalizing the action to

$$S = -\frac{1}{2}\int d^4x \left[\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 2V(\phi)\right] \tag{2.5.2}$$

where $V(\phi)$ is an arbitrary scalar field potential, one possibility for which would be

$$V(\phi) = \sum_{n=2} g_n\phi^n(x) \tag{2.5.3}$$

so that $g_2 = m^2c^2/2\hbar^2$ would represent the "mass term" of the Klein-Gordon equation. If $g_2 = 0$ we say that the field is "massless". The equation of motion

$$\Box\phi(x) + V'(\phi) = 0, \tag{2.5.4}$$

where $V'(\phi) = dV/d\phi$, is not linear if $g_n \neq 0$ for $n > 2$ showing that all the higher order terms represent interactions of the field with itself. For example, the anharmonic potential

$$V(\phi) = g_2\phi^2 + g_4\phi^4 \tag{2.5.5}$$

is useful in particle phenomenology and condensed matter physics.[10]

If all the g_n vanish, with the possible exception of g_4, the action has an interesting global **scaling** symmetry: it is invariant under the transformation $x \to x' = \lambda x$ and $\phi(x) \to \phi'(x') = \lambda^{-1}\phi(x)$. As in the case of fluid dynamics, position and time are required to scale in the same way for any *relativistic* field theory, otherwise Lorentz invariance would be violated. As an example of scaling in a theory that is not Lorentz invariant, consider the action whose extremization would lead to the Schroedinger equation,

$$S_{\text{Sch}} = \int dt \int d^3\vec{r} \left[\frac{i\hbar}{2}\psi^* \overleftrightarrow{\partial}_t\psi - \frac{\hbar^2}{2m}(\vec{\nabla}\psi^*)\cdot(\vec{\nabla}\psi) - V(t,\vec{r})\psi^*\psi\right]. \tag{2.5.6}$$

One can check that it is invariant under the scaling transformation

$$\vec{r} \to \vec{r}' = \lambda\vec{r}, \quad t \to t' = \lambda^2 t, \quad V(t,\vec{r}) \to V'(t'\vec{r}') = \lambda^{-2}V(t,r), \tag{2.5.7}$$

and

$$\psi(t,\vec{r}) \to \psi'(t',\vec{r}') = \lambda^{-3/2}\psi(t,\vec{r}) \tag{2.5.8}$$

In fact, because space and time decouple in Galilean relativity, there is no required *à priori* relationship between the scaling of space and the scaling of time. When space and time scale in the same way the scaling is said to be **isotropic** and only isotropic scaling is compatible with Lorentz invariance. **Anisotropic** scaling is allowed only in non-relativistic models.

Other choices of $V(\phi)$ are often of interest: for example, taking $V(\phi) = \alpha[1-\cos(\phi/\beta)]$, we get the **Sine-Gordon** equation[11]

$$\Box\phi(x) + \frac{\alpha}{\beta}\sin(\phi/\beta) = 0, \tag{2.5.9}$$

where the constants α and β have mechanical dimensions $[\alpha] = m/lt^2$ and $[\beta] = \sqrt{ml}/t$ respectively. Here all powers of ϕ occur in the potential via the cosine funtion.[12]

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[10] Potentials with $g_n \neq 0$ for $n > 4$ do not satisfy the "perturbative renormalizability" criterion and are not considered to yield self-consistent quantum field theories. However, they are perfectly good classical field theories.

[11] The Sine-Gordon theory is "renormalizable".

[12] Problem: Find an exact traveling solution to the Sine-Gordon equation in two dimensions as follows:

The real scalar field may be generalized to the complex scalar field, described by the action

$$S = -\int d^4x \left[\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi + V(|\phi|)\right], \tag{2.5.10}$$

where $\phi^*(x)$ is the complex conjugate of $\phi(x)$ and $|\phi(x)|$ is its magnitude. The equation of motion, obtained by varying the action independently with respect to the field and its conjugate, is the same as before:

$$\Box\phi(x) + V'(|\phi|)\phi(x) = 0 \tag{2.5.11}$$

(and likewise for the conjugate field), where $V'(|\phi|) = dV(|\phi|)/d|\phi|^2$. Furthermore, because the action is real, we have the additional global "**gauge**" symmetry: $x \to x' = x$, $\phi(x) \to \phi'(x') = e^{i\alpha}\phi(x)$ for any real constant, α.

2.5.2 Non-Relativistic Limit

Before proceeding with an analysis of the scalar field, let us see how the Schroedinger equation emerges from the Klen-Gordon equation as its non-relativistic limit. The mechanical energy of a particle, $\mathcal{E} = E - mc^2$, is much less than its rest mass energy in this limit, so define the Schoredinger wave function by

$$\phi(t,\vec{r}) = \frac{\hbar}{\sqrt{2m}} e^{-\frac{imc^2t}{\hbar}}\psi(t,\vec{r}) \tag{2.5.12}$$

- Assume that the scalar field is of the form $\phi(t,x) = \phi(u)$ where $u = x - vt$.
- Show that the Sine-Gordon equation turns into

$$\frac{d^2\phi}{du^2} = \phi'\frac{d\phi'}{d\phi} = \sigma V'(\phi),$$

  where $\phi' = d\phi/du$ and determine the constant σ.
- Integrate once to get $\phi'^2(u) = 2\sigma V(\phi) + C$, where C is a constant.
- Ask for a "localized" solution, *i.e.,* one for which the energy density vanishes at infinity, or $\lim_{|u|\to\infty}\phi'(u) = \lim_{|u|\to\infty} V(\phi) = C = 0$. This says that $\phi(\infty) = 2n\pi\beta$ for integer n.
- Integrate this equation for $\phi'(u)$ and find the solution

$$\phi_\pm(u) = 4\beta\tan^{-1}\left\{\exp\left[\pm\frac{\sqrt{\sigma\alpha}}{\beta}(u-u_0)\right]\right\}$$

  This is the Sine-Gordon soliton, with $\phi(u_0) = \pi\beta$; describe the solutions. The solution with the positive sign, which approaches $4m\pi\beta$ as $u \to -\infty$ and $2(2m+1)\pi\beta$ as $u \to +\infty$, where m is an integer, is called a "**kink**". The one with a negative sign approaches $4m\pi\beta$ as $u \to +\infty$ and $2(2m+1)\pi\beta$ as $u \to -\infty$, and is called an "**anti-kink**".

(the pre-factor ensures that the dimension ψ is $l^{-3/2}$) and take the potential to be of the form

$$V(\phi) = \frac{m^2c^2}{\hbar^2}|\phi|^2 + \frac{2m}{\hbar^2}\widetilde{V}_1(|\phi|) = \frac{mc^2}{2}|\psi|^2 + \widetilde{V}(|\psi|). \tag{2.5.13}$$

We have

$$\dot{\phi} = \frac{\hbar}{\sqrt{2m}} e^{-\frac{imc^2 t}{\hbar}} \left[\dot{\psi} - \frac{imc^2}{\hbar}\psi\right], \tag{2.5.14}$$

and therefore the kinetic term in the action for $\phi(x)$ turns to

$$-\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi = \frac{\hbar^2}{2mc^2}|\dot{\psi}|^2 + \frac{i\hbar}{2}\left(\psi^*\dot{\psi} - \dot{\psi}^*\psi\right) - \frac{\hbar^2}{2m}|\vec{\nabla}\psi|^2 + \frac{mc^2}{2}|\psi|^2. \tag{2.5.15}$$

We ignore the term $|\dot{\psi}|^2/mc^2$ and write[13]

$$-\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi \approx \frac{i\hbar}{2}\left(\psi^*\dot{\psi} - \dot{\psi}^*\psi\right) - \frac{\hbar^2}{2m}|\vec{\nabla}\psi|^2 + \frac{mc^2}{2}|\psi|^2, \tag{2.5.16}$$

so the non-relativistic action can now be written as

$$S = \int dt \int d^3\vec{r} \left[\frac{i\hbar}{2}\psi^* \overleftrightarrow{\partial}_t \psi - \frac{\hbar^2}{2m}|\vec{\nabla}\psi|^2 - \widetilde{V}(|\psi|)\right]. \tag{2.5.17}$$

The action yielding the Schroedinger equation in (2.5.6) is obtained by taking $\widetilde{V}(|\psi|) = V(t,\vec{r})|\psi|^2$.

2.5.3 Conservation Laws for the Free Scalar Field

Provided the scalar potential satisfies the required scaling properties, the action will be invariant under the global scale transformation,

$$x \to x' = \lambda x, \quad \phi(x) \to \phi'(x') = \lambda^{-1}\phi(x). \tag{2.5.18}$$

This gives

$$\delta x^\nu = x^\nu \delta\lambda, \quad \delta\phi = -\phi(x)\delta\lambda \tag{2.5.19}$$

---

[13] The dispersion relation that leads to the Klein Gordon equation is $-E^2 + \vec{p}^2c^2 + m^2c^4 = 0$ (a). Now consider the same equation, written in terms of ψ, *i.e.,*

$$\frac{1}{c^2}\left[\ddot{\psi} - \frac{2imc^2}{\hbar}\dot{\psi}\right] - \vec{\nabla}^2\psi = 0$$

which follows from the dispersion relation $-\mathcal{E}^2/c^2 - 2m\mathcal{E} + \vec{p}^2 = 0$ (b). The energies are related by $\mathcal{E} = E - mc^2$, *i.e.,* substituting for $\mathcal{E}$ in (b) returns the original dispersion relation in (a). Thus $\mathcal{E}$ is the mechanical energy, which is required to be much less than mc^2 in the non-relativistic limit and it follows that the first term in (2.5.15) is much smaller than the second.

(*i.e.*, $G^\nu = x^\nu$, $G^{(\phi)} = -\phi$) and therefore Noether's first theorem assures us that the **dilatation** current

$$\begin{aligned} j^\mu &= \left(\mathfrak{L}\delta^\mu_\nu - \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi\right) x^\nu - \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi)}\phi(x) \\ &= \phi\partial^\mu\phi + \Theta^\mu{}_\nu x^\nu \end{aligned} \tag{2.5.20}$$

is conserved, where

$$\Theta^\mu{}_\nu = \partial^\mu\phi\partial_\nu\phi + \delta^\mu{}_\nu\mathfrak{L}. \tag{2.5.21}$$

Translational invariance will soon be seen to yield the conserved energy momentum tensor for the scalar field,

$$\Theta^\mu{}_\nu = \mathfrak{L}\delta^\mu{}_\nu - \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi)}\partial_\nu\phi = \partial^\mu\phi\partial_\nu\phi + \delta^\mu{}_\nu\mathfrak{L} \tag{2.5.22}$$

according to (2.3.3).[14]

The energy momentum tensor, $\Theta^{\mu\nu}$, is symmetric. From it we can construct the momentum density $\wp^\mu = \Theta^{0\mu}$ and the total momentum carried by the field,

$$\mathrm{P}^\mu = \int d^3\vec{r}\ \wp^\mu. \tag{2.5.23}$$

Again, because $\phi(x)$ transforms as a scalar under Lorentz transformations, $G^A_{\alpha\beta}$ vanishes identically and with it so does the intrinsic angular momentum, $S^\mu{}_{\alpha\beta}$. The orbital angular momentum tensor density,

$$L^\mu{}_{\alpha\beta} = \frac{1}{2}\left(\Theta^\mu{}_\alpha x_\beta - \Theta^\mu{}_\beta x_\alpha\right) \tag{2.5.24}$$

is conserved and it gives the field angular momentum tensor

$$\mathrm{L}^{\nu\lambda} = \int d^3\vec{r}\ L^{0\nu\lambda} = \int d^3\vec{r}\ (x^\nu\wp^\lambda - x^\lambda\wp^\nu) \tag{2.5.25}$$

as before.

In the case of the free, complex scalar field, we should find once again that the energy momentum tensor is symmetric,

$$\Theta_{\mu\nu} = (\partial_\mu\phi^*\partial_\nu\phi + \partial_\nu\phi^*\partial_\mu\phi) + \eta_{\mu\nu}\mathfrak{L} \tag{2.5.26}$$

and that the field carries no intrinsic angular momentum (spin). This means that the total momentum and the orbital angular momentum can be defined in precisely the same way

[14] Problem: Show that the simultaneous conservation of the energy momentum tensor and the dilatation current on shell implies that $V(\phi) = \lambda\phi^4$, where λ is a constant.

as they were for the real scalar field. However, there is the additional gauge symmetry. Global gauge transformations do not involve any transformation of the coordinates (*i.e.,* $x \to x' = x$) but only a transformation of the fields according to

$$\phi(x) \to \phi'(x') = e^{ig\delta\alpha}\phi(x) = (1 + ig\delta\alpha)\phi(x) \Rightarrow \delta\phi = ig\delta\alpha\phi \tag{2.5.27}$$

where $\delta\alpha$ is constant (similarly for the conjugate field). Thus there is a conserved current given by

$$j^\mu = i\frac{\partial\mathfrak{L}}{\partial(\partial_\mu\phi)}\phi - i\phi^*\frac{\partial\mathfrak{L}}{\partial(\partial_\mu\phi^*)} = i\phi^*\overleftrightarrow{\partial^\mu}\phi. \tag{2.5.28}$$

2.6 Local Gauge Invariance

Global symmetries can often be turned into a local symmetries by introducing a new field. The action for the complex scalar field is not invariant under a *local* gauge transformation, *i.e.,* when α depends on x, because of the derivative term, which picks up derivatives of $\alpha(x)$ as well. This can be remedied as follows:

- Introduce a new real, vector field $A_\mu(x)$, called a gauge field, which simultaneously transforms under the local gauge transformation,

$$x \to x' = x, \quad \phi(x) \to \phi'(x') = e^{ig\alpha(x)}\phi(x), \tag{2.6.1}$$

  where g is a constant, according to

$$A_\mu(x) \to A'_\mu(x') = A_\mu(x) + \partial_\mu\alpha(x) \tag{2.6.2}$$

- Define the new action

$$S = -\int d^4x \left[\eta^{\mu\nu}(D_\mu\phi)^* D_\nu\phi + V(|\phi|)\right], \tag{2.6.3}$$

  where D_μ is a new derivative operator,

$$D_\mu = \partial_\mu - igA_\mu. \tag{2.6.4}$$

With this definition,

$$D_\mu\phi(x) \to D'_\mu\phi'(x') = e^{ig\alpha(x)}D_\mu\phi(x) \tag{2.6.5}$$

and

$$(D_\mu\phi(x))^* \to (D'_\mu\phi'(x'))^* = e^{-ig\alpha(x)}(D_\mu\phi(x))^* \tag{2.6.6}$$

and therefore the new action remains invariant under the local gauge transformation. The operator D_μ is called a "(gauge) covariant" derivative. It satisfies the Leibnitz rule, but this is not immediately obvious. It follows from the fact that the precise form of the covariant derivative depends on the transformation property of the quantity on which it acts (characterized by g). Thus, consider the product of two functions with transformations

$$\phi(x) \to \phi'(x') = e^{ig_1\alpha(x)}\phi(x),$$
$$\psi(x) \to \psi'(x') = e^{ig_2\alpha(x)}\phi(x),$$

then it is straightforward that

$$D_\mu(\psi\phi) = \{\partial_\mu - i(g_1+g_2)A_\mu\}(\psi\phi) = \psi(D_\mu\phi) + (D_\mu\psi)\phi.$$

A_μ is called the "gauge connection".

The covariant derivative does not commute with itself as the ordinary derivative does. In fact

$$[D_\mu, D_\nu]\phi = -igF_{\mu\nu}\phi, \tag{2.6.7}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.6.8}$$

which is called the **field strength** tensor and which you may recognize as the Maxwell tensor of Electrodynamics. It can be checked directly that $F_{\mu\nu}$ is invariant under the gauge transformations in (2.6.2). This is indeed quite remarkable, saying that the electromagnetic interaction can be understood as being a direct consequence of the gauge invariance of the action describing matter. To include a gauge invariant "kinetic" term for the gauge field, turning it into a dynamical field in its own right, we write the total action as

$$S = -\int d^4x \left[\eta^{\mu\nu}(D_\mu\phi)^* D_\nu\phi + V(|\phi|) + \frac{gc}{4}F_{\mu\nu}F^{\mu\nu}\right], \tag{2.6.9}$$

and vary with respect to both $\phi(x)$ and $A_\mu(x)$.[15],[16] What we have obtained is the action describing scalar electrodynamics. Varying with respect to $\phi^*(x)$ gives us the equation of

[15] Problem: Show that the mechanical dimension of A_i must be ml/t and of A_0 is ml^2/t^2. Then show that the mechanical dimension of g is t/ml^2. For this reason we may set $g = e^2/4\pi\alpha\hbar$, where both e and α are dimensionless, e is taken to be the electric charge and α is the fine structure constant. Indeed, comparing with the action for the electromagnetic field, we see that $gc = 1/\mu_0$, where $\mu_0 = 4\pi\times 10^{-7}$ kg·m is the permeability of the vacuum. This gives $g = 2.6526\times 10^{-3}$ $\text{J}^{-1}\cdot\text{s}^{-1}$ and $\alpha = 7.2783\times 10^{-3} \approx 1/137$. In units in which $c = 1 = \hbar$ (in these units the mechanical dimensions of l and t are the same and $m \sim l^{-1}$), the coupling constant g is dimensionless and the gauge field has the dimension of mass.

[16] Problem: It is sometimes convenient to rescale the gauge fields according to $A'_\mu(x) = \sqrt{gc\hbar}A_\mu(x)$ and redefine the coupling constant as $g' = \sqrt{g/\hbar c}$. show that, in terms of the rescaled fields and coupling constant, we may write

$$S = -\int d^4x \left[\eta^{\mu\nu}(D_\mu\phi)^* D_\nu\phi + V(|\phi|) + \frac{1}{4\hbar}F'_{\mu\nu}F'^{\mu\nu}\right],$$

where $F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$ and $D_\mu = \partial_\mu - ig'A'_\mu$. Veryfy the mechanical dimensions: $[A'_i] = \sqrt{m^2l^3/t^3}$, $[A'_0] = \sqrt{m^2l^5/t^5}$ and $[g'] = \sqrt{t^3/m^2l^5}$.

motion for the scalar field,

$$- D^\mu D_\mu \phi(x) + V'(|\phi|)\phi(x) = 0 \tag{2.6.10}$$

and varying with respect to $A_\mu(x)$ gives Maxwell's equations with the scalar field as the source[17]

$$\partial_\alpha F^{\alpha\mu} = \frac{i}{c}[\phi^* D^\mu \phi - (D^\mu \phi)^* \phi] = \frac{i}{c}\phi^* \overleftrightarrow{D^\mu}\phi = \frac{1}{c}\left(j^\mu + 2g|\phi|^2 A^\mu\right). \tag{2.6.11}$$

The first term in the source for the electromagnetic field, on the right hand side of the above equation, is the Noether current associated with global gauge invariance. The second term is characteristic of what one sees for a massive vector field (imagine $|\phi|^2$ is constant), for example in (2.7.4). We will exploit this fact in a later chapter, when we examine spontaneous symmetry breaking. Local invariance is stronger than global invariance since an action that is invariant under local transformations is invariant under global transformations but not vice-versa.

2.7 Vector Fields

The electromagnetic field discussed above is a real vector field with an interesting symmetry: gauge invariance, as captured in (2.6.2). The price we pay for gauge invariance is that the field is massless.

2.7.1 Action and Symmetries

If we add a "mass term" to the action for a gauge field in analogy with the scalar field,

$$S = -c \int d^4x \left(\frac{g}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2 c^2}{2\hbar^3} A_\mu A^\mu \right), \tag{2.7.1}$$

then the action ceases to be gauge invariant. Gauge invariance is not an essential symmetry for an arbitrary vector field (as is, for example, Lorentz invariance) and we may take (2.7.1) to be the action describing a massive vector field. This is the **Wentzel-Pauli** action and the equation of motion,

$$- \partial_\mu F^{\mu\nu} + \frac{m^2 c^2}{g\hbar^3} A^\nu = 0 \tag{2.7.2}$$

is the **Proca** equation for massive vector fields. Now if we expand the Maxwell tensor, we find that the equations of motion read

$$\Box_x A^\nu + \partial^\nu (\partial \cdot A) + \frac{m^2 c^2}{g\hbar^3} A^\nu = 0 \tag{2.7.3}$$

[17] Problem: Demonstrate the consistency of (2.6.11) by showing that that on shell *i.e.,* subject to the equations of motion, $\partial_\mu(\phi^* \overleftrightarrow{D}^\mu \phi) = D_\mu(\phi^* \overleftrightarrow{D}^\mu \phi) = 0$.

and if we take the divergence of the above equation we find $m^2(\partial \cdot A) = 0$, so that, as long as $m \neq 0$, the condition $\partial \cdot A = 0$ holds and the vector field equations of motion

$$\Box_x A^\nu + \frac{m^2 c^2}{g\hbar^3} A^\nu = 0 \tag{2.7.4}$$

look exactly like those of four massive, real scalar fields (or D real scalar fields, where D is the dimension of space-time).[18]

Gauge invariance of the massless vector field tells us that any solution of the field equations is physically equivalent to another solution differing from the first by the gradient of a scalar function. We are free to choose any one of the infinite possible solutions that differ from each other in this way. Another way to think about this is to realize that we are free to impose one additional condition on the field. This condition is called "a gauge choice". The argument goes as follows: for any solution, A_μ, of the field equations that does *not* obey the gauge condition, a scalar function, $\alpha(x)$, always exists such that the gauge transformed field,

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x) \tag{2.7.5}$$

does obey the gauge condition. Now the transformed field is also a solution of Maxwell's equations because the latter are invariant under gauge transformations, so the gauge condition could be imposed from the start. How the gauge condition is chosen is irrelevant, so long as it is not gauge invariant. It is generally chosen in such a way as to simplify one's computations. A popular gauge choice is

$$\partial \cdot A = 0 \tag{2.7.6}$$

because it is linear in the vector field and Lorentz invariant but not gauge invariant. This is called the **Lorenz** gauge. In this gauge, Maxwell's equations for the massless vector field are just $\Box_x A_\mu = 0$. Suppose, for example, that we have found a solution, $A_\mu(x)$, of the field equations that does not satisfy the Lorentz condition. Consider the gauge transformation of $A_\mu(x)$ as given in (2.7.5) and take the divergence of both sides,

$$\partial \cdot A' = \partial \cdot A - \Box\alpha. \tag{2.7.7}$$

We could choose α to be a solution of the Poisson equation, $\Box\alpha = \partial \cdot A$ and work with $A'_\mu(x)$ from the start.

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[18] Theories with massive vector fields are not perturbatively renormalizable unless the mass is generated by interactions with another field. The way to generate renormalizable massive gauge theories was first described by P.W. Anderson in 1962 and a relativistic generalization was developed a couple of years later by various authors. It has come to be known as the **Higgs** mechanism (after P. Higgs) in relativistic field theory.

One might have supposed that the natural action for a vector field would take a similar form as the action for four scalar fields (or D of them, in D dimensions),

$$S = -\frac{c}{2}\int d^4x \left[g(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{m^2c^2}{2\hbar^3} A_\mu A^\mu \right]. \tag{2.7.8}$$

What is wrong with this action is that it leads to an indefinite energy density, Θ^{00}, which occurs via the indefinite metric signature. This is a serious problem for any field theory as it signals the presence of tachyons (particles moving faster than light). It is corrected by adding the term

$$\mathfrak{L}_{\text{corr}} = \frac{gc}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) \tag{2.7.9}$$

to the Lagrangian of (2.7.8), which then ends up becoming the Wentzel-Pauli Lagrangian.

2.7.2 Plane Waves

If the sources are very far away or in the absence of sources, we could expand the vector field in Fourier modes that look like plane waves

$$A_\mu(k,x) = A_\mu(\vec{k})e^{ik\cdot x} \tag{2.7.10}$$

where $k_\mu = (-\omega_k, \vec{k})$ and $\omega_k = \pm c\sqrt{\vec{k}^2 + m^2c^2/g\hbar^3}$. The Lorentz condition implies that $k \cdot A = 0$, and therefore $A_\mu(\vec{k})$ has three independent components, or three independent polarization states, which we categorize as two transverse states with $A_0^{\text{T}} = 0$ and $\vec{A}^{\text{T}} \perp \vec{k}$, and one longitudinal state with $A_0^{\text{L}} \neq 0$ and $\vec{A}^{\text{L}} \parallel \vec{k}$. Letting $\vec{A}^{\text{L}} = A\widehat{k}$, the longitudinal state must satisfy the condition

$$A_0^{\text{L}} = -\frac{c^2|\vec{k}|A}{\omega_k} \tag{2.7.11}$$

All three polarization states are spacelike, but the longitudinal state has norm

$$-\eta^{\mu\nu}A_\mu^{\text{L}}A_\nu^{\text{L}} = -\frac{A^2}{\omega_k^2}\frac{m^2c^2}{g\hbar^3}, \tag{2.7.12}$$

which means that it becomes null in the massless limit.

If the field is massless, there is a residual gauge invariance even after the Lorentz condition is imposed. To see how this comes about, recall that any solution of the field equations can be transformed to a field satisfying the Lorentz gauge condition if we choose $\alpha(x)$ to be the solution of a particular Poisson equation, specifically (2.7.7). Solutions of Poisson's equation are not unique because one can add to them any solution of Laplace's equation to get a new solution. Thus, suppose we have two solutions of the field equations, $A_\mu(x)$ and $A'_\mu(x)$, both satisfying the Lorentz gauge condition, and related by

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \beta(x), \quad \Box\beta(x) = 0 \tag{2.7.13}$$

The function $\beta(x) = b(\vec{k})e^{ik\cdot x}$ automatically satisfies Laplace's equation because $\omega_k = \pm|\vec{k}|c$. Therefore,

$$A'_\mu(\vec{k}) = A_\mu(\vec{k}) + ib(\vec{k})k_\mu \tag{2.7.14}$$

We could use this equation to eliminate one more component of $A'_\mu(\vec{k})$, leaving it with just two independent polarization states. For example, we could set $A'_0(\vec{k}) = 0$ by taking $b(\vec{k}) = -iA_0(\vec{k})/\omega_k$. In that case, the Lorentz condition reads $\vec{k}\cdot\vec{A}'(\vec{k}) = 0$, showing that only the two transverse polarization states survive.

2.7.3 Conservation Laws and the Bianchi Identities

It is easy to see that the massless vector field Lagrangian is also invariant under isotropic scaling provided that $A_\mu(x) \to A'_\mu(x') = \lambda^{-1}A_\mu(x)$. This is identical to the scaling symmetry of the massless scalar field treated above and we can expect the conserved dilatation current

$$\begin{aligned} j^\mu &= \left(\mathfrak{L}\delta^\mu_\nu - \frac{\partial\mathfrak{L}}{\partial(\partial_\mu A_\alpha)}\partial_\nu A_\alpha\right)x^\nu - \frac{\partial\mathfrak{L}}{\partial(\partial_\mu A_\alpha)}A_\alpha(x) \\ &= gc\left[F^{\mu\alpha}\partial_\nu A_\alpha - \frac{1}{4}F^2_{\alpha\beta}\delta^\mu_\nu\right]x^\nu + \frac{gc}{2}F^{\mu\alpha}A_\alpha \\ &= gcF^{\mu\alpha}A_\alpha + x_\nu\Theta^{\mu\nu}, \end{aligned} \tag{2.7.15}$$

where

$$\Theta^{\mu\nu} = gc\left[F^{\mu\alpha}\partial^\nu A_\alpha - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right] \tag{2.7.16}$$

is the canonical energy momentum tensor, which is neither symmetric nor gauge invariant. From here one can construct the orbital angular momentum tensor density,

$$L^{\mu\alpha\beta} = \frac{1}{2}\left(\Theta^{\mu\alpha}x^\beta - \Theta^{\mu\beta}x^\alpha\right) \tag{2.7.17}$$

and the orbital angular momentum tensor,

$$\mathrm{L}_{\alpha\beta} = \int d^3\vec{r}\; L^0{}_{\alpha\beta}. \tag{2.7.18}$$

This tensor has six independent components, three of which are spatial. The spatial components of the tensor can be viewed as components of the angular momentum vector. Defining the electric and magnetic fields in the usual way by

$$E_i = -F_{0i}, \quad B_i = \frac{1}{2}\epsilon_{ijk}F^{jk}, \tag{2.7.19}$$

we find

$$\mathrm{L}^i = -\epsilon^{ijk}\int d^3\vec{r}\; L^0{}_{jk} = \frac{g}{c}\int d^3\vec{r}\; \vec{E}^k(\vec{r}\times\vec{\nabla})^i A_k. \tag{2.7.20}$$

The falure of the orbital angular momentum to be gauge invariant follows from the failure of the canonical energy momentum tensor to be gauge invariant. To determine the intrinsic angular momentum of the vector field we must evaluate the change in A_μ under a Lorentz transformation,

$$A^\mu(x) \to A'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\alpha}A^\alpha(x) \Rightarrow \delta A^\mu = \frac{1}{2}(\delta^\mu_\alpha\eta_{\beta\lambda} - \delta^\mu_\beta\eta_{\alpha\lambda})A^\lambda\delta\omega^{\alpha\beta} \tag{2.7.21}$$

(upon antisymmetrizing) which says that

$$G_{\mu\alpha\beta} = \frac{1}{2}(\eta_{\mu\alpha}\eta_{\beta\lambda} - \eta_{\mu\beta}\eta_{\alpha\lambda})A^\lambda \tag{2.7.22}$$

and gives the intrinsic spin tensor density

$$S^\mu{}_{\alpha\beta} = -gcF^{\mu\nu}G_{\nu\alpha\beta} = -\frac{gc}{2}\left[F^\mu{}_\alpha A_\beta - F^\mu{}_\beta A_\alpha\right]. \tag{2.7.23}$$

As we did with the orbital angular momentum, we now define the spin tensor as

$$\mathrm{S}_{\alpha\beta} = \int d^3\vec{r}\; S^0{}_{\alpha\beta} = -\frac{gc}{2}\int d^3\vec{r}\;\left[F^0{}_\alpha A_\beta - F^0{}_\beta A_\alpha\right]. \tag{2.7.24}$$

which also as six independent components, three of which are spatial,

$$\mathrm{S}_{jk} = -\frac{g}{2c}\int d^3\vec{r}\;\left[E_jA_k - E_kA_j\right], \tag{2.7.25}$$

and can equivalently define a spin density *vector* by contraction with the Levi-Civita tensor,

$$\mathrm{S}^i = -\epsilon^{ijk}\mathrm{S}_{jk} = \frac{g}{c}\int d^3\vec{r}\;\vec{E}\times\vec{A}. \tag{2.7.26}$$

S^i is also not gauge invariant.

Translational invariance yields the weak conservation of $\Theta^{\mu\nu}$, but it is *not* a symmetric tensor and therefore one cannot directly construct the momentum density of the field, as we did earlier. To symmetrize the stress tensor we follow the prescription laid out in section 3, *i.e.,* we look for a third rank tensor, $k^{\lambda\mu\nu}$ which is antisymmetric in (λ, μ). An obvious choice is to apply the Belinfante-Rosenfeld prescription in (2.3.16) with the intrinsic spin given in (2.7.23). We find $k^{\lambda\mu\nu} = gcF^{\lambda\mu}A^\nu$ and

$$\Delta^{\mu\nu} = gc\partial_\lambda(F^{\lambda\mu}A^\nu) = gc\{(\partial_\lambda F^{\lambda\mu})A^\nu + F^{\lambda\mu}\partial_\lambda A^\nu\}. \tag{2.7.27}$$

The first term vanishes by the vacuum Maxwell equations, so $\Delta^{\mu\nu} = gcF^{\lambda\mu}\partial_\lambda A^\nu$. Adding this term to $\Theta^{\mu\nu}$ gives the Belinfante-Rosenfeld tensor,

$$t^{\mu\nu} = gc\left[F^{\mu\alpha}F^\nu{}_\alpha - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right], \tag{2.7.28}$$

which is both symmetric in its indices and gauge invariant. We can use $t^{\mu\nu}$ to define the momentum density, $\wp^\mu = t^{0\mu}$, so that

$$\wp^0 = t^{00} = gcF^{0i}F^0{}_i + \frac{g}{4c}F_{\alpha\beta}F^{\alpha\beta} = \frac{g}{2c}\left(\frac{\vec{E}^2}{c^2} + \vec{B}^2\right) = \frac{\mathcal{E}}{c^2} \tag{2.7.29}$$

where $F_{0i} = -E_i$ is the electric field, F_{ij} is given in terms of the magnetic field, B_i as $F_{ij} = \epsilon_{ijk}B^k$ and $\mathcal{E}$ represents the energy density of the electromagnetic field.[19]

The field momentum density is

$$\wp^i = t^{0i} = gcF^{0j}F^i{}_j = \frac{g}{c}\epsilon^i{}_{jk}E^jB^k = \frac{g}{c}\vec{E}\times\vec{B}, \tag{2.7.30}$$

which will be recognized as the Poynting vector and determines the energy flux of the vector field.

Consider, also, the modified angular momentum tensor density,

$$\widetilde{L}^{\alpha\mu\nu} = \frac{1}{2}(t^{\alpha\mu}x^\nu - t^{\alpha\nu}x^\mu) \tag{2.7.31}$$

and the angular momentum tensor

$$\widetilde{\mathrm{L}}^{\mu\nu} = \int d^3\vec{r}\;\widetilde{L}^{0\mu\nu} = \frac{1}{2}\int d^3\vec{r}\;(\wp^\mu x^\nu - \wp^\nu x^\mu), \tag{2.7.32}$$

whose three spatial components,

$$\widetilde{\mathrm{L}}^{ij} = \frac{1}{2}\int d^3\vec{r}\;(\wp^i x^j - \wp^j x^i) \tag{2.7.33}$$

[19] Problem: Show that the energy density of the vector field given by the action (2.7.8) is indefinite.

represent the orbital angular momentum carried by the field. Using the three dimensional Levi-Civita tensor we can now define the modified orbital angular momentum vector as

$$\widetilde{\mathrm{L}}^i = -\int d^3\vec{r}\,\varepsilon^{ijk}\widetilde{L}^0_{jk} = \int d^3\vec{r}\,(\vec{r}\times\vec{\wp}) = \frac{g}{c}\int d^3\vec{r}\,\left[\vec{r}\times(\vec{E}\times\vec{B})\right]^i, \tag{2.7.34}$$

which is gauge invariant.[20]

We may also ask what Noether's theorems say about local gauge transformations of the massless vector field. If the transformations are taken to be arbitrary, parameterized by the infinitesimal function $\delta\alpha(x)$ then

$$\delta x^\mu = 0, \quad \delta A_\mu = \partial_\mu \delta\alpha(x) \tag{2.7.35}$$

determine $G^\mu = 0 = G^{(A)}{}_\mu$ and $T^{(A)}{}^\mu_\nu = \delta^\mu_\nu$. Insert these into (2.2.14) to find

$$j^\mu = gc\left[\partial_\lambda F^{\mu\lambda}\delta\alpha(x) + F^{\mu\lambda}\partial_\lambda\delta\alpha(x)\right] = gc\partial_\lambda(F^{\mu\lambda}\delta\alpha(x)). \tag{2.7.36}$$

This is a conserved current, which is guaranteed by the antisymmetry of $F^{\mu\nu}$.

The **Bianchi** identities are a set of *algebraic* relations obeyed by the first derivatives of the field strength tensor. These algebraic identities follow directly from the Jacobi identity,

$$\{[[D_\mu, D_\nu], D_\alpha] + [[D_\alpha, D_\mu], D_\nu] + [[D_\nu, D_\alpha], D_\mu]\}\phi = 0, \tag{2.7.37}$$

and from the definition of the tensor; specifically we find that

$$\partial_\alpha F_{\mu\nu} + \partial_\nu F_{\alpha\mu} + \partial_\mu F_{\nu\alpha} \equiv 0. \tag{2.7.38}$$

In four dimensions there are four independent identities and we can contract the left hand side with the four dimensional Levi-Civita tensor, writing the relation as

$$\epsilon^{\sigma\alpha\mu\nu}\partial_\alpha F_{\mu\nu} = \partial_\alpha {}^*F^{\sigma\alpha} = 0 \tag{2.7.39}$$

where ${}^*F^{\sigma\alpha} = \epsilon^{\sigma\alpha\mu\nu}F_{\mu\nu}$ is the dual of the field strength tensor. Expanding the above into time and space components one sees that they are simply the "sourceless" Maxwell equations, *viz.,* Faraday's law and the absence of magnetic monopoles.[21]

[20] Problem: Show that the modified spin angular momentum is

$$\widetilde{S}^i = \frac{g}{c}\int d^3\vec{r}\,\left[\vec{E}\times\vec{A} + \vec{r}\times(\vec{E}\cdot\vec{\nabla})\vec{A}\right]^i = \frac{g}{c}\int d^3\vec{r}\,\vec{\nabla}\cdot\left[\vec{E}\,(\vec{r}\times\vec{A})^i\right],$$

in the absence of sources. Therefore the spin vector associated with the Belinfante-Rosenfeld tensor vanishes for fields that decay fast enough.

[21] Problem: Show that (2.7.38) and (2.7.39) are identical. Then show that they reduce to

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0, \quad \vec{\nabla}\cdot\vec{B} = 0$$

2.8 Conservation Laws for Interacting Fields

Interactions are described by adding terms to the Lagrangian for the free fields. Collectively, these terms are generally referred to as the interaction Lagrangian. The Noether currents get modified by the interaction Lagrangian but, because the currents are linear in $\mathfrak{L}$, these modifications are always additive. If we think of the Lagrangian as made up of a "free" or non-interacting part plus an interacting part it is necessary only to compute the modifications arising from the interaction Lagrangian and add them to the free currents. For example, for the complex scalar field interacting with the gauge field in (2.6.9) write

$$\mathfrak{L} = \mathfrak{L}_\phi + \mathfrak{L}_A + \mathfrak{L}_{(\text{int})} \tag{2.8.1}$$

where $\mathfrak{L}_\phi$ is found in (2.5.10), $\mathfrak{L}_A$ in (2.7.1) (with $m = 0$) and

$$\mathfrak{L}_{(\text{int})} = -ig\phi^* \overleftrightarrow{\partial_\mu} \phi A^\mu - g^2 A^2 |\phi|^2 = -g(j \cdot A) - g^2 A^2 |\phi|^2, \tag{2.8.2}$$

where j^μ is the free scalar field current density given in (2.5.28). To compute the modifications to the stress-energy tensor arising from the interactions, one would determine

$$\begin{aligned} \Theta^\mu{}_{\nu(\text{int})} &= \mathfrak{L}_{(\text{int})} \delta^\mu{}_\nu - \partial_\nu \phi^* \frac{\partial \mathfrak{L}_{(\text{int})}}{\partial(\partial_\mu \phi^*)} - \frac{\partial \mathfrak{L}_{(\text{int})}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \frac{\partial \mathfrak{L}_{(\text{int})}}{\partial(\partial_\mu A_\alpha)} \partial_\nu A_\alpha \\ &= \left\{ -g(j \cdot A) - g^2 A^2 |\phi|^2 \right\} \delta^\mu{}_\nu + g A^\mu j_\nu \end{aligned} \tag{2.8.3}$$

and find

$$\Theta^\mu{}_\nu = \Theta^\mu{}_{\nu(\phi)} + \Theta^\mu{}_{\nu(A)} + \Theta^\mu{}_{\nu(\text{int})}. \tag{2.8.4}$$

A similar approach determines the contributions of the interaction to the spin and angular momentum tensors painlessly.

According to (2.2.14), the Noether current corresponding to local gauge invariance, with non-vanishing variations at the boundaries,

$$\delta\phi(x) = ig\delta\alpha(x)\phi(x), \quad \delta\phi^*(x) = -ig\delta\alpha(x)\phi(x), \quad \delta A_\lambda(x) = \partial_\lambda(\delta\alpha(x)), \tag{2.8.5}$$

will be

$$\begin{aligned} J^\mu &= c\left[\partial_\lambda F^{\lambda\mu} \delta\alpha(x) + F^{\lambda\mu} \partial_\lambda \delta\alpha(x)\right] \\ &= c\partial_\lambda \left(F^{\lambda\mu} \delta\alpha(x)\right). \end{aligned} \tag{2.8.6}$$

The terms involving the scalar field cancel, so this is identical to (2.7.36) even though the matter fields have been taken into account in applying (2.2.14)! Since $\delta\alpha(x)$ is arbitrary we could take it to be constant (say, unity), in which case, using the equations of motion,

$$J^\mu = c\partial_\lambda F^{\lambda\mu} = (i\phi^* \overleftrightarrow{D^\mu} \phi) = j^\mu. \tag{2.8.7}$$

where j^μ is the source current for the electromagnetic field. Thus the conserved charge is exclusively due to the source currents

$$Q = -g \int d^3\vec{r}\; J^0 \sim -g \int d^3\vec{r}\; j^0 = -gc \int d^3\vec{r}\; \partial_i F^{i0} = +\frac{g}{c} \oint_S \vec{E} \cdot d\vec{S} \tag{2.8.8}$$

so that for fields that are generated by localized charges and currents the source charge can be converted to a surface integral at infinity, which is finite and non-zero in general. This is Gauss' law. It is a remarkable property of the minimal coupling that the Noether current associated with gauge symmetry can be expressed in terms of the field strengths with no explicit reference to the source (scalar) field. The infinite family of conserved charges generated by (2.8.6) is redundant.

2.9 Hamiltonian Description of Fields

The Hamiltonian dynamics of fields parallels the Hamiltonian dynamics of point particles with obvious differences that owe to the functional nature of the field Lagrangian. Suppose that we are given a Lagrangian density function

$$\mathfrak{L} = \mathfrak{L}(\phi^A(x), \partial_\mu \phi^A(x), x), \tag{2.9.1}$$

and a Lagrangian functional

$$\mathcal{L}[\phi^A, \partial_\mu \phi^A, t] = \int d^3\vec{r}\; \mathfrak{L}(\phi^A(\vec{r},t), \partial_\mu \phi^A(\vec{r},t), \vec{r}, t). \tag{2.9.2}$$

We define the momentum conjugate to the field ϕ^A as

$$\pi_A(x) = \frac{\partial}{\partial \dot{\phi}^A(x)} \mathfrak{L}(\phi^A(x), \partial_\mu \phi^A(x), x) \tag{2.9.3}$$

and the Hamiltonian *density* by the Legendre transformation

$$\mathfrak{H}(\pi_A(x), \phi^A(x), \partial_i \phi^A(x), x) = \pi_A(\vec{r},t)\dot{\phi}^A(\vec{r},t) - \mathfrak{L}. \tag{2.9.4}$$

The Hamiltonian of the system is constructed from the Hamiltonian density in the same way as the Lagrangian is constructed from the Lagrangian density,

$$\mathcal{H}[\pi_A, \phi^A, \partial_i \phi^A, t] = \int d^3x\; \mathfrak{H} = \int d^3\vec{r} \left[\pi_A(\vec{r},t)\dot{\phi}^A(\vec{r},t) - \mathfrak{L}\right]. \tag{2.9.5}$$

and it is not difficult to see that $\mathfrak{H}$ and likewise $\mathcal{H}$ are independent of $\dot{\phi}^A$; for instance taking a derivative of $\mathfrak{H}$ w.r.t. $\dot{\phi}^A$ yields

$$\frac{\partial \mathfrak{H}}{\partial \dot{\phi}^A(x)} = \pi_A(x) - \frac{\partial \mathfrak{L}}{\partial \dot{\phi}^A(x)} \equiv 0. \tag{2.9.6}$$

As was the case for point particles, the equations of motion are recovered by requiring that the action,

$$\mathcal{S} = \int_1^2 d^4x \left[\pi_A(x)\dot{\phi}^A(x) - \mathfrak{H}(\pi_A(x), \phi^A(x), \partial_i\phi^A(x), x)\right] \tag{2.9.7}$$

be stationary under independent variations of the field variables $(\pi_A(x), \phi^A(x))$, keeping *both* fixed at the boundaries "1" and "2". This gives the canonical equations

$$\dot{\phi}^A(x) = \frac{\delta\mathcal{H}}{\delta\pi_A(x)} = \frac{\partial\mathfrak{H}}{\partial\pi_A}, \quad \dot{\pi}_A(x) = -\frac{\delta\mathcal{H}}{\delta\phi^A(x)} = -\frac{\partial\mathfrak{H}}{\partial\phi^A} + \partial_i\left(\frac{\partial\mathfrak{H}}{\partial(\partial_i\phi^A)}\right) \tag{2.9.8}$$

For example, for the Lagrangian density (2.0.10) describing the elementary excitations of the rod, we have (suppressing the x dependence)

$$\pi = \mu\dot{\eta}, \quad \mathfrak{H} = \pi\dot{\eta} - \mathfrak{L} = \frac{\pi^2}{2\mu} + \frac{Y}{2}\eta'^2 \tag{2.9.9}$$

giving the equations of motion

$$\dot{\eta} = \frac{\partial\mathfrak{H}}{\partial\pi} = \frac{\pi}{\mu}, \quad \dot{\pi} = -\frac{\partial\mathfrak{H}}{\partial\eta} + \left(\frac{\partial\mathfrak{H}}{\partial\eta'}\right)' = Y\eta'' \tag{2.9.10}$$

To obtain the Lagrangian equation of motion from the above canonical ones, use the first to replace π with $\mu\dot{\eta}$ in the second and get

$$\mu\ddot{\eta} - Y\eta'' = 0 \tag{2.9.11}$$

as we had before.

As in the case of point particles, the canonical equations can also be given in terms of Poisson brackets. Let $\mathcal{A}[\pi, \phi, t]$ and $\mathcal{B}[\pi, \phi, t]$ be two functionals of the phase space, $\{\phi, \pi\}$, then define the Poisson brackets between them as

$$\{\mathcal{A}, \mathcal{B}\}_{P.B}^{t=t'} = \int d^3\vec{r}\left[\frac{\delta\mathcal{A}}{\delta\phi^A(\vec{r},t)}\frac{\delta\mathcal{B}}{\delta\pi_A(\vec{r},t)} - \frac{\delta\mathcal{A}}{\delta\pi_A(\vec{r},t)}\frac{\delta\mathcal{B}}{\delta\phi^A(\vec{r},t)}\right] \tag{2.9.12}$$

If we compare the above expression with the definition we used for point particles, we will notice that only two things have changed, *viz.,* (a) there is an additional integration over space and (b) the ordinary derivatives have been replaced by functional derivatives. Both changes are dictated by the fact that we are now dealing with Lagrangian *functionals.* Note, finally, that the Poisson brackets are defined *at equal times.*

Consider the functional

$$\mathcal{F}^A[\phi] = \int d^3\vec{r}\ \phi^A(\vec{r}, t) \tag{2.9.13}$$

then

$$\{\mathcal{F}^A, \mathcal{H}\}_{P.B.}^{t=t'} = \int d^3\vec{r} \frac{\delta \mathcal{H}}{\delta \pi_A(\vec{r},t)} = \int d^3\vec{r}\ \dot{\phi}^A(\vec{r},t) \tag{2.9.14}$$

and likewise,

$$\mathcal{F}_A[\pi] = \int d^3\vec{r}\ \pi_A(\vec{r},t) \tag{2.9.15}$$

leads to

$$\{\mathcal{F}_A, \mathcal{H}\}_{P.B.}^{t=t'} = -\int d^3\vec{r} \frac{\delta \mathcal{H}}{\delta \phi^A(\vec{r},t)} = \int d^3\vec{r}\ \dot{\pi}_A(\vec{r},t) \tag{2.9.16}$$

From these equations it will be clear that we could also define the fundamental Poisson bracket relations

$$\begin{aligned} \{\phi^A(\vec{r},t), \phi^B(\vec{r}',t')\}_{P.B.}^{t'=t} &= 0 = \{\pi_A(\vec{r},t), \pi_B(\vec{r}',t')\}_{P.B.}^{t'=t} \\ \{\phi^A(\vec{r},t), \pi_B(\vec{r}',t')\}_{P.B.}^{t'=t} &= \delta^A_B \delta^3(\vec{r}-\vec{r}') \end{aligned} \tag{2.9.17}$$

and write

$$\begin{aligned} \dot{\phi}^A(\vec{r},t) &= \{\phi^A(\vec{r},t), \mathcal{H}(t')\}_{P.B.}^{t'=t} \\ \dot{\pi}_A(\vec{r},t) &= \{\pi_A(\vec{r},t), \mathcal{H}(t')\}_{P.B.}^{t'=t} \end{aligned} \tag{2.9.18}$$

They would yield the canonical equations of (2.9.8).[22]

2.9.1 Scalar Fields

The algorithm to construct the Hamiltonian is straightforward for the complex scalar field. Using the Lagrangian functional in (2.5.10),

$$\mathcal{L} = -\int d^3\vec{r}\,[\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi + V(|\phi|)] \tag{2.9.19}$$

we compute the field momentum densities, one conjugate to each of ϕ and ϕ^*,

$$\pi^* = \frac{\partial \mathfrak{L}}{\partial \dot{\phi}^*} = \frac{1}{c^2}\dot{\phi}, \quad \pi = \frac{\partial \mathfrak{L}}{\partial \dot{\phi}} = \frac{1}{c^2}\dot{\phi}^* \tag{2.9.20}$$

and construct the Hamiltonian density

$$\mathfrak{H} = \pi^*\dot{\phi}^* + \pi\dot{\phi} - \mathfrak{L} = c^2\pi^*\pi + \{\vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + V(|\phi|)\} \tag{2.9.21}$$

[22] Problem: Show this!

and Hamiltonian

$$\mathcal{H}[\phi, \phi^*, \pi, \pi^*] = \int d^3\vec{r}\mathfrak{H} = \int d^3\vec{r}\left[c^2\pi^*\pi + \{\vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + V(|\phi|)\}\right]. \tag{2.9.22}$$

Notice that the Hamiltonian density is simply the time-time component of the canonical energy momentum tensor, Θ^{00}.[23]

The canonical (Hamilton) equations of motion become

$$\begin{aligned} \dot{\phi}(x) &= \{\phi(x), \mathcal{H}\}_{P.B} = \frac{\delta\mathcal{H}}{\delta\pi(x)} = c^2\pi^*(x), \\ \dot{\pi}(x) &= \{\pi(x), \mathcal{H}\}_{P.B.} = -\frac{\delta\mathcal{H}}{\delta\phi(x)} = \{\vec{\nabla}^2\phi^*(x) - V'(|\phi|)\phi^*(x)\}, \end{aligned} \tag{2.9.23}$$

together with their hermitean conjugates, and the second order (Lagrangian) equations are recovered in the usual way, for example

$$\dot{\pi}^* = \frac{1}{c^2}\ddot{\phi} = \{\vec{\nabla}^2\phi - V'(|\phi|)\phi\} \Rightarrow \Box_x\phi(x) + V'(|\phi|)\phi(x) = 0 \tag{2.9.24}$$

and its hermitean conjugate.

2.9.2 Massless Vector Fields

Straightforward though it is for an ordinary scalar field, there are some subtleties to keep in mind when dealing with massless vector fields. Let us begin with the Lagrangian in (2.7.1), taking $m = 0$, and see that

$$\pi^\alpha = \frac{\partial\mathfrak{L}}{\partial(\partial_t A_\alpha)} = gcF^{\alpha 0} \tag{2.9.25}$$

so $\pi^0 = 0$ (the momentum conjugate to A_0 vanishes!) is a primary constraint and $\pi^i = -gE^i/c$. We want the "velocities", $\dot{A}_i$, in terms of the momenta, so we write

$$\pi_i = gF_i{}^0 = -\frac{g}{c}F_{i0} = -\frac{g}{c}(\partial_i A_0 - \partial_0 A_i) \tag{2.9.26}$$

and solve to get

$$\partial_0 A_i = \dot{A}_i = \frac{c}{g}\pi_i + \partial_i A_0. \tag{2.9.27}$$

The Lagrangian density can now be expressed as

$$\mathfrak{L} = -\frac{gc}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{gc}{4}\left[2F_{i0}F^{i0} + F_{ij}F^{ij}\right] = \frac{c}{2g}\pi_i\pi^i - \frac{gc}{4}F_{ij}F^{ij}, \tag{2.9.28}$$

[23] Problem: Verify the energy localization conditions we used to find the kink/anti-kink solutions for the Sine-Gordon equation and compute the energy density for each solution.

so that we define the primary Hamiltonian density as

$$\mathfrak{H}_p = \pi^i \dot{A}_i + \pi^0 \mu - \mathfrak{L} = \frac{c}{2g}\pi_i \pi^i + \pi^0 \mu + \pi^i \partial_i A_0 + \frac{gc}{4} F_{ij} F^{ij}, \tag{2.9.29}$$

where we have introduced the Lagrange multiplier μ to enforce the single primary constraint $\Phi = \pi^0 \approx 0$. Integrating the above over space, we get Hamiltonian

$$\mathcal{H}_p = \int d^3\vec{r} \left[\frac{c}{2g}\pi_i \pi^i + \pi^0 \mu + \frac{gc}{4} F_{ij} F^{ij} + \pi^i \partial_i A_0\right] \tag{2.9.30}$$

and, again, integrating the last term by parts,

$$\mathcal{H}_p = \int d^3\vec{r} \left[\frac{c}{2g}\pi_i \pi^i + \frac{gc}{4} F_{ij} F^{ij} + \pi^0 \mu - A_0 \partial_i \pi^i\right], \tag{2.9.31}$$

where we have assumed that the fields fall off rapidly enough at infinity or vanish at the boundaries so that the surface terms disappear.

The first two terms in the Hamiltonian density will be recognized as the energy density of the vector field,

$$\frac{c}{2g}\pi_i \pi^i + \frac{gc}{4} F_{ij} F^{ij} = \frac{gc}{2}\left[\frac{\vec{E}^2}{c^2} + \vec{B}^2\right], \tag{2.9.32}$$

and may be compared with $\mathcal{E}$ in (2.7.29). We may also define the Poisson brackets of the fundamental fields,

$$\{A_\mu(\vec{r}, t), \pi^\nu(\vec{r}', t)\}_{P.B.} = \delta^\nu_\mu \delta^3(\vec{r} - \vec{r}') \tag{2.9.33}$$

and derive the canonical equations of motion from these relations via

$$\dot{A}_i(t, \vec{r}) = \{A_i(t, \vec{r}), \mathcal{H}_p\}_{P.B.} = \frac{c}{g}\pi_i + \partial_i A_0, \tag{2.9.34}$$

which is just the relation we had before for the “velocities”, and

$$\dot{\pi}^i(t, \vec{r}) = \{\pi^i(t, \vec{r}), \mathcal{H}_p\}_{P.B.} = -gc\partial_j F^{ji}. \tag{2.9.35}$$

We also have

$$\begin{aligned} \dot{A}_0 &= \{A_0(t, \vec{r}), \mathcal{H}_p\}_{P.B.} = \mu \\ \dot{\pi}^0 &= \{\pi^0(t, \vec{r}), \mathcal{H}_p\}_{P.B.} = \partial_i \pi^i \approx 0 \end{aligned} \tag{2.9.36}$$

(Gauss’ law is a secondary constraint). There are no further constraints, so we end up with one primary constraint ($\pi^0 \approx 0$) and one secondary constraint ($\partial_i \pi^i \approx 0$), both of which are first class. Thus the Dirac brackets between observables coincide with the Poisson brackets, the massless vector field has two (local) degrees of freedom and μ is not determined. The constraints generate gauge transformations.

2.10 Some Classical Solutions

Let us take a look at some useful, non-perturbative solutions of the theories we have studied so far. We will be interested in localized solutions of the classical (Euler-Lagrange) equations of motion. By "localized" we mean that the energy density vanishes fast enough at infinity so that the total energy of the system is finite. As we will see, there is a conserved charge, derivable from a locally defined current, associated with these solutions. However, this current does not arise out of a symmetry of the action and is therefore not a Noether current. It is topological current, constructed so that its conservation is guaranteed by the dimensionality of space-time.

2.10.1 Scalar Solitons in 1+1 dimensions

To warm up we return to the 1+1 dimensional sine-Gordon soliton ("solitary wave") of footnote 12, Chapter 2. Upon integrating once we had found

$$\phi'^2 - 2\sigma V(\phi) = C \tag{2.10.1}$$

where C is an integration constant. To get a feeling for what the interesting solutions might look like, we interpret C as the "energy" of a "particle" moving in the inverted potential $U(\phi) = -2\sigma\alpha(1 - \cos\phi/\beta)$ and u as a "time". For periodic motion we must require that this "energy" lies between the minima and maxima of $U(\phi)$, *i.e.,*

$$-4\sigma\alpha \leq C \leq 0. \tag{2.10.2}$$

The case $C > 0$ would correspond to unbounded motion. It is convenient to parametrize C as $C = -4\sigma\alpha(1-k^2)$ where $k \in [0,1]$. Then (2.10.1) turns into

$$\phi'^2 = 4\alpha\sigma(k^2 - \cos^2\phi/2\beta)$$

and, when $k = 1$ ($C = 0$), we obtain the solutions we had earlier,

$$\phi_\pm(u) = 4\beta\tan^{-1}\left\{\exp\left[\pm\frac{\sqrt{\alpha}}{\beta}\frac{u-u_0}{\sqrt{1-v^2/c^2}}\right]\right\} \tag{2.10.3}$$

where $u = x - vt$.[24] The energy density of these solutions,

$$\mathcal{H} = \frac{1}{2}\left[c^2\pi^2 + \phi'^2 + 2V(\phi)\right] = \frac{4\alpha}{1 - v^2/c^2}\text{sech}^2\left(\frac{\sqrt{\alpha}}{\beta}\frac{(u-u_0)}{\sqrt{1-v^2/c^2}}\right), \tag{2.10.4}$$

vanishes exponentially at $\pm\infty$. Integrating the energy density, we verify that their total energy is finite as required,

$$E = \int_{-\infty}^{\infty} dx\ \mathcal{H} = \frac{8\beta\sqrt{\alpha}}{\sqrt{1-v^2/c^2}}, \tag{2.10.5}$$

and can be used to define the kink/anti-kink mass, M, according to

$$E \stackrel{\text{def}}{=} \frac{Mc^2}{\sqrt{1-v^2/c^2}}. \tag{2.10.6}$$

At the other extreme, when $k = 0$, the only solutions are the constants $\phi = (2n+1)\pi\beta$ for integer n. Their energy density is constant, at the maximum of the sine-Gordon potential, so they are not localized. All the solutions in between, with $k \in (0, 1)$, will be periodic but not localized, and described by the Jacobi elliptic functions.

Let us concentrate on the $k = 1$ solutions. One can think of these solutions as taking us from one minimum of the potential (equivalently a maximum of the inverted potential) at $u \to -\infty$ to another minimum (maximum of the inverted potential) at $u \to \infty$. However, at the minima, the velocity of the field is zero and so are all of its higher derivatives! We can see that this is true for *any* potential satisfying the required condition at infinity as follows:

$$\phi' = \sqrt{2V(\phi)},\ \ \phi'' = V'(\phi),\ \ \ \phi''' = V''(\phi)\phi',\ \ \ldots \tag{2.10.7}$$

Therefore if $\phi(u)$ starts out in one of the minima, say, $\phi = 4n\pi\beta$, the kink solution transitions the field to the neighboring minimum at $\phi = 2(2n+1)\pi\beta$, but cannot get past it.

Although they possess the same energy density, the kink and anti-kink solutions are inequivalent. We say that two solutions are topologically inequivalent if *one of them*

[24] $\phi(u)$ is also obtained by Lorentz boosting the static solution ($v = 0$)

$$\phi(x) = 4\beta\tan^{-1}\left\{\exp\left[\pm\frac{\sqrt{\alpha}}{\beta}(x - x_0)\right]\right\},$$

which can be obtained by extremizing the energy functional

$$E[\phi] = \frac{1}{2}\int dx\left[\phi'^2 + 2V(\phi)\right].$$

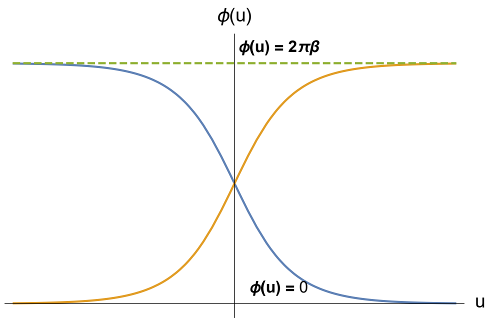

Figure 2.5: Kink and anti-kink solutions in the sine-Gordon theory.

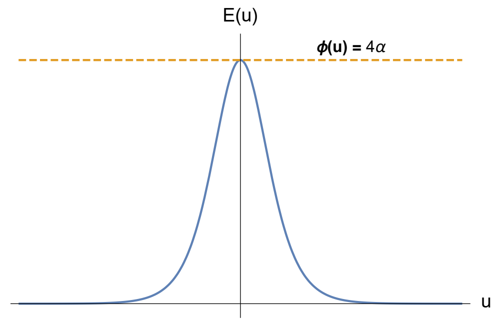

Figure 2.6: Energy density.

cannot be continuously deformed into the other without passing through a barrier of infinite action. To see that the kink and anti-kink solutions are inequivalent, consider only the static solutions and a deformation

$$\Phi(\eta, x) = g(\eta)\phi_+(x) + h(\eta)\phi_-(x) \tag{2.10.8}$$

where $g(\eta)$ and $h(\eta)$ are smooth functions of $\eta \in [0,1]$, satisfying $g(0) = 1$, $g(1) = 0$, $h(0) = 0$, $h(1) = 1$. These conditions ensure that $\Phi(0,x) = \phi_+$ and $\Phi(1,x) = \phi_-$. Consider the static action,

$$S(\eta) = -\frac{1}{2}\int dx\left[\Phi'^2 + 2\alpha\left(1 - \cos\Phi/\beta\right)\right]. \tag{2.10.9}$$

Using

$$\phi'_+ = +2\sqrt{\alpha}\sin\frac{\phi_+}{\beta}, \quad \phi'_- = -2\sqrt{\alpha}\sin\frac{\phi_-}{\beta}$$

it is then easy to show that

$$\Phi'^2 = \frac{4\alpha(g-h)^2}{\cosh^2\frac{\sqrt{\alpha}}{\beta}(x-x_0)}, \tag{2.10.10}$$

which ensures that the integral of the first term in (2.10.9) is finite. However, the integral of the second term,

$$2\alpha\int_{-\infty}^{\infty} dx\ \sin^2\left[2g\tan^{-1}e^{\frac{\sqrt{\alpha}}{\beta}(x-x_0)} + 2h\tan^{-1}e^{\frac{-\sqrt{\alpha}}{\beta}(x-x_0)}\right],$$

diverges unless $g = 0, 1$ and $h = 0, 1$, which conditions cannot be met by the deforming functions. Thus the two solutions are inequivalent.

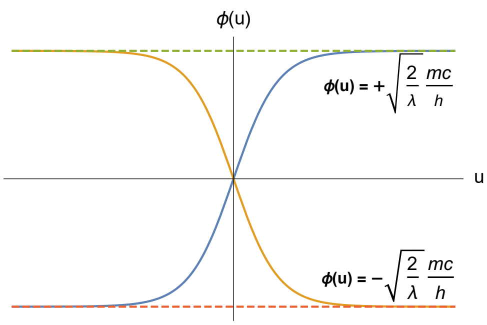

Figure 2.7: Kink and anti-kink solutions in the $\lambda\phi^4$ theory.

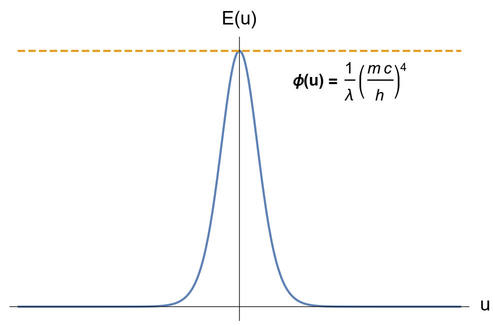

Figure 2.8: Energy density.

In two dimensions, a simpler way to see that the kink and anti-kink solutions are inequivalent is to define the conserved current,

$$J^\mu = \epsilon^{\mu\nu}\partial_\nu\phi, \tag{2.10.11}$$

where $\epsilon^{\mu\nu}$ is the two dimensional Levi-Civita symbol. Conservation of J^μ is due to the antisymmetry of the Levi-Civita tensor and is independent of the particular model being considered. J^μ is therefore a topological current not a Noether current, because it does not arise out of any continuous symmetry. It defines a (topological) charge,

$$Q = \int_{-\infty}^{\infty} dx\ J^0 = -\int_{-\infty}^{\infty} dx\ \phi'(u) = \phi|_{-\infty}^{\infty} = 0, \pm 2\pi\beta, \tag{2.10.12}$$

where we have included the charge, $Q = 0$, associated with the trivial solution $\phi(u) = 2n\pi\beta$. The positive charge is associated with the kink solution and the negative charge with the anti-kink. The sine-Gordon equation admits more solutions of the form

$$\phi(t,x) = 4\beta\tan^{-1}\left(\frac{f(x)}{g(t)}\right), \tag{2.10.13}$$

but we leave the development of such solutions and their interpretation for a more specialized study.

Kink and anti-kink solutions are also found in the "$\lambda\phi^4$" theory with negative m^2, for which the equation of motion may be written as

$$\square\phi - \frac{m^2c^2}{\hbar^2}\phi + \lambda\phi^3 = 0 \tag{2.10.14}$$

and the potential has two neighboring global minima at $\phi_{\text{min}} = \pm\sqrt{\frac{2}{\lambda}}\frac{mc}{\hbar}$, separated by a local maximum at $\phi_{\text{max}} = 0$. It can be viewed as a small field expansion about $\phi = 0$ of the Sine-Gordon equation, with the appropriate identifications of the constants ($\alpha < 0$), or as the Euler equation for the field action

$$S = -\frac{1}{2}\int d^2x \left[\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m^2c^2}{\hbar^2}\phi^2 + \frac{\lambda}{4}\phi^4\right] \tag{2.10.15}$$

By adding a suitable constant to the action, the potential may be redefined as

$$2V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{2m^2c^2}{\lambda\hbar^2}\right)^2 \tag{2.10.16}$$

and localized solutions obtained from

$$\int \frac{d\phi}{\sqrt{2V(\phi)}} = \pm\sqrt{\sigma}(u-u_0), \tag{2.10.17}$$

will be of the form

$$\phi_\pm(u) = \pm\sqrt{\frac{2}{\lambda}}\frac{mc}{\hbar}\tanh\frac{mc}{\sqrt{2}\hbar}\frac{(u-u_0)}{\sqrt{1-v^2/c^2}}. \tag{2.10.18}$$

Therefore the kink, ϕ_+, takes one from the left vacuum, $\phi = -\sqrt{\frac{2}{\lambda}}\frac{mc}{\hbar}$, at $u \to -\infty$ to the right vacuum, $\phi = +\sqrt{\frac{2}{\lambda}}\frac{mc}{\hbar}$, at $u \to +\infty$ and the anti-kink, ϕ_-, does precisely the reverse. Both solutions possess a localized energy density,

$$\mathcal{H} = \frac{1}{\lambda}\left(\frac{mc}{\hbar}\right)^4 \frac{\operatorname{sech}^4\left[\frac{mc}{\sqrt{2}\hbar}\frac{(u-u_0)}{\sqrt{1-v^2/c^2}}\right]}{1-v^2/c^2} \tag{2.10.19}$$

and a finite total energy,

$$E = \int_{-\infty}^{\infty} dx\ \mathcal{H} = \frac{4}{3\lambda\sqrt{1-v^2/c^2}}\left(\frac{mc}{\hbar}\right)^3 \stackrel{\text{def}}{=} \frac{M'c^2}{\sqrt{1-v^2/c^2}}, \tag{2.10.20}$$

but they are not equivalent, as seen from the topological charge,

$$Q = \int_{-\infty}^{\infty} dx\ J^0 = \phi|_{-\infty}^{\infty} = 0,\ \pm 2\sqrt{\frac{2}{\lambda}}\frac{mc}{\hbar}. \tag{2.10.21}$$

As before, the zero charge is associated with the trivial solution $\phi = 0$, the positive charge with the kink, ϕ_+, and the negative charge with the anti-kink, ϕ_-.

Can the above two dimensional solutions be extended to higher dimensions? The answer is "no", according to **Derrick's theorem**. The reason is that the finite total energy condition cannot be met in more than two spatial dimensions. To see that this is so, note that, in D spatial dimensions, the total energy corresponding to the field configuration $\phi(\vec{r})$ is

$$E[\phi] = \int d^D\vec{r}\ \mathcal{H} = \frac{1}{2}\int d^D\vec{r}\left[\delta^{ij}\partial_i\phi(\vec{r})\partial_j\phi(\vec{r}) + 2V(\phi)\right] \stackrel{\text{def}}{=} E_1[\partial\phi] + E_2[\phi], \quad (2.10.22)$$

Suppose that $\phi_1(\vec{r})$ is a localized solution of $\delta E[\phi] = 0$. A necessary condition for the solution to be stable is that the second variation of the energy function should be greater than or equal to zero, $\delta^2 E[\phi] \geq 0$. To find out if $\delta^2 E[\phi_1] \geq 0$, consider the field configurations

$$\phi_\lambda(\vec{r}) = \phi_1(\lambda\vec{r}) \quad (2.10.23)$$

where $\lambda > 0$ is a scale factor. $\phi_\lambda(\vec{r})$ is not, in general, a solution of the field equations, except when $\lambda = 1$. The energy functional for ϕ_λ is

$$E[\phi_\lambda] = E_\lambda = \frac{1}{2}\int d^D\vec{r}\left[\delta^{ij}\partial_i\phi_\lambda(\vec{r})\partial_j\phi_\lambda(\vec{r}) + 2V(\phi_\lambda)\right] \quad (2.10.24)$$

With $\vec{r}' = \lambda\vec{r}$, $y^i = \lambda x^i$ we write it as

$$\begin{aligned} E_\lambda &= \frac{1}{2}\int d^D\vec{r}'\lambda^{-D}\left[\lambda^2\delta^{ij}\frac{\partial\phi_1(\vec{r}')}{\partial y^i}\frac{\partial\phi_1(\vec{r}')}{\partial y^j} + 2V(\phi_1(\vec{r}'))\right] \\ &= \lambda^{-D+2}E_1[\partial\phi_1] + \lambda^{-D}E_2[\phi_1] \end{aligned} \quad (2.10.25)$$

E_λ is extreme when

$$\left.\frac{dE_\lambda}{d\lambda}\right|_{\lambda=1} = (2-D)E_1[\partial\phi_1] - DE_2[\phi_1] = 0 \;\Rightarrow\; E_2[\phi_1] = \frac{2-D}{D}E_1[\partial\phi_1]$$

and stability requires that

$$\left.\frac{d^2E_\lambda}{d\lambda^2}\right|_{\lambda=1} = (2-D)(1-D)E_1[\partial\phi_1] + D(D+1)E_2[\phi_1] \geq 0 \quad (2.10.26)$$

Now $E_1[\phi_1]$ is certainly non-negative and, if $V(\phi_1)$ is non-negative as well, so is $E_2[\phi_1]$. The first condition implies that $D \leq 2$, in which case the stability requirement also holds. Moreover,

- if $D = 1$ (the case we have been examining) then $E_1 = E_2$ and
- if $D = 2$ then $E_2 = 0$ ($V(\phi) = 0$).

There are no solutions when $D > 2$.

It is interesting to ask what is the significance of the potential with $m^2 < 0$. If the coefficient of the quadratic term in the potential were positive then it has the interpretation of a mass term and the potential energy would admit just one lowest energy configuration, *i.e.,* at $\phi = 0$. However, a negative quadratic term cannot be interpreted as a mass term and now the potential energy admits two non trivial lowest energy configurations corresponding to the values of ϕ given above. In the case of a complex scalar field, it gives an infinite number of lowest energy configurations, each differing from the other by a phase. When coupled to a gauge field, the complex scalar field may be expanded about one of its lowest energy configurations,

$$\phi = \phi_{\text{min}} + H$$

and the Lagrangian, written in terms of perturbations, H, about the lowest energy configuration, now exhibits a mass term for the gauge fields. Any particular choice of field configuration (amounting to a particular choice of the "vacuum" as indicated above) is not gauge invariant. Therefore, in expanding the scalar field about one of its lowest energy configurations, one of the scalar field degrees of freedom gets donated to the gauge field which, upon acquiring the additional degree of freedom, becomes massive. This is called **spontaneous symmetry breaking** (the $U(1)$ gauge symmetry is spontaneously broken by the choice of vacuum) and the process by which the gauge field acquires a mass is the **Higgs** mechanism. We will examine this in greater detail in a forthcoming chapter.

2.10.2 The Abelian Higgs Model

Here we want to consider some non-perturbative solutions of the "Abelian-Higgs" model which is described by the Lagrange density (in D dimensions)

$$\mathfrak{L} = -\eta^{\mu\nu}(D_\mu\phi)^* D_\nu\phi - V(|\phi|) - \frac{gc}{4}F^{\mu\nu}F_{\mu\nu} \tag{2.10.27}$$

We are looking for static configurations, so let us begin by noting that it is always possible to pick a gauge in which $A_0 = 0$ (the "temporal" gauge). If we begin with any gauge configuration, $A_\mu(\vec{r}, t)$, and apply a gauge transformation, $U = e^{ig\alpha(\vec{r},t)}$, then $A_0(\vec{r}, t) \to A'_0(\vec{r}, t) = A_0(\vec{r}, t) + \partial_t\alpha(\vec{r}, t)$. Setting $A'_0(\vec{r}, t) = 0$ we find that

$$\partial_t\alpha(t) = -A_0(x), \quad \Rightarrow \; U = e^{-ig\int^t A_0(x,t')dt'}.$$

So we will work in the gauge in which $D_t\phi = 0$ and, for static solutions, the energy density will be

$$\mathcal{H} = \eta^{ij}(D_i\phi)^*(D_j\phi) + V(|\phi|) + \frac{gc}{4}F^{ij}F_{ij}. \tag{2.10.28}$$

The total energy can therefore be written as the sum of three terms

$$E[\phi, A] = E_1[D\phi] + E_2[\phi] + E_3[A] \tag{2.10.29}$$

Let us repeat Derrick's scaling argument and assume that we already know a solution pair $\{\phi_1(\vec{r}), A_{(1)\mu(\vec{r})}\}$ of the equations of motion. If we consider the family of functions (not solutions except when $\lambda = 1$)

$$\phi_\lambda(\vec{r}) = \phi(\lambda\vec{r}), \quad A_{(\lambda)\mu}(\vec{r}) = \lambda A_\mu(\lambda\vec{r}) \tag{2.10.30}$$

then, following the same argument we had for the scalar field, we should find that the energy functional scales as

$$E_\lambda = \lambda^{-D}\left[\lambda^2 E_1[D\phi] + E_2[\phi] + \lambda^4 E_3[A]\right] \tag{2.10.31}$$

and therefore that for stable solutions the following conditions must hold:

$$\begin{aligned} \left.\frac{dE_\lambda}{d\lambda}\right|_{\lambda=1} &= (2-D)E_1[D\phi] - DE_2[\phi] + (4-D)E_3[A] = 0 \\ \left.\frac{d^2E_\lambda}{d\lambda^2}\right|_{\lambda=1} &= (2-D)(1-D)E_1[D\phi] + D(D+1)E_2[\phi] + (4-D)(3-D)E_3[A] \geq 0. \end{aligned} \tag{2.10.32}$$

- In $D = 1$ we find $E_1 - E_2 + 3E_3 = 0$ but there are no propagating gauge fields in one dimension, so $E_3 = 0$, in which case we recover the $D = 1$ result for the scalar field alone: $E_1 = E_2 \geq 0$.
- In $D = 2$ we find $E_2 = E_3 \geq 0$. This is known as a "vortex".
- In $D = 3$ the conditions read $E_1 + 3E_2 = E_3$, $E_1 + 6E_2 \geq 0$. This is a "monopole" solution.
- In $D = 4$, $E_1 + 2E_2 = 0$ and $E_2 \geq 0$, but since $E_1 \geq 0$ these conditions can only hold if $E_1 = 0 = E_2$. The solution consists of the gauge field alone.

We will examine some of these solutions below.

A $U(1)$ Vortex in 2+1 dimensions

We are interested in static solutions of the Euler equations. For an arbitrary potential and in the temporal gauge, we must then solve the equations

$$-D^i D_i\phi + V'(|\phi|)\phi = 0$$

$$\nabla_j F^{ij} + \frac{i}{c}\phi^* \overleftrightarrow{D}^i \phi = 0 \tag{2.10.33}$$

where $D_i = \nabla_i - igA_i$. It will be convenient to work in polar coordinates in which $A_i = A_i(r)$ is required for cylindrical symmetry and, for the scalar field, we make the ansatz $\phi(r,\theta) = f(r)e^{i\sigma(\theta)}$, where $\sigma(\theta + 2\pi) = \sigma(\theta) + 2k\pi$ for integer k is required so that $\phi(r,\theta)$ is single valued.[25]

Begin with the simplest choice (obeying the single valuedness condition) for the phase function, $\sigma(\theta) = -k\theta$ (integer k); Maxwell's tensor has just one independent spatial component in two spatial dimensions

$$F_{r\theta} = \partial_r A_\theta - \partial_\theta A_r = \partial_r A_\theta = -F_{\theta r} = rB \tag{2.10.34}$$

where B is the "magnetic" field. The vector field equations then read

$$\begin{aligned} &A_r = 0 \\ &A_\theta'' - \frac{A_\theta'}{r} + \frac{2}{c}\left(k + gA_\theta\right) f^2 = 0 \end{aligned} \tag{2.10.35}$$

(assuming $f(r) \neq 0$) and, for the scalar field,

$$f'' + \frac{f'}{r} - \frac{1}{r^2}(k + gA_\theta)^2 f + V'(f) = 0 \tag{2.10.36}$$

The energy density of the system is

$$\mathcal{H} = f'^2 + \frac{gc}{2}\frac{A_\theta'^2}{r^2} + \frac{(k + gA_\theta)^2}{r^2} f^2 + V(f), \tag{2.10.37}$$

therefore, to ensure that the total energy,

$$E = 2\pi \int dr\ r \left[f'^2 + \frac{gc}{2}\frac{A_\theta'^2}{r^2} + \frac{(k + gA_\theta)^2}{r^2} f^2 + V(f) \right], \tag{2.10.38}$$

[25] In all that follows we define the components of a vector as its *coordinate* components, so that the line integral is

$$\int_C \vec{A} \cdot d\vec{r} = \int_C A_i dy^i$$

where y^i are the spatial coordinates. For example, if the spatial coordinate system is spherical, with coordinates (r, θ, ϕ), we define the components by

$$\int_C \vec{A} \cdot d\vec{r} = \int_C A_r dr + A_\theta d\theta + A_\phi d\phi$$

and so on (see Chapter 6).

is finite we adopt the following boundary conditions on the fields:[26]

$$\lim_{r\to\infty} f = \text{const.}, \quad \lim_{r\to\infty} V(f) = 0, \quad \lim_{r\to\infty} A_\theta = -\frac{k}{g}$$
$$\lim_{r\to 0} f = 0, \quad \lim_{r\to 0} V(f) = \text{const.}, \quad \lim_{r\to 0} \frac{A_\theta}{r} = 0. \tag{2.10.39}$$

We can define a topological current in much the same way as we did for the scalar field in 1+1 dimensions,

$$J^\mu = -\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda. \tag{2.10.40}$$

As before, its conservation is a result of the antisymmetry of the Levi-Civita symbol. The current leads to a (conserved) topological charge

$$Q = \int d^2\vec{r}\; J^0 = -\int d^2\vec{r}\; \epsilon^{0ij}\partial_i A_j = -2\pi \int_0^\infty dr \partial_r A_\theta = -2\pi\; A_\theta|_0^\infty\,, \tag{2.10.41}$$

where we used the fact that the Levi-Civita symbol is a density of weight -1.[27] The boundary conditions in (2.10.39) therefore determine this charge and we find

$$Q = \frac{2\pi k}{g}. \tag{2.10.42}$$

The integral determining Q is the magnetic flux through the disk bounded by the circle at infinity and therefore, for $k \neq 0$, it represents a quantized magnetic charge.

For definiteness, we will now concentrate on the $|\phi|^4$ model, taking

$$V(|\phi|) = \frac{\lambda}{8}(\phi^*\phi - v^2)^2 = \frac{\lambda}{8}(f^2 - v^2)^2 \tag{2.10.43}$$

so that, requiring $V(f) \to 0$ as $r \to \infty$ implies that $f \to \pm v$ in that limit. Also, $V(f) \to \lambda v^4/8$ as $r \to 0$.

To get a better feeling for the asymptotic behavior of the solutions, take $f(r) = v + h(r)$, $A_\theta(r) = -k/g + A(r)$ as $r \to \infty$ and linearize the Euler equations to get

$$h'' + \frac{h'}{r} + \lambda v^2 h = 0$$

[26] The boundary conditions at infinity ensure that $\lim_{r\to\infty} D_i\phi = 0$ so that the scalar kinetic term vanishes on the boundary.

[27] Problem: In Cartesian coordinates, x^α, the Levi-Civita tensor is just the permutation symbol, *i.e.,* $\epsilon^{\alpha\beta\gamma} = [\alpha,\beta,\gamma]$, with the convention $\epsilon^{012} = [0,1,2] = +1$. In a general coordinate system, y^μ, one must transform according to the general rules for tensors,

$$\epsilon^{\mu\nu\lambda} = \frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta}\frac{\partial y^\lambda}{\partial x^\gamma}[\alpha,\beta,\gamma].$$

Show that in polar coordinates, $\epsilon^{0r\theta} \equiv \epsilon^{r\theta} = \frac{1}{r}$ and that $\epsilon_{0r\theta} \equiv \epsilon_{r\theta} = r$.

$$A'' - \frac{A'}{r} + \frac{2gv^2}{c}A = 0. \tag{2.10.44}$$

Solutions obeying the required boundary conditions are

$$\begin{aligned} f(r) &= v + C_1 J_0(\sqrt{\lambda} v r) \\ A_\theta &= -\frac{k}{g} + C_2 r J_1\left(\sqrt{\frac{2g}{c}} v r\right) \end{aligned} \tag{2.10.45}$$

It is very difficult to obtain exact solutions to the coupled, second order equations above.

We get some insight into the behavior of solutions by examining particular cases. Let us discuss an interesting limit in which the energy of the system is minimum and determined only by the topological charge; in this limit, the system will reduce to a set of (coupled) first order equations, but we will still not be able to get exact, analytical solutions. First, we apply Derrick's argument, by which $E_2[\phi] = E_3[A]$. Thus

$$\int d^2x \left[\frac{gc}{4} F_{ij} F^{ij} - \frac{\lambda}{8}(|\phi|^2 - v^2)^2\right] = 0, \tag{2.10.46}$$

or

$$\frac{gc}{4}\int d^2x \left[F_{ij} + \frac{\epsilon_{ij}}{2}\sqrt{\frac{\lambda}{gc}}(|\phi|^2 - v^2)\right]\left[F^{ij} - \frac{\epsilon^{ij}}{2}\sqrt{\frac{\lambda}{gc}}(|\phi|^2 - v^2)\right] = 0 \tag{2.10.47}$$

so, if we ask for solutions satisfying

$$F_{ij} + \frac{\epsilon_{ij}}{2}\sqrt{\frac{\lambda}{gc}}(|\phi|^2 - v^2) = 0, \tag{2.10.48}$$

we guarantee that Derrick's condition is satisfied. Consider then the total energy, written as

$$\begin{aligned} E &= \int d^2x \left[\frac{gc}{4}\left(F_{ij} + \frac{\epsilon_{ij}}{2}\sqrt{\frac{\lambda}{gc}}(|\phi|^2 - v^2)\right)^2 \right. \\ &\qquad \left. - \frac{\sqrt{\lambda gc}}{4}\epsilon^{ij} F_{ij}(|\phi|^2 - v^2) + \eta^{ij}(D_i\phi)^*(D_j\phi)\right] \\ &= \int d^2x \left[-\frac{\sqrt{\lambda gc}}{4}\epsilon^{ij} F_{ij}(|\phi|^2 - v^2) + \eta^{ij}(D_i\phi)^*(D_j\phi)\right] \\ &= \int d^2x \left[-\frac{\sqrt{\lambda gc}}{4}\epsilon^{ij} F_{ij}|\phi|^2 + \eta^{ij}(D_i\phi)^*(D_j\phi)\right] - \frac{\sqrt{\lambda gc}}{8} Q v^2 \end{aligned} \tag{2.10.49}$$

where Q is the topological charge in (2.10.42).

Turning to the kinetic term for the scalar field, we write

$$\eta^{ij}(D_i\phi)^*(D_j\phi) = \frac{1}{2}\eta^{ij}(\{\eta_{il}+i\epsilon_{il}\}D^l\phi)^*(\{\eta_{jm}+i\epsilon_{jm}\}D^m\phi) - i\epsilon^{ij}(D_i\phi)^*(D_j\phi) \quad (2.10.50)$$

which can be checked directly by expanding the right hand side. Moreover,

$$-i\int d^2x\epsilon^{ij}(D_i\phi)^*(D_j\phi) = +\frac{i}{2}\int d^2x\epsilon^{ij}\phi^*[D_i, D_j]\phi = +\frac{g}{2}\int d^2x\epsilon^{ij}F_{ij}|\phi|^2 \quad (2.10.51)$$

follows from an integration by parts, therefore

$$\int d^2x\eta^{ij}(D_i\phi)^*(D_j\phi) = \frac{1}{2}\int d^2x\left[\eta^{ij}(\{\eta_{il}+i\epsilon_{il}\}D^l\phi)^*(\{\eta_{jm}+i\epsilon_{jm}\}D^m\phi) + \frac{g}{2}\epsilon^{ij}F_{ij}|\phi|^2\right]. \quad (2.10.52)$$

We may now rewrite what's left of the energy function as

$$E = \int d^2x\left[\left(\frac{g}{2} - \frac{\sqrt{\lambda g c}}{4}\right)\epsilon^{ij}F_{ij}|\phi|^2 + \frac{1}{2}\eta^{ij}(\{\eta_{il}+i\epsilon_{il}\}D^l\phi)^*(\{\eta_{jm}+i\epsilon_{jm}\}D^m\phi)\right] - \frac{\sqrt{\lambda g c}}{8}Qv^2 \quad (2.10.53)$$

The second term in the integrand is non-negative, but the first term can go either way depending on the relative sizes of g and λ. If we consider the special case in which

$$g = \frac{\lambda c}{4} \quad (2.10.54)$$

then this term vanishes and we are left with a non-negative integral plus a topological term. The lowest energy solution will then satisfy

$$(\eta_{il}+i\epsilon_{il})D^l\phi = 0. \quad (2.10.55)$$

Equations (2.10.48) and (2.10.55) are the Bogomol'nyi-Prasad-Sommerfield **(BPS)** equations. The energy of this solution is $E = -\frac{1}{2}\pi k v^2$ for the ansatz we have been using.

One must now verify that these first order equations are solutions of the second order equations of motion. In polar coordinates, (2.10.48) reads

$$\frac{A'_\theta}{r} = -\frac{1}{c}(f^2 - v^2) \quad (2.10.56)$$

and (2.10.55)

$$f' + \frac{(k + gA_\theta)}{r}f = 0. \quad (2.10.57)$$

Indeed, the Euler (field) equations (2.10.35) and (2.10.36) are verified directly by taking a derivative of each of the above.

2.10.3 The 3+1 dimensional Dirac Monopole

Let us briefly review Dirac's treatment of the magnetic monopole. Suppose that a point-like particle is endowed with a magnetic charge, q_m. The magnetic flux due to the magnetic charge would be

$$\Phi_B = \oint_S \vec{B} \cdot d\vec{S} = \frac{q_m}{4\pi} \;\Rightarrow\; B_r = \frac{q_m}{4\pi r^2} \tag{2.10.58}$$

and all other components would vanish. B_r is defined everywhere except at the origin and is isotropic, as is to be expected. If we insist that the vector potential exists as a description of the theory, then $B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}$ gives

$$F_{r\theta} = F_{r\phi} = 0$$

$$F_{\theta\phi} = \frac{q_m}{4\pi}\sin\theta \tag{2.10.59}$$

The well-known solution that vanishes at the north pole is

$$A_\phi = \frac{q_m}{4\pi}\left(1-\cos\theta\right), \tag{2.10.60}$$

which is, however, singular on the half line, $\theta = \pi$,[28] so the magnetic field obtained by taking the curl of $\vec{A}$ coincides with the magnetic field in (2.10.58) everwhere except on the half line itself. That is because, physically, this is precisely the vector potential of a semi-infinite and infinitesimally thin solenoid from infinity to the origin. This solenoid is the Dirac "string".

Now, at the classical level, the vector potential itself is not given any intrinsic meaning. One can think of it as a convenient device with which to write the field equations in such a way as to make the symmetries of the theory manifest. Thus one could consider two gauge fields, both of which are solutions of the field equations (Wu and Yang)

$$A_\phi = \begin{cases} \frac{q_m}{4\pi}\left(1-\cos\theta\right) & 0 \le \theta < \pi-\epsilon \\ -\frac{q_m}{4\pi}\left(1+\cos\theta\right) & \epsilon < \theta \le \pi \end{cases} \tag{2.10.62}$$

[28]This becomes clear if we consider the components of the vector potential in the traditional basis, $\{\hat{r}, \hat{\theta}, \hat{\varphi}\}$,

$$\vec{A} = a_r\hat{r} + a_\theta\hat{\theta} + a_\phi\hat{\varphi} \tag{2.10.61}$$

Then by comparing the expression for the line integral,

$$\int_C \vec{A}\cdot d\vec{r} = \int_C (a_r dr + r a_\theta d\theta + r\sin\theta a_\phi d\phi) = \int_C A_r dr + A_\theta d\theta + A_\phi d\phi$$

we arrive at $a_r = A_r$, $a_\theta = A_\theta/r$ and

$$a_\phi = \frac{A_\phi}{r\sin\theta} = \frac{q_m}{4\pi r}\frac{(1-\cos\theta)}{\sin\theta} = \frac{q_m}{4\pi r}\tan\left(\frac{\theta}{2}\right).$$

or, in the $\{\widehat{r}, \widehat{\theta}, \widehat{\varphi}\}$ basis,

$$a_\phi = \begin{cases} \frac{q_m}{4\pi r} \tan(\theta/2) & 0 \le \theta < \pi - \epsilon \\ -\frac{q_m}{4\pi r} \cot(\theta/2) & \epsilon < \theta \le \pi \end{cases} \tag{2.10.63}$$

The first (we call it $\vec{A}^{\rm up}$) covers the upper hemisphere and a portion of the lower hemisphere, and the second (which we call $\vec{A}^{\rm down}$) covers the lower hemisphere, but only a portion of the upper. Where they overlap, they are required to agree up to a gauge transformation and it is easy to see that $A_\phi^{\rm up} - A_\phi^{\rm down} = \frac{q_m}{2\pi}$ so that $\alpha(x) = \frac{q_m \phi}{2\pi}$ does the trick. The gauge transformation is therefore

$$U = e^{ig\alpha(x)} = e^{i\frac{gq_m}{2\pi}\phi}, \tag{2.10.64}$$

which, being single valued, implies that

$$gq_m = 2n\pi \;\Rightarrow\; q_m = \frac{2n\pi}{g}, \quad n \in \mathbb{Z} \tag{2.10.65}$$

Thus we arrive at Dirac's conclusion that the magnetic charge on a monopole is quantized in integer units of $2\pi/g$.

We can take an alternative, more physical approach by asking if this quantization of magnetic charge is actually required for consistency by any measurement. After all, if the gauge potential is really just an artifact of our description, should it not be impossible to detect the Dirac string in the first place? Not in the quantum theory. Consider the Schroedinger equation for an electron moving in the neighborhood of the string: we have

$$E\psi(\vec{r}) = -\frac{\hbar^2}{2m}\left(\vec{\nabla} - ig\vec{A}(\vec{r})\right)^2 \psi(\vec{r}) + V(\vec{r})\psi(\vec{r}) \tag{2.10.66}$$

where g is effectively the electric charge on the electron. The solution can be given as

$$\psi(\vec{r}) = \exp\left[ig\int_{\vec{r}_0}^{\vec{r}} \vec{A}\cdot d\vec{r}\right]\psi_0(\vec{r}) \tag{2.10.67}$$

where $\psi_0(\vec{r})$ satisfies the free Schroedinger equation! Therefore, the electron will pick up a phase as it moves around the Dirac string. If we consider two paths passing the string on opposite sides of it (see figure 2.9), there will be a difference in phase acquired along the paths and this phase difference can lead to intereference fringes. The phase difference will be

$$g\oint \vec{A}\cdot d\vec{r} = gq_m \tag{2.10.68}$$

taking the integral over an infinitesimal circle around the string ($\theta \to \pi$). For the string to be unobservable,

$$gq_m = 2n\pi \quad \Rightarrow q_m = \frac{2n\pi}{g} \quad n \in \mathbb{Z} \tag{2.10.69}$$

as before, and the monopole cannot have an arbitrary charge: only quanta of $2\pi/g$.

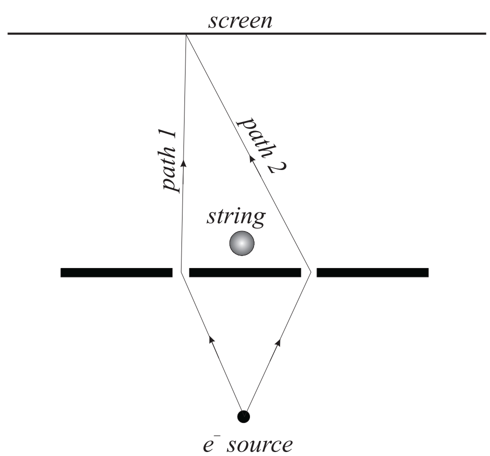

Figure 2.9: A violation of the magnetic charge quantization condition would lead to a detectable interference pattern on the screen due to the magnetic flux.

Chapter 3

Spinor Fields

Dirac, still thinking in terms of single particle relativistic quantum mechanics, attempted to find an equation that was first order in time so as to overcome the problems associated with an indefinite probability density. While he succeeded with a most remarkable first order equation, negative energy states were still present, so we will interpret Dirac's equation as the equation of a *classical* field, the Dirac field, in keeping with our general view. The Dirac field turns out to describe Fermions. Together with the scalar field and the non-abelian gauge fields, which generalize the abelian gauge field and will be introduced in the next chapter, this will give us the ingredients necessary to construct a Lagrangian description of the Standard Model of particle physics.

3.1 The Dirac Equation

Recall the problems associated with the scalar field of the previous chapter if the function $\phi(x)$ were to be interpreted as a wave function. In (2.4.4), we found the conserved current,

$$j_\mu = -\frac{i}{\hbar}(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*), \tag{3.1.1}$$

whose conservation law $\partial_\mu j^\mu = 0$ should then be interpreted as a continuity equation,

$$\partial_t\rho + \vec{\nabla}\cdot\vec{j} = 0, \tag{3.1.2}$$

with $\rho(t,\vec{r})d^3\vec{r} = j^0(t,r)d^3\vec{r}$ representing the probability of finding the Klein-Gordon particle in the volume $d^3\vec{r}$, according to the Born interpretation. However,

$$\rho(t,\vec{r}) = j^0(t,\vec{r}) = \frac{i}{\hbar c^2}(\phi^*\dot{\phi} - \phi\dot{\phi}^*) \tag{3.1.3}$$

is *not* positive definite, as is required for the Born interpretation. The problem can be traced to the fact that the Klein Gordon equation is second order in the time derivative, which also means that two initial data, *i.e.,* the value of ϕ *and* its first derivative at one instant must be specified for a unique solution of the equation.

To avoid the problems associated with the second order (in time) equation, Dirac proposed that the Hamiltonian of the free relativistic particle should instead be treated as

$$\mathcal{H} = \pm\sqrt{\vec{p}^2 c^2 + m^2 c^4} \stackrel{\text{def}}{=} \left(\widehat{\alpha} \cdot \vec{p}\, c + \widehat{\beta} m c^2\right). \tag{3.1.4}$$

If we agree to go with this view then $\widehat{\alpha}$ and $\widehat{\beta}$ are evidently no longer real numbers but matrices. This becomes clear if we square both sides of the above equation,

$$\vec{p}^2 c^2 + m^2 c^4 = \frac{c^2}{2}\{\widehat{\alpha}^i, \widehat{\alpha}^j\} p_i p_j + m^2 c^4 \widehat{\beta}^2 + \frac{c^3}{2}\{\widehat{\alpha}^i, \widehat{\beta}\} m p_i \tag{3.1.5}$$

where the braces, in eg., $\{\widehat{\alpha}^i, \widehat{\alpha}^j\}$, represent the anticommutator $\widehat{\alpha}^i \widehat{\alpha}^j + \widehat{\alpha}^j \widehat{\alpha}^i$. Because the Hamiltonian operator is required to be Hermitean (to ensure real eigenvalues), we should take $\widehat{\alpha}^i$ and $\widehat{\beta}$ to be Hermitean as well. Furthermore, comparing the left and right hand sides, consistency requires

$$\{\widehat{\alpha}^i, \widehat{\alpha}^j\} = 2\eta^{ij}, \quad \{\widehat{\alpha}^i, \widehat{\beta}\} = 0, \quad \widehat{\beta}^2 = 1 \tag{3.1.6}$$

Assuming that this algebra is self-consistent, we can then apply the Dirac quantization rule, letting $p_i \to -i\hbar\partial_i$ and $\mathcal{H} \to i\hbar\partial_t$ to get

$$i\hbar\partial_t \psi = \left(-ic\hbar\widehat{\alpha} \cdot \partial + mc^2 \widehat{\beta}\right)\psi, \tag{3.1.7}$$

which is the **Dirac equation**. It is convenient, however, to multiply this equation throughout by $\widehat{\beta}$ on the left and to define $c\gamma^0 = \widehat{\beta}$, $\gamma^i = \widehat{\beta}\widehat{\alpha}^i$.[1] This implies then that γ^0 is Hermitean but γ^i is anti-Hermitean because $\gamma^{i\dagger} = \widehat{\alpha}^{i\dagger}\beta^\dagger = \widehat{\alpha}^i \beta = -\beta\widehat{\alpha}^i = -\gamma^i$. In terms of the "$\gamma$ matrices", the equation above can be written as

$$(ic\hbar\gamma^\mu \partial_\mu - mc^2)\psi = 0. \tag{3.1.8}$$

This is the most commonly used form of the Dirac equation.

The γ^μ satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}\mathbf{1}, \tag{3.1.9}$$

where $\mathbf{1}$ is the identity matrix. This is a particular example of a **Clifford Algebra**. Notice that if we multiply the Dirac equation on the left by the operator $-(ic\hbar\gamma \cdot \partial + mc^2)$, then we will recover the Klein Gordon equation,

$$-(ic\hbar\gamma \cdot \partial + mc^2)(ic\hbar\gamma \cdot \partial - mc^2)\psi \;=\; -\left(-\frac{\hbar^2 c^2}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu - m^2 c^4\right)\psi = 0$$

[1] Although each γ is not a number we drop the "hats" for ease of notation.

$$\Rightarrow \quad \left(\square_x + \frac{m^2c^2}{\hbar^2}\right)\psi \quad = \quad 0, \tag{3.1.10}$$

as in (2.4.1).

Being first order in the time derivative, one could construct a (positive definite) probability density out of the Dirac equation in the usual way as $\rho = \psi^\dagger\psi$. This fixes the mechanical dimension of ψ to be $l^{-3/2}$. Furthermore, we expect that this probability density will be the "time" component of a four vector current density and this in turn leads us to suspect that the current density should have the form

$$j^\mu = c^2\psi^\dagger\gamma^0\gamma^\mu\psi. \tag{3.1.11}$$

The combination "$c^2\gamma^0\gamma^\mu$" is necessary for the desired expression, $\rho = j^0 = \psi^\dagger\psi$ and it makes j^μ Hermitean.[2] We must show that it is conserved so, calling $\overline{\psi} = c^2\psi^\dagger\gamma^0$, we must evaluate $\partial_\mu(\overline{\psi}\gamma^\mu\psi)$. To do this we have to determine the equation of motion for $\overline{\psi}$, which is obtained from the Hermitean conjugate of the Dirac equation,

$$c\hbar(i\partial_\mu\psi)^\dagger\gamma^{\mu\dagger} = mc^2\psi^\dagger. \tag{3.1.12}$$

Note that $(i\partial_\mu\psi)^\dagger = -i\partial_\mu\psi^\dagger$ and so the above equation becomes

$$-ic\hbar\partial_\mu\psi^\dagger\gamma^{\mu\dagger} = mc^2\psi^\dagger. \tag{3.1.13}$$

Now multiply on the right by γ^0 and introduce unity in the form $c^2{\gamma^0}^2$ as follows:

$$-ic\hbar\partial_\mu\psi^\dagger \underbrace{c^2\gamma^0\gamma^0}\gamma^{\mu\dagger}\gamma^0 = mc^2\psi^\dagger\gamma^0 \tag{3.1.14}$$

then, because

$$\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu/c^2, \tag{3.1.15}$$

(see the footnote below), one gets the following equation of motion for $\overline{\psi}$

$$-ic\hbar\partial_\mu\overline{\psi}\gamma^\mu = mc^2\overline{\psi} \tag{3.1.16}$$

and it follows immediately that $\partial_\mu j^\mu = 0$. The conservation equation can once again be interpreted as a continuity equation for the probability current density, but this time the probability density is positive definite, satisfying the requirements of the Born interpretation.

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[2] Problem: Show explicitly that $\gamma^{\mu\dagger} = c^2\gamma^0\gamma^\mu\gamma^0$. Then show that j^μ is hermitean.

3.2 The Clifford Algebra

A realization of the Clifford Algebra in (3.1.9) can be given in terms of the unit 2×2 matrix and the Pauli sigma matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.2.1}$$

which are Hermitean and satisfy two very interesting properties, *viz.*,

$$\{\sigma^i, \sigma^j\} = 2\eta^{ij} \quad \text{and} \quad [\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k \tag{3.2.2}$$

as can be checked explicitly. If we now define the 4×4 **Dirac matrices** as

$$\gamma^0 = \frac{1}{c}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \stackrel{\text{def}}{=} \frac{1}{c}\sigma^3 \otimes \mathbf{1}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \stackrel{\text{def}}{=} i\sigma^2 \otimes \sigma^i, \tag{3.2.3}$$

where we used the Kronecker product of matrices, then (3.1.9) can be verified by direct computation.[3]

The matrix representations above, which we use in these notes, form the "Dirac" basis. Other common realizations of the Clifford algebra are (a) the "Weyl" (or chiral) basis,

$$\gamma^0 = \frac{1}{c}\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \frac{1}{c}\sigma^1 \otimes \mathbf{1}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = i\sigma^2 \otimes \sigma^i \tag{3.2.4}$$

and the (b) "Majorana" basis

$$\begin{aligned} \gamma^0 &= \frac{1}{c}\begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} = \frac{1}{c}\sigma^1 \otimes \sigma^2, \quad \gamma^1 = i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} = i\mathbf{1} \otimes \sigma^3, \\ \gamma^2 &= \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} = -i\sigma^2 \otimes \sigma^2, \quad \gamma^3 = -i\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} = -i\mathbf{1} \otimes \sigma^1. \end{aligned} \tag{3.2.5}$$

---

[3]In (3.2.3) we used the "**Kronecker**" or "tensor" product of two matrices, which is defined in such a way as to make the block structure of the result manifest in a compact way: if $\widehat{a}$ is $n \times m$ and $\widehat{b}$ is $p \times q$ then,

$$\widehat{a} \otimes \widehat{b} = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ a_{21} & \dots & a_{2m} \\ \dots & & \\ a_{n1} & \dots & a_{nm} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & \dots & b_{1q} \\ b_{21} & \dots & b_{2q} \\ \dots & & \\ b_{p1} & \dots & b_{pq} \end{pmatrix} = \begin{pmatrix} a_{11}\widehat{b} & \dots & a_{1m}\widehat{b} \\ a_{21}\widehat{b} & \dots & a_{2m}\widehat{b} \\ \dots & & \\ a_{n1}\widehat{b} & \dots & a_{nm}\widehat{b} \end{pmatrix}$$

is an $np \times mq$ matrix. So, for example, the Kronecker product of the two 2×2 Pauli matrices,

$$i\sigma^2 \otimes \sigma^i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \sigma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

is a four dimensional (Dirac γ) matrix.

Each basis satisfies the defining relation in (3.1.9). However, the Majorana basis is special in that it is purely imaginary, *i.e.,* it also satisfies the condition

$$\gamma^{\mu\star} = -\gamma^{\mu} \tag{3.2.6}$$

and so the Dirac operator, like the Klein Gordon operator, becomes real. Therefore it is possible to find real (carrying no charge) solutions for ψ, which is not possible with the Dirac and the Weyl representations.

What then is the relationship between different solutions of the Dirac equation, obtained from different representations of the γ matrices? Indeed, in even dimensions, the representations of our Clifford Algebra are unique up to a similarity transformation: if γ^μ and $\widetilde{\gamma}^\mu$ are two representations of (3.1.9) then

$$\gamma^{\mu} = U\widetilde{\gamma}^{\mu}U^{\dagger} \tag{3.2.7}$$

and

$$\psi = U\widetilde{\psi} \tag{3.2.8}$$

where U is a unitary matrix.[4] Therefore, if $\widetilde{\psi}$ is a solution of the Dirac equation in the Majorana representation then in any other representation the reality condition for Majorana spinors would read

$$(U^{\dagger}\psi)^{\star} = U^{\dagger}\psi, \quad \Rightarrow \quad U^{t}\psi^{\star} = U^{\dagger}\psi \quad \Rightarrow \quad \psi = (UU^{t})\psi^{\star} \tag{3.2.9}$$

where U transforms from the Majorana representation.

Let us now determine sixteen linearly independent 4×4 matrices that can be built with the γ matrices so that we will have a complete basis in which to expand *any* 4×4 matrix. In addition to the four γ matrices, we can build six matrices in terms of the antisymmetric combination

$$\sigma_{\mu\nu} = \frac{1}{2!}\epsilon_{\mu\nu\alpha\beta}\gamma^{\alpha}\gamma^{\beta} \tag{3.2.10}$$

then σ_{0i} is anti Hermitean but σ_{ij} is Hermitean. In the same spirit, define the four additional matrices

$$\chi_{\mu} = \frac{1}{3!}\epsilon_{\mu\nu\alpha\beta}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta} \tag{3.2.11}$$

[4] Problem: Show that the unitary matrix

$$U_{D\to W} = \frac{1}{\sqrt{2}}\left(\mathbf{1}\otimes\mathbf{1} - i\sigma^{2}\otimes\mathbf{1}\right)$$

transforms the Dirac basis to the Weyl basis. Show also that

$$U_{D\to M} = \frac{1}{\sqrt{2}}\left(\sigma^{3}\otimes\mathbf{1} + \sigma^{1}\otimes\sigma^{2}\right)$$

transforms the Dirac basis to the Majorana basis. Find the matrix that transforms the Weyl basis to the Majorana.

of which χ_0 is Hermitean but χ_i is anti Hermitean, and finally define the Hermitean matrix

$$\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{1}{c}\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \frac{1}{c}\sigma^1 \otimes \mathbf{1}. \tag{3.2.12}$$

in the Dirac basis. In the Weyl basis,

$$\gamma^5 = \frac{1}{c}\begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} = -\frac{1}{c}\sigma^3 \otimes \mathbf{1} \tag{3.2.13}$$

and in the Majorana basis

$$\gamma^5 = \frac{1}{c}\begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} = \frac{1}{c}\sigma^3 \otimes \sigma^2. \tag{3.2.14}$$

It is easy to see that $\chi_\mu = i\eta_{\mu\nu}\gamma^\nu\gamma^5$ and, in this way, we have constructed sixteen matrices,

$$\Gamma_I = \left\{\mathbf{1}, \gamma^\mu, \sigma_{\mu\nu}, i\eta_{\mu\nu}\gamma^\nu\gamma^5, \gamma^5\right\}. \tag{3.2.15}$$

We will now argue that they are linearly independent, *i.e.*, $c_I = 0 \Leftrightarrow \sum_I c_I\Gamma_I = 0,\ \forall\ I$ so that any (complex) 4×4 matrix may be written as a linear combination of them. That $c_I = 0$ is sufficient for the sum to vanish is obvious; to prove that it is necessary as well, let us first consider some "trace identities" that the Γ_I satisfy. Using the following three trace relations,

$$\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$$

$$\text{Tr}(rA) = r\text{Tr}(A), \quad r \in \mathbb{R}$$

$$\text{Tr}(ABC) = \text{Tr}(CAB) = \text{Tr}(BCA)$$

which hold for any matrices, A, B and C and any real number r, together with (3.1.9), one can establish the following:

- $\text{Tr}(\mathbf{1}) = 4$,
- $\text{Tr}(\gamma^\mu) = 0 = \text{Tr}(\gamma^5)$,
- $\text{Tr}(\gamma^\mu\gamma^\nu) = -4\eta^{\mu\nu}$,
- $\text{Tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha})$,
- $\text{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{2n}}) =$
- $\text{Tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^5) = -\frac{4i}{c}\ \epsilon^{\mu\nu\alpha\beta}$,

- $\text{Tr}(\sigma_{\mu\nu}) = 0$,
- $\text{Tr}(\chi_\mu) = 0$,
- $\text{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{2n+1}}) = 0$,
- $\text{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{2n+1}}\gamma^5) = 0$.

Now we see that $\text{Tr}(\Gamma_I) = 0$ for all I and $\text{Tr}(\Gamma_I\Gamma_J) \sim \delta_{IJ}$, so consider

$$\sum_I c_I\Gamma_I = 0 \tag{3.2.16}$$

and multiply by Γ_K, for some fixed K. We get

$$\sum_I c_I(\Gamma_I\Gamma_K) = 0 = \sum_{I\neq K} c_I\Gamma_I\Gamma_K + c_K\Gamma_K^2. \tag{3.2.17}$$

Taking the trace we conclude that $c_K = 0$ and, since K was arbitrarily chosen, we have shown that $c_K = 0$ for all K is necessary for the sum to vanish.

3.3 Properties of the Dirac Particle

3.3.1 Spin

Recall that the Hamiltonian of the Dirac particle can be written as

$$\mathcal{H} = \alpha\cdot\vec{p}c + \beta mc^2 = c^2\gamma^0(\vec{\gamma}\cdot\vec{p} + mc) \tag{3.3.1}$$

so it is quite clear that $[\vec{p},\mathcal{H}] = 0$ and therefore that momentum is conserved. However if we consider the orbital angular momentum, $L = \vec{r}\times\vec{p}$, we discover that it is not conserved because

$$[L_i,\mathcal{H}] = \epsilon_{ijk}[x_jp_k,\mathcal{H}] = \epsilon_{ijk}[x_j,\mathcal{H}]p_k = i\hbar\epsilon_{ijk}c^2\gamma^0\gamma^l\delta_{jl}p_k = i\hbar c^2\gamma^0[\vec{\gamma}\times\vec{p}]_i \neq 0, \tag{3.3.2}$$

so the question that arises is this: is it possible to find some vector, other than the orbital angular momentum, whose commutator with the Hamiltonian exactly cancels the above commutator of the orbital angular momentum with $\mathcal{H}$? If this were possible then the sum of the angular momentum and this vector would represent a conserved "total angular momentum". Such a vector does indeed exist and is constructed exclusively out of the γ matrices as

$$\vec{S} = \frac{i\hbar}{4}\,\vec{\gamma}\times\vec{\gamma} \tag{3.3.3}$$

(we will derive it later, as a consequence of Noether's first theorem).[5] Therefore

$$\vec{J} = \vec{L} + \vec{S} = \vec{r} \times \vec{p} + \frac{i\hbar}{4}\, \vec{\gamma} \times \vec{\gamma} \tag{3.3.4}$$

can be thought of as the conserved, total angular momentum of the particle. It includes an "intrinsic" part, $\vec{S}$, which is interpreted as the "**spin**" of the Dirac particle. Its components can be written as

$$S_i = \frac{i\hbar}{2}\sigma_{0i} = \frac{\hbar}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} = \frac{c^2\hbar}{2}\gamma^0\gamma^i\gamma^5. \tag{3.3.5}$$

The spin obeys commutation relations typical of the angular momentum operator in quantum mechanics,

$$[S_i, S_j] = -\frac{\hbar^2}{4}[\gamma^i, \gamma^j] = -i\hbar\epsilon_{ijm}S_m \tag{3.3.6}$$

and, moreover,

$$\vec{S} \cdot \vec{S} = \vec{S}^2 = \frac{\hbar^2}{4}\left[(\gamma^0\gamma^i\gamma^5)(\gamma^0\gamma^i\gamma^5)\right] = \frac{3\hbar^2}{4}\mathbf{1} \tag{3.3.7}$$

so, putting this together with (3.3.6), we can conclude that $\vec{S}$ represents an intrinsic spin of 1/2. The "**helicity operator**" measures the component of the spin vector in the direction of the momentum, $H = \frac{1}{\hbar}\vec{S} \cdot \hat{p}$; it commutes with the Hamiltonian, so the helicity is also a conserved quantity.

3.3.2 Continuous Symmetries

We now seek the action of a Lorentz transformation on the wave function using only the principle of covariance. Suppose that under the transformation $x^\mu \to x'^\mu = L^\mu{}_\nu x^\nu$, the wave function transforms as

$$\psi(x) \to \psi'(x') = \hat{S}\psi(x) \tag{3.3.8}$$

where $\widehat{S}$ is a 4×4 matrix to be determined. By the principle of covariance, the Dirac equation must have the same form in the primed frame, so

$$(ic\hbar\gamma^\mu\partial'_\mu - mc^2)\psi'(x') = 0 = (ic\hbar\gamma^\mu(L^{-1})^\alpha{}_\mu\partial_\alpha - mc^2)\widehat{S}\psi \tag{3.3.9}$$

(where L is a boost or a rotation) or

$$ic\hbar\widehat{S}^{-1}\gamma^\mu\widehat{S}(L^{-1})^\alpha{}_\mu\frac{\partial\psi}{\partial x^\alpha} - mc^2\psi = 0 \tag{3.3.10}$$

and it follows that

$$(L^{-1})^\alpha{}_\mu\widehat{S}^{-1}\gamma^\mu\widehat{S} = \gamma^\alpha \;\Rightarrow\; \widehat{S}^{-1}\gamma^\mu\widehat{S} = L^\mu{}_\alpha\gamma^\alpha. \tag{3.3.11}$$

[5] Problem: Show that $[L_i + S_i, \mathcal{H}] = 0$.

This is the equation we want to solve for $\widehat{S}(L)$. Any function that transforms as (3.3.8), with $\widehat{S}$ given by (3.3.11) is a **Dirac Spinor**.

To determine $\widehat{S}$, consider infinitesimal transformations,

$$L^\mu{}_\alpha = \delta^\mu{}_\alpha + \delta\omega^\mu{}_\alpha$$

$$\widehat{S} = \mathbf{1} + \delta\widehat{S} \tag{3.3.12}$$

then, by expanding both sides of (3.3.11) one has

$$[\gamma^\mu, \delta\widehat{S}] = \delta\omega^\mu{}_\alpha\gamma^\alpha. \tag{3.3.13}$$

The solution is

$$\delta\widehat{S} = -\frac{1}{8}\delta\omega_{\alpha\beta}[\gamma^\alpha, \gamma^\beta], \tag{3.3.14}$$

which can be verified using the anti-commutation relations satisfied by the γ matrices. Defining

$$\Sigma^{\alpha\beta} = \frac{1}{2}[\gamma^\alpha, \gamma^\beta],$$

the infinitesimal form of $\widehat{S}$ is

$$\widehat{S} = \mathbf{1} - \frac{1}{4}\delta\omega_{\alpha\beta}\Sigma^{\alpha\beta} \tag{3.3.15}$$

and from this infinitesimal form one can construct a finite transformation in the usual way, by exponentiation,

$$\widehat{S} = \lim_{N\to\infty}\left(\mathbf{1} - \frac{\omega_{\alpha\beta}}{4N}\Sigma^{\alpha\beta}\right)^N = \exp\left[-\frac{1}{4}\omega_{\alpha\beta}\Sigma^{\alpha\beta}\right] \tag{3.3.16}$$

where we think of the finite transformation as an infinite series of infinitesimal transformations applied successively, each with parameter $\delta\omega_{\alpha\beta} = \omega_{\alpha\beta}/N$.

As an example, consider a boost determined by an infinitesimal velocity parameter, $\delta\vec{v}$, then taking into account only terms that are linear in $\delta\vec{v}$, we have $\delta\omega_{i0} = -\delta\omega_{0i} = \delta v_i$ and thus

$$\delta\omega_{\alpha\beta}\Sigma^{\alpha\beta} = \delta v_i[\gamma^i, \gamma^0] = -\frac{2\delta v_i}{c}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{3.3.17}$$

giving

$$\widehat{S} = \exp\left[\frac{v_i}{2c}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}\right]. \tag{3.3.18}$$

Expanding the exponential according to the usual definition,

$$e^{\widehat{A}} = \sum_{n=0}^\infty \frac{\widehat{A}^n}{n!} \tag{3.3.19}$$

we find that

$$\widehat{S} = \cosh\left(\frac{|\vec{v}|}{2c}\right) + \frac{v_i}{|\vec{v}|}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \sinh\left(\frac{|\vec{v}|}{2c}\right) \tag{3.3.20}$$

where we made repeated use of the properties of the Pauli spin matrices in (3.2.1).

Again, consider a rotation for which $\delta\omega_{ij} = \delta\theta_k [\epsilon^k]_{ij}$ and which gives

$$\delta\omega_{\alpha\beta}\Sigma^{\alpha\beta} = \delta\theta_k \epsilon^k{}_{ij}\gamma^i\gamma^j. \tag{3.3.21}$$

Now it is not difficult to see that

$$\epsilon^k{}_{ij}\gamma^i\gamma^j = -\epsilon^k{}_{ij}\begin{pmatrix} \sigma^i\sigma^j & 0 \\ 0 & \sigma^i\sigma^j \end{pmatrix} = -\frac{1}{2}\epsilon^k{}_{ij}\begin{pmatrix} [\sigma^i, \sigma^j] & 0 \\ 0 & [\sigma^i, \sigma^j] \end{pmatrix} = -2i\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \tag{3.3.22}$$

once again exploiting the properties of the Pauli matrices. Therefore

$$\widehat{S} = \exp\left[+\frac{i\theta_k}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}\right] = \cos\left(\frac{|\vec{\theta}|}{2}\right) + \frac{i\theta_k}{|\vec{\theta}|}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}\sin\left(\frac{|\vec{\theta}|}{2}\right), \tag{3.3.23}$$

where we set $|\vec{\theta}| = \sqrt{\theta_k\theta_k}$. If, for example, we consider a rotation about the z axis by an angle φ,

$$\widehat{S} = \cos\left(\frac{\varphi}{2}\right) + i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\sin\left(\frac{\varphi}{2}\right) \tag{3.3.24}$$

and a rotation through and angle $\varphi = 2\pi$ is seen to lead to $\widehat{S} = -\mathbf{1}$, *i.e.,* the wave function suffers a phase change of π. A rotation through 4π is required to return the wavefunction to its original value.

The matrices $\Sigma^{\mu\nu}$ provide a finite dimensional reprsentation of the Lorentz algebra. The block diagonal matrices, Σ^{ij}, generate rotations. and provide a unitary representation of the rotation group. On the other hand, the boost generators, Σ^{0i} do not provide a unitary representation of the boosts. This is expected as the Lorentz group is not compact and therefore has no unitary, finite dimensional representation.

3.3.3 Discrete Symmetries

There are three discrete transformations, *viz.,* Parity (P), or space inversion, Time Reversal (T) and Charge Conjugation (C). Each one of them is an improper transformation *i.e.,* the determinant of the transformation matrix is -1 and cannot, therefore, be reduced to a combination of rotations and/or boosts. In considering their action on fields, it is useful to begin by appealing to our basic intuitions in mechanics.

Parity (P)

Parity is the transformation that reflects the spatial coordinates, as if in a mirror, so that

$$\vec{r} \to \vec{r}' = -\vec{r}, \quad t \to t' = t \tag{3.3.25}$$

The transformation may be represented by $x^\mu \to x'^\mu = P^\mu{}_\nu x^\nu$, where

$$P^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.3.26}$$

If we assume that the fundamental laws of mechanics are invariant under these transformations, then since $\vec{p} \to \vec{p}' = -\vec{p}$, we expect $\vec{F} \to \vec{F}' = -\vec{F}$. From the Lorentz force law,

$$\vec{F} = e[\vec{E} + \vec{v} \times \vec{B}] \tag{3.3.27}$$

it then follows that the electric and magnetic fields behave respectively as $\vec{E} \to -\vec{E}$ and $\vec{B} \to \vec{B}$, or in terms of the vector potential, A^μ, that

$$\vec{A}(t,\vec{r}) \to \vec{A}'(t',\vec{r}') = -\vec{A}(t,\vec{r}), \quad A^0(t,\vec{r}) \to A'^0(t',\vec{r}') = A^0(t,\vec{r}) \tag{3.3.28}$$

Now consider the effect of parity on the complex Klein-Gordon field, by considering its equation of motion when coupled to an electromagnetic field:

$$\left[-\left(\partial^\mu - igA^\mu\right)\left(\partial_\mu - igA_\mu\right) + \frac{m^2c^2}{\hbar^2}\right]\phi(t,\vec{r}) = 0. \tag{3.3.29}$$

Because A^μ transforms just as x^μ, the relative sign between the derivative and the electromagnetic potential remains the same and the operator is unchanged by the transformation. Requiring covariance of the equation of motion, we conclude that

$$\phi(t,\vec{r}) \to \phi'(t,-\vec{r}) = \widehat{\mathcal{P}}\phi(t,\vec{r}) = \lambda_P\phi(t,\vec{r}) \tag{3.3.30}$$

where λ_P is some internal phase associated with the parity transformation. If $\lambda_P = 1$ then the field is a scalar, if $\lambda_P = -1$ it is a pseudo-scalar.

The transformation is expected to be linear, so we can use the relation (3.3.11) to discuss the effects of a parity transformation on the Dirac spinor,

$$\widehat{\mathcal{P}}^{-1}\gamma^\mu\widehat{\mathcal{P}} = P^\mu{}_\alpha\gamma^\alpha. \tag{3.3.31}$$

Finding $\widehat{\mathcal{P}}$ for this transformation is trivial, since $c\gamma^0$ is easily seen to satisfy the equation:

$$c^2\gamma^0\gamma^\mu\gamma^0 = P^\mu{}_\alpha\gamma^\alpha$$

Thus $\widehat{\mathcal{P}} = c\lambda_P\gamma^0$ is the "parity operator" and we determine that

$$\psi(t,\vec{r}) \to \psi'(t,-\vec{r}) = \widehat{\mathcal{P}}\psi(t,\vec{r}) = c\lambda_P\gamma^0\psi(t,\vec{r}), \tag{3.3.32}$$

where λ_P is an internal phase associated with the transformation as before.

Time Reversal (T)

Time reversal is more subtle because it is not a linear transformation, as we will now see. As the name suggests, time reversal reflects only the time coordinate according to $x^\mu \to x'^\mu = T^\mu{}_\nu x^\nu$, where

$$T^\mu{}_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.3.33}$$

Under this transformation it should be clear that $\vec{p} \to \vec{p}' = -\vec{p}$ and therefore that $\vec{F} \to \vec{F}' = \vec{F}$. Now returning to the Lorentz force this would imply by arguments similar to the ones used earlier that $\vec{E} \to \vec{E}' = \vec{E}$ and $\vec{B} \to \vec{B}' = -\vec{B}$. Therefore, we get the result in (3.3.28) for the electromagnetic potential. Consider what happens to the Klein-Gordon equation under this transformation: because the space-time derivative and the electromagnetic potential transform in an opposite way, the equation of motion for $\phi'(-t, \vec{r})$ will now read

$$\left[-\left(\partial^\mu + igA^\mu\right)\left(\partial_\mu + igA_\mu\right) + \frac{m^2c^2}{\hbar^2}\right]\phi'(-t, \vec{r}) = 0. \tag{3.3.34}$$

But this is precisely the equation of motion for the conjugate field, $\phi^*(t, \vec{r})$ and covariance requires that

$$\phi(t, \vec{r}) \to \phi'(-t, \vec{r}) = \widehat{\mathcal{T}}\phi^*(t, r) = \lambda_T \phi^*(t, r), \tag{3.3.35}$$

where λ_T is an internal phase associated with time reversal.

Time reversal is therefore an *anti*-linear transformation and, for this reason, one should not expect to use (3.3.11) to determine the operator that induces a time reversal on the Dirac spinor. Instead we ask for

$$\psi(t, \vec{r}) \to \psi'(-t, \vec{r}) = \widehat{\mathcal{T}}\psi^*(t, \vec{r}). \tag{3.3.36}$$

Going through the arguments of the previous section, we find the condition

$$\widehat{\mathcal{T}}^{-1}\gamma^\mu \widehat{\mathcal{T}} = -T^\mu{}_\alpha \gamma^{\alpha *} \tag{3.3.37}$$

which is solved by $\widehat{\mathcal{T}} = \lambda_T \gamma^1\gamma^3$. Thus

$$\psi(t, \vec{r}) \to \psi'(-t, \vec{r}) = \widehat{\mathcal{T}}\psi(t, \vec{r}) = \lambda_T \gamma^1\gamma^3\psi^*(t, \vec{r}) \tag{3.3.38}$$

is the transformation of the Dirac spinor under time reversal.

Charge Conjugation (C)

Charge conjugation, unlike the previous two discrete transformations, is not a space-time symmetry but instead changes the sign of the electromagnetic charge. The electromagnetic force will not change only if $\vec{E} \to \vec{E}' = -\vec{E}$ and $\vec{B} \to \vec{B}' = -\vec{B}$ (according to the Lorentz force law), *i.e.,* $A^\mu(x) \to A'^\mu(x) = -A^\mu(x)$. It is then clear that the Klein Gordon equation turns into

$$\left[-\left(\partial^\mu + igA^\mu\right)\left(\partial_\mu + igA_\mu\right) + \frac{m^2c^2}{\hbar^2}\right]\phi'(t,\vec{r}) = 0, \tag{3.3.39}$$

just as it did in the case of time reversal. This transformation is therefore also antilinear and we have

$$\phi(t,\vec{r}) \to \phi'(t,\vec{r}) = \widehat{\mathcal{C}}\phi^*(t,\vec{r}) = \lambda_{\mathcal{C}}\phi^*(t,\vec{r}). \tag{3.3.40}$$

For the Dirac spinor we ask for

$$\psi(t,\vec{r}) \to \psi'(t,\vec{r}) = \widehat{\mathcal{C}}\psi^*(t,\vec{r}) \tag{3.3.41}$$

which gives the condition analogous to (3.3.37), with $L^\mu{}_\alpha = \delta^\mu{}_\alpha$,

$$\widehat{\mathcal{C}}^{-1}\gamma^\mu\widehat{\mathcal{C}} = -\gamma^{\mu*} \tag{3.3.42}$$

and is solved by $\widehat{\mathcal{C}} = \lambda_{\mathcal{C}}\gamma^2$, where $\lambda_{\mathcal{C}}$ is the arbitrary phase associated with charge conjugation.

The CPT theorem

The importance of the discrete symmetries discussed above is captured by the **CPT** theorem, which states that:

- *Every local, Lorentz invariant quantum field theory (QFT) with a hermitean hamiltonian is CPT invariant.*

Therefore, Lorentz invariance may be preserved while the discrete symmetries are broken but the combined transformation,

$$\Theta = CPT$$

is always a symmetry of a local, Lorentz invariant QFT with a Hermitean Hamiltonian.[6]

[6]While Quantum Electrodynamics (QED) respects each of C, P and T separately, certain interaction terms in Quantum Chromodynamics (QCD) break both T and P. Electroweak (EW) interactions violate P maximally and also T weakly (in Kaon oscillations).

3.4 Solutions of the Dirac Equation

The problem of negative energies lingers in Dirac's theory and we have seen that this leads to serious problems applying the Born interpretation to the wave function. The arguments given for interpreting the Klein-Gordon equation as describing a field theory may be applied here. To get a better feeling for the physics of the Dirac equation, we now look for solutions. With the Dirac representation, this means solving equation (3.1.8) for $\psi(x)$ in the representation given by (3.2.3). We will exploit the block diagonal form of the γ matrices in what follows.

As with the Klein-Gordon equation, a general solution of the Dirac equation may be given as

$$\psi(x) = \int \frac{d^3\vec{p}}{2E_k}\left[c(p)u_p^{(+)}(x) + d^*(p)u_p^{(-)}(x)\right] \tag{3.4.1}$$

where $p_\mu = (-E_p, \vec{p})$, $u_p^{(-)}(x) = u_{-p}^{(+)}(x)$ and $d^*(p) = -c(-p)$. The mode functions must satisfy the Dirac equation, so we take them to be of the form

$$u_p^{(\pm)}(x) = u^{(\pm)}(E_p, \vec{p})e^{\pm\frac{i}{\hbar}(\vec{p}\cdot\vec{r} - E_p t)}, \tag{3.4.2}$$

It is easy to verify that $v_p^{(\pm)}(x)$ represent positive and negative energy waves respectively because

$$\widehat{H}\ u_p^{(\pm)}(x) = i\hbar\frac{\partial u_p^{(\pm)}(x)}{\partial t} = \pm E_p u_p^{(\pm)}(x). \tag{3.4.3}$$

Applying the Dirac operator to the wave function $v_p^{(\pm)}(x)$, we find that

$$-(\pm c\gamma\cdot p + mc^2)u^{(\pm)}(E_p, \vec{p}) = \begin{pmatrix} \pm E_p - mc^2 & \mp c\vec{\sigma}\cdot\vec{p} \\ \pm c\vec{\sigma}\cdot\vec{p} & \mp E_p - mc^2) \end{pmatrix} u^{(\pm)}(E_p, \vec{p}) = 0. \tag{3.4.4}$$

Evidently $u^{(+)}(E_p, \vec{p})$ has four components, which we write in terms of two component (Pauli) spinors explicitly as

$$u^{(\pm)}(E_p, \vec{p}) = \mathcal{N}\begin{pmatrix} u_L^{(\pm)}(E_p, \vec{p}) \\ u_S^{(\pm)}(E_p, \vec{p}) \end{pmatrix}. \tag{3.4.5}$$

where $\mathcal{N}$ is some normalization factor. The superscripts "L" and "S" stand for "long" and "short". Then we have

$$\begin{aligned} (\pm E_p - mc^2)u_L^{(\pm)} \mp (c\vec{\sigma}\cdot\vec{p})u_S^{(\pm)} &= 0 \\ \pm(c\vec{\sigma}\cdot\vec{p})u_L^{(\pm)} - (\pm E_p + mc^2)u_S^{(\pm)} &= 0, \end{aligned} \tag{3.4.6}$$

Two non-trivial conditions arise when $\vec{p} = 0$, *viz.,*

$$(\pm E_p - mc^2)u_L^{(\pm)} = 0$$
$$(\pm E_p + mc^2)u_S^{(\pm)} = 0. \tag{3.4.7}$$

Thus $u_S^{(+)} = 0$ and $u_L^{(-)} = 0$, because $E_p > 0$ by definition, and the solutions will be linear combinations of

$$u^{(+)}(E_p, 0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad u^{(+)}(E_p, 0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.4.8}$$

For negative energy solutions, we must take $u_L^{(-)} = 0$ and they would be linear combinations of

$$u^{(-)}(E_p, 0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u^{(-)}(E_p, 0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.4.9}$$

In fact, (3.4.6) says that $u_S^{(+)} \to 0$ even if the particle is not at rest but moving slowly compared to the speed of light, so in the non-relativistic approximation we will have

$$u_p^{(+)}(x) = \mathcal{N} \begin{pmatrix} u_L^{(+)} \\ 0 \end{pmatrix} e^{\frac{i}{\hbar} p \cdot x}. \tag{3.4.10}$$

Likewise, $u_L^{(-)} \to 0$ for negative energy solutions in the non-relativistic limit and it also follows that

$$u_p^{(-)}(x) = \mathcal{N} \begin{pmatrix} 0 \\ u_S^{(-)} \end{pmatrix} e^{-\frac{i}{\hbar} p \cdot x}. \tag{3.4.11}$$

In each case the wave function is completely determined by a single Pauli spinor, u_L or u_S, which corresponds with our expectation that the non-relativistic Dirac particle has two spin states.

Beyond the non-relativistic approximation, a suitable eigenbasis for a general spinor would be the normalized eigenstates of the helicity operator, $\vec{\sigma} \cdot \hat{p}$. This is called the **helicity basis**. Let us express $\hat{p}$ in polar coordinates,

$$\hat{p} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),$$

where θ is the polar angle and φ the azimuthal angle. The eigenvalues of $\vec{\sigma} \cdot \hat{p}$ are determined to be ± 1 and we easily find expressions for the normalized eigenstates of this operator,

$$\chi_{+1} = \begin{pmatrix} \cos(\theta/2)e^{-i\varphi/2} \\ \sin(\theta/2)e^{i\varphi/2} \end{pmatrix}, \quad \chi_{-1} = \begin{pmatrix} -\sin(\theta/2)e^{-i\varphi/2} \\ \cos(\theta/2)e^{i\varphi/2} \end{pmatrix}$$

with eigenvalues +1 and −1 respectively. For example, when $\theta = 0 = \varphi$, $\hat{p}$ is directed along the $z-$ axis and

$$\chi_{+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.4.12}$$

are just the eigenstates of σ^3. For the positive energy solutions, take $u_L^{(+)} = a_+\chi_{+1} + a_-\chi_{-1}$ with $|a_+|^2 + |a_-|^2 = 1$, and $u_S^{(+)}$ will then be

$$u_S^{(+)} = \frac{c\vec{\sigma} \cdot \vec{p}}{E_p + mc^2}(a_+\chi_{+1} + a_-\chi_{-1}). \tag{3.4.13}$$

The positive energy modes are then of the form

$$u_p^{(+)}(x) = \mathcal{N}^+ \begin{pmatrix} u_L^{(+)} \\ u_S^{(+)} \end{pmatrix} e^{ip\cdot x} = \mathcal{N} \begin{pmatrix} 1 \\ \frac{c\vec{\sigma}\cdot\vec{p}}{E+mc^2} \end{pmatrix} \otimes (a_+\chi_{+1} + a_-\chi_{-1}) e^{\frac{i}{\hbar}p\cdot x} \tag{3.4.14}$$

Once again, we can normalize the wave function in a large box of volume V, using the inner product determined by $j^0(x)$, which requires that

$$|\mathcal{N}|^2 V \left(u_L^{(+)\dagger} u_L^{(+)} + u_S^{(+)\dagger} u_S^{(+)} \right) = 1 \tag{3.4.15}$$

and, because u^L is already normalized,

$$u^{S\dagger}u^S = \frac{|\vec{p}|^2 c^2}{(E + mc^2)^2}, \tag{3.4.16}$$

therefore, according to (3.4.15),

$$|\mathcal{N}|^2 V \left(1 + \frac{E_p - mc^2}{E_p + mc^2} \right) = 1 \;\Rightarrow\; |\mathcal{N}| = \sqrt{\frac{E_p + mc^2}{2E_p V}} \tag{3.4.17}$$

As was the case for the Klein-Gordon modes, this is not a Lorentz invariant normalization because $u^\dagger u$ is the time component of the current vector and the volume is not a Lorentz scalar. An alternative normalization uses the bilinear $\overline{u}\ u = u^\dagger\gamma^0 u$, which *is* a Lorentz invariant as we show shortly. The normalization condition (still not Lorentz invariant because of the volume) then reads

$$|\mathcal{N}|^2 V \left(u_L^{(+)\dagger} u_L^{(+)} - u_S^{(+)\dagger} u_S^{(+)} \right) = 1 \tag{3.4.18}$$

and we have

$$|\mathcal{N}|^2 V \left(1 - \frac{E_p - mc^2}{E_p + mc^2} \right) = 1 \;\Rightarrow\; |\mathcal{N}| = \sqrt{\frac{E_p + mc^2}{2mc^2 V}} \tag{3.4.19}$$

which, of course, is not applicable to massless particles. Therefore,

$$u_p^{(+)}(x) = \sqrt{\frac{E_p + mc^2}{2mc^2 V}} \begin{pmatrix} u_L^{(+)} \\ u_S^{(+)} \end{pmatrix} e^{\frac{i}{\hbar} p\cdot x} \tag{3.4.20}$$

and, by an identical treatment,

$$u_p^{(-)}(x) = \sqrt{\frac{E_p + mc^2}{2mc^2 V}} \begin{pmatrix} u_L^{(-)} \\ u_S^{(-)} \end{pmatrix} e^{-\frac{i}{\hbar} p\cdot x}, \tag{3.4.21}$$

where

$$\begin{aligned} u_S^{(-)} &= b_+ \chi_{+1} + b_- \chi_{-1} \\ u_L^{(-)} &= \frac{c\vec{\sigma}\cdot\vec{p}}{E_p^2 + mc^2} u_S^{(-)}. \end{aligned} \tag{3.4.22}$$

To summarize, a basis of positive energy solutions can be given as

$$u_{\pm 1}^{(+)}(x) = \sqrt{\frac{E_p + mc^2}{2mc^2 V}} \begin{pmatrix} 1 \\ \frac{c\vec{\sigma}\cdot\vec{p}}{E_p + mc^2} \end{pmatrix} \otimes \chi_{\pm 1} e^{\frac{i}{\hbar} p\cdot x} \tag{3.4.23}$$

and a basis of negative energy solutions as

$$u_{\pm 1}^{(-)}(x) = \sqrt{\frac{E_p + mc^2}{2mc^2 V}} \begin{pmatrix} \frac{c\vec{\sigma}\cdot\vec{p}}{E_p + mc^2} \\ 1 \end{pmatrix} \otimes \chi_{\mp 1} e^{-\frac{i}{\hbar} p\cdot x}. \tag{3.4.24}$$

It is not difficult to establish the following orthonormality conditions:

$$\begin{aligned} \langle \psi_{s\vec{p}}^{(+)} | \psi_{s'\vec{p}\,'}^{(+)} \rangle &= \frac{(2\pi\hbar)^3}{V} \delta_{ss'} \delta(\vec{p} - \vec{p}\,') = -\langle \psi_{s\vec{p}}^{(-)} | \psi_{s'\vec{p}\,'}^{(-)} \rangle \\ \langle \psi_{s\vec{p}}^{(+)} | \psi_{s'\vec{p}\,'}^{(-)} \rangle &= 0 = \langle \psi_{s\vec{p}}^{(-)} | \psi_{s'\vec{p}\,'}^{(+)} \rangle \end{aligned} \tag{3.4.25}$$

where $s = \pm 1$. Each eigenstate is defined by its energy, E_p, momentum, $\vec{p}$, and helicity, s.[7]

3.5 Particles and Antiparticles

Dirac sought to exploit the Pauli exclusion principle for fermions and proposed that the vacuum is a multiparticle state in which *all the negative energy eigenstates are filled* (the so-called **Dirac sea**). Since each state can be occupied by at most one particle no transitions

[7] Problem: Write out the general solutions in the basis:

$$u_{L(S)} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

for positive (negative) energy solutions.

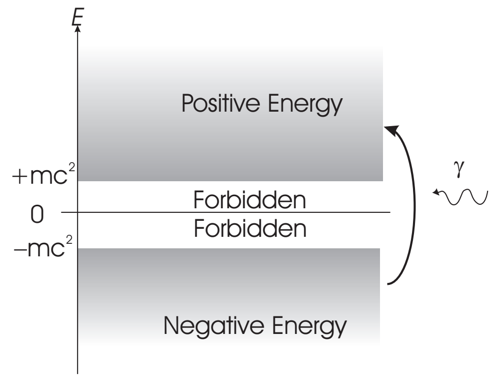

Figure 3.1: The Dirac Sea

to negative energy states can occur. This interpretation also gets rid of the single particle interpretation of the Dirac equation, for a photon with sufficient energy can excite one of the fermions in a negative energy state leaving behind a "hole" in the Dirac sea. This hole behaves as a particle with a positive mass and the opposite charge, so interactions will produce particle-hole pairs out of the vacuum. This led Dirac to posit the existence of oppositely charged partners of all fermions, called "antiparticles". The positively charged partner of the electron, for example, is the positron, which was discovered by Anderson (1932) soon after Dirac posited its existence.

This interpretation is not entirely satisfactory because (a) it is restricted to fermions and (b) it results in an infinite negative energy contribution to the vacuum energy. One could argue that this is not a problem since absolute energy is unobservable, but this is no longer true in Einstein's theory of general relativity. This infinite negative energy therefore must be canceled by an infinite positive contribution from somewhere. The infinite charge and current densities from the sea must likewise be canceled by an infinite contribution of the opposite sign.

A more general interpretation of the negative energy states, that does not rely on the Pauli exclusion principle to bring stability to the energy spectrum, emerges when we realize that the transformation $p_\mu \to -p_\mu$ takes us from positive to negative energy states. Causality requires that positive energy states, with time dependence e^{-iEt}, propagate forwards in time, but the fact that

$$e^{-iEt} = e^{-i(-E)(-t)}$$

suggests that causality is retained if we require negative energy states to propagate only backwards in time. In this picture the emission of a negative energy particle with momentum p_μ at time t is equivalent to the absorption of a positive energy particle with

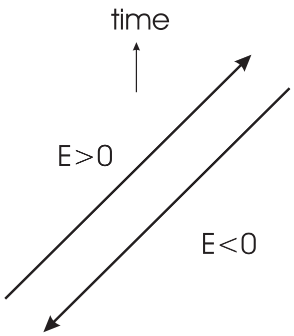

Figure 3.2: Stückelberg-Feynman picture of antiparticles.

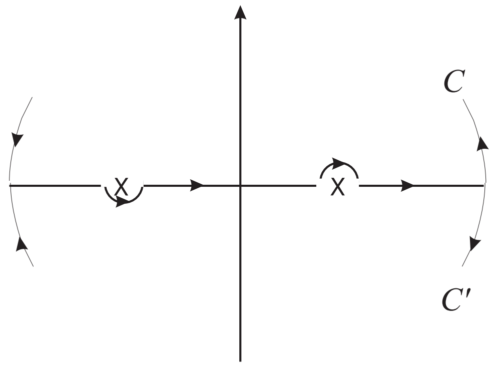

Figure 3.3: Contour for the Feynman Green's function.

momentum $-p_\mu$ at an earlier time, t', and vice-versa. Again, the electric current density of a positive energy scalar particle carrying a charge $+e$,

$$j_\mu^{(+)}(+e) = (-ie/\hbar)\phi^* \overleftrightarrow{\partial_\mu}\phi = \frac{ep_\mu c^2}{E_p V}$$

is identical to the current density, $j_\mu^{(-)}(-e)$, for a *negative* energy scalar particle carrying the opposite charge, $-e$. The **Stückelberg-Feynman** interpretation of an antiparticle is as a negative energy particle propagating backwards in time. The antiparticle has the same mass as the particle and the opposite charge. This interpretation necessitates a new choice contour when defining the Green's function. The new contour is shown in figure 3.3.

3.6 Projection Operators

Let us now construct a few projection operators that will become useful later These are operators that project wave functions onto (a) the positive energy and negative energy states respectively, (b) states of definite helicity and (c) states of definite chirality.

3.6.1 Energy

We can use the orthonormality and completeness of the eigenfunctions in (3.4.23) and (3.4.24) to construct the projectors onto positive and negative energy states as follows

$$\Lambda_\pm(p) = \sum_s \psi_s^{(\pm)}(p) \otimes \overline{\psi}_s^{(\pm)}(p) = \frac{\mp\gamma \cdot p + mc}{2mc}. \tag{3.6.1}$$

from which it follows that

$$\begin{aligned}\Lambda_+\psi_s^{(+)}(p) &= \psi_s^{(+)}(p)\\ \Lambda_+\psi_s^{(-)}(p) &= 0\\ \Lambda_-\psi_s^{(+)}(p) &= 0\\ \Lambda_-\psi_s^{(-)}(p) &= \psi_s^{(-)}(p).\end{aligned} \tag{3.6.2}$$

They satisfy the relations

$$\begin{aligned}\Lambda_\pm^2(p) &= \Lambda_\pm(p)\\ \mathrm{Tr}\Lambda_\pm(p) &= 2\\ \Lambda_\pm\Lambda_\mp &= 0\\ \Lambda_+(p)+\Lambda_-(p) &= \mathbf{1}\end{aligned} \tag{3.6.3}$$

so $\Lambda_+(p)$ projects onto particle states and $\Lambda_-(p)$ projects onto antiparticle states.

3.6.2 Spin

To construct the projectors onto states of definite helicity is straightforward because they will be eigenstates of the helicity operator,

$$H = \frac{1}{\hbar}\vec{S}\cdot\hat{p} = \frac{c^2}{2}\gamma^0(\vec{\gamma}\cdot\hat{p})\gamma^5 = \frac{1}{2}\begin{pmatrix}\vec{\sigma}\cdot\hat{p} & 0\\ 0 & \vec{\sigma}\cdot\hat{p}\end{pmatrix}, \tag{3.6.4}$$

which commutes with the Hamiltonian. In fact, it is clear that

$$\begin{aligned}H\psi_{\pm1}^{(+)} &= \pm\frac{1}{2}\psi_{\pm1}^{(+)}\\ H\psi_{\pm1}^{(-)} &= \mp\frac{1}{2}\psi_{\pm1}^{(-)}.\end{aligned} \tag{3.6.5}$$

A positive eigenvalue is to be interpreted as denoting the state for which the spin is parallel to the direction of propagation ("**right handed**") and a negative eigenvalue denotes the state for which the spin is antiparallel to the direction of propagation ("**left handed**"). Notice that for antiparticles it is $\psi_{-1}^{(-)}$ that is of positive helicity; we can understand why this is so if we remember that, for a particle momentum $\vec{p}$ the antiparticle momentum is $-\vec{p}$, so for right handed antiparticles both the momentum and spin are antiparallel with the *particle* momentum (and therefore themselves parallel).

We can now define the **spin projection** operator as follows

$$\Sigma_\pm = \frac{1}{2}(1\pm 2H) \tag{3.6.6}$$

from which its action on the helicity eigenstates

$$\begin{aligned}
&\Sigma_+\psi^{(+)}_{+1} = \psi^{(+)}_{+1}, \quad \Sigma_+\psi^{(+)}_{-1} = 0\\
&\Sigma_+\psi^{(-)}_{+1} = 0, \quad \Sigma_+\psi^{(-)}_{-1} = \psi^{(-)}_{-1}\\
&\Sigma_-\psi^{(+)}_{+1} = 0, \quad \Sigma_-\psi^{(+)}_{-1} = \psi^{(+)}_{-1}\\
&\Sigma_-\psi^{(-)}_{+1} = \psi^{(-)}_{+1}, \quad \Sigma_-\psi^{(-)}_{-1} = 0
\end{aligned} \tag{3.6.7}$$

and the identities

$$\begin{aligned}
&\Sigma^2_\pm = \Sigma_\pm\\
&\mathrm{Tr}(\Sigma_\pm) = 2\\
&\Sigma_\pm\Sigma_\mp = 0\\
&\Sigma_+ + \Sigma_- = 1
\end{aligned} \tag{3.6.8}$$

are straightforward.

3.6.3 Chirality

Because γ^5 anticommutes with γ^μ, the solutions of the massless Dirac equation,

$$ic\hbar\gamma^\mu\partial_\mu\psi = 0 \tag{3.6.9}$$

will also be eigentstates of γ^5. Now because ${\gamma^5}^2 = 1/c^2$, it has two eigenvalues, *viz.,* $\pm 1/c$, which will therefore characterize the independent solutions of the massless Dirac equation. The eigenfunctions of γ^5 form the **chirality basis**. But what are the solutions of the massless Dirac equation? Notice that one could not simply take the limit as $m \to 0$ of our solutions in (3.4.23) and (3.4.24) as the normalization would not make much sense in this limit. For massless particles, we use a different normalization condition (which could also be used for massive particles if desired) that reads

$$\int \frac{d^3\vec r}{(2\pi\hbar)^3}\psi^{(+)\dagger}_{\vec p}(t,\vec r)\psi^{(+)}_{\vec p\,'}(t,\vec r) = 2|E|\delta(\vec p - \vec p\,') = \int \frac{d^3\vec r}{(2\pi\hbar)^3}\psi^{(-)\dagger}_{\vec p}(t,\vec r)\psi^{(-)}_{\vec p\,'}(t,\vec r) \tag{3.6.10}$$

(here we do not use $\overline{\psi}_p(x)\psi_{p'}(x)$ because this would be undefined; we use $2|E|$ in the normalization because $\psi^\dagger\psi$ transforms as the time component of a vector density). The massless eigenstates are then

$$\begin{aligned}
\psi^{(+)}_{\pm 1}(x) &= \sqrt{|E|}\begin{pmatrix}\chi_{\pm 1}\\ \pm\chi_{\pm 1}\end{pmatrix} e^{ip\cdot x}\\
\psi^{(-)}_{\pm 1}(x) &= \sqrt{|E|}\begin{pmatrix}\chi_{\mp 1}\\ \mp\chi_{\mp 1}\end{pmatrix} e^{ip\cdot x}
\end{aligned} \tag{3.6.11}$$

and we see that

$$\begin{aligned} c\gamma^5\psi^{(+)}_{\pm 1}(x) &= \pm\psi^{(+)}_{\pm 1}(x) \\ c\gamma^5\psi^{(-)}_{\pm 1}(x) &= \mp\psi^{(-)}_{\pm 1}(x) \end{aligned} \tag{3.6.12}$$

so for massless particles (only!) the helicity and chirality bases coincide. Define the chiral projector

$$P_\pm = \frac{1}{2}\left(1 \pm c\gamma^5\right) \tag{3.6.13}$$

then

$$\begin{aligned} &P_+\psi^{(+)}_{+1} = \psi^{(+)}_{+1}, \quad P_+\psi^{(+)}_{-1} = 0 \\ &P_+\psi^{(-)}_{+1} = 0, \quad P_+\psi^{(-)}_{-1} = \psi^{(-)}_{-1} \\ &P_-\psi^{(+)}_{+1} = 0, \quad P_-\psi^{(+)}_{-1} = \psi^{(-)}_{-1} \\ &P_-\psi^{(-)}_{+1} = \psi^{(-)}_{+1}, \quad P_-\psi^{(-)}_{-1} = 0 \end{aligned} \tag{3.6.14}$$

and, moreover,

$$\begin{aligned} &P_\pm^2 = P_\pm \\ &\mathrm{Tr}(P_\pm) = \frac{1}{2} \\ &P_\pm P_\mp = 0 \\ &P_+ + P_- = 1 \end{aligned} \tag{3.6.15}$$

are straightforward.

The chirality operator does not commute with the Hamiltonian of a massive particle and so chiral eigenstates will not be solutions of Dirac's equation if $m \neq 0$. However, because of the fourth identity in (3.6.15), we could decompose any solution of the Dirac equation for massive fermions into chiral components,

$$\psi = P_+\psi + P_-\psi = \psi_+ + \psi_- \tag{3.6.16}$$

(but neither ψ_+ nor ψ_- would be solutions of the massive Dirac equation). One can show that

$$j^\mu = \overline{\psi}\gamma^\mu\psi = \overline{\psi}_+\gamma^\mu\psi_+ + \overline{\psi}_-\gamma^\mu\psi_- = j^\mu_+ + j^\mu_- \tag{3.6.17}$$

becomes the sum of chiral currents, but the term

$$\overline{\psi}\psi = \overline{\psi}_+\psi_- + \overline{\psi}_-\psi_+ \tag{3.6.18}$$

is necessarily mixed. Thus the mass term can be viewed as arising from an interaction that transforms the chiral components of the spinor into one another. Indeed one can define the axial current vector as

$$j^\mu_A = j^\mu_+ - j^\mu_- = \overline{\psi}\gamma^\mu\gamma^5\psi, \tag{3.6.19}$$

which satisfies the conservation equation

$$\partial_\mu j^\mu_A = 2im\overline{\psi}\gamma^5\psi \tag{3.6.20}$$

on shell, and so is conserved only if the field is massless. The current, j^μ, however, is always conserved.

3.7 Lagrangian Description

We now want to make our way to a Lagrangian description of the Dirac field. We will begin by asking ourselves about all possible bilinear covariants that can be constructed from the wave function $\psi(x)$, its hermitean conjugate and the gamma matrices. Then we will propose a Lagrangian density from which the Dirac equation can be derived by Euler's equations.

3.7.1 Bilinear Covariants

Starting with the linear transformation of $\psi(x)$ in (3.3.8) and the equation for $\widehat{S}$ in (3.3.11), we have

$$\begin{aligned} &\psi(x) \to \psi'(x') = \widehat{S}\psi(x) \\ &\overline{\psi}(x) \to \overline{\psi}'(x') = \psi^\dagger \underbrace{(\gamma^0\gamma^0)}\widehat{S}^\dagger\gamma^0 = \overline{\psi}(c^2\gamma^0\widehat{S}^\dagger\gamma^0) \end{aligned} \tag{3.7.1}$$

Now a very useful property of γ^0 is that

$$c^2\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger} \tag{3.7.2}$$

as can be checked explicitly. This property obviously translates to arbitrary products of the gamma matrices

$$c^2\gamma^0(\gamma^{\mu_1}\ldots\gamma^{\mu_n})\gamma^0 = c^{2n}(\gamma^0\gamma^{\mu_1}\gamma^0)(\gamma^0\gamma^{\mu_2}\ldots\gamma^{\mu_n}\gamma^0) = \gamma^{\mu_1\dagger}\ldots\gamma^{\mu_n\dagger} \tag{3.7.3}$$

and, by extension, to $\Sigma^{\alpha\beta}$. We could then could use either (3.3.15) or (3.3.16) to show that[8]

$$c^2\gamma^0\widehat{S}^\dagger\gamma^0 = \widehat{S}^{-1} \tag{3.7.4}$$

and so conclude that

$$\overline{\psi}(x) \to \overline{\psi}'(x') = \overline{\psi}(x)\widehat{S}^{-1}. \tag{3.7.5}$$

In this way we obtain the following bilinear covariants:

[8] Problem: Prove this.

- $\overline{\psi}(x)\psi(x)$ is an invariant, or scalar.
- $\overline{\psi}'(x')\gamma^\mu\psi'(x') = \overline{\psi}(x)\widehat{S}^{-1}\gamma^\mu\widehat{S}\psi(x) = L^\mu{}_\alpha\overline{\psi}(x)\gamma^\alpha\psi(x)$ and therefore $\overline{\psi}\gamma^\mu\psi$ is a vector.
- $\overline{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = L^\mu{}_\alpha L^\nu{}_\beta\overline{\psi}(x)\sigma^{\alpha\beta}\psi(x)$ is an antisymmetric tensor.
- Consider $\overline{\psi}'(x')\gamma^5\psi'(x') = \overline{\psi}(x)(\widehat{S}^{-1}\gamma^5\widehat{S})\psi(x)$; now

$$\begin{aligned}\widehat{S}^{-1}\gamma^5\widehat{S} &= \frac{i}{4!}\epsilon_{\mu\nu\alpha\beta}\widehat{S}^{-1}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\widehat{S}\\ &= \frac{i}{4!}\epsilon_{\mu\nu\alpha\beta}(\widehat{S}^{-1}\gamma^\mu\widehat{S})\ldots(\widehat{S}^{-1}\gamma^\beta\widehat{S})\\ &= \frac{i}{4!}\epsilon_{\mu\nu\alpha\beta}L^\mu{}_\lambda\ldots L^\beta{}_\kappa(\gamma^\lambda\ldots\gamma^\kappa) = \||L\||\gamma^5 \qquad (3.7.6)\end{aligned}$$

  so its sign depends on whether $\widehat{L}$ is a proper or an improper Lorentz transformation. Under a proper transformation, $\overline{\psi}\gamma^5\psi$ transforms as a scalar. However, under a parity transformation $\overline{\psi}'(x')\gamma^5\psi'(x') \to -\overline{\psi}(x)\gamma^5\psi(x)$, *i.e.,* $\overline{\psi}\gamma^5\psi$ transforms as a **pseudoscalar**.
- An identical argument will show that $\overline{\psi}\gamma^\mu\gamma^5\psi$ transforms as a **pseudovector**.

3.7.2 Action

It is now easy to see that the Dirac equation may be derived from an action principle by Hamilton's principle, exactly as were the equations describing the scalar and electromagnetic fields. The required action is

$$S = -\hbar c^2\int d^4x\,\overline{\psi}\left(\frac{i}{2}\overleftrightarrow{\partial\!\!\!/} - \frac{mc}{\hbar}\right)\psi \qquad (3.7.7)$$

where we have introduced a new (and commonly used) notation, which we henceforth follow: $\partial\!\!\!/ \stackrel{\text{def}}{=} \gamma\cdot\partial$. From here it follows that the mechanical dimension of ψ is $[\psi] = l^{-3/2}$. The action is invariant under Lorentz transformations, Parity and Time Reversal.

It is also invariant under global gauge transformations, $x \to x' = x$, $\psi(x) \to \psi'(x') = e^{i\alpha}\psi(x)$, where α is constant and can be made invariant under local gauge transformations (α depends on x) by the same procedure we employed for the scalar field. If we introduce the gauge field, $A_\mu(x)$, which transforms as (2.6.2), and the covariant derivative as in (2.6.4) then it is straightforward to show that

$$S = -\hbar c^2\int d^4x\,\overline{\psi}\left(\frac{i}{2}\overleftrightarrow{D\!\!\!\!/} - \frac{mc}{\hbar}\right)\psi, \qquad (3.7.8)$$

where

$$\overline{\psi}\overleftrightarrow{\not{D}}\psi = \overline{\psi}(\overrightarrow{\not{\partial}} - ig\not{A})\psi - \overline{\psi}(\overleftarrow{\not{\partial}} + ig\not{A})\psi \tag{3.7.9}$$

is indeed invariant. The combined action of the spinor field and the interacting electromagnetic field would then be

$$S = -\hbar c^2 \int d^4x \, \left[\overline{\psi}\left(\frac{i}{2}\overleftrightarrow{\not{D}} - \frac{mc}{\hbar}\right)\psi + \frac{g}{4\hbar c}F_{\mu\nu}F^{\mu\nu}\right], \tag{3.7.10}$$

where $F_{\mu\nu}$ is, of course, the Maxwell tensor of the electromagnetic field. The equations of motion follow by Hamilton's principle,

$$(i\hbar\overrightarrow{\not{D}} - mc)\psi = 0$$

$$\overline{\psi}(i\hbar\overleftarrow{\not{D}} + mc) = 0$$

$$\partial_\mu F^{\mu\nu} = \hbar c\overline{\psi}\gamma^\nu\psi \tag{3.7.11}$$

where the first two equations follow respectively by varying with respect to $\overline{\psi}$ and ψ, whereas the last follows by varying with respect to A_μ. The first pair of equations are the Dirac equation for ψ and $\overline{\psi}$ respectively in the presence of an interacting electromagnetic field. The last equation is familiar from electrodynamics, when

$$j^\mu = \hbar c^2\overline{\psi}\gamma^\mu\psi$$

is interpreted as the matter current four vector density. This term arises as a consequence of the global gauge invariance of the free field action as we will see in the following section.

3.7.3 Non-Relativistic limit

Let us take a closer look at the first equation (drop the arrow indicating the direction of the derivative's action). First, we rewrite it as

$$-\frac{i}{\hbar}\not{D}\,(i\hbar\not{D} - mc)\psi = 0 = \gamma^\mu\gamma^\nu D_\mu D_\nu + \frac{m^2c^2}{\hbar^2}\psi, \tag{3.7.12}$$

where we use $i\hbar\not{D}\,\psi = mc\psi$. Moreover, because

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\left(\{\gamma^\mu, \gamma^\nu\} + [\gamma^\mu, \gamma^\nu]\right) = -\eta^{\mu\nu} + \Sigma^{\mu\nu} \tag{3.7.13}$$

it follows that (3.7.12) becomes

$$-D^2 + \Sigma^{\mu\nu}D_\mu D_\nu\psi + \frac{m^2c^2}{\hbar^2}\psi = 0. \tag{3.7.14}$$

Again, $\Sigma^{\mu\nu}$ is antisymmetric so

$$\Sigma^{\mu\nu} D_\mu D_\nu = \frac{1}{2}\Sigma^{\mu\nu}[D_\mu, D_\nu] = -\frac{ig}{2}\Sigma^{\mu\nu}F_{\mu\nu} \tag{3.7.15}$$

and the Dirac equation can be rewritten as

$$\left(-D^2 - \frac{ig}{2}\Sigma^{\mu\nu}F_{\mu\nu} + \frac{m^2c^2}{\hbar^2}\right)\psi = 0. \tag{3.7.16}$$

We could go back to expressing this in terms of the momentum and energy operators; if we set $\hbar g = e$ (the "electric charge") and

$$D_\mu \to \frac{i}{\hbar}(p_\mu - eA_\mu), \tag{3.7.17}$$

then (3.7.16) turns into

$$\left[(p_\mu - eA_\mu)^2 - \frac{ie\hbar}{2}\Sigma^{\mu\nu}F_{\mu\nu} + m^2c^2\right]\psi = 0. \tag{3.7.18}$$

This is not the Klein-Gordon equation in the presence of an electromagnetic field. The difference lies in the second term, which is an entirely new feature introduced by the Dirac equation.

We get a feeling for its meaning when we look at the non-relativistic limit of this equation. Noting that

$$(p_\mu - eA_\mu)^2 = -\frac{1}{c^2}(E - e\phi)^2 + (\vec{p} - e\vec{A})^2, \tag{3.7.19}$$

where ϕ is the electric potential and $\vec{A}$ the vector potential, let us set $E = mc^2 + E_{\text{n.r.}}$, where $E_{\text{n.r.}}$ is the non-relativistic energy of the Dirac particle which we take to be much larger than $e\phi$. Then

$$(E - e\phi)^2 = (mc^2 + E_{\text{n.r.}} - e\phi)^2 \approx m^2c^4 + 2mc^2(E_{\text{n.r.}} - e\phi) \tag{3.7.20}$$

and therefore

$$\left[-2m(E_{\text{n.r.}} - e\phi) + (\vec{p} - e\vec{A})^2 - \frac{ie\hbar}{2}\Sigma^{\mu\nu}F_{\mu\nu}\right]\psi = 0 \tag{3.7.21}$$

or

$$E_{\text{n.r.}}\psi = \left[\frac{(\vec{p} - e\vec{A})^2}{2m} + e\phi - \frac{ie\hbar}{4m}\Sigma^{\mu\nu}F_{\mu\nu}\right]\psi. \tag{3.7.22}$$

The first two terms on the right can be identified with the Hamiltonian of a scalar particle and would lead to a Schroedinger equation of the form we are familiar with. The last term on the right represents a coupling of the spin of a Dirac particle to the external electromagnetic field. In the non-relativistic limit it is sufficient to consider only the spatial components in the sum,

$$\frac{i\hbar}{2}\Sigma^{\mu\nu}F_{\mu\nu} \approx \frac{i\hbar}{2}\Sigma^{ij}F_{ij} = \frac{i\hbar}{2}\epsilon_{ijk}\Sigma^{ij}B^k = 2\vec{S}\cdot\vec{B} \tag{3.7.23}$$

where $\vec{S}$ is the particle's spin. Thus we have ended up with

$$E_{\text{n.r.}}\psi = [\mathcal{H}_0 + \mathcal{H}_{\text{int}}]\psi = \left[\left(\frac{(\vec{p}-e\vec{A})^2}{2m} + e\phi\right) + \left(-\frac{e}{m}\vec{S}\cdot\vec{B}\right)\right]\psi \tag{3.7.24}$$

and if we recall that in this limit the four dimensional Dirac spinor effectively reduces to the two dimensional u^L, we find the Schroedinger equation

$$i\hbar\frac{\partial u^L}{\partial t} = \left[\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - e\vec{A}\right)^2 + e\phi - \frac{e\hbar}{2m}\vec{\sigma}\cdot\vec{B}\right]u^L, \tag{3.7.25}$$

which will be recognized as the Pauli equation! The interactions with the electromagnetic field are more complicated in the relativistic case and we would then use the full Dirac equation (first in (3.7.11)). Continuing with the non-relativistic limit, we see that the interaction with the magnetic field takes the form $\mathcal{H}_{\text{int}} = -\vec{\mu}\cdot\vec{B}$, where

$$\vec{\mu} = \frac{e\mathfrak{g}\vec{S}}{2m} \tag{3.7.26}$$

is the "magnetic moment". Above, we introduced the quantity "$\mathfrak{g}$", called the $g-$factor. While, according to the classical theory, $\mathfrak{g} = 2$, there are small corrections due to quantum electrodynamic effects. Experimentally $\mathfrak{g} = 2\times(1 + 0.00115965241 \pm 0.0000030020)$ for the electron. For the proton, on the other hand, $\mathfrak{g} = 2\times(1 + 1.79328\ldots)$. This large discrepancy from the expected value is due to (and indeed an indication of) the fact that the proton has structure.

3.8 Conservation Laws

Let us begin with the action for a free Dirac particle in (3.7.7). In the massless case there is the obvious scaling symmetry,

$$x \to x' = \lambda x, \quad \psi(x) \to \psi'(x') = \lambda^{-3/2}\psi(x). \tag{3.8.1}$$

Therefore, applying Noether's theorem with

$$G^\nu = \frac{\delta x^\nu}{\delta\lambda} = x^\nu, \quad G^{(\psi)} = \frac{\delta\psi}{\delta\lambda} = -\frac{3}{2}\psi, \quad G^{(\overline{\psi})} = -\frac{3}{2}\overline{\psi} \tag{3.8.2}$$

gives the (conserved) dilation current density, $\mathrm{j}^\mu = \Theta^{\mu\nu}x_\nu$, where

$$\Theta^{\mu\nu} = \frac{i\hbar c^2}{2}\overline{\psi}\gamma^\mu \overleftrightarrow{\partial^\nu}\psi + \mathfrak{L}\eta^{\mu\nu}, \tag{3.8.3}$$

as in the case of the Klein-Gordon field, is the conserved energy momentum tensor for the Dirac field and can be explicitly obtained by demanding translation invariance (when the equations of motion are satisfied (on-shell), the Lagrangian of the Dirac field vanishes identically). We could write the orbital angular momentum tensor density as

$$L^{\mu\alpha\beta} = \frac{1}{2}\left(\Theta^{\mu\alpha}x^\beta - \Theta^{\mu\beta}x^\alpha\right) \tag{3.8.4}$$

but it is not conserved. The orbital angular momentum vector is determined from $L^{\mu\alpha\beta}$ to have its usual form,

$$L^i = -\epsilon^{ijk}\int d^3\vec{r}\; L^0{}_{jk} = \frac{i\hbar}{2}\int d^3\vec{r}\; \psi^\dagger(\vec{r}\times \overleftrightarrow{\nabla})\psi. \tag{3.8.5}$$

We can also determine the intrinsic spin of the Dirac particle. Using the change in ψ by a Lorentz transformation as given in (3.3.15),

$$\frac{\delta\psi}{\delta\omega_{\alpha\beta}} = -\frac{1}{4}\Sigma^{\alpha\beta}\psi \tag{3.8.6}$$

the spin density turns out to be just

$$S^{\mu\alpha\beta} = \frac{\partial\mathfrak{L}}{\partial(\partial_\mu\psi)}\frac{\delta\psi}{\delta\omega_{\alpha\beta}} + \frac{\delta\overline{\psi}}{\delta\omega_{\alpha\beta}}\frac{\partial\mathfrak{L}}{\partial(\partial_\mu\overline{\psi})} = \frac{i\hbar c}{8}\overline{\psi}\left\{\gamma^\mu, \Sigma^{\alpha\beta}\right\}\psi. \tag{3.8.7}$$

Therefore the spin vector will be

$$S^i = -\epsilon^{ijk}\int d^3\vec{r}\; S^0{}_{jk} = \frac{i\hbar}{4}\int d^3\vec{r}\; \psi^\dagger(\vec{\gamma}\times\vec{\gamma})^i\psi \tag{3.8.8}$$

and in this way we have recovered, by an application of Noether's theorem, the spin operator that we had derived in (3.3.3). The spin is not separately conserved because $\Theta^{\mu\nu}$ is not symmetric but the sum of the orbital angular momentum and the intrinsic spin

$$M^{\mu\alpha\beta} = L^{\mu\alpha\beta} + S^{\mu\alpha\beta} \tag{3.8.9}$$

will be conserved.

To construct an orbital angular momentum tensor and a spin tensor that are *separately* conserved, we must first determine the Belinfante tensor. The energy momentum tensor is made symmetric by applying the prescription laid out in the previous chapter, *i.e.,* by the addition of $\partial_\lambda k^{\lambda\mu\nu}$ to $\Theta^{\mu\nu}$ in (3.8.3), with[9]

$$k^{\lambda\mu\nu} = -\frac{i\hbar c^2}{16}\overline{\psi}\left\{\gamma^\lambda, [\gamma^\mu, \gamma^\nu]\right\}\psi. \tag{3.8.10}$$

Using the Clifford algebra one can show without too much difficulty that

$$k^{\lambda\mu\nu} = -\frac{i\hbar c^2}{4}\overline{\psi}\left[\gamma^\lambda\gamma^\mu\gamma^\nu + \eta^{\mu\nu}\gamma^\lambda - \eta^{\nu\lambda}\gamma^\mu + \eta^{\mu\lambda}\gamma^\nu\right]\psi, \tag{3.8.11}$$

is antisymmetric in (λ, μ). Then, using the equations of motion one has

$$\partial_\lambda k^{\lambda\mu\nu} = -\frac{i\hbar c}{4}\left[-\overline{\psi}\overleftarrow{\partial^\nu}\gamma^\mu\psi + \overline{\psi}\overleftarrow{\partial^\mu}\gamma^\nu\psi + \overline{\psi}\gamma^\mu\overrightarrow{\partial^\nu}\psi - \overline{\psi}\gamma^\nu\overrightarrow{\partial^\mu}\psi\right] \tag{3.8.12}$$

and, adding this tensor $\Delta^{\mu\nu} = \partial_\lambda k^{\lambda\mu\nu}$ to (3.8.3) we end up with the symmetric tensor

$$t^{\mu\nu} = \frac{i\hbar c^2}{4}\left[\overline{\psi}\gamma^\mu\overleftrightarrow{\partial^\nu}\psi + \overline{\psi}\gamma^\nu\overleftrightarrow{\partial^\mu}\psi\right] \tag{3.8.13}$$

because the Lagrangian density also vanishes on-shell. With this symmetric tensor we might identify

$$\wp^0 = t^{00} = \frac{i\hbar}{2}\left[\psi^\dagger\overleftrightarrow{\partial^t}\psi\right] \tag{3.8.14}$$

as the energy density carried by the Dirac field and

$$\wp^i = t^{0i} = \frac{i\hbar c^2}{4}\left[\overline{\psi}\gamma^0\overleftrightarrow{\partial^i}\psi + \overline{\psi}\gamma^i\overleftrightarrow{\partial^t}\psi\right] \tag{3.8.15}$$

as its momentum density.

Having the symmetric energy momentum tensor, we could define the angular momentum density and the angular momentum tensor in the same way as we have done for the electromagnetic field. In this way, we arrive at the modified orbital angular momentum tensor density,

$$\widetilde{L}^{\mu\alpha\beta} = \frac{1}{2}\left(t^{\mu\alpha}x^\beta - t^{\mu\beta}x^\alpha\right), \tag{3.8.16}$$

which *is* conserved and, from $\widetilde{L}^{\mu\alpha\beta}$, the modified orbital angular momentum

$$\widetilde{L}^i = \int d^3\vec{r}\ (\vec{r}\times\vec{\wp}), \tag{3.8.17}$$

[9]Problem: Prove this.

which is also conserved. Similarly, the *modified* spin density tensor,

$$\widetilde{S}^{\mu\alpha\beta} = S^{\mu\alpha\beta} - \frac{1}{2}\left(\Delta^{\mu\alpha}x^{\beta} - \Delta^{\mu\beta}x^{\alpha}\right) \tag{3.8.18}$$

is separately conserved.

Infinitesimal global gauge transformations of the spinor field,

$$\psi(x) \to \psi'(x') = (1 + ig\delta\alpha)\psi(x) \;\Rightarrow G = \frac{\delta\psi}{g\delta\alpha} = i\psi \tag{3.8.19}$$

and analogously for $\overline{\psi}$, leads to the existence of the conserved current

$$j^{\mu} = -i\overline{\psi}\frac{\partial\mathfrak{L}}{\partial(\partial_{\mu}\overline{\psi})} + i\frac{\partial\mathfrak{L}}{\partial(\partial_{\mu}\psi)}\psi = \hbar c^2\overline{\psi}\gamma^{\mu}\psi. \tag{3.8.20}$$

The Lagrangian density for spinors interacting with the electromagnetic field therefore takes the form

$$\mathfrak{L} = \mathfrak{L}_{\text{free}} + \mathfrak{L}_{\text{int}} \tag{3.8.21}$$

where

$$\mathfrak{L}_{\text{int}} = -g(j \cdot A). \tag{3.8.22}$$

The interaction Lagrangian density does not contain terms quadratic in the gauge fields as did the scalar field.

Local gauge invariance of the action in (3.7.10) leads to the strongly conserved current

$$J^{\mu} = c\partial_{\lambda}(F^{\lambda\mu}\delta\alpha(x)), \tag{3.8.23}$$

which should be compared with (2.8.6). Again, since $\alpha(x)$ is arbitrary, we take it to be a constant and discover that

$$J^{\mu} = c\delta\alpha\partial_{\lambda}F^{\lambda\mu} = (\hbar c^2\overline{\psi}\gamma^{\mu}\psi)\delta\alpha \stackrel{\text{def}}{=} j^{\mu}\delta\alpha, \tag{3.8.24}$$

where $j^{\mu} = \hbar c^2\overline{\psi}\gamma^{\mu}\psi$ is the source current for the electromagnetic field in the equations of motion. One sees a generic feature of minimal coupling once again: the Noether currents associated with gauge symmetry can be expressed in terms of the field strengths with no explicit reference to the source. Defining the Noether charge,

$$Q = -g\int d^3\vec{r}\; j^0 = -gc\int d^3\vec{r}\;\partial_i F^{i0} = \frac{g}{c}\oint_S \vec{E}\cdot d\vec{S}, \tag{3.8.25}$$

we recover Gauss' law. (As before the infinite family of currents generated via local gauge invariance is redundant.)

3.9 Hamiltonian

If we begin with the Lagrangian for the Dirac field and determine the momenta in the usual way, we find that the momenta do not involve the field "velocities",[10]

$$\pi = \frac{\partial \mathfrak{L}}{\partial \dot{\psi}} = -\frac{i\hbar c^2}{2}\overline{\psi}\gamma^0, \quad \overline{\pi} = \frac{\partial \mathfrak{L}}{\partial \dot{\overline{\psi}}} = \frac{i\hbar c^2}{2}\gamma^0\psi. \tag{3.9.1}$$

This was to be expected because the equation of motion is first order in time and not second order, but it means that the field "velocities" cannot be determined in terms of the momenta and therefore that the above should be viewed as primary constraints,

$$\Phi = \overline{\pi} - \frac{i\hbar c^2}{2}\gamma^0\psi, \quad \overline{\Phi} = \pi + \frac{i\hbar c^2}{2}\overline{\psi}\gamma^0. \tag{3.9.2}$$

We must apply the Dirac-Bergman algorithm. Begin with the canonical Hamiltonian density[11]

$$\mathfrak{H}_c = \dot{\overline{\psi}}\overline{\pi} + \pi\dot{\psi} - \mathfrak{L} = \hbar c^2 \overline{\psi}\left(\frac{i}{2}\gamma^i \overleftrightarrow{\partial_i} - \frac{mc}{\hbar}\right)\psi, \tag{3.9.3}$$

and write the primary Hamiltonian by introducing two multipliers, μ and $\overline{\mu}$,

$$\mathcal{H}_p = \int d^3\vec{r}\,(\overline{\mu}\Phi + \overline{\Phi}\mu + \mathfrak{H}_c) = \int d^3x \left[\overline{\mu}\Phi + \overline{\Phi}\mu + \hbar c^2\overline{\psi}\left(\frac{i}{2}\gamma^i\overleftrightarrow{\partial_i} - \frac{mc}{\hbar}\right)\psi\right]. \tag{3.9.4}$$

This gives the canonical equations

$$\begin{aligned}
\dot{\psi} &= \mu, \\
\dot{\overline{\psi}} &= \overline{\mu}, \\
\dot{\pi} &= \hbar c^2\left[\frac{i}{2}\overline{\mu}\gamma^0 + i\overline{\psi}\gamma^i\overleftarrow{\partial}_i + \frac{mc}{\hbar}\overline{\psi}\right], \\
\dot{\overline{\pi}} &= \hbar c^2\left[-\frac{i}{2}\gamma^0\mu - i\gamma^i\overrightarrow{\partial}_i\psi + \frac{mc}{\hbar}\psi\right]
\end{aligned} \tag{3.9.5}$$

Consistency requires that $\dot{\Phi} = 0 = \dot{\overline{\Phi}}$, which give the conditions that fix μ and $\overline{\mu}$ (they do not lead to secondary constraints). Straightforwardly, one finds

$$\chi = i\gamma^0\mu + i\gamma^i\overrightarrow{\partial}_i\psi - \frac{mc}{\hbar}\psi \approx 0 \;\Rightarrow\; \frac{\mu}{c^2} \approx -\gamma^0\gamma^i\overrightarrow{\partial}_i\psi - \frac{imc}{\hbar}\gamma^0\psi$$

[10]Observe that $\overline{\pi} = c^2\gamma^0\pi^\dagger$.

[11]Problem: Show by using the equation of motion that $\mathfrak{H}_c$ is the energy density, $\wp^0$, obtained in (3.8.14)

$$\overline{\chi} = i\overline{\mu}\gamma^0 + i\overline{\psi}\gamma^i\overleftarrow{\partial}_i + \frac{mc}{\hbar}\overline{\psi} \approx 0 \ \Rightarrow \ \frac{\overline{\mu}}{c^2} = -\overline{\psi}\gamma^i\gamma^0\overleftarrow{\partial}_i + \frac{imc}{\hbar}\overline{\psi}\gamma^0 \tag{3.9.6}$$

Inserting these expressions for μ and $\overline{\mu}$ into $\mathcal{H}_p$, we get

$$\begin{aligned}\mathcal{H}_p &= \int d^3\vec{r}\left[c^2\left(-\overline{\psi}\gamma^i\gamma^0\overleftarrow{\partial}_i + \frac{imc}{\hbar}\overline{\psi}\gamma^0\right)\left(\overline{\pi} - \frac{i\hbar c^2}{2}\gamma^0\psi\right)\right.\\ &\quad + c^2\left(\pi + \frac{i\hbar c^2}{2}\overline{\psi}\gamma^0\right)\left(-\gamma^0\gamma^i\overrightarrow{\partial}_i\psi - \frac{imc}{\hbar}\gamma^0\psi\right)\\ &\quad \left. + \hbar c^2\overline{\psi}\left(\frac{i}{2}\gamma^i\overleftrightarrow{\partial_i} - \frac{mc}{\hbar}\right)\psi\right]\\ &= c^2\int d^3\vec{r}\left[-\overline{\psi}\gamma^i\gamma^0\overleftarrow{\partial}_i\overline{\pi} - \pi\gamma^0\gamma^i\overrightarrow{\partial}_i\psi + \frac{imc}{\hbar}\left(\overline{\psi}\gamma^0\overline{\pi} - \pi\gamma^0\psi\right)\right] \end{aligned} \tag{3.9.7}$$

The canonical equations of motion now coincide with the Euler-Lagrange equations.

A simple calculation confirms that the Poisson Brackets between the constraints do not vanish, so they are all second class. This means that if we want to quantize the field then the Poisson brackets must be replaced by Dirac brackets.[12] We can also determine the number of local degrees of freedom carried by the Dirac field. The Dirac spinor has 4 complex components or 8 real ones, so $n = 8$. There are no first class constraints, but there are 4 complex second class constraints, or 8 real ones, so we have $8 - 8/2 = 4$ real, local degrees of freedom. Two of these are associated with the Dirac particle and two with the antiparticle.

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[12] Problem: Determine the 2×2 complex valued matrix $C_{\rho\lambda}$ of Poisson brackets between the constraints and then evaluate the fundamental Dirac brackets for the Dirac field.

Chapter 4

Yang-Mills Fields

The need for locally interacting, dynamical fields that act as "carriers" of forces between particles follows directly from the requirement of Lorentz invariance. The only question is what properties the force carrying fields must have to best describe the kinds of interactions found in nature.

We have seen in previous chapters that just as the need for force carrying fields follows from Lorentz invariance, both the most salient properties as well as the dynamics of at least one of these fields, the electromagnetic field, can be determined by requiring yet another kind of symmetry: local gauge invariance. It is therefore natural to think of the symmetry as fundamental and of the electromagnetic field as arising as a consequence of the symmetry. Following this line of reasoning, we ask if the gauge symmetry used to "derive" the existence and properties of the electromagnetic field can be generalized to describe other fields that carry information about the other fundamental forces of nature. The answer is affirmative; Yang-Mills fields are the generalization we seek for two more of the four known fundamental forces of nature, *viz.,* the "weak" and "strong" interactions. Thus three of the fundamental forces are "gauge" theories and Gravitation, as we currently understand it, must be dealt with separately.

4.1 Gauge Groups

The relevant generalization of the gauge symmetry of electrodynamics makes use of direct products of compact, simple Lie groups and $U(1)$ factors. In the simplest case of electrodynamics, this is just $U(1)$. In the standard model of particle physics (excluding gravity) it is $SU(3) \times SU(2) \times U(1)$ and describes the strong, weak and electromagnetic interactions in that order. All compact, simple Lie groups can be represented by finite dimensional matrices and fall into the following categories:

- $SU(N)$, with $N \geq 2$ can be minimally represented by N dimensional unitary matrices of determinant one (this is called the fundamental representation).

- $O(N)$, with $N \geq 2$ can be minimally represented by N dimensional, real orthogonal matrices. Requiring the matrices to have unit determinant leads to the subgroup $SO(N)$.
- $Sp(2N)$, with $N \geq 1$ can be minimally represented by $2N$ dimensional real, symplectic matrices, *i.e.,* matrices, $\hat{O}$ satisfying the property

$$\hat{O}^T \hat{\omega} \hat{O} = \hat{\omega}, \quad \hat{\omega} = \begin{pmatrix} 0 & \mathbf{1}_{N\times N} \\ -\mathbf{1}_{N\times N} & 0 \end{pmatrix}$$

Any group element, $\mathfrak{g}$, that is connected to the identity can always be represented as

$$\hat{U}(\mathfrak{g}) = e^{ig\alpha^a \hat{t}_a} \tag{4.1.1}$$

where g is a constant, α^a are the **parameters** of the group and $\hat{t}_a$ are hermitean matrices called the **generators** of the group. The number of parameters required determines the **dimension** of the Lie group, $\dim(G)$. The number of generators must obviously be the same as $\dim(G)$ and is not to be confused with the dimension, N, of the matrix representation, $\hat{U}(\mathfrak{g})$ of $\mathfrak{g}$. The generators satisfy commutation relations

$$[\hat{t}_a, \hat{t}_b] = f_{ab}^c \hat{t}_c, \tag{4.1.2}$$

called the **Lie Algebra** of the group G. The constants f_{ab}^c are called the **structure constants** of the group and they are antisymmetric

$$f_{ab}^c = -f_{ba}^c \tag{4.1.3}$$

as follows directly from their definition. There is another identity that the structure constants must satisfy and this one follows from the Jacobi identity

$$[\hat{t}_a, [\hat{t}_b, \hat{t}_c]] + [\hat{t}_c, [\hat{t}_a, \hat{t}_b]] + [\hat{t}_b, [\hat{t}_c, \hat{t}_a]] = 0, \tag{4.1.4}$$

which implies that

$$f_{ad}^e f_{bc}^d + f_{cd}^e f_{ab}^d + f_{bd}^e f_{ca}^d = 0. \tag{4.1.5}$$

It is now easy to see that any set of structure constants will determine one set of generators of G by taking $[\hat{T}_a]^b{}_c = f_{ac}^b = [f_a]^b{}_c$.[1] The dimension of this representation is $\dim(G)$ and it is called the **adjoint representation** of the group G.

In this chapter we will begin by discussing pure Yang-Mills fields. Then we will go on to discuss the adjustments that must be made to describe the actual interactions observed

[1] Problem: Use the Jacobi identitity to prove that the generators $\hat{T}_a$ satisfy the Lie algebra of G.

in nature. We consider a set of N fields that transform locally as a vector under some irreducible $N \times N$ representation $\hat{U}(\mathfrak{g})$ of a compact Lie group G.[2] For convenience, begin with a set of scalar fields,

$$\vec{\phi}(x) = \begin{pmatrix} \phi^1(x) \\ \phi^2(x) \\ \cdots \\ \cdots \\ \cdots \\ \phi^N(x) \end{pmatrix} \tag{4.1.6}$$

and let

$$\vec{\phi}(x) \to \vec{\phi}'(x') = \hat{U}(g)\vec{\phi}(x) = e^{ig\alpha^a(x)\hat{t}_a}\vec{\phi}(x) \tag{4.1.7}$$

($\vec{\phi}$ transforms as a vector under the action of G in the $N \times N$ representation of $\{\hat{t}_a\}$). Infinitesimally,

$$\delta\phi^i(x) = ig\alpha^a(x)t^i_{aj}\phi^j(x) \tag{4.1.8}$$

where $\hat{t}_a$ are the generators of the group G. Our immediate purpose is to construct Lagrangians that are invariant under *local* transformations of the type in (4.1.7).

4.2 Gauge Invariance

The electromagnetic field, discussed earlier, arose by requiring an invariant action for the scalar field under transformations that are a special case of the transformations in (4.1.7). There, the Lie group G is just the abelian group, $U(1)$, of unitary transformations of the field. Taking a cue from our construction for the electromagnetic field, we realize that we should introduce a gauge field, A^a_μ, that should transform simultaneously under a gauge transformation of $\vec{\phi}$. But how must it transform? To answer this, we must construct a gauge covariant derivative, which we define to be

$$\hat{D}_\mu = \partial_\mu \mathbf{1} - igA^a_\mu\hat{t}_a \stackrel{\text{def}}{=} \partial_\mu\mathbf{1} - ig\hat{A}_\mu \tag{4.2.1}$$

where $\hat{A}_\mu = A^a_\mu\hat{t}_a$, in components,

$$D_\mu\phi^i = \partial_\mu\phi^i - igA^a_\mu t^i_{aj}\phi^j = \partial_\mu\phi^i - igA^i_{\mu j}\phi^j \tag{4.2.2}$$

with the requirement that, under the gauge transformation of (4.1.7),

$$\hat{D}_\mu\vec{\phi}(x) \to \hat{D}'_\mu\vec{\phi}'(x') = \hat{U}D_\mu\vec{\phi}(x). \tag{4.2.3}$$

[2] An irreducible representation of a group has no nontrivial invariant subspaces. Any representation of a semisimple Lie group can be decomposed into a direct sum of irreducible representations.

Then under the general non-abelian gauge transformation of (4.1.7),

$$\hat{D}_\mu\vec{\phi} \to \hat{D}'_\mu\vec{\phi}'(x') = [\partial_\mu\mathbf{1} - ig\hat{A}'_\mu]\hat{U}\vec{\phi} = \hat{U}\left[\partial_\mu\vec{\phi} + \hat{U}^{-1}(\partial_\mu\hat{U})\vec{\phi} - ig\hat{U}^{-1}\hat{A}'_\mu\hat{U}\vec{\phi}\right] \tag{4.2.4}$$

so covariance requires that

$$\hat{U}^{-1}\hat{A}'_\mu\hat{U} = \hat{A} - \frac{i}{g}(\hat{U}^{-1}\partial_\mu\hat{U}) \tag{4.2.5}$$

or

$$\hat{A}'_\mu = \hat{U}\hat{A}_\mu\hat{U}^{-1} + \frac{i}{g}\hat{U}\partial_\mu\hat{U}^{-1}, \tag{4.2.6}$$

which reduces to the familiar gauge transformation of electromagnetism if $\hat{U} = e^{ig\alpha(x)}$.

For an infinitesimal transformation, *i.e.,* retaining the parameter α^a only up to first order,

$$\delta A^a_\mu = \partial_\mu\alpha^a - igf^a_{bc}A^b_\mu\alpha^c. \tag{4.2.7}$$

The first term is, of course, familiar from electrodynamics. The last term is new and owes to the non-abelian character of the symmetry group G. Notice that the right hand side of (4.2.7) has the same structure as (4.2.2) because the structure constants serve as generators in the adjoint representation, *i.e.,* $f^a_{bc} = [\hat{T}_b]^a{}_c$ and we could set $A^b_\mu f^a_{bc} = A^a_{\mu c}$. This means that α^a transforms as a vector in the adjoint representation of the group. Indeed, the covariant derivative acting on vectors V^a in the adjoint representation of G (disregarding any space-time indices that may be present) can be given as

$$\hat{\mathcal{D}}_\mu = \partial_\mu\mathbf{1} - igA^b_\mu\hat{T}_b \stackrel{\text{def}}{=} \partial_\mu\mathbf{1} - ig\hat{A}_\mu, \tag{4.2.8}$$

where $[\hat{A}_\mu]^a{}_c = A^b_\mu f^a_{bc} \equiv A^a_{\mu c}$. In components,

$$\mathcal{D}_\mu V^a = (\partial_\mu V^a - igf^a_{bc}A^b_\mu V^c) \tag{4.2.9}$$

and, in particular, (4.2.7) becomes

$$\delta A^a_\mu = \mathcal{D}_\mu\alpha^a. \tag{4.2.10}$$

The covariant derivative of a vector in the adjoint representation can always be expressed in matrix form as

$$\hat{\mathcal{D}}_\mu\hat{V} = \partial_\mu\hat{V} - ig[\hat{A}_\mu, \hat{V}]. \tag{4.2.11}$$

where $\hat{V}$ is defined as $\hat{V} = V^b\hat{T}_b$. This, of course, is equivalent to (4.2.9).

With the covariant derivative in (4.2.1), the following action for the complex scalar field,

$$S = -\int d^4x\left[\eta^{\mu\nu}(\hat{D}_\mu\vec{\phi})^\dagger(\hat{D}_\nu\vec{\phi}) + V(|\vec{\phi}|)\right], \tag{4.2.12}$$

is guaranteed to be gauge invariant provided that $U^\dagger U = \mathbf{1}$. Any group G that satisfies this condition is a **unitary group**. The unitary groups that satisfy the additional condition $\|\hat{U}\| = 1$ are the **special unitary** groups, $SU(N)$ (here the N refers to the smallest dimension of the matrix representation of a group element). Henceforth we will confine our attention to unitary and special unitary groups. We must now construct an action for the gauge field. Following the prescription laid out by our treatment of the electromagnetic field we determine the Maxwell tensor for this field,

$$[\hat{D}_\mu, \hat{D}_\nu]\vec{\phi} \stackrel{\text{def}}{=} -ig\hat{F}_{\mu\nu}\vec{\phi}, \tag{4.2.13}$$

which easily gives

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - ig[\hat{A}_\mu, \hat{A}_\nu] \tag{4.2.14}$$

or ($\hat{F}_{\mu\nu} = F^a_{\mu\nu}\hat{t}_a$)

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - igf^a_{bc}A^b_\mu A^c_\nu. \tag{4.2.15}$$

Notice that, according to its definition, the Maxwell tensor must transform under a gauge transformation as[3]

$$\hat{F}_{\mu\nu}(x) \to \hat{F}'_{\mu\nu}(x') = \hat{U}\hat{F}_{\mu\nu}(x)\hat{U}^{-1} \tag{4.2.16}$$

i.e., as a mixed, second rank tensor. Therefore the quadratic form $\text{Tr}(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu})$ is a gauge invariant Lorentz scalar. This is precisely what we need to complete the scalar field action,

$$S = -\int d^4x \left[\eta^{\mu\nu}(\hat{D}_\mu\vec{\phi})^\dagger(\hat{D}_\nu\vec{\phi}) + V(|\vec{\phi}|) + \frac{gc}{4}\text{Tr}(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu})\right]. \tag{4.2.17}$$

The last term can be rewritten using

$$\text{Tr}(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}) = F^a_{\mu\nu}F^{b\mu\nu}\text{Tr}(\hat{t}_a\hat{t}_b) = \kappa_{ab}F^a_{\mu\nu}F^{b\mu\nu} \tag{4.2.18}$$

showing that the symmetric matrix $\kappa_{ab} = \text{Tr}(\hat{t}_a\hat{t}_b)$ and its inverse can be used to lower and raise indices according to, for example, $F^{\mu\nu}_a = \kappa_{ab}F^{b\mu\nu}$. The matrix κ_{ab} must be taken to be real for the Lagrangian density to be real.

Using κ_{ab}, consider the structure constants with all lowered indices, defined as

$$f_{abc} \stackrel{\text{def}}{=} \kappa_{am}f^m_{bc} \tag{4.2.19}$$

It turns out that for all semi-simple Lie algebras the structure constants will satisfy the cyclic property,

$$f_{abc} = f_{cab} = f_{bca}. \tag{4.2.20}$$

[3]Problem: While this transformation is obvious from the definition of $\hat{F}_{\mu\nu}$, it is instructive to show this by starting with the definition in (4.2.14) and applying the gauge transformation of $\hat{A}_\mu$ in (4.2.6).

We can prove this as follows:

$$[\hat{t}_a, \hat{t}_b] = f_{ab}^m \hat{t}_m \Rightarrow [\hat{t}_a, \hat{t}_b]\hat{t}_n = f_{ab}^m \hat{t}_m \hat{t}_n \tag{4.2.21}$$

and taking the trace of both sides gives

$$f_{nab} = \mathrm{Tr}(\hat{t}_a \hat{t}_b \hat{t}_n - \hat{t}_b \hat{t}_a \hat{t}_n) \tag{4.2.22}$$

so (4.2.20) is merely a consequence of the cyclic property of the trace. Taken with the antisymmetry of the second two indices, this guarantees that f_{abc} is totally antisymmetric.[4]

The field equations that follow from the action in (4.2.17) are

$$-\hat{D}^\mu \hat{D}_\mu \vec{\phi} + V'(|\vec{\phi}|)\vec{\phi} = 0$$

$$\mathcal{D}_\alpha F_a^{\alpha\mu} \stackrel{\text{def}}{=} \partial_\alpha F_a^{\alpha\mu} + ig f_{na}^m A_\alpha^n F_m^{\alpha\mu} = \frac{i}{c}\left[(\hat{t}_a \vec{\phi})^\dagger (\hat{D}^\mu \vec{\phi}) - (\hat{D}^\mu \vec{\phi})^\dagger (\hat{t}_a \vec{\phi})\right]. \tag{4.2.23}$$

The left hand side will be recognized as the covariant derivative in the adjoint representation and the result generalizes (2.6.11) to the non-abelian case.[5]

That a similar construction can be developed for spinor fields should be quite obvious. Starting with N spinors,

$$\vec{\psi}(x) = \begin{pmatrix} \psi^1(x) \\ \psi^2(x) \\ \cdots \\ \cdots \\ \cdots \\ \psi^N(x) \end{pmatrix} \tag{4.2.24}$$

and repeating all the steps performed with the scalar field, we would end up with the Lagrangian

$$S = -\hbar c^2 \int d^4x \left[\vec{\bar{\psi}} \left(\frac{i}{2} \overleftrightarrow{\hat{\not{D}}} - \frac{\hat{m}c}{\hbar} \right) \vec{\psi} + \frac{g}{4\hbar c} \mathrm{Tr}(\hat{F}_{\mu\nu} \hat{F}^{\mu\nu}) \right] \tag{4.2.25}$$

where $\hat{m} = m_i \delta_{ij}$ is the "mass matrix" of the fermions. The equations of motion are analogous to those we obtained before with abelian gauge invariance

$$(i\hbar \overrightarrow{\hat{\not{D}}} - \hat{m}c)\vec{\psi} = 0$$

[4] Problem: Use the cycic property to show that

$$\mathcal{D}_\mu V_a = (\partial_\mu V_a + ig f_{ba}^c A_\mu^b V_c).$$

for any dual vector in the adjoint representation.

[5] A consequence of the result in the previous footnote is that the covariant derivative of the Maxwell tensor can also be written in matrix form as

$$\mathcal{D}_\alpha F_a^{\alpha\mu} = \partial_\alpha F_a^{\alpha\mu} + ig[\hat{A}_\alpha, \hat{F}_a^{\alpha\mu}].$$

$$\vec{\bar{\psi}}(i\hbar\overleftarrow{\hat{\not{D}}} + \hat{m}c) = 0$$

$$\mathcal{D}_\alpha F_a^{\alpha\mu} = \hbar c \vec{\bar{\psi}}\, \hat{t}_a \gamma^\mu \vec{\psi} \tag{4.2.26}$$

thus generalizing (3.7.11).

4.3 Conservation Laws

The conserved Noether currents for the free scalar and spinor fields corresponding to translation and Lorentz invariance (stress energy and total angular momentum respectively) are straightforward generalizations of their counterparts in the previous chapters.

The canonical energy momentum tensor, the total angular momentum tensor and the orbital and spin angular momenta of the non-abelian gauge fields all have expressions that are intimately connected to the corresponding expressions we have derived for the electromagnetic field. Thus, for example, the canonical energy momentum tensor of the gauge field,

$$\Theta^{\mu\nu} = gc\left[F_a^{\mu\alpha}\partial^\nu A_\alpha^a - \frac{1}{4}\eta^{\mu\nu}F_a^{\alpha\beta}F_{\alpha\beta}^a\right], \tag{4.3.1}$$

is neither gauge invariant nor symmetric. We may define the non-abelian electric and magnetic fields as

$$E_i^a = -F_{0i}^a, \quad B_i^a = \frac{1}{2}\epsilon_{ijk}F^{ajk}, \tag{4.3.2}$$

or, in vector form,

$$\begin{aligned} \vec{E}^a &= -\vec{\nabla}\phi^a - \partial_t\vec{A}^a - igf_{bc}^a\phi^b\vec{A}^c \\ \vec{B}^a &= \vec{\nabla}\times\vec{A}^a + \frac{ig}{2}f_{bc}^a\vec{A}^b\times\vec{A}^c, \end{aligned} \tag{4.3.3}$$

where $\phi^a = -A_0^a$ are the scalar potentials.[6] They are not gauge invariant, as are their abelian counterparts. In terms if them, however, we recover familiar expressions for the orbital angular momentum

$$L^i = \frac{g}{c}\int d^3\vec{r}\; E_a^k(\vec{r}\times\vec{\nabla})^i A_k^a \tag{4.3.4}$$

and the spin angular momentum

$$S^i = \frac{g}{c}\int d^3\vec{r}\;(\vec{E}^a\times\vec{A}_a)^i \tag{4.3.5}$$

[6] Problem: Rewrite the sourced Maxwell equations in terms of the non-abelian fields, $\vec{E}^a$ and $\vec{B}^a$, and obtain their integral form.

following the methods of the previous chapters. As before, neither is gauge invariant and neither is separately conserved.

The symmetric, Belinfante tensor is also constructed in a similar way,

$$t^{\mu\nu} = gc\left[F_a^{\mu\alpha}F^{a\nu}{}_\alpha - \frac{1}{4}\eta^{\mu\nu}F_a^{\alpha\beta}F^a_{\alpha\beta}\right] \tag{4.3.6}$$

and from it one may obtain the energy density,

$$\mathcal{E} = \frac{gc}{2}\left(\frac{1}{c^2}\vec{E}^a\cdot\vec{E}_a + \vec{B}^a\cdot\vec{B}_a\right), \tag{4.3.7}$$

the momentum density,

$$\vec{\wp} = \frac{g}{c}(\vec{E}^a\times\vec{B}_a) \tag{4.3.8}$$

and the conserved angular momentum

$$L^i = \frac{g}{c}\int d^3\vec{r}\ [\vec{r}\times(\vec{E}^a\times\vec{B}_a)]^i \tag{4.3.9}$$

of the gauge field.

Local gauge invariance will lead to a strongly conserved current. Consider, for example, the action describing a scalar field coupled to a non-abelian gauge field in (4.2.17). For the scalar field, we have

$$\delta\phi^i = igt^i_{aj}\delta\alpha^a\phi^j,\quad \delta\phi^{*i} = -igt^i_{aj}\delta\alpha^a\phi^{*j} \tag{4.3.10}$$

and for the gauge fields,

$$\delta A^a_\nu = \partial_\nu\alpha^a - igf^a_{bc}A^b_\nu\delta\alpha^c. \tag{4.3.11}$$

Carefully inserting these transformations into (2.2.14) we find

$$J^\mu = c\partial_\lambda(F_a^{\lambda\mu}\delta\alpha^a), \tag{4.3.12}$$

which, of course, obeys $\partial\cdot J = 0$ because of the antisymmetry of $F_a^{\lambda\mu}$. Remarkably this is *no different* from its abelian counterpart. It implies, however, that if we take $\delta\alpha^a$ to be constant and let j^μ_a represent the source currents for the gauge fields then

$$J^\mu = c(\partial_\lambda F_a^{\lambda\mu})\delta\alpha^a = (j^\mu_a - igcf^c_{ba}A^b_\lambda F_c^{\lambda\mu})\delta\alpha^a \tag{4.3.13}$$

on-shell. The conserved charge, which we define as

$$Q_a = -g\int d^3\vec{r}J^0_a = -gc\int d^3\vec{r}\ \partial_iF_a^{i0} = +\frac{g}{c}\oint_S\vec{E}_a\cdot d\vec{S}, \tag{4.3.14}$$

is now *not* exclusively from the matter fields, as it was in the abelian case, but receives a contribution from the gauge potentials as well,

$$Q_a = -g \int d^3\vec{r} j_a^0 + \frac{ig^2}{c} f_{ba}^c \int d^3\vec{r}\ \vec{A}^b \cdot \vec{E}_c. \tag{4.3.15}$$

The first term represents the contribution from the matter fields while the second is a contribution from the gauge fields themselves. This is to be expected because the gauge fields also carry non-abelian charge and therefore contribute to the total charge.

The Bianchi identities will follow (as usual) from the Jacobi identity,

$$[\hat{D}_\mu, [\hat{D}_\nu, \hat{D}_\lambda]]\vec{\phi} + [\hat{D}_\lambda, [\hat{D}_\mu, \hat{D}_\nu]]\vec{\phi} + [\hat{D}_\nu, [\hat{D}_\lambda, \hat{D}_\mu]]\vec{\phi} = 0. \tag{4.3.16}$$

Using the definition of the Maxwell tensor in (4.2.13), we find

$$\mathcal{D}_\mu F^a_{\nu\lambda} + \mathcal{D}_\lambda F^a_{\mu\nu} + \mathcal{D}_\nu F^a_{\lambda\mu} = 0. \tag{4.3.17}$$

Contracting the identity with the Levi-Civita tensor, as we did in the abelian case, we recover the homogeneous equations,

$$\mathcal{D}_\alpha {}^*F_a^{\alpha\mu} = 0, \tag{4.3.18}$$

for the non-abelian gauge field.

4.4 Examples

The first use of a non-abelian gauge theory was made by Shaw, Yang and Mills to describe the proton and the neutron in the nuclei of atoms. It was based on the observation that the proton and the neutron possess (almost) the same mass and play an identical role in strong interaction processes. According to them, the proton and the neutron could be regarded as a doublet,

$$\psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}. \tag{4.4.1}$$

This is analogous to the spin $\frac{1}{2}$ doublet and is called an **isospin** doublet. Conservation of isospin leads to invariance under isospin rotations,

$$\psi \to \psi' = U(\mathfrak{g})\psi \tag{4.4.2}$$

where $\mathfrak{g}$ is an element of $SU(2)$. If this invariance is required to be local we end up with an $SU(2)$ gauge field theory. The Pauli matrices provide a basis for the fundamental representation of $SU(2)$ transformations, which we take to be

$$\hat{t}_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{t}_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{t}_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.4.3}$$

satisfying

$$[\hat{t}_a, \hat{t}_b] = -i\epsilon_{ab}{}^c \hat{t}_c, \tag{4.4.4}$$

so $f^c_{ab} = -i\epsilon_{ab}{}^c$. It is easy to see that the raising/lowering matrix κ_{ab} is given by $\kappa_{ab} = \text{Tr}(\hat{t}_a, \hat{t}_b) = \frac{1}{2}\delta_{ab}$ and that the generators in the adjoint representation are $[T_a]^c{}_b = -i\epsilon_{ab}{}^c$, or

$$\hat{T}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad \hat{T}_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \hat{T}_3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.4.5}$$

which are both antisymmetric and imaginary.

Today the proton and neutron are known not to be elementary, but isospin invariance still plays a very big role in particle physics. The left handed electron, muon and tauon (or tau lepton) form isospin doublets with their neutrino counterparts, the electron-neutrino, the mu-neutrino and the tau-neutrino,

$$\begin{pmatrix} \nu_l \\ l \end{pmatrix}_L \equiv \left\{ \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \right\} \tag{4.4.6}$$

but their corresponding right handed particles do not suffer these "weak interactions". Likwise, the six quark flavors (labeled up, down, charm, strange, top and bottom) form isospin doublets according to

$$\begin{pmatrix} q_{\text{u}} \\ q_{\text{d}} \end{pmatrix}_L \equiv \left\{ \begin{pmatrix} u \\ d \end{pmatrix}_L, \quad \begin{pmatrix} c \\ s \end{pmatrix}_L, \quad \begin{pmatrix} t \\ b \end{pmatrix}_L \right\}. \tag{4.4.7}$$

Local $SU(2)$ invariance requires three gauge bosons to mediate the "flavor" interaction. Because we're working with a non-abelian gauge theory, the gauge bosons will interact among themselves.

Quarks possess an additional quantum number called color, causing them to undergo "strong interactions". "Chromodynamics", the theory of the color or strong interactions, is described by the symmetry group $SU(3)$. Each quark comes in three colors (say red, green and blue), which are arranged in a color triplet of fermionic fields, q^i. The Lagrangian density

$$\mathfrak{L} = -\hbar c^2 \overline{q}^i \left(\frac{i}{2} \overleftrightarrow{\not\partial} \mathbf{1} - \frac{\hat{m}c}{\hbar} \right) q^j \tag{4.4.8}$$

is clearly invariant under global $SU(3)$ transformations. If this invariance is made local we will obtain local color interactions mediated by $SU(3)$ gauge bosons (called **gluons**) and described by an action of the type in (4.2.25). $SU(3)$ is eight dimensional, so there are eight gluons and, because the gauge group is non-abelian, the gluons carry color charge and interact among themselves.

A basis for the fundamental representation of $SU(3)$ can be given in terms of the 3×3 complex Gell-Mann matrices:

$$\hat{t}_i = \frac{1}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & 0\end{pmatrix}, \quad i \in \{1,2,3\},$$

$$\hat{t}_4 = \frac{1}{2}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, \quad \hat{t}_5 = \frac{1}{2}\begin{pmatrix}0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix},$$

$$\hat{t}_6 = \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad \hat{t}_7 = \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix},$$

$$\hat{t}_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}. \tag{4.4.9}$$

It's easy to see that $\mathrm{Tr}(\hat{t}_a, \hat{t}_b) = \frac{1}{2}\delta_{ab}$. The structure constants (equivalently, the adjoint representation) can be determined by direct computation.

4.5 Hamiltonian

The Hamiltonian description of non-abelian gauge fields is a straightforward generalization of its abelian counterpart and one gets very similar results. Beginning with the action for a pure non Abelian gauge field,

$$S = -\frac{gc}{4}\int F^a_{\mu\nu}F^{\mu\nu}_a \tag{4.5.1}$$

we compute the momenta conjugate to A^a_μ and find

$$\pi^\mu_a = \frac{\partial\mathfrak{L}}{\partial(\partial_t A^a_\mu)} = gcF^{\mu 0}_a \tag{4.5.2}$$

showing that $\pi^0_a = 0$ is a primary constraint and relating the electric fields to the momenta, $\pi^i_a = -gE^i_a/c$. The Lagrangian density can now be written as

$$\mathfrak{L} = \frac{c}{2g}\pi^i_a\pi^a_i - \frac{gc}{4}F^a_{ij}F^{ij}_a \tag{4.5.3}$$

and the "velocities" obtained as before:

$$\dot{A}_i^a = \frac{c}{g}\pi_i^a + \partial_i A_0^a - igf_{bc}^a A_i^b A_0^c \tag{4.5.4}$$

We may now write down the primary Hamiltonian density as

$$\mathfrak{H}_p = \pi_a^i \dot{A}_i^a + \pi_a^0 \mu^a - \mathfrak{L} = \frac{c}{2g}\pi_a^i \pi_i^a + \pi_a^0 \mu^a + \pi_a^i(\partial_i A_0^a - igf_{bc}^a A_i^b A_0^c) + \frac{gc}{4} F_{ij}^a F_a^{ij} \tag{4.5.5}$$

where the lagrange multipliers μ^a enforce the primary constraints, $\Phi_a = \pi_a^0 \approx 0$. We therefore get the primary Hamiltonian,

$$\mathcal{H}_p = \int d^3\vec{r} \left[\frac{c}{2g}\pi_a^i \pi_i^a + \frac{gc}{4} F_{ij}^a F_a^{ij} + \pi_a^0 \mu^a - A_0^a \mathcal{D}_i \pi_a^i\right], \tag{4.5.6}$$

after an integration by parts, where $\mathcal{D}_i$ is the covariant derivative acting on the dual vector (in the adjoint representation), π_a^i. It follows that $\mathcal{D}_i \pi_a^i \approx 0$ are secondary constraints and, again as before, the first two terms in the primary Hamiltonian yield the energy density

$$\frac{c}{2g}\pi_a^i \pi_i^a + \frac{gc}{4} F_{ij}^a F_a^{ij} = \frac{gc}{2}\left[\frac{1}{c^2} E_i^a E_a^i + B_i^a B_a^i\right]. \tag{4.5.7}$$

The fundamental Poisson brackets,

$$\{A_\mu^a(\vec{r},t), \pi_\nu^b(\vec{r}',t)\}_{P.B.} = \delta_\nu^\mu \delta_b^a \delta^3(\vec{r}-\vec{r}') \tag{4.5.8}$$

allow us to derive the canonical equations of motion,

$$\begin{aligned}
\dot{A}_i^a(\vec{r},t) &= \{A_i^a(\vec{r},t), \mathcal{H}_p\}_{P.B.} = \frac{c}{g}\pi_i^a + \partial_i A_0^a - igf_{bc}^a A_i^b A_0^c \\
\dot{A}_0^a(\vec{r},t) &= \{A_0^a(\vec{r},t), \mathcal{H}_p\}_{P.B.} = \mu^a \\
\dot{\pi}_a^i(\vec{r},t) &= \{\pi_a^i(\vec{r},t), \mathcal{H}_p\}_{P.B.} = -gc\mathcal{D}_j F_a^{ji} \\
\dot{\pi}_a^0(\vec{r},t) &= \{\pi_a^0(\vec{r},t), \mathcal{H}_p\}_{P.B.} = \mathcal{D}_j \pi_a^j \approx 0.
\end{aligned} \tag{4.5.9}$$

Both the primary and secondary constraints are first class, as is easy to verify. The Lagrange multipliers μ^a therefore cannot be determined. There are then $2 \times \dim(G)$ first class constraints, so the number of degrees of freedom is $(4-2)\times\dim(G)$, *i.e.,* two degrees of freedom for each generator of the Lie group.

4.6 The 't Hooft-Polyakov Monopole

The Dirac monopole is a singular solution of an Abelian gauge theory (Maxwell's equations with magnetic sources). The 't Hooft Polyakov monopole is a singularity free, topological soliton that arises in spontaneously broken Yang-Mills theories. Before considering the problem of solitonic solutions in non-Abelian gauge theories, however, let's look at what has occurred in the examples we have worked out through a slightly different lens.

In the "$\lambda\phi^4$" theory in one space dimension, both spatial infinity and the space of solutions consist of two points, which we can identify with the zero dimensional sphere, S^0. Thus the boundary condition, $\lim_{|x|\to\infty}\phi(x) = \pm v$ can be thought of as a map from the boundary of space to the space of boundary configurations: the kink maps the point at $-\infty$ to the solution $\phi = -v$ and point at $+\infty$ to the solution $\phi = +v$ and the anti-kink does the opposite. Then, of course, there is the trivial solution that maps both infinities to a single point in the space of boundary configurations. As we have seen, the maps cannot be deformed into one another, neither can they be deformed into the trivial map that takes both spatial infinities to the same vacuum. This is guaranteed by the fact that each of the solutions possesses a distinct conserved charge, Q. Likewise, in the vortex solution of the Abelian Higgs model, the boundary of two dimensional space is the circle S^1 and so is the space of vacua, defined by $|\phi| = v$. Thus the boundary condition, $\lim_{r\to\infty}|\phi| = v$ can be thought of as a map from the spatial boundary, S^1 at infinity, to the S^1 of the boundary configurations $|\phi| = v$. Maps of this kind are characterized by their "homotopy" which counts the number of topologically inequivalent maps possible. In the case of the maps from $S^1 \to S^1$ distinct maps are characterized by integers, as in $\phi \to ve^{ik\theta}$ for any integer k. This map takes a single turn around the boundary S^1 to k turns around the S^1 of configurations $|\phi| = v$ and the dependence of Q on the integer k indicates that maps with differing values of k are inequivalent.

In fact, inequivalent maps from the n sphere, S^n, to the n sphere, S^n, for any $n \geq 1$, are always characterized by a single integer. So, in 3+1 dimensions, where the spatial boundary can be thought of as the two sphere, S^2, scalar fields whose space of boundary configurations is M, will be characterized by the homotopy of maps from $S^2 \to M$. If M is also S^2 then they will, once again, be characterized by integers. Thus, for the simplest example, we are led to consider a gauge theory of real vector fields, ϕ^a, transforming under $SO(3)$ and governed by the Lagrangian in (4.2.17) with

$$V(|\phi|) = \frac{\lambda}{8}\left(|\vec{\phi}|^2 - v^2\right)^2. \tag{4.6.1}$$

As before, we will pick the temporal gauge, $A_0^a = 0$, and assume time independence, which together lead to the equations of motion,

$$-\hat{D}^i\hat{D}_i\vec{\phi} + V'(|\vec{\phi}|)\vec{\phi} = 0$$

$$\nabla_j F_a^{ji} + igf_{na}^m A_j^n F_m^{ji} = \frac{i}{c}\left[(\hat{t}_a\vec{\phi})^\dagger(\hat{D}^i\vec{\phi}) - (\hat{D}^i\vec{\phi})^\dagger(\hat{t}_a\vec{\phi})\right]. \tag{4.6.2}$$

The identity map from S^2 to S^2 indicates that we should choose an asymptotic behavior

$$\lim_{r\to\infty}\vec{\phi} = v\hat{r}, \tag{4.6.3}$$

and seek solutions of the form

$$\vec{\phi} = vf(r)\hat{r} \tag{4.6.4}$$

with $f(r) \to 1$ as $r \to \infty$. The scalar field points raidially, so this ansatz is sometimes referred to as the "hedgehog" configuration.

For the Higgs model, the scalar potential term in the expression for the energy of the configuration

$$E = \int d^3\vec{r}\left[\eta^{ij}(\hat{D}_i\vec{\phi})^\dagger(\hat{D}_j\vec{\phi}) + V(|\vec{\phi}|) + \frac{gc}{4}\text{Tr}\left(\hat{F}^{ij}\hat{F}_{ij}\right)\right] \tag{4.6.5}$$

vanishes at infinity with our choice in (4.6.4). To have the scalar kinetic term also vanish (in the interest of finiteness) at infinity, we must require that $\hat{D}_i\vec{\phi} = 0$, or $\partial_i\phi^r - ig(A^a\hat{t}_a)^r{}_s\phi^s = 0$ as $r \to \infty$. We find:[7]

$$\lim_{r\to\infty} A_i^a = -\frac{i}{g}\epsilon^a{}_{ij}\frac{x^j}{r^2} \tag{4.6.6}$$

(is pure gauge) and therefore make the ansatz

$$A_i^a = -\frac{i}{g}\epsilon^a{}_{ij}A(r)\frac{x^j}{r^2} \tag{4.6.7}$$

where $A(r) \to 1$ as $r \to \infty$. The scalar and vector field equations then read,

$$\begin{aligned} f'' + \frac{2}{r}f' + \left[\mu^2 - \frac{2}{r^2}(A-1)^2\right]f - \mu^2 f^2 = 0 \\ A'' - \frac{A}{r^2}\left(2 - 3A + A^2\right) + \frac{2gv^2}{c}(1-A)f^2 = 0 \end{aligned} \tag{4.6.8}$$

where $\mu^2 = \lambda v^2/4$. As in the case of the vortex, these equations are second order and difficult to solve, but, as we have seen, essential properties of the solutions can be determined without explicit solutions.

Let us now consider the BPS equations. According to Derrick's conditions (2.10.32), in $D = 3$ we must have $E_3 - (E_1 + 3E_2) = 0$ and $E_1 + 6E_2 \geq 0$. The first requires $E_2 = \frac{1}{3}(E_3 - E_1)$ so that

$$E = \frac{2}{3}\left(E_1[D\vec{\phi}] + 2E_3[\vec{A}]\right) = \frac{2}{3}\int d^3\vec{r}\left[\eta^{ij}(D_i\vec{\phi})^\dagger\cdot(D_j\vec{\phi}) + \frac{gc}{2}\vec{F}^{ij}\cdot\vec{F}_{ij}\right] \tag{4.6.9}$$

[7]Problem: Prove this. Show that A_i^a is pure gauge and find the gauge transformation, $\hat{U}$.

which can be written as

$$E = \frac{gc}{3}\int d^3\vec{r}\left[\left(\vec{F}^{ij} \pm \frac{1}{\sqrt{gc}}\epsilon^{ijk}D_k\vec{\phi}\right)^2 \mp \frac{2}{\sqrt{gc}}\epsilon^{ijk}\vec{F}_{ij}\cdot D_k\vec{\phi}\right]. \quad (4.6.10)$$

As the first term is clearly non-negative, we have a lower bound on the energy of the system,

$$E \geq \mp\frac{2\sqrt{gc}}{3}\int d^3\vec{r}\ \epsilon^{ijk}\vec{F}_{ij}\cdot D_k\vec{\phi}, \quad (4.6.11)$$

which is saturated by solutions satisfying

$$\vec{F}^{ij} \pm \frac{1}{\sqrt{gc}}\epsilon^{ijk}D_k\vec{\phi} = 0 \quad (4.6.12)$$

These are the BPS states.

't Hooft has argued that the antisymmetric field strength tensor

$$\mathcal{F}_{\mu\nu} = \vec{F}_{\mu\nu}\cdot\hat{\phi} - \frac{i}{g}\epsilon_{abc}\hat{\phi}^a D_\mu\hat{\phi}^b D_\nu\hat{\phi}^c \quad (4.6.13)$$

may be identified as the field strength tensor of a $U(1)$ gauge field, where $\hat{\phi}^a = \phi^a/|\vec{\phi}|$. The new field strength can be written as

$$\mathcal{F}_{\mu\nu} = M_{\mu\nu} + H_{\mu\nu} \quad (4.6.14)$$

where

$$\begin{aligned} M_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \\ B_\mu &= A^a_\mu\hat{\phi}^a \\ H_{\mu\nu} &= \frac{i}{g}\epsilon_{abc}\hat{\phi}^a\partial_\mu\hat{\phi}^b\partial_\nu\hat{\phi}^c \end{aligned} \quad (4.6.15)$$

and we may define the magnetic current as

$$J^\mu = -\frac{i}{2}\partial_\nu{}^*\mathcal{F}^{\mu\nu} = \frac{1}{2}\partial_\nu{}^*H^{\mu\nu} = \frac{1}{2g}\epsilon_{abc}\epsilon^{\mu\nu\alpha\beta}\partial_\nu\hat{\phi}^a\partial_\alpha\hat{\phi}^b\partial_\beta\hat{\phi}^c \quad (4.6.16)$$

It is immediately obvious that J^μ is a topological current as its conservation is a consequence of the antisymmetry of the Levi-Civita tensor. The conserved charge is

$$Q = \int d^3\vec{r}\ J^0 = \frac{1}{2g}\epsilon^{ijk}\epsilon_{abc}\int d^3\vec{r}\ \partial_i\hat{\phi}^a\partial_j\hat{\phi}^b\partial_k\hat{\phi}^c = \frac{1}{2g}\int d^3\vec{r}\ \epsilon^{ijk}\epsilon_{abc}\partial_i\left(\hat{\phi}^a\partial_j\hat{\phi}^b\partial_k\hat{\phi}^c\right) \quad (4.6.17)$$

and, by Gauss' law, the last is re-expressed and evaluated as the surface integral

$$Q = \frac{1}{2g}\int_{S^2_\infty} dS\ \widehat{n}_i \epsilon^{ijk}\epsilon_{abc}\hat{\phi}^a\partial_j\hat{\phi}^b\partial_k\hat{\phi}^c = \frac{4\pi}{g}, \tag{4.6.18}$$

using the asymptotic behavior of $\phi(x)$ in (4.6.3).[8] This result is reminiscent of the vortex solution in the two dimensional Abelian-Higgs model.

We learn that the magnetic charge in non-Abelian gauge theories also does not originate in any local symmetry of the action but is rather (in the case of $SO(3)$ gauge theory) an invariant of a mapping between two spheres, *viz.,* the spatial boundary at infinity and the boundary configuration of the field. The mapping is provided by the scalar fields and the result in (4.6.18) is for the identity map. In general, let ξ^α, $\alpha \in \{1, 2\}$, be a parametrization of S^2_∞, then $x^i = x^i(\xi^\alpha)$ and

$$dS\ \widehat{n}_i = \frac{1}{2}\epsilon_{ijk}\epsilon^{\alpha\beta}\partial_\alpha x^i\partial_\beta x^j d^2\xi, \tag{4.6.19}$$

so we find that

$$Q = \frac{1}{2g}\int_{S^2_\infty} d^2\xi\epsilon^{\alpha\beta}\epsilon_{abc}\hat{\phi}^a\partial_\alpha\hat{\phi}^b\partial_\beta\hat{\phi}^c. \tag{4.6.20}$$

It is now easy to see that the integrand is simply the solid angle subtended *by the boundary configuration.* While the coordinates ξ^α cover the two sphere once, the field configuration may cover it an integer number of times (so that the field may be single valued). Therefore, for a general, single valued map,

$$Q = \frac{4\pi k}{g}. \tag{4.6.21}$$

where $k \in \mathbb{Z}$, in line with (2.10.42).

Finally, as $r \to \infty$, $D_\mu\hat{\phi} \to 0$ and $\mathcal{F}_{\mu\nu} = \vec{F}_{\mu\nu}\cdot\hat{\phi}$. The lower bound of the energy in (4.6.11) can be expressed as

$$E \geq \mp\frac{2\sqrt{gc}}{3}\int d^3\vec{r}\ \epsilon^{ijk}\partial_k(\vec{F}_{ij}\cdot\vec{\phi}) = \mp\frac{2v\sqrt{gc}}{3}\int_{S^2_\infty} dS\ \widehat{n}_k\epsilon^{ijk}\mathcal{F}_{ij}, \tag{4.6.22}$$

If we identify the $U(1)$ magnetic field, B^k, with $\frac{1}{2}\epsilon^{ijk}\mathcal{F}_{ij}$ then

$$E \geq \mp\frac{4v\sqrt{gc}}{3}\Phi_B = \mp\frac{Qv\sqrt{gc}}{3\pi} = \frac{4kv}{3}\sqrt{\frac{c}{g}} \tag{4.6.23}$$

where $\Phi_B = Q/4\pi$ is the magnetic flux.

[8] Problem: Using (4.6.3) and (4.6.6) derive the charge Q.

Chapter 5

The Standard Model

Gauge invariance requires the gauge bosons that mediate the force field to be massless. This is observed for electrodynamics (the photon) and chromodynamics (the gluons) but it is not observed in the case of the flavor changing interactions. Experimentally, this "weak" force is short range and mediated by three massive particles, two of which, the $W^{\pm}$, carry electric charge and the third, the Z^0, is neutral. The problem is to build a theory of particle interactions mediated by massive spin one bosons. One of the great intellectual accomplishments of the twentieth century, among many, was the discovery that the flavor changing interactions *are* in fact described by a gauge field theory even though the gauge bosons are massive. The central idea is that "particles" of any field theory are actually excitations of the field around some vacuum, which is generally chosen to minimize an effective potential. If the minimum of the effective potential is non-trivial, *i.e.,* if the field acquires a non zero value in its minimum configuration then one may ask for an effective description of the theory by perturbations of the field around this vacuum. It turns out that although the Lagrangian of the system may be invariant under a certain set of symmetries, the lowest order pertrubations about the non trivial vacuum do not necessarily obey all those symmetries. An effective description of the field and its interactions, built out of these lowest order perturbations, generates masses for those gauge bosons that correspond to the "broken" symmetries. This is the underlying mechanism of "**spontaneous symmetry breaking**", which we now describe in some detail.

5.1 Spontaneous Symmetry Breaking: Toy Models

Consider first a single, real scalar field, $\phi(x)$, with potential $V(\phi(x))$ and action

$$S = -\frac{1}{2}\int d^4x[\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)]$$

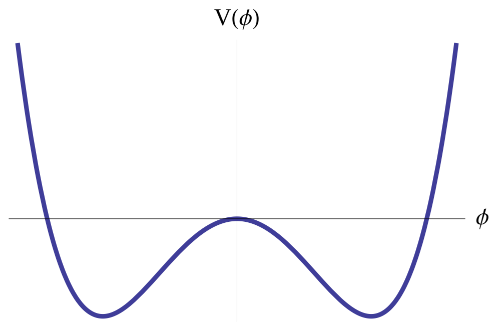

Figure 5.1: Potential for spontaneously broken symmetry (real scalar field)

The classical minimum energy configuration is given by the minimum of the potential,

$$\frac{\partial V}{\partial \phi} = 0, \quad \frac{\partial^2 V}{\partial \phi^2} > 0. \tag{5.1.1}$$

Take, for example, the ϕ^4 theory, for which the action is invariant under the discrete transformation $\phi \to -\phi$ and

$$V(\phi) = g_2\phi^2 + g_4\phi^4. \tag{5.1.2}$$

For there to be a global minimum energy configuration we must require g_4 to be positive. If g_2 is also positive there is a single global minimum at $\phi = 0$. If, however, g_2 is negative (it can no longer be interpreted as related to the mass of the field), then there is a local maximum at $\phi = 0$ and two global minima located at

$$\phi_0 = v = \pm\sqrt{\frac{|g_2|}{2g_4}} \tag{5.1.3}$$

The configuration $\phi(x) = \pm v$ corresponds to the lowest energy and therefore most stable configuration of the scalar field and the constant v is called the **vacuum expectation value** or VEV of the scalar field. Taking v as the background value of $\phi(x)$, consider the field of small perturbations of the scalar field around this value, setting

$$\phi(x) = v + H(x) \tag{5.1.4}$$

then, because v is constant, the Lagrangian density for the small perturbations, $H(x)$, reads

$$\mathfrak{L} = -\frac{1}{2}\left[\eta^{\mu\nu}\partial_\mu H \partial_\nu H + 2|g_2|H^2 + \sqrt{8|g_2|g_4}H^3 + g_4 H^4\right] + \text{const.} \tag{5.1.5}$$

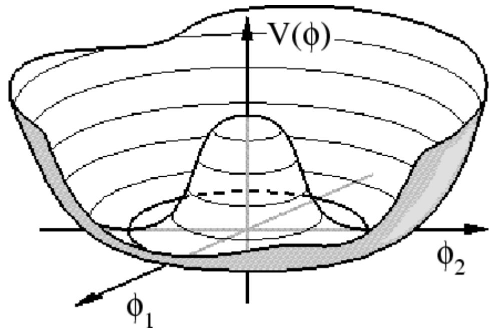

Figure 5.2: Potential for spontaneously broken symmetry (complex scalar field)

which describes a scalar field, H, of positive mass and both H^3 and H^4 potential terms. Notice that the action for H, although derived from the action for ϕ, is not invariant under the transformation $H \to -H$ on account of the appearance of a cubic term in the potential. The original reflection symmetry of the action ($\phi \to -\phi$) does not survive in the effective description but only in the relationship between the three coefficients of the polynomial potential. Thus although the original Lagrangean theory is reflection symmetric, the vacuum of the theory is not and one says that this symmetry has been "spontaneously broken".[1]

This gets interesting when we consider a complex scalar field with a continuous global $U(1)$ symmetry,

$$S = -\int d^4x \, [\eta^{\mu\nu}(\partial_\mu \phi^*)(\partial_\nu \phi) + V(|\phi|)] \tag{5.1.6}$$

where we will take

$$V(|\phi|) = g_2|\phi|^2 + g_4|\phi|^4. \tag{5.1.7}$$

An infinite number of global minima of the potential exist at

$$|\phi| = v = \sqrt{\frac{|g_2|}{2g_4}}, \tag{5.1.8}$$

whenever g_2 is negative and they are all connected by $U(1)$ transformations, *i.e.,* rotations in the complex ϕ plane. Let us define the (two) real fields of small perturbations about

[1]This is what happens in Landau's mean field description of ferromagnetism. For example, the zero field Landau free energy of a one dimensional, ferromagnetic material would have a ϕ^4 potential like the one we have dealt with here, where ϕ represents the magnetization of the material (an "order" parameter) and the coefficents g_2 and g_4 depend on the temperature. At high enough temperatures both g_2 and g_4 are positive and the ground state is completely disordered ($\phi = 0$). At temperatures below a critical temperature, T_c, the coefficient g_2 becomes negative and the system settles into one of two possible magnetized states.

the vacuum as

$$\phi(x) = [v + H(x)]e^{i\theta(x)} \tag{5.1.9}$$

The VEV, v, represents is a fixed vector in the complex ϕ plane oriented along the real axis and so it destroys the $U(1)$ symmetry of the vacuum.

Rewriting the potential term in terms of $H(x)$ and $\theta(x)$ we find that only the former appears,

$$V(|\phi|) = -[g_2(v + H)^2 + g_4(v + H)^4] \tag{5.1.10}$$

and the potential looks precisely like (5.1.5). The scalar field $\theta(x)$ appears through the kinetic term, which now reads

$$-\eta^{\mu\nu}\left[\partial_\mu H\partial_\nu H + (v + H)^2\partial_\mu\theta\partial_\nu\theta\right] \tag{5.1.11}$$

and involves derivative interactions between $H(x)$ and $\theta(x)$. Notice that the field $H(x)$, which represents "radial" fluctuations, about the trough of the minimum, has the usual mass term, whereas $\theta(x)$, which represents fluctuations along a direction in which the potential is "flat" *i.e.,* in the direction of neighboring vacua, is massless. $\theta(x)$ is called a **Nambu-Goldstone boson** and this is an example of Goldstone's theorem, which states:

- *For every generator of a continuous global symmetry that is spontaneously broken, there will appear one massless scalar particle.*

This theorem was at first considered to be a serious problem with spontaneously broken symmetries, but it was discovered, a few years later, that if the broken symmetries were local instead of global then the degrees of freedom associated with the massless Nambu-Goldstone bosons become associated with the zero helicity modes of the gauge bosons, giving them a mass.

In light of this, consider the action for scalar electrodynamics,

$$S = -\int d^4x\left[\eta^{\mu\nu}(D_\mu\phi^*)(D_\nu\phi) + V(|\phi|) + \frac{gc}{4}F_{\mu\nu}F^{\mu\nu}\right]. \tag{5.1.12}$$

Take the scalar field potential to be the same as before,

$$V(|\phi|) = g_2|\phi|^2 + g_4|\phi|^4 \tag{5.1.13}$$

and, as before, assuming that g_2 is negative, pick $\phi(x) = [v + H(x)]e^{i\theta(x)}$, with $v = \sqrt{|g_2|/2g_4}$. This time, consider the kinetic term for the scalar field:

$$(D_\mu\phi^*)(D^\mu\phi) = \partial_\mu H\partial^\mu H + (v+H)^2\partial_\mu\theta\partial^\mu\theta - 2g(v+H)^2A^\mu\partial_\mu\theta + g^2A^2(v+H)^2 \tag{5.1.14}$$

The first three are just the kinetic energy terms of the fields $H(x)$ and $\theta(x)$ as well as some derivative interactions between the two. The last term, apart from the interactions between $A_\mu(x)$ and $H(x)$, contains a term that has the form of a canonical mass term for the gauge fields. This is

$$\mathfrak{L}_{\text{gauge mass}} = -g^2 v^2 A^2 \tag{5.1.15}$$

corresponding to a gauge field mass

$$m_A^2 = \frac{2g^2\hbar^4}{c^2} v^2. \tag{5.1.16}$$

that depends on the VEV of the scalar field. Notice that this is ultimately scalar electrodynamics, which is known to be a consistent quantum field theory. By spontaneous symmetry breaking and a simple field redefinition to fluctuations about the new global minimum of the potential, we are able to have a description involving massive gauge fields. The theory continues to be a consistent quantum field theory, but now with massive gauge fields in its spectrum. We do not know of any other way in which a renormalizable theory of massive gauge fields can be obtained. It comes at the cost of introducing the Goldstone bosons and a range of unusual interactions between the fields.

Now a massive vector field has three, not two, degrees of freedom so it appears that an extra degree of freedom has sneaked into our system. This would, of course, be inconsistent, so let's show that it is not the case. The action for the scalar field is invariant under the gauge transformations

$$\phi(x) \to \phi'(x) = e^{i\Lambda(x)}\phi(x) = [v + H(x)]e^{i[\Lambda(x)+\theta(x)]}.$$

We could pick a gauge, $\Lambda(x) = -\theta(x)$ and let $A_\mu - \frac{1}{g}\partial_\mu\theta(x) = B_\mu(x)$. In this gauge the action becomes

$$\begin{aligned} S \;\; = \;\; -\int d^4x \Big[& \eta^{\mu\nu}\partial_\mu H \partial_\nu H + 2|g_2|H^2 + \frac{g}{4\hbar}F_{\mu\nu}F^{\mu\nu} + g^2v^2B^2 + \\ & + g^2(H^2 + 2vH)B^2 + g_4(4vH^3 + H^4)\Big] + \text{const.}, \end{aligned} \tag{5.1.17}$$

where $F_{\mu\nu} = \partial_{[\mu}B_{\nu]}$, and the Goldstone boson has been "gauged away". In this way we end up with only one massive scalar particle, *i.e.,* H, a massive vector field, B_μ, and no Goldstone boson. The total number of degrees of freedom have not changed: we began with four degrees of freedom, two for the massless vector field, and two for the complex scalar field. We end up with four degrees of freedom, one for the massive scalar field and three for the massive gauge field: one of the original scalar degrees of freedom "turned into" the longitudinal degree of freedom of the vector field! The gauge in which only the real degrees of freedom survive in the Lagrangian of the effective theory is called the **unitary gauge**.

5.2 Goldstone's Theorem

It is important to keep in mind that in choosing a vacuum and expanding about it we have not in fact lost the symmetries of the Lagrangian. While the full theory remains invariant under the the original symmetry group, it is the choice of vacuum that breaks the symmetry. Different choices of vacua are related by transformations of the full symmetry group. This distinction will become useful in generalizing the results above. We will be interested in counting real degrees of freedom, so let us examine a set of N real scalar fields transforming as a vector under a real representation of some Lie group, G, and described by the action

$$S = -\frac{1}{2}\int d^4x[\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi_i + V(\phi^i)], \tag{5.2.1}$$

which is invariant under the global transformations

$$\vec{\phi} \to \vec{\phi}' = e^{i\alpha^a \hat{t}_a}\vec{\phi}, \tag{5.2.2}$$

where $\hat{t}_a$ are the generators of G. Because we are considering a real, N dimensional representation of G, the generators are all imaginary. Moreover, if G is compact then we can choose the representation to be unitary and therefore orthogonal so that $\hat{t}_{aij}$ is both imaginary and antisymmetric.

Before proceeding, it will be necessary to formalize what we mean by the mass of a field excitation. Suppose that the vacuum state of the field is given by $\phi^i = v^i$, for some constant vector $\vec{v}$. The matrix

$$\mathcal{M}_{ij} = \frac{1}{2!}\left.\frac{\partial^2 V}{\partial\phi^i\partial\phi^j}\right|_{\vec{\phi}=\vec{v}} \tag{5.2.3}$$

is called the **mass matrix** of the theory. We do not expect $\widehat{\mathcal{M}}$ to be diagonal in general but, if it is diagonalizable, let λ_i be its eigenvalues. If the vacuum is chosen to be a minimum of the potential we are guaranteed that $\widehat{\mathcal{M}}$ will have only non-negative eigenvalues. In this case, we set $\lambda_i = m_i^2c^2/\hbar^2$ for real m_i. Now, because $\widehat{\mathcal{M}}$ is a constant, symmetric matrix, it can be diagonalized by a constant, orthogonal transformation, $\widehat{S}$. Therefore, let $\widehat{M}^D = \widehat{S}\widehat{\mathcal{M}}\widehat{S}^{-1} = (m_i^2c^2/\hbar^2)\delta_{ij}$ be the diagonal form of $\widehat{\mathcal{M}}$. Expanding $V(\phi^i)$ about the vacuum, *i.e.*, letting $\phi^i = v^i + \varphi^i$, we would get

$$V(\phi^i) = \text{const.} + \varphi^i\mathcal{M}_{ij}\varphi^j + \ldots\ \text{(interactions)} \tag{5.2.4}$$

because the first derivative vanishes at the minimum of the potential. The term that is quadratic in φ can be put in the form $\varphi'^i\mathcal{M}^D_{ij}\varphi'^j = m_i^2\varphi'^i\varphi'^i$, where $\vec{\varphi}' = \widehat{S}\vec{\varphi}$ are called **mass eigenstates**. Mass eigenstates have masses determined by the eigenvalues of the mass matrix.

The kinetic term in (5.2.1) is manifestly invariant under the action of the group. For the potential also to be invariant,

$$\delta V = i\alpha^a \frac{\partial V}{\partial \phi^i} t^i_{aj} \phi^j = 0 \tag{5.2.5}$$

must hold as well. A global minimum exists at $\phi^i = v^i$ if and only if both

$$\left.\frac{\partial V}{\partial \phi^i}\right|_{v_i} = 0 \tag{5.2.6}$$

and the mass matrix

$$\mathcal{M}_{ij} = \frac{1}{2!} \left.\frac{\partial^2 V}{\partial \phi^i \partial \phi^j}\right|_{\vec{v}} \tag{5.2.7}$$

has no negative eigenvalues. We will now show that to every symmetry of the Lagrangian that is not a symmetry of the vacuum state $\phi^i = v^i$ (a broken symmetry) there must exist a zero eigenvalue of $\widehat{\mathcal{M}}$. This is the mathematical statement of Goldstone's theorem. Taking the derivative of the invariance condition in (5.2.5) with respect to ϕ^k, we find

$$i\alpha^a \frac{\partial^2 V}{\partial \phi^k \partial \phi^i} t^i_{aj} \phi^j + i\alpha^a \frac{\partial V}{\partial \phi^i} t^i_{ak} = 0 \tag{5.2.8}$$

so at the global minimum of the potential we would have the relation

$$i\alpha^a \left.\frac{\partial^2 V}{\partial \phi^k \partial \phi^i}\right|_{\vec{v}} t^i_{aj} v^j = 0 = 2\mathcal{M}_{ki} \delta v^i, \tag{5.2.9}$$

where we set $\delta v^i = i\alpha^a t^i_{aj} v^j$. If it happens that $t^i_{aj} v^j \neq 0$, for some a, then it follows that $t^i_{aj} v^j$ is an eigenvector of $\mathcal{M}_{ij}$ with zero eigenvalue. Therefore the mass matrix has a zero eigenvalue for every broken symmetry, which proves Goldstone's theorem.

5.2.1 Examples

As an example, take a set of three real scalar fields transforming under global $SO(3)$ transformations and let

$$\vec{\phi} = \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \end{pmatrix}. \quad \vec{\phi} \to \vec{\phi}' = e^{\frac{i}{2}\alpha^a \hat{t}_a} \vec{\phi}, \tag{5.2.10}$$

where $\hat{t}_a$, $a \in \{1, 2, 3\}$ are the generators of $SO(3)$. Imagine that the potential is given as before by

$$V(|\vec{\phi}|) = g_2 \vec{\phi} \cdot \vec{\phi} + g_4 (\vec{\phi} \cdot \vec{\phi})^2 \tag{5.2.11}$$

where $g_2 < 0$. It will have a minimum at

$$|\vec{\phi}| = \sqrt{\frac{|g_2|}{2g_4}} = v \tag{5.2.12}$$

as before and we can take the vacuum to be, for example,

$$\vec{\phi}_0 = \vec{v} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} . \tag{5.2.13}$$

Only $\hat{t}_3\vec{v} = 0$ (rotations in the "xy plane"), so the other two generators (which involve rotations of the "z" axis) are broken symmetries of the theory and we expect two Goldstone bosons. Indeed, if we compute the mass matrix we find that

$$\mathcal{M}_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2|g_2| \end{pmatrix} . \tag{5.2.14}$$

has two zero eigenvalues.[2]

Consider also a doublet of complex scalar fields transforming under global transformations belonging to the group $SU(2)$,

$$\phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} , \quad \vec{\phi} \to \vec{\phi}' = e^{\frac{i}{2}\alpha^a \hat{\sigma}_a} \vec{\phi}. \tag{5.2.15}$$

where $\hat{\sigma}_a$ are the Pauli sigma matrices. We take

$$V(\phi, \phi^\dagger) = g_2 \phi^\dagger \phi + g_4 (\phi^\dagger \phi)^2, \tag{5.2.16}$$

which has a minimum at

$$\phi^\dagger \phi = \frac{|g_2|}{2g_4} = v^2 \tag{5.2.17}$$

if $g_2 < 0$, and the vacuum state can be taken to be

$$\vec{\phi}_0 = \vec{v} = \begin{pmatrix} 0 \\ v \end{pmatrix} . \tag{5.2.18}$$

[2] Problem: Take

$$\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} , \quad a^2 + b^2 + c^2 = v^2,$$

and determine $\mathcal{M}_{ij}$. Find the matrix $\widehat{S}$ that brings $\widehat{\mathcal{M}}$ to diagonal form.

Evidently, $\hat{\sigma}_a\vec{v} \neq 0$ for all a, so by Goldstone's theorem we expect the mass matrix to have three zero eigenvalues (three Goldstone bosons), considering that $SU(2)$ is three dimensional. One could take the following "polar" parametrization of the small perturbations about the vacuum,

$$\phi = \begin{pmatrix} Ke^{i\theta_1} \\ (v+H)e^{i\theta_2} \end{pmatrix}, \tag{5.2.19}$$

then expanding the scalar potential we find

$$V(\phi, \phi^\dagger) = g_2(v+H)^2 + g_4(v+H)^4 + 2g_4K^2(H^2 + 2vH) + g_4K^4, \tag{5.2.20}$$

which shows that H appears with a mass of $m_H = 2|g_2|$ and K, θ_1 and θ_2 are all Goldstone bosons.

5.3 Non-Abelian Gauge Groups

Let us now couple the real scalar fields to local gauge fields by requiring local gauge invariance. The action we must consider is then

$$S = -\int d^4x \left[\frac{1}{2}\eta^{\mu\nu}D_\mu\phi^T D_\nu\phi + \frac{1}{2}V(\phi^T\phi) + \frac{gc}{4}F^a_{\mu\nu}F^{\mu\nu}_a\right], \tag{5.3.1}$$

where "T" indicates the transpose, and we expand the kinetic term about some VEV, v^i, of $\phi^i(x)$, as we did in the $U(1)$ case. We will find

$$-\frac{1}{2}[\partial_\mu\varphi^T\partial^\mu\varphi + 2igA^\mu_{ij}(v^i+\varphi^i)\partial_\mu\varphi^j + g^2(v^i+\varphi^i)A^\mu_{ik}A^k_{\mu j}(v^j+\varphi^j)], \tag{5.3.2}$$

which contains the canonical mass term for the gauge field,

$$\mathfrak{L}_{\text{gauge mass}} = -\frac{1}{4}g^2v^j\{\hat{t}_a, \hat{t}_b\}_{jk}v^kA^{\mu a}A^b_\mu \tag{5.3.3}$$

along with several derivative interaction terms and defines the the mass matrix

$$(m_A^2)_{ab} = \frac{g^2\hbar^4}{2c^2}v^j\{\hat{t}_a, \hat{t}_b\}_{jk}v^k. \tag{5.3.4}$$

This mass matrix makes it clear that gauge fields associated with unbroken symmetries, *i.e.,* with generators for which $\hat{t}_a\vec{v} = 0$, get no mass whereas those associated with broken symmetries, $\hat{t}_a\vec{v} \neq 0$, become massive. Had our two previous examples been local gauge theories, two of the three $SO(3)$ gauge fields would have acquired a mass and one remained massless, whereas all three of the $SU(2)$ gauge fields would become massive.

The entire action can be rearranged as

$$S = -\int d^4x \left[\frac{1}{2}D_\mu\varphi^T D^\mu\varphi + \frac{1}{2}V(v+\varphi) + \frac{g}{4\hbar}F^a_{\mu\nu}F^{\mu\nu}_a \right. \\ \left. + \frac{c^2}{2\hbar^4}(m_A^2)_{ab}A^a_\mu A^{b\mu} + igA^\mu_{ij}v^i(D_\mu\varphi^j)\right] \tag{5.3.5}$$

The last term contains interactions that vanish for the unbroken symmetries, since $A^i_{\mu j}v^j = A^a_\mu\ t^i_{aj}v^j = 0$ for those symmetries. They are therefore interactions that are peculiar to the gauge bosons that have acquired a mass by spontaneous symmetry breaking.

A naïve count of the degrees of freedom makes it appear that we have increased the number of degrees of freedom by precisely the number of newly massive gauge bosons simply by the our choice of vacuum. This cannot be correct of course and we want to argue, as we did before, that the Goldstone bosons are spurious degrees of freedom that can be eliminated by a suitable choice of gauge. Suppose we represent by $\hat{\tau}_\alpha$ the generators of G that correspond to broken symmetries (*i.e.,* $\hat{\tau}_\alpha\vec{v} \neq 0$, let there be p of these) and by $\hat{t}_A$ the generators that correspond to the unbroken ones (there will be $\dim(G) - p$ of these). It is easy to see that the generators that preserve the symmetries of the vacuum form a subalgebra of the algebra of G and generate a subgroup, H, of G, called the **little group** of the vacuum, since

$$[\hat{t}_A, \hat{t}_B]\vec{v} = f^m_{AB}\hat{t}_m\vec{v} = 0, \ \ \Rightarrow \ \ \hat{t}_m \ \in \ H \tag{5.3.6}$$

i.e., $f^\alpha_{AB} = 0$. This means that all three of $f_{\alpha AB}$, $f_{A\alpha B}$ and $f_{AB\alpha}$ vanish because of the antisymmetry. Thus we also conclude that

$$[\hat{t}_A, \hat{t}_B] = f^C_{AB}\hat{t}_C$$
$$[\hat{t}_A, \hat{\tau}_\alpha] = f^\gamma_{A\alpha}\hat{\tau}_\gamma$$
$$[\hat{\tau}_\alpha, \hat{\tau}_\beta] = f^A_{\alpha\beta}\hat{t}_A + f^\gamma_{\alpha\beta}\hat{\tau}_\gamma \tag{5.3.7}$$

To generalize the "gauging away" of the Goldstone boson to the non-abelian case, we begin by expressing $\vec{\phi}(x)$ as a G transformation of fields, $\vec{\varphi}(x)$, that do *not* contain any Goldstone bosons. In other words we express $\vec{\phi}(x)$ as

$$\phi^i(x) = U^i{}_j(x)(v^j + \varphi^j(x)) \tag{5.3.8}$$

where $\hat{U} \in G$. Now any element in G can always be parameterized as

$$\hat{U}(x) = e^{i\theta^\alpha(x)\hat{\tau}_\alpha}e^{i\theta^A(x)\hat{t}_A} \tag{5.3.9}$$

and the statement that $\vec{\varphi}(x)$ should contain no Goldstone bosons is the statement that $\vec{\varphi}(x)$ is orthogonal to all the massless eigenvectors of the mass matrix. But, according to Goldstone's theorem, the massless eigenvectors of the mass matrix are just linear combinations of $\tau^{\alpha}_{ij} v^j$, therefore we should have the condition

$$(v^i + \varphi^i)\tau^{\alpha}_{ij} v^j = 0 = \varphi^i \tau^{\alpha}_{ij} v^j \tag{5.3.10}$$

(because τ^{α}_{ij} is antisymmetric).[3] These are p conditions on the fields, so there will be $N-p$ independent fields contained in $\vec{\varphi}(x)$. Furthermore, any $\vec{\varphi}\,'(x)$ related to $\vec{\varphi}(x)$ by an H transformation will also satisfy this orthogonality condition and therefore

$$e^{i\theta^A \hat{t}_A}(\vec{v} + \vec{\varphi}) = \vec{v} + \vec{\varphi}\,' \tag{5.3.11}$$

simply amounts to a redefinition of $\varphi(x)$. The only elements of G that yield new, independent fields in (5.3.8) must belong to G/H, *i.e.,* we can take

$$\vec{\phi}(x) = e^{i\theta^{\alpha}(x)\hat{\tau}_{\alpha}}(\vec{v} + \vec{\varphi}(x)) \tag{5.3.12}$$

In this form, it is easy to see that the fields $\theta^{\alpha}(x)$ can be eliminated by a gauge transformation, as in the $U(1)$ case. In fact, (5.3.1) is invariant under the transformation

$$\phi(x) \to \phi'(x) = \hat{U}\vec{\phi}(x) = e^{i\Lambda^a \hat{t}_a}\vec{\phi}(x)$$

together with

$$\hat{A}_{\mu} \to \hat{B}_{\mu} = \hat{U}\hat{A}_{\mu}\hat{U}^{-1} + \frac{i}{g}\hat{U}\partial_{\mu}\hat{U}^{-1},$$

so with the choice $\Lambda^{\alpha} = -\theta^{\alpha}$ and $\Lambda^A = 0$ (the unitary gauge), direct substitution into (5.3.1) will give

$$S = -\int d^4x \left[\frac{1}{2}(D_{\mu}\overline{\phi})^T(D^{\mu}\overline{\phi}) + \frac{1}{2}V(|\overline{\phi}|) + \frac{g}{4\hbar}F^a_{\mu\nu}F^{\mu\nu}_a + \frac{c^2}{2\hbar^4}(m_B^2)_{ab}B^a_{\mu}B^{b\mu}\right] \tag{5.3.13}$$

where

$$\overline{\phi}^i(x) = v^i + \varphi^i(x) \tag{5.3.14}$$

and$(m_A^2)_{ab}$ is the mass matrix

$$(m_A^2)_{ab} = \frac{g^2\hbar^4}{2c^2} v^j \{\hat{t}_a, \hat{t}_b\}_{jk} v^k \tag{5.3.15}$$

[3]One solution is obviously $\varphi^i(x) = H(x)v^i$. This solution is also left invariant by the action of $\hat{t}_A \in H$, since they annihilate $\vec{v}$, so there will be at least one massive vector field.

and $F^a_{\mu\nu} = \partial_{[\mu} B^a_{\nu]} + igf^A_{bc} B^b_\mu B^c_\nu$. There is no longer any mixing term because $\varphi^i t^a_{ij} v^j = 0$ for all a. The spectrum consists of $\dim(G) - p$ massless gauge bosons corresponding to the little group, H, and p massive gauge bosons corresponding to the p broken symmetries interacting with $N-p$ massive, real scalar fields generally referred to as the **Higgs** bosons. All interactions are determined by the original gauge theory and the choice of VEV, v^i, of ϕ. The masses of the massive gauge bosons are determined by the eigenvalues of $(m^2)_{ab}$ and by the VEV. The masses of the Higgs fields are determined by the scalar potential that fixes the vaccum state.

5.3.1 The Example of $SU(2) \times U(1)_Y$

Consider a local gauge theory of the four dimensional product group $SU(2) \times U(1)_Y$ with scalar fields and the usual ϕ^4 potential we have been studying. The subscript "Y" is used to indicate that the $U(1)$ factor is not directly connected with electromagnetic charge. It is usually referred to as the "**hypercharge**" factor. Let the scalar fields form a vector of the 2×2 complex representation of the group. The group generators satisfy

$$[\hat{\sigma}_i, \hat{\sigma}_j] = -i\epsilon_{ijk}\hat{\sigma}_k$$

$$[\hat{Y}, \hat{\sigma}_i] = 0 \tag{5.3.16}$$

and can be chosen as the three Pauli matrices in (4.4.3) together with

$$\hat{Y} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.3.17}$$

for the hypercharge. We will represent $\phi(x)$ by the complex doublet of fields

$$\phi(x) = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix}, \quad \phi^\dagger = (\phi_1^*, \phi_2^*), \tag{5.3.18}$$

where φ_i, $i \in \{1, 2, 3, 4\}$ are real fields, and take the vacuum of the theory (after spontaneous symmetry breaking) to be given by

$$\phi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{5.3.19}$$

with real $v = \sqrt{|g_2|/2g_4}$. All degenerate vacua will be obtained by acting on ϕ_0 by an $SU(2) \times U(1)_Y$ transformation.[4] The chosen vacuum state breaks the $SU(2) \times U(1)$ symmetry leaving the little group $H = U(1)_Q$; we readily verify that

$$e^{i\alpha(\hat{\sigma}_3 + \hat{Y})}\begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{5.3.20}$$

[4]The scalar potential of the ϕ^4 theory has a symmetry group that is actually the group $O(4) \sim SU(2) \times SU(2)$ of rotations of the four real scalar fields φ_i.

is the only transformation that leaves the vacuum invariant, so

$$\hat{Q} = \hat{\sigma}_3 + \hat{Y}. \tag{5.3.21}$$

generates $U(1)_Q$.

Three of the four symmetries are broken, so we expect a gauge spectrum containing three massive gauge bosons and one massless gauge boson. The scalar spectrum will consist of one massive Higgs field, $H(x)$, and three Goldstone bosons, $\theta^\alpha(x)$; we take the scalar field to have the form

$$\phi(x) = e^{i(\theta^1(x)\hat{\sigma}_1 + \theta^2(x)\hat{\sigma}_2 + \theta^-(x)\hat{\sigma}_-)} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \tag{5.3.22}$$

where

$$\hat{\sigma}_- = \hat{\sigma}_3 - \hat{Y} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.3.23}$$

To simplify the algebra, we will now used rescaled fields as discussed in the problem of §2.6. In the unitary gauge, our Lagrangian density will read

$$\mathfrak{L} = -\int d^4x \left[\frac{1}{2}(D_\mu\overline{\phi})^\dagger(D^\mu\overline{\phi}) + \frac{1}{2}V(|\overline{\phi}|) + \frac{1}{4\hbar}W^a_{\mu\nu}W_a^{\mu\nu} + \frac{1}{4\hbar}B_{\mu\nu}B^{\mu\nu} \right], \tag{5.3.24}$$

where we have defined

$$\begin{aligned}
&\overline{\phi}(x) = \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \\
,\quad &W^a_{\mu\nu} = \partial_{[\mu}W^a_{\nu]} + igf^a_{bc}W^b_\mu W^c_\nu, \qquad \hat{\sigma}_a \in SU(2) \\
&B_{\mu\nu} = \partial_{[\mu}B_{\nu]} \\
&D_\mu\overline{\phi} = (\partial_\mu - igW^a_\mu\hat{\sigma}_a - ig'B_\mu\hat{Y})\overline{\phi}, \qquad \hat{Y} \in U(1)_Y.
\end{aligned} \tag{5.3.25}$$

The mass matrix for the gauge fields is found to be

$$(m^2)_{rs} = \frac{\hbar^3 v^2}{4c^2} \begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & -gg' \\ 0 & 0 & -gg' & g'^2 \end{pmatrix}, \tag{5.3.26}$$

but we are interested in expressing our effective Lagrangian in terms the mass eigenstates of the gauge bosons, since these are what we observe. To do so we should diagonalize the mass matrix.

This is easily accomplished by the orthogonal transformation

$$S_{rs} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_W & \sin\theta_W \\ 0 & 0 & -\sin\theta_W & \cos\theta_W \end{pmatrix} \tag{5.3.27}$$

with

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}, \quad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}. \tag{5.3.28}$$

The angle θ_W is called the **Weinberg** or **weak** mixing angle (hence the subscript). It depends in the coupling strengths of the $SU(2)$ and $U(1)$ factors according to

$$\tan\theta_W = \frac{g'}{g}.$$

and the diagonal form of m^2 will be

$$(m_D^2)_{rs} = \frac{\hbar^3 v^2}{4c^2}\begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2+g'^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.3.29}$$

This leads us to define the linear combinations

$$\begin{aligned} Z_\mu &= \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu \\ A_\mu &= \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu \end{aligned} \tag{5.3.30}$$

so that Z_μ will end up having a mass proportional to $g^2+g'^2$ and A_μ will be massless. Further, considering that $W_\mu^{1,2}$ have the same mass, we could also define

$$W_\mu^\pm = \frac{1}{2}(W_\mu^1 \pm iW_\mu^2) \tag{5.3.31}$$

With these field definitions, let us rewrite our original Lagrangian density.

Beginning with the scalar field kinetic term, we have

$$\begin{aligned} D_\mu\overline{\phi} &= \left(\partial_\mu \mathbf{1} - ig(W_\mu^1\hat{\sigma}_1 + W_\mu^2\hat{\sigma}_2) - igW_\mu^3\hat{\sigma}_3 - ig'B_\mu\hat{Y}\right)\begin{pmatrix} 0 \\ v+H \end{pmatrix} \\ &= \begin{pmatrix} -igW_\mu^-(v+H) \\ \partial_\mu H + i(gW_\mu^3 - g'B_\mu)(v+H) \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} -igW_\mu^-(v+H) \\ \partial_\mu H + i\sqrt{g^2+g'^2}Z_\mu(v+H) \end{pmatrix} \tag{5.3.32}$$

therefore

$$(D_\mu\overline{\phi})^\dagger(D^\mu\overline{\phi}) = (\partial_\mu H)^\dagger(\partial^\mu H) + g^2W_\mu^+W^{-\mu}(v+H)^2 + (g^2+g'^2)Z_\mu Z^\mu(H+v)^2. \tag{5.3.33}$$

In this term we see the expected masses of the $W^\pm$ and Z vector bosons

$$\begin{aligned} M_W &= \frac{\hbar^{3/2}vg}{2c} \\ M_Z &= \frac{\hbar^{3/2}v}{2c}\sqrt{g^2+g'^2} = \frac{M_W}{\cos\theta_W}. \end{aligned} \tag{5.3.34}$$

The scalar potential expands to

$$V(v+H) = -|g_2|(v+H)^2 + g_4(v+H)^4, \tag{5.3.35}$$

from where we also recover the scalar Higgs boson mass, $M_H = 2|g_2|$. The Higgs boson does not interact with the massless gauge boson, A_μ. If we think of A_μ as mediating the electromagnetic force, this is telling us that the Higgs boson carries no electric charge. It does, however, interact with the massive gauge bosons of $SU(2)$.

Finally, consider the gauge kinetic terms. While the algebra is tedious, it is in fact quite straightforward to find that the two terms may be rewritten in terms of the fields introduced so far as

$$\begin{aligned} -c\Bigg[&\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + ig\{(\cos\theta_W Z^\mu + \sin\theta_W A^\mu)(W^+_{\mu\nu}W^{-\nu} - W^-_{\mu\nu}W^{+\nu}) \\ &\quad -(\sin\theta_W F_{\mu\nu} + \cos\theta_W Z_{\mu\nu})W^{+\mu}W^{-\nu}\} \\ &-g^2\{\cos^2\theta_W[Z^2(W^+\cdot W^-) - (Z\cdot W^+)(Z\cdot W^-)] \\ &\quad +\sin^2\theta_W[A^2(W^+\cdot W^-) - (A\cdot W^+)(A\cdot W^-)]\}] \\ &\quad -\sin\theta_W\cos\theta_W[2(Z\cdot A)(W^+\cdot W^-) - (Z\cdot W^+)(A\cdot W^-) - (Z\cdot W^-)(A\cdot W^+)] \\ &-\frac{1}{2}[(W^+\cdot W^-) - W^{+2}W^{-2}]\}\Bigg], \end{aligned} \tag{5.3.36}$$

where $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$, $Z_{\mu\nu} = \partial_{[\mu}Z_{\nu]}$ and $W^\pm_{\mu\nu} = \partial_{[\mu}W^\pm_{\nu]}$. The effective theory therefore describes

- One massless, neutral vector boson, $A_\mu(x)$, corresponding to the only unbroken gauge symmetry we called $U(1)_Q$,
- One massive, neutral vector boson, $Z_\mu(x)$,
- One (conjugate) pair of massive, charged vector bosons, $W^\pm(x)$, and
- A neutral scalar boson, $H(x)$, (the Higgs field).

The theory has three parameters, *viz.,* g, θ_W and v, therefore if the masses of the W, Z and Higgs bosons are known, they should be sufficient to determine these parameters. Experimentally, it has been found that $M_W = 80.385 \pm 0.015$ GeV, $M_Z = 91.1876 \pm 0.0021$ GeV and $M_H \approx 126.0 \pm 0.4$ GeV so, directly from the ratio of the W and Z masses we find $\theta_W \approx 28.17°$. All interactions involving $Z_\mu(x)$ have a strength of $g\cos\theta_W$ and interactions involving $A_\mu(x)$ have a strength of $g\sin\theta_W$. If we identify $g\sin\theta_W$ with the strength of the electromagnetic interactions then it follows that the strength of the weak interactions is stronger than the electromagnetic interactions by a factor of $\csc\theta_W \approx 2.1182$. Thus the weak interactions are *intrinsically* stronger than the electromagnetic interactions, but they are effectively weakened by the large masses of the gauge bosons that carry this "force", which shortens the range of the interactions.

This gauge fixed $SU(2)\times U(1)_Y$ theory in fact describes the physical degrees of freedom of the bosonic sector of the standard model of particles except for the gluons of the strong interactions. To arrive at the Standard Model of particle physics, all that is left to add to this model are the leptons, the quarks and the strong interactions.

5.4 The Glashow-Weinberg-Salam Model

The Standard Model refers to the field theory that describes matter and its interactions as observed in nature, with the important exception of the gravitational interaction. In our discussion of $SU(2)\times U(1)_Y$ we discovered a portion of the bosonic degrees of freedom consisting of three massive gauge bosons, one massless gauge boson and one scalar Higgs boson. The $W^\pm$ carry isospins of ± 1 (in electronic units) respectively, because they transform in the adjoint representation of $SU(2)$, but they carry no hypercharge. Therefore they are electrically charged, with charges of ± 1 according to (5.3.21). On the other hand, the Z boson carries neither weak isospin nor hypercharge, therefore it is electrically neutral. The Higgs boson carries a weak isospin of $-\frac{1}{2}$ and a hypercharge of $+\frac{1}{2}$, so it is also electrically neutral.

An essential feature of the electroweak interactions is that only left handed fermions interact weakly. This was first established in an experiment at the National Bureau of Standards by the low temperature experimentalist C.S Wu at the suggestion of theoretical particle physicists T.D. Lee and C.N. Yang. The experiment studied the β decay of polarized Co nuclei,

$$ {}^{27}_{60}\text{Co} \to {}^{28}_{60}\text{Ni} + e^- + \overline{\nu}_e $$

or

$$ n \to p + e^- + \overline{\nu}_e, $$

which occurs by the transformation of one of the d quarks into an u quark according to

$$ d \to u + W^- \to u + e^- + \overline{\nu}_e $$

The spins of the Cobalt nuclei were aligned by the application of an external magnetic field. The original Cobalt nuclei carried $J = 5$ oriented along the positive z axis (say) by the magnetic field, whereas the residual Ni nuclei carried $J = 4$ also oriented by the same magnetic field. The difference of $J_z = 1$ was carried away together by the electron and the anti-neutrino. Since the neutrino and the electron are both spin $\frac{1}{2}$ particles, this is explained by requiring both their spins to align with the magnetic field. It was discovered, however, that the electrons preferred to be emitted opposite the direction of the nuclear spins and the anti-neutrinos preferred the direction of the nuclear spins (this held true even when the magnetic field was reversed). Thus left handed electrons and right handed anti-neutrinos were the preferred decay mode. Many experiments have since confirmed the conclusion that only left handed leptons (and right handed anti-leptons) participate in weak interactions.

The fermions of the standard model come in two families, the leptons and the quarks, each family consisting of six leptons or six quarks. Each family is further separated into left handed and right handed fermions. The right handed fermions are subject only to a local $U(1)$ symmetry (there are no right handed neutrinos in the standard model, which is why the neutrino is massless), but the left handed fermions are conveniently arranged in three generations of isospin ($SU(2)$) doublets according to their masses (increasing from left to right) as follows:

$$ f = \begin{cases} \begin{pmatrix} \nu_l \\ l \end{pmatrix}_L \equiv \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \\ \\ \begin{pmatrix} \vec{q}_u \\ \vec{q}_d \end{pmatrix}_L \equiv \begin{pmatrix} \vec{u} \\ \vec{d} \end{pmatrix}_L, \begin{pmatrix} \vec{c} \\ \vec{s} \end{pmatrix}_L, \begin{pmatrix} \vec{t} \\ \vec{b} \end{pmatrix}_L \end{cases} \tag{5.4.1} $$

Left handed leptons carry a hypercharge of $-\frac{1}{2}$. Neutrinos carry a weak isospin of $+\frac{1}{2}$, which makes them electrically neutral, whereas each of the "bottom" left handed leptons, e, μ and τ carries a weak isospin of $-\frac{1}{2}$, therefore all of them possess an electric charge of -1. All left handed quarks quarks carry a hypercharge of $+\frac{1}{6}$. Combining this with the weak isospin of $+\frac{1}{2}$ for the "up" quarks (again, left handed quarks only) and of $-\frac{1}{2}$ for the "down" quarks, we conclude that the "up" quarks are positively charged at $+\frac{2}{3}$ and the "down" quarks are negatively charged at $-\frac{1}{3}$. Thus a neutron, for example, which is made out of three quarks: two "downs" and an "up", $n \sim (udd)$, has a weak isospin of $-\frac{1}{2}$, a hypercharge of $+\frac{1}{2}$ and is electrically neutral, whereas a proton, which is made out of two "ups" and a "down", $p \sim (uud)$, carries a weak isospin of $+\frac{1}{2}$, a hypercharge of $+\frac{1}{2}$ and a positive charge of $+1$. Right handed quarks and leptons carry no isospin (they do not form isospin doublets) and, in each case, their hypercharge is equal to their electric charge. We may summarize this information in the tables below.

Particle	$\hat{\sigma}_3$	$\hat{Y}$	$\hat{Q}$
W^+	$+1$	0	$+1$
W^-	-1	0	-1
Z	0	0	0
H	$-\frac{1}{2}$	$+\frac{1}{2}$	0

Particle	$\hat{\sigma}_3$	$\hat{Y}$	$\hat{Q}$
u_L	$+\frac{1}{2}$	$+\frac{1}{6}$	$+\frac{2}{3}$
d_L	$-\frac{1}{2}$	$+\frac{1}{6}$	$-\frac{1}{3}$
u_R	0	$+\frac{2}{3}$	$+\frac{2}{3}$
d_R	0	$-\frac{1}{3}$	$-\frac{1}{3}$

Particle	$\hat{\sigma}_3$	$\hat{Y}$	$\hat{Q}$
$\nu_{l,L}$	$+\frac{1}{2}$	$-\frac{1}{2}$	0
l_L	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
$\nu_{l,R}$	0	0	0
l_R	0	-1	-1

Each of the fermions has its corresponding antiparticle and the quarks possess an additional charge, called **color**, by which they interact strongly among themselves forming bound states. These strong interactions between quarks are described by an unbroken gauge theory of $SU(3)$. There are therefore three colors (hence the vector over each quark flavor) and, because the dimension of $SU(3)$ is eight, the color force is mediated by eight (themselves colored) gluons. There are, in all, $N_q = 3 \times 6 = 18$ quarks and consequently eighteen antiquarks. Likewise, there are six leptons and six antileptons, therefore the total number of fermions in the standard model is forty eight. In the bosonic sector, we have one photon, three massive $SU(2)$ gauge bosons and eight gluons together with one massive scalar boson for a total of thirteen bosons. Thus the standard model describes the mutual interactions of sixty one elementary particles. It has passed every experimental test to which it has been subjected to date.

5.5 Fermions in the Standard Model

The gauge theory of $SU(2) \times U(1)_Y$ is an example of a unification of the electromagnetic and weak interactions. Later it was extended to include quark matter as well by postulating the existence of the charm quark in addition to the u, d and s quarks that we already known at the time. But, because quarks interact via the strong interactions as well, we must now consider the gauge group $SU(3) \times SU(2) \times U(1)_Y$ acting on quarks and leptons subject to the experimental constraints described in the introduction to this section. Because of the parity violation associated with $SU(2)$ it will be necessary to consider separately the left and right handed fermions, with the left handed ones forming isospin doublets and the right handed ones as isospin singlets. Thus the "matter content" of the standard model will be

$$L_- = \begin{pmatrix} \nu_{l-} \\ l_- \end{pmatrix}, \quad L_+ = \{\nu_{l+},\ l_+\}, \quad Q_- = \begin{pmatrix} \vec{u}_- \\ \vec{d}_- \end{pmatrix}, \quad Q_+ = \{\vec{u}_+,\ \vec{d}_+\}, \tag{5.5.1}$$

(we include right handed neutrinos here although they were not included in the original Standard Model. We include them because more recent experimental evidence strongly suggests that neutrinos do carry a small, but non-zero mass, which would be impossible without the existence of right handed neutrinos). The fact that the right handed particles are all singlets under $SU(2)$ implies that they will not interact with the $W^\pm$ and Z bosons. Now this separation into left and right handed particles only makes sense if they are massless (the chiral projection operator does not commute with the Dirac Hamiltonian for the massive case), so we will have to begin with massless particles and turn to the problem of fermion masses afterwards.

5.5.1 Kinetic Terms

It is straightforward to write down the Lagrangian for massless fermions:

$$\mathfrak{L} = -\frac{i\hbar c^2}{2} \sum_{\text{fermions}} \overline{\psi} \overleftrightarrow{\hat{\not{D}}}\ \psi \tag{5.5.2}$$

where, taking into account the isospin and hypercharge assignments,

$$\begin{aligned}
\hat{D}_\mu &= \partial_\mu - ig_s G^a_\mu(\mathbf{1}\otimes\hat{t}_a) - igW^a_\mu(\hat{\sigma}_a\otimes\mathbf{1}) - \frac{1}{3}ig' B_\mu(\hat{Y}\otimes\mathbf{1}), && \text{for } Q_- \\
\hat{D}_\mu &= \partial_\mu - ig_s G^a_\mu \hat{t}_a - \frac{2}{3}ig' B_\mu \mathbf{1}, && \text{for } u_+ \\
\hat{D}_\mu &= \partial_\mu - ig_s G^a_\mu \hat{t}_a + \frac{1}{3}ig' B_\mu \mathbf{1}, && \text{for } d_+ \\
\hat{D}_\mu &= \partial_\mu - igW^a_\mu \hat{\sigma}_a + ig' B_\mu \hat{Y}, && \text{for } L_- \\
D_\mu &= \partial_\mu, && \text{for } \nu_{l+} \\
D_\mu &= \partial_\mu + ig' B_\mu, && \text{for } l_+
\end{aligned} \tag{5.5.3}$$

The tensor product of generators in the first of the covariant derivatives arises because each of the u and d in

$$Q_- = \begin{pmatrix} \vec{u}_- \\ \vec{d}_- \end{pmatrix}$$

represents a color triplet.

Let us now expand the covariant derivatives. To avoid the (purely algebraic) complications of the extra $SU(3)$ gauge invariance carried by the quarks, begin with the leptons,

$$\mathfrak{L} = -\frac{i\hbar c^2}{2}\sum_{\text{leptons}} \overline{\psi}\overleftrightarrow{\hat{\not{D}}}\,\psi = -\frac{i\hbar c^2}{2}\sum_l \left[\overline{L}_-\overleftrightarrow{\hat{\not{D}}}\,L_- + \overline{\nu}_{l+}\overleftrightarrow{\hat{\not{\partial}}}\,\nu_{l+} + \overline{l}_+\overleftrightarrow{\not{D}}\,l_+\right] \tag{5.5.4}$$

and fully expand the derivative terms in square brackets:

$$(\overline{\nu}_{l-}, \overline{l}_-)\begin{pmatrix} \not{\partial} - \frac{ig}{2}\left(\not{W}^3 - \tan\theta_W \not{B}\right) & -ig\not{W}^- \\ -ig\not{W}^+ & \not{\partial} + \frac{ig}{2}\left(\not{W}^3 + \tan\theta_W \not{B}\right) \end{pmatrix}\begin{pmatrix}\nu_{l-} \\ l_-\end{pmatrix}$$

$$\overline{l}_+(\not{\partial} + ig\tan\theta_W \not{B})l_+ + \overline{\nu}_{l+}\not{\partial}\,\nu_{l+} \tag{5.5.5}$$

or

$$\overline{\nu}_{l-}\left\{\not{\partial} - \frac{ig}{2}\left(\not{W}^3 - \tan\theta_W\not{B}\right)\right\}\nu_{l-} + \overline{l}_-\left\{\not{\partial} + \frac{ig}{2}\left(\not{W}^3 + \tan\theta_W\not{B}\right)\right\}l_-$$
$$-ig\overline{l}_-\not{W}^+\nu_{l-} - ig\overline{\nu}_{l-}\not{W}^-l_- + \overline{l}_+(\not{\partial} + ig\tan\theta_W\not{B})l_+ + \overline{\nu}_{l+}\not{\partial}\,\nu_{l+} \tag{5.5.6}$$

The kinetic terms clearly reduce to the expected

$$\mathfrak{L}^{\text{lepton}}_{\text{kin}} = \sum_l\left(\overline{\nu}_l\not{\partial}\,\nu_l + \overline{l}\not{\partial}\,l\right) \tag{5.5.7}$$

(modulo the constant prefactor $-i\hbar c^2/2$) if one makes use of the fact that for any Dirac spinor $\overline{\psi}\not{\partial}\,\psi = \overline{\psi}_-\not{\partial}\,\psi_- + \overline{\psi}_+\not{\partial}\,\psi_+$. Let us address the interactions. Replace the W^3 and B fields with their mass eigenstates using

$$\begin{aligned} W^3_\mu &= \cos\theta_W Z_\mu + \sin\theta_W A_\mu \\ B_\mu &= -\sin\theta_W Z_\mu + \cos\theta_W A_\mu, \end{aligned}$$

then we have

$$\begin{aligned}\mathfrak{L}^{\text{lepton}}_{\text{int}} = \sum_l\Bigg(&-\frac{ig}{2\cos\theta_W}\overline{\nu}_{l-}\not{Z}\,\nu_{l-} - ig\overline{l}_-\not{W}^+\nu_{l-} - ig\overline{\nu}_{l-}\not{W}^-l_- \\ &+ig\sin\theta_W\overline{l}\not{A}\,l + \frac{ig\cos 2\theta_W}{2\cos\theta_W}\overline{l}_-\not{Z}\,l_- - \frac{ig\sin^2\theta_W}{\cos\theta_W}\overline{l}_+\not{Z}\,l_+\Bigg)\end{aligned} \tag{5.5.8}$$

where we used $\overline{l}_-\not{A}\,l_- + \overline{l}_+\not{A}\,l_+ = \overline{l}\not{A}\,l$. In fact, we can get rid of the left and right handed spinors entirely by simply replacing

$$\psi_\pm = \frac{1}{2}(1 \pm c\gamma^5)\psi$$

and we find, after a bit of algebra, that

$$\begin{aligned}\mathfrak{L}_{\text{int}}^{\text{lepton}} &= \sum_l \left(-\frac{ig}{4\cos\theta_W} \left\{ \overline{\nu}_l \not{Z}\ (1 - c\gamma^5)\nu_l - \bar{l}\not{Z}\ (1 - 4\sin^2\theta_W - c\gamma^5) l \right\} \right. \\ & \left. -\frac{ig}{2} \left\{ \bar{l}\not{W}^+ (1 - c\gamma^5)\nu_l + \overline{\nu}_l \not{W}^- (1 - c\gamma^5) l \right\} + ig\sin\theta_W \bar{l}\not{A}\, l \right)\end{aligned} \tag{5.5.9}$$

In the language of particle physics:

- The first term represents the coupling of left handed neutrinos to the weak, neutral vector boson, Z_μ.
- The second term represents the coupling of both left and right handed l to the weak, neutral vector boson, Z_μ. Note that both the first and second terms have the general form
$$-\frac{ig}{4\cos\theta_W}\overline{f}\not{Z}\ (2I(1 - c\gamma^5) - 4Q\sin^2\theta_W) f,$$
where Q is the fermion charge and I is its isospin.
- The next two terms permit transitions between left handed ν_l and l via the emission of a weak, charged vector boson, $W_\mu^\pm$.
- The last term is the standard coupling of l with the electromagnetic field, A_μ.

(Almost) identical expressions for the kinetic and interaction terms of the quarks will exist, with the added strong interaction. We could write

$$\mathfrak{L} = -\frac{i\hbar c^2}{2} \sum_{i,\text{quarks}} \left[\overline{Q}_{i-} \overleftrightarrow{\hat{\not{D}}}\ Q_{i-} + \overline{u}_{i+} \overleftrightarrow{\not{D}}\ u_{i+} + \overline{d}_{i+} \overleftrightarrow{\not{D}}\ d_{i+} \right] \tag{5.5.10}$$

where the sum is over all colors, i, and over all generations of up and down quarks. Taking into account the isospins and hypercharges and repeating the algebra above, we will arrive at the kinetic term

$$\mathfrak{L}_{\text{kin}}^{\text{quark}} = \sum_{i,u,d} \left(\overline{u}_i \not{\partial}\, u_i + \overline{d}_i \not{\partial}\, d_i \right) \tag{5.5.11}$$

and the interaction terms

$$\begin{aligned}\mathfrak{L}_{\text{int}}^{\text{quark}} &= -ig_s \sum_i \overline{q}_i \not{G}_{ij}\ q_j \\ & + \sum_{i,u,d} \left(-\frac{ig}{4\cos\theta_W} \left\{ \overline{u}_i \not{Z}\ \left(1 + \frac{8}{3}\sin^2\theta_W - c\gamma^5 \right) u_i \right. \right.\end{aligned}$$

$$
\begin{aligned}
-\overline{d}_i \not{Z} \left(1 - \frac{4}{3}\sin^2\theta_W - c\gamma^5\right) d_i \Big\} \\
-\frac{ig}{2}\left\{\overline{d}_i \not{W}^{+}(1-c\gamma^5)u_i + \overline{u}_i \not{W}^{-}(1-c\gamma^5)d_i\right\} \\
-\frac{2ig}{3}\sin\theta_W \overline{u}_i \not{A}\, u_i + \frac{ig}{3}\sin\theta_W \overline{d}_i \not{A}\, d_i \Big)
\end{aligned}
\tag{5.5.12}
$$

where the first sum (over i) is over the color index carried by each quark and the second sum is over all colors and all generations of up and down quarks. Again, in the language of particle physics:

- The first term represents the standard, unbroken gauge coupling with the gluon field (strong interactions).
- The second and third terms represent the coupling of both left and right handed up and, respectively, down quarks to the weak, neutral vector boson, Z_μ.
- The fourth and fifth terms permit transitions between left handed up and down quarks via the emission of the weak, charged vector boson, $W^{\pm}_{\mu}$.
- The last two terms are the standard coupling to the electromagnetic field, A_μ, with strengths that depend on the fractional charges of the quarks.

5.5.2 Lepton Masses

In the original standard model right handed neutrinos were excluded because there was no evidence at the time that neutrinos were massive. Today, however, there is mounting evidence that neutrinos do indeed have a very small, even by particle physics standards. Neutrino masses have never actually been measured directly, but a strong upper bound of about 0.3 eV for the summed masses of the three neutrino generations comes from cosmology by careful examinations of the Cosmic Microwave Background (CMB), Galaxy surveys and the Lyman Alpha forrest. Results from the Super-Kamiokande detector have shown that neutrinos can oscillate, *i.e.,* transform from one type to another. This is only possible if they carry a mass and results show that there should be at least one neutrino type with a mass of at least 0.04 eV.

To begin, we will assume that neutrinos are massless and ask how it is possible, within the standard model, to have massive electrons, muons and tauons. One cannot simply introduce mass terms because they mix left and right handed particles, which have been assigned to different multiplets of $SU(2)$, so a term such as

$$m(\bar{l}_- l_+ + \bar{l}_+ l_-)$$

is not gauge invariant. On the other hand, gauge invariance permits an interaction term involving a left handed lepton doublet, the scalar Higgs doublet and a right handed lepton, such as

$$\sum_l G_l \overline{L}^i_- \Phi_i l_+ + \text{h.c.} \tag{5.5.13}$$

where "h.c." refers to "hemitean conjugate". Such an term added to the action (by hand) is called a **Yukawa interaction**. Expanding, in the unitary gauge, we find

$$\sum_l G_l [\bar{l}_-(v+H)l_+ + \bar{l}_+(v+H)l_-] \tag{5.5.14}$$

and we see straightaway that the VEV of the scalar field gives the desired structure,

$$\sum_l G_l v[\bar{l}_- l_+ + \bar{l}_+ l_-] = \sum_l G_l v \bar{l} l,$$

of a mass term for each l. Thus if m_l is the mass of l, then

$$m_l = \frac{\hbar G_l v}{c} \Rightarrow G_l = \frac{m_l c}{\hbar v} = \frac{g m_l \hbar}{2M_W} \tag{5.5.15}$$

and the Yukawa interaction also implies the following interaction between the l and the Higgs,

$$\frac{g m_l \hbar}{2M_W} \; \bar{l} \; H \; l \tag{5.5.16}$$

whose strength is proportional to the lepton mass, m_l.

Now in principle nothing prevents us from generalizing this to

$$\sum_{l,l'} G_{ll'} \overline{L}^i_- \Phi_i l'_+ + \text{h.c.}, \tag{5.5.17}$$

where $G_{ll'}$ is a general complex 3×3 matrix mixing different generations of leptons. Such a term would also be gauge invariant and it has the expanded form

$$\sum_{l,l'} (v+H)\bar{l}_- G_{ll'} l'_+ + \text{h.c.} \tag{5.5.18}$$

Let us use the fact that any matrix, such as $G_{ll'}$, can be diagonalized by a pair of unitary matrices and may be expressed as,

$$\widehat{G} = \widehat{U}_1^\dagger \; \widehat{G}_D \; \widehat{U}_2 \tag{5.5.19}$$

where $\widehat{G}_D$ is a diagonal matrix, so our Yukawa interaction term looks like

$$\sum_{l,l'} (v+H)\bar{l}_- \; (\widehat{U}_1^\dagger \; \widehat{G}_D \; \widehat{U}_2)_{ll'} \; l'_+ + \text{h.c.} \tag{5.5.20}$$

One can now define the mixtures

$$\begin{aligned}\widetilde{l}_+ &= (U_2)_{ll'} l'_+, \\ \widetilde{l}_- &= (U_1)_{ll'} l'_- \\ \widetilde{\nu}_{l-} &= (U_1)_{ll'} \nu_{l'-}\end{aligned} \tag{5.5.21}$$

(we have made use of our freedom to define $\widetilde{\nu}_{l-}$ as we please) and re-express the rest of the lepton Lagrangian in (5.5.9) in terms of them as well. We see that this transformation does not affect the other terms (this is easiest to see through (5.5.8)) but has the effect of putting the Yukawa terms in the diagonal form

$$\sum_{l,l'} (v+H) G_l \overline{\widetilde{l}}_- \widetilde{l}_+ + \text{h.c.} \tag{5.5.22}$$

where G_l are the eigenvalues of $G_{ll'}$. The mass eigenstates, $\widetilde{l}$ and $\widetilde{\nu}_l$, are the physical, propagating states of the theory.

The situation is somewhat different for quarks. Both up and down quarks are known to be massive, which brings in some added considerations that we will consider below. We will have to deal with the same issues for standard model extensions with massive neutrinos.

5.5.3 Quark Masses

As for leptons, we could write a gauge invariant Yukawa coupling of quarks as follows:

$$\sum_{u,d} G_d \overline{Q}^{\alpha}_{i-} \Phi_\alpha d_{i+} + \text{h.c.}, \tag{5.5.23}$$

where we have explicitly intoduced the flavor index "α". This would give coupling terms for the down quark

$$\sum_d G_d d_{i-} (v+H) d_{i+} + \text{h.c.} \tag{5.5.24}$$

but *not* for the up quarks. To get a coupling term for the up quarks we need

$$\sum_{u,d} F_u \epsilon_{\alpha\beta} \overline{Q}^{\alpha}_{i-} \Phi^{\beta} u_{i+} + \text{h.c.}, \tag{5.5.25}$$

where (α, β) label the quark generations and $\epsilon_{\alpha\beta}$ is the Levi Civita tensor in two dimensions. This term is also gauge invariant and expands to

$$\sum_u F_u \overline{u}_{i-} (v+H) u_{i+} + \text{h.c.} \tag{5.5.26}$$

Both Yukawa couplings therefore give rise to couplings between the quarks and the Higgs with strengths that are proportional to their masses, since

$$F_u = \frac{g m_u \hbar}{2M_W}, \quad G_d = \frac{g m_d \hbar}{2M_W} \tag{5.5.27}$$

as in the case of the leptons.

Generalizing as before, nothing prevents us from writing the two terms

$$\sum_{u,u'} F_{uu'} \overline{u}_{i-}(v+H) u'_{i+} + \text{h.c.}, \quad \sum_{d,d'} G_{dd'} \overline{d}_{i-}(v+H) d'_{i+} + \text{h.c.} \tag{5.5.28}$$

for arbitrary complex matrices $\widehat{F}$ and $\widehat{G}$. Again, each could be diagonalized by two unitary matrices

$$\widehat{F} = \widehat{U}_1^{F\dagger}\ \widehat{F}_D\ \widehat{U}_2^F, \quad \widehat{G} = \widehat{U}_1^{G\dagger}\ \widehat{G}_D\ \widehat{U}_2^G \tag{5.5.29}$$

where $\widehat{F}_D$ and $\widehat{G}_D$ are diagonal. Therefore there are now *four* unitary matrices required to diagonalize the mass matrices. Suppose we define

$$\begin{aligned} \widetilde{u}_- &= (U_1^F)_{uu'} u'_- \\ \widetilde{u}_+ &= (U_2^F)_{uu'} u'_+ \\ \widetilde{d}_- &= (U_1^G)_{dd'} d'_- \\ \widetilde{d}_+ &= (U_2^G)_{dd'} d'_+ \end{aligned} \tag{5.5.30}$$

then we will end up with diagonal Yukawa terms

$$\begin{aligned} &\sum_u F_u \overline{\widetilde{u}}_{i-}(v+H)\widetilde{u}_{i+} + \text{h.c.} \\ &\sum_u G_d \overline{\widetilde{d}}_{i-}(v+H)\widetilde{d}_{i+} + \text{h.c.} \end{aligned} \tag{5.5.31}$$

for the quarks. However, if we now go back to the quark kinetic and interaction terms in (5.5.12) and replace all the up and down quarks, $(u_{i\pm}, d_{i\pm})$, by $(\widetilde{u}_{i\pm}, \widetilde{d}_{i\pm})$, then we find that the terms permitting transitions between the (left handed) up and down quarks via the emission of a charged, weak vector boson become

$$-ig\left\{\overline{\widetilde{d}}_-(\widehat{U}_1^G \widehat{U}_1^{F\dagger}) \not{W}^+ \widetilde{u}_- + \overline{\widetilde{u}}_- \not{W}^- (\widehat{U}_1^F \widehat{U}_1^{G\dagger}) \widetilde{d}_-\right\} \tag{5.5.32}$$

while all the other terms remain form invariant. The unitary matrix

$$\widehat{V}_{\text{CKM}} = \widehat{U}_1^F \widehat{U}_1^{G\dagger} \tag{5.5.33}$$

is called the **Cabbibo-Kobayashi-Maskawa** (or CKM) mixing matrix. Since the physical, propagating states are always mass eigenstates, it says that either (i) the weak eigenstates of the down quarks should be considered as mixtures of the mass eigenstates; explicitly,

$$\begin{pmatrix} d \\ s \\ b \end{pmatrix} = \widehat{V}_{\rm CKM} \begin{pmatrix} \widetilde{d} \\ \widetilde{s} \\ \widetilde{b} \end{pmatrix} \tag{5.5.34}$$

taking the weak eigenstates of the up quarks to be identical to their mass eigenstates, or (ii) the weak eigenstates of the up quarks should be considered as mixtures of the mass eigenstates; explicitly,

$$\begin{pmatrix} u \\ c \\ t \end{pmatrix} = \widehat{V}^\dagger_{\rm CKM} \begin{pmatrix} \widetilde{u} \\ \widetilde{c} \\ \widetilde{t} \end{pmatrix} \tag{5.5.35}$$

taking the weak eigenstates of the down quarks to be identical to their mass eigenstates. Either choice (our's will be the first) is merely a convention. This mixing will only play a role in the terms

$$-\frac{ig}{2}\left\{\overline{\widetilde{d}}\,\widehat{V}^\dagger_{\rm CKM}\not{W}^{+}(1-c\gamma^5)\widetilde{u} + \overline{\widetilde{u}}\,\not{W}^{-}(1-c\gamma^5)\widehat{V}_{\rm CKM}\widetilde{d}\right\} \tag{5.5.36}$$

but the effect is dramatic: up quarks of one generation can now have transitions with down quarks of a different generation by emission or absorption of a charged vector boson, whereas no such inter-generational transitions can occur via emission or absorption of a neutral vector boson because the CKM matrix is unitary. In the case of leptons, of course, neutrinos will *always* interact only with their partner (an electron neutrino with an electron, etc.) so long as the neutrino is massless.

5.6 The CKM Matrix

Consider the general case of n quark generations, because it is somewhat instructive (the standard model has $n = 3$). The CKM matrix is therefore an $n \times n$ unitary matrix and will be completely characterized by n^2 real numbers if we count the real and imaginary parts of a complex element as two real numbers. How many of these real numbers are phases? We know that a *real* unitary matrix would have $n(n-1)/2$ real parameters because this is the number of parameters of $SO(n)$ given that there are $n(n-1)/2$ independent rotations in n dimensions. It follows that

$$n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

of them are phases. Not all these phases may be observable because there is always a possibility of absorbing some of them by a field redefinition.

Now with n generations there are $2n$ quarks, so suppose we re-phase the quarks by letting

$$u_{\alpha\pm} \to e^{i\phi_\alpha} u_{\alpha\pm}, \quad d_\alpha \to e^{i\theta_\alpha} d_{\alpha\pm} \tag{5.6.1}$$

where ϕ_α and θ_α are completely arbitrary. The mass terms are clearly invariant under this transformation, since both left and right handed quarks are rotated by the same phase. All the other terms, except the one in which the CKM mixing occurs, will also be invariant. In (5.5.36), the net effect of this is to transform $\widehat{V}_{\text{CKM}}$ according to

$$V^{\text{CKM}}_{\alpha\beta} \to e^{i(\theta_\beta - \phi_\alpha)} V^{\text{CKM}}_{\alpha\beta} \tag{5.6.2}$$

A judicious re-phasing of the quarks should therefore be capable of absorbing some of the phases of the CKM matrix. The question is: how many phases of the CKM matrix can be absorbed in this way? Now, although there are $2n$ possible angles, $(\phi_\alpha, \theta_\alpha)$, corresponding to $2n$ quarks, only $2n-1$ of the differences will be independent and can actually be used to eliminate phases in the CKM matrix.[5] This leaves

$$\frac{n(n+1)}{2} - (2n-1) = \frac{(n-1)(n-2)}{2}$$

phases in $\widehat{V}_{\text{CKM}}$ that cannot be absorbed by a re-phasing. For three generations, the CKM matrix will therefore have three real parameters and one phase. If there were just two generations (as was originally believed) there would be one real parameter and *all* the phases of the CKM matrix could have been absorbed by re-phasing the quarks. Complex phases in the CKM matrix lead to processes that violate CP. Thus, if there were two quark generations, the standard model would exhibit no CP violation.[6]

It remains to parameterize the CKM matrix. Naturally, there are many possibilities but we will use the one recommended by the Particle Data Group:

$$\widehat{V}_{\text{CKM}} = \widehat{R}^1(\theta_1) \cdot \widehat{U}^2(\theta_2, \delta) \cdot \widehat{R}^3(\theta_3) \tag{5.6.3}$$

where

$$\widehat{R}^1(\theta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

---

[5] Problem: Given two sets of n unknown numbers, $\{x_i\}$ and $\{y_j\}$ form the differences $x_i - y_j = a_{ij}$ where the differences, a_{ij}, are assumed given. Show that only $2n-1$ of the difference equations are independent. Then show that the other equations are consistency conditions on the differences a_{ij}.

[6] The third generation of quarks was in fact predicted by Kobayashi and Maskawa (in 1973) to explain the observed CP violation as the neutral K meson ($K^0 = (d, \bar{s})$) decays to its antiparticle, $\overline{K}^0 = (\bar{d}, s)$. It is currently being studied for neutral B mesons ($B^0 = (d, \bar{b})$).

$$\begin{aligned}\widehat{U}^2(\theta_2,\delta) &= \begin{pmatrix}\cos\theta_2 & 0 & \sin\theta_2 e^{-i\delta}\\ 0 & 1 & 0\\ -\sin\theta_2 e^{i\delta} & 0 & \cos\theta_2\end{pmatrix}\\ \widehat{R}^3(\theta_3) &= \begin{pmatrix}\cos\theta_3 & \sin\theta_3 & 0\\ -\sin\theta_3 & \cos\theta_3 & 0\\ 0 & 0 & 1\end{pmatrix}\end{aligned} \tag{5.6.4}$$

This gives

$$\widehat{V}_{\text{CKM}} = \begin{pmatrix} c_2c_3 & c_2s_3 & s_2e^{-i\delta}\\ -c_1s_3 - s_1s_2c_3e^{i\delta} & c_1c_3 - s_1s_2s_3e^{i\delta} & s_1c_2\\ s_1s_3 - s_2c_1c_3e^{i\delta} & -s_1c_3 - c_1s_2s_3e^{i\delta} & c_1c_2\end{pmatrix} \tag{5.6.5}$$

where $s_i = \sin\theta_i$ and $c_i = \cos\theta_i$.

Chapter 6

More general coordinate systems

6.1 Introduction

There are many examples of physical systems whose symmetries may sometimes make their mathematical description much more difficult in Cartesian coordinates. Imagine, for example, how difficult the problem of determining the gravitational field of a spherical matter distribution would be in Cartesian coordinates. In such situations we search for a different set of coordinates, selecting one that is best adapted to the symmetries (for example it is much easier to find the gravitational field of a spherical mass distribution in spherical coordinates). A judicious choice of coordinates can not only lead to a mathematically simpler description of the problem but also to a more transparent one. Similarly, alternate coordinate systems may be advantageous if the physical system is constrained, or if it turns out to be difficult to implement the appropriate boundary conditions in Cartesian coordinates. If the system is constrained we first attempt to solve the constraints in parametric form and the parameters then turn into a new set of coordinates. The new coordinates are not usually Cartesian (for example, think of the problem of determining geodesics on a sphere). Yet again, if the boundary conditions are not rectangular, we turn to coordinate systems that incorporate the symmetries of the boundary. A generic feature of such systems – and one that is exploited in the physical problem – is that one or more of the coordinate surfaces are curved. They are therefore called "curvilinear" systems.

The use of curvilinear coordinate systems becomes all the more relevant when the effects of gravity cannot be ignored. According to Einstein's theory of general relativity, the gravitational field is properly described by the curvature of space-time. Space-time becomes a "dynamical manifold", whose curvature is determined by the matter distribution and determines, in turn, how matter moves.[1] In contrast to the examples discussed

[1]Quote: "Space-time tells matter how to move; matter tells space-time how to curve", in "Geons, Black Holes, and Quantum Foam: A Life in Physics" by K.W. Ford and J.A. Wheeler.

in the previous paragraph it is then impossible to introduce a single Cartesian system or, often, even any single curvilinear system, in all of space-time.

If the space-time is everywhere Minkowski then it is possible to introduce a global system of Cartesian coordinates, x^a, $a \in \{0, 1, 2, 3\}$, and a corresponding set of four constant, linearly independent vectors, or **tetrad** $\{\vec{u}_{(a)}\}$,[2] such that an infinitesimal displacement is given by

$$d\vec{s} = \vec{u}_{(a)} dx^a, \quad ds^2 = d\vec{s} \cdot d\vec{s} = (\vec{u}_{(a)} \cdot \vec{u}_{(b)}) dx^a dx^b = -\eta_{ab} dx^a dx^b. \tag{6.1.1}$$

To get a handle on curved space-times, we assume that every curved space-time is "locally flat", resembling Minkowski space in small enough neighborhoods of each point. To be more precise, by "locally flat" is meant that in some neighborhood of every point, P, it is possible to find local coordinates, x^a, and a tetrad, $\{\vec{u}_{(a)}\}$, so that an infinitesimal displacement within that neighborhood can be represented by

$$d\vec{s} = \vec{u}_{(a)} dx^a, \quad ds^2 = d\vec{s} \cdot d\vec{s} = (\vec{u}_{(a)} \cdot \vec{u}_{(b)}) dx^a dx^b = -g^{(0)}_{ab}(P) dx^a dx^b, \tag{6.1.2}$$

where

$$\begin{aligned} g^{(0)}_{ab}(P) &= \eta_{ab} \\ \frac{\partial}{\partial x^c} g^{(0)}_{ab}(P) &= 0 \end{aligned} \tag{6.1.3}$$

and η_{ab} is the ordinary Minkowski metric. Higher order derivatives of $g^{(0)}_{ab}$ are not required to vanish. Coordinates that satisfy the conditions above are called **Riemann Normal** coordinates and the tetrad $\{\vec{u}_{(a)}\}$ is called a **Local Lorentz frame** (LLF). The normal coordinates and the LLFs in two neighborhoods are not required to agree even in their overlap, but the existence of a smooth (*i.e.,* differentiable as many times as desired), local Lorentz transformation from one to the other in the overlap is assumed. Physically, the LLFs correspond to families of "freely falling" observers (therefore they are sometimes called the Local *Inertial* frames, LIFs). The unit tangent vectors to the worldlines of these freely falling observers provides a time-like vector for the LLF and at each point one may define three orthogonal spacelike vectors to complete that frame.

Whether we are dealing with non-rectangular symmetries of a physical system, complicated boundary conditions, constrained systems or curved space-times, the idea is to introduce a new set of "curvilinear" coordinates, ξ^μ, that cover some finite region of space-time.[3] We may then define another tetrad $\{\vec{u}_{(\mu)}\}$, called the **coordinate frame**, so that

$$d\vec{s} = \vec{u}_{(\mu)} d\xi^\mu, \quad ds^2 = d\vec{s} \cdot d\vec{s} = (\vec{u}_{(\mu)} \cdot \vec{u}_{(\nu)}) d\xi^\mu d\xi^\nu \stackrel{\text{def}}{=} -g_{\mu\nu}(\xi) d\xi^\mu d\xi^\nu \tag{6.1.4}$$

[2]We assume a four dimensional space-time, but all the results of this chapter can be easily generalized to any dimension.

[3]Henceforth we'll use the following notation: indices from the beginning of the alphabet, $a, b, c, ...$, will represent a LLF and greek indices $\mu, \nu, ...$ will represent a general (curvilinear) system.

In general several coordinate systems may be necessary to cover the entire space-time; we then require the coordinate functions to transform smoothly into one another in the overlap.

What will be relevant in the following discussion is that it is *always* possible to find (at least) one LLF in some neighborhood of every point. Here we will never actually need the LLF itself, nor will we construct it explicitly, although this is sometimes useful. Our aim will be to develop some general techniques to describe physics in curvilinear coordinate systems. Later in this chapter we will address the problem of distinguishing quantities that represent the actual dynamics of a "curved" space-time from quantities that appear merely because we have chosen to use curvilinear coordinates.

6.2 "Flat" Space-time

By "flat" we will mean that it is possible to introduce a single Lorentz frame, with coordinates x^a and Lorentz metric η^{ab}, everywhere in space-time, *i.e.,* a *global* Lorentz frame. Suppose we perform a coordinate transformation from the coordinates x^a to a set of curvilinear coordinates, ξ^μ. We will take the new coordinates to be smooth, *i.e.,* differentiable as many times as we like, invertible functions of the x^a, so that we are given $\xi^\mu = \xi^\mu(x)$ and $x^a = x^a(\xi)$ (think of the transformations leading to spherical or cylindrical coordinates). In the Minkowski system, the invariant distance between two infinitesimally separated points is given by the Lorentz metric

$$ds^2 = -\eta_{ab} dx^a dx^b, \tag{6.2.1}$$

but, using the coordinate transformations, we can relate the differential of x^a to the differential of ξ^μ by the chain rule,

$$dx^a = \frac{\partial x^a}{\partial \xi^\mu} d\xi^\mu, \tag{6.2.2}$$

and express the distance in terms of the new coordinates, $\xi^\mu(x)$, as follows

$$ds^2 = -\eta_{ab} dx^a dx^b = -\left(\eta_{ab} \frac{\partial x^a}{\partial \xi^\mu} \frac{\partial x^b}{\partial \xi^\nu}\right) d\xi^\mu d\xi^\nu = -g_{\mu\nu} d\xi^\mu d\xi^\nu, \tag{6.2.3}$$

where

$$g_{\mu\nu} = \eta_{ab} \frac{\partial x^a}{\partial \xi^\mu} \frac{\partial x^b}{\partial \xi^\nu} \tag{6.2.4}$$

now gives the distance between infinitesimally separated points labeled by the new curvilinear coordinates and plays the role of a metric. In contrast with the metric η_{ab}, the symmetric matrix $g_{\mu\nu}$ is not constant. As a symmetric matrix it can be expected to have ten component functions. However, because our Minkowski system x^a is global, $g_{\mu\nu}$ is determined by just the four coordinate functions, $\xi^\mu(x)$, and so the ten components are not all independent. This simplification comes about only because we started with a global Lorentz frame.

If we define the matrix

$$e^a_\mu(\xi) = \frac{\partial x^a}{\partial \xi^\mu} \tag{6.2.5}$$

then we have

$$dx^a = e^a_\mu d\xi^\mu, \quad \frac{\partial}{\partial \xi^\mu} = e^a_\mu \frac{\partial}{\partial x^a} \tag{6.2.6}$$

by the chain rule. The mixed index object e^a_μ is called the **vierbein** ("four legs") and, like the metric, it is generally a function of position.

Let $\{\vec{u}_{(a)}\}$ be a tetrad representing the directions of a global Lorentz frame, say $u_{(0)} = (1,0,0,0)$, $u_{(1)} = (0,1,0,0)$, $u_{(2)} = (0,0,1,0)$, $u_{(3)} = (0,0,0,1)$. Then

$$\vec{e}_{(\mu)} = u_{(a)} e^a_\mu \tag{6.2.7}$$

will represent the coordinate frame since an infinitesimal displacement will have the form

$$d\vec{s} = \vec{u}_{(a)} dx^a = (\vec{u}_{(a)} e^a_\mu) d\xi^\mu = \vec{e}_{(\mu)} d\xi^\mu. \tag{6.2.8}$$

The metric in (6.2.4) is then the matrix whose components are the inner products of the $\vec{e}_{(\mu)}$, *i.e.*,[4]

$$g_{\mu\nu}(\xi) = \eta_{ab} e^a_\mu e^b_\nu = (\vec{u}_{(a)} \cdot \vec{u}_{(b)}) e^a_\mu e^b_\nu = \vec{e}_{(\mu)} \cdot \vec{e}_{(\nu)} \tag{6.2.9}$$

and it encodes the vierbein. It is manifestly a scalar under Lorentz transformations and is invertible as a matrix if the transformation $x \leftrightarrow \xi$ is invertible. In this case we can define the inverse metric, $g^{\mu\nu}$, by the condition $g^{\mu\nu} g_{\nu\kappa} = \delta^\mu_\kappa$ and find

$$g^{\mu\nu} = \eta^{ab} \frac{\partial \xi^\mu}{\partial x^a} \frac{\partial \xi^\nu}{\partial x^b} \stackrel{\text{def}}{=} \eta^{ab} E^\mu_a E^\nu_b \tag{6.2.10}$$

where

$$E^\mu_a = \frac{\partial \xi^\mu}{\partial x^a}. \tag{6.2.11}$$

We will then have relations analogous to (6.2.6),

$$d\xi^\mu = E^\mu_a dx^a, \quad \frac{\partial}{\partial x^a} = E^\mu_a \frac{\partial}{\partial \xi^\mu}, \tag{6.2.12}$$

and so E^μ_a transforms the coordinate frame into the the Lorentz frame according to

$$\vec{u}_{(a)} = E^\mu_a \vec{e}_{(\mu)}. \tag{6.2.13}$$

[4] Problem: Check that the identity transformation leads to $g_{\mu\nu} = \eta_{\mu\nu}$

E_a^μ is called the *inverse* vierbein because of the orthonormality relations

$$e_\mu^a E_b^\mu = \frac{\partial x^a}{\partial \xi^\mu}\frac{\partial \xi^\mu}{\partial x^b} = \delta_b^a, \quad \text{and} \quad e_\mu^a E_a^\nu = \frac{\partial x^a}{\partial \xi^\mu}\frac{\partial \xi^\nu}{\partial x^a} = \delta_\mu^\nu, \tag{6.2.14}$$

which follow by repeatedly using the chain rule.

Transformations of $\{\vec{u}_{(a)}\}$ must preserve the Lorentz metric and are therefore Lorentz transformations,

$$\vec{u}'_{(a)} = \frac{\partial x^b}{\partial x'^a}\vec{u}_{(b)} = \vec{u}_{(b)}(L^{-1})^b{}_a. \tag{6.2.15}$$

On the other hand, if we consider a transformation from one set of curvilinear coordinates, ξ, to another set, ξ', which are invertible functions of ξ, then

$$\vec{e}'_{(\mu)} = \frac{\partial \xi^\lambda}{\partial \xi'^\mu}\vec{e}_{(\lambda)} = \vec{e}_{(\lambda)}(\Lambda^{-1})^\lambda{}_\mu. \tag{6.2.16}$$

The vierbein and its inverse have "mixed" indices, therefore:

- e_μ^a transforms as a contravariant vector under Lorentz transformations,

$$e'^a_\mu = L^a{}_b e^b_\mu \tag{6.2.17}$$

  whereas E_a^μ will transform as a covariant vector

$$E'^\mu_a = E^\mu_b (L^{-1})^b{}_a \tag{6.2.18}$$

  under the same transformations and

- under general coordinate transformations, $\xi^\mu \to \xi'^\mu$, the vierbein will transform as

$$e'^a_\mu = e^a_\nu (\Lambda^{-1})^\nu{}_\mu \tag{6.2.19}$$

  and the inverse as

$$E'^\mu_a = \Lambda^\mu{}_\nu E^\nu_a. \tag{6.2.20}$$

(Remember that $\hat{\Lambda}$, unlike $\hat{L}$, is not necessarily a constant matrix. These transformation properties imply that the metric in (6.2.4) transforms (under coordinate transformations) as

$$g'_{\mu\nu} = g_{\alpha\beta}(\Lambda^{-1})^\alpha{}_\mu (\Lambda^{-1})^\beta{}_\nu \tag{6.2.21}$$

and

$$g'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g^{\alpha\beta} \tag{6.2.22}$$

but is a Lorentz *scalar*.

6.3 "Curved" Space-time

When no global Lorentz frame exists, we take $\{\vec{u}_{(a)}\}$ to be the tetrad defined by a Local Lorentz Frame at each point, p, on the space-time. We can no longer take it to be global, as we did in the previous secition, but all that we have said before carries over in infinitesimal neighborhoods of points if one simply *defines* the vierbein to be the (invertible) matrix that transforms the LLF into the coordinate frame, $\{\vec{e}_{(\mu)}\}$, according to (6.2.7). Infinitesimal displacements are still given by (6.2.8) and the metric in the LLF, which is

$$d\vec{s} \cdot d\vec{s} = (\vec{u}_{(a)} \cdot \vec{u}_{(b)}) dx^a dx^b = -\eta_{ab} dx^a dx^b, \tag{6.3.1}$$

transforms, in our curvilinear system, to

$$d\vec{s} \cdot d\vec{s} = (\vec{e}_{(\mu)} \cdot \vec{e}_{(\nu)}) d\xi^\mu d\xi^\nu = (-\eta_{ab} e^a_\mu e^b_\nu) d\xi^\mu d\xi^\nu = -g_{\mu\nu} d\xi^\mu d\xi^\nu \tag{6.3.2}$$

i.e., $g_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu$ as before. The metric continues to encode the vierbein. The difference in this case is that, because a single tetrad frame is no longer defined throughout the space-time, there are six degrees of freedom in choosing it *at every point*. These are, of course, the three rotations and three boosts, all of which keep the Lorentz metric invariant. Add this to the freedom to choose four coordinate functions and we find that all ten of the metric components are now independent.

6.4 Vectors and Tensors

Whether or not a global Lorentz frame exists, we will always have two frames, *viz.,* the LLF and the coordinate frame at every point p. Any vector $\vec{A}$ could be expanded in the tetrad frame

$$\vec{A} = A^a \vec{u}_{(a)}, \tag{6.4.1}$$

where A^a are its contravariant components in this frame. We will refer to them as the LLF components of $\vec{A}$. Of course, $\vec{A}$ itself is independent of the basis in which we choose to express it and we could use the coordinate basis at each point instead. Then

$$\vec{A} = A^\mu \vec{e}_{(\mu)}, \tag{6.4.2}$$

where A^μ are its contravariant components in the coordinate frame. We will call these its coordinate components. Thus $\vec{A}$ can be given by specifying either the components A^a or the components A^μ. Using the vierbein to transform from one frame to the other, we find relations between the contravariant components,

$$A^a = e^a_\mu A^\mu \tag{6.4.3}$$

One could also introduce the dual vector space and a dual basis in the usual way: $\vec{\theta}^{(a)}$ and $\vec{E}^{(\mu)} = E^\mu_a \vec{\theta}^{(a)}$ satisfying

$$\theta^{(b)}(\vec{u}_{(a)}) = \theta^{(b)} \cdot \vec{u}_{(a)} = \delta^b_a, \quad \vec{E}^{(\nu)}(\vec{e}_{(\mu)}) = \vec{E}^{(\nu)} \cdot \vec{e}_{(\mu)} = \delta^\nu_\mu, \tag{6.4.4}$$

and express any dual vector as

$$\vec{\omega} = \omega_\mu \vec{E}^{(\mu)} = \omega_a \vec{\theta}^{(a)}. \tag{6.4.5}$$

where ω_μ are its covariant components. Then there are relations between the covariant components of dual vectors as well,

$$\omega_a = \omega_\mu E^\mu_a, \tag{6.4.6}$$

in complete correspondence with the relations for contravariant vectors.

It should be clear that the LLF components, A^a (A_a), of any vector $\vec{A}$ transform under Lorentz transformations according to the usual rules but they do not transform under coordinate transformations. Likewise, the components A^μ (A_μ) will transform under general coordinate transformations but not under Lorentz transformations. The latter transform under the same rules as the LLF components, but with $\hat{\Lambda}$ instead of $\hat{L}$. As a vector should not depend on the basis in which it is expanded,

$$\vec{A} = A'^\mu \; \vec{e}'_{(\mu)} = A'^\mu (\Lambda^{-1})^\nu{}_\mu \vec{e}_{(\nu)} = A^\nu \vec{e}_{(\nu)} \tag{6.4.7}$$

implying obviously that

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu. \tag{6.4.8}$$

A completely analogous argument shows that

$$A'_\mu = A_\nu (\Lambda^{-1})^\nu{}_\mu \tag{6.4.9}$$

is the transformation property of the covariant components. The contravariant components and the covariant components transform inversely to one another, so they must be related by the metric

$$\begin{aligned} A_\mu &= g_{\mu\nu} A^\nu \\ A^\mu &= g^{\mu\nu} A_\nu \end{aligned} \tag{6.4.10}$$

because the metric and its inverse have precisely the required transformation properties.

Knowing how the components of a vector transform, we now wish to know now how to construct scalars under general coordinate transformations. Given two vectors $\vec{A}$ and $\vec{B}$, we know that

$$-\vec{A} \cdot \vec{B} = -\eta_{ab} A^a B^b \tag{6.4.11}$$

is a scalar. Rewrite the above using (6.4.6),

$$\vec{A}\cdot\vec{B} = (A^\mu \vec{e}_\mu)\cdot(B^\nu \vec{e}_\nu) = A^\mu B^\nu \vec{e}_\mu \cdot \vec{e}_\nu = g_{\mu\nu} A^\mu B^\nu = A^\mu B_\mu. \tag{6.4.12}$$

That this is a scalar also follows directly from

$$A'^\mu B'_\mu = \Lambda^\mu{}_\alpha A^\alpha B_\beta (\Lambda^{-1})^\beta{}_\mu = \delta^\beta_\alpha A^\alpha B_\beta = A^\alpha B_\alpha. \tag{6.4.13}$$

As usual we will define tensors as copies of vectors, their components in any basis being given by

$$\mathbb{T} = T^{\mu\nu\lambda\ldots} \vec{e}_{(\mu)} \otimes \vec{e}_{(\nu)} \otimes \vec{e}_{(\lambda)} \ldots = T_{\mu\nu\lambda\ldots} \vec{E}^{(\mu)} \otimes \vec{E}^{(\nu)} \otimes \vec{E}^{(\lambda)} \ldots \tag{6.4.14}$$

where $T^{\mu\nu\lambda\ldots}$ and $T_{\mu\nu\lambda\ldots}$ are the contravariant and covariant components of $\mathbb{T}$ respectively. We could also consider "mixed" components,

$$\mathbb{T} = T^{\mu\nu\ldots}{}_{\alpha\beta\ldots}\ \vec{e}_{(\mu)} \otimes \vec{e}_{(\nu)} \ldots \otimes \vec{E}^{(\alpha)} \otimes \vec{E}^{(\beta)} \ldots \tag{6.4.15}$$

Components with m contravariant and n covariant indices are said to be of rank (m,n). The transformation properties of contravariant and covariant tensor components are given by

$$T'^{\mu\nu\lambda\ldots} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\lambda{}_\gamma T^{\alpha\beta\gamma\ldots} \tag{6.4.16}$$

and

$$T'_{\mu\nu\lambda\ldots} = T_{\alpha\beta\gamma\ldots} (\Lambda^{-1})^\alpha{}_\mu (\Lambda^{-1})^\beta{}_\nu (\Lambda^{-1})^\gamma{}_\lambda \tag{6.4.17}$$

respectively. Just as for vectors, the covariant and contravariant components of a tensor are related by the metric (tensor):

$$T^{\mu\nu\lambda\ldots} = g^{\mu\alpha} g^{\nu\beta} g^{\lambda\gamma} \ldots\ T_{\alpha\beta\gamma\ldots} \tag{6.4.18}$$

and

$$T_{\mu\nu\lambda\ldots} = g_{\mu\alpha} g_{\nu\beta} g_{\lambda\gamma} \ldots\ T^{\alpha\beta\gamma\ldots} \tag{6.4.19}$$

and one can interpolate between components in the LLF frame and in the coordinate frame by simply applying the vierbein and its inverse, just as we did for vectors

$$\begin{aligned} T^{ab\ldots} &= e^a_\mu e^b_\nu \ldots\ T^{\mu\nu\ldots}, \qquad T^{\mu\nu\ldots} = E^\mu_a E^\nu_b \ldots\ T^{ab\ldots} \\ T_{ab\ldots} &= E^\mu_a E^\nu_b \ldots\ T_{\mu\nu\ldots}, \qquad T_{\mu\nu\ldots} = e^a_\mu e^b_\nu \ldots\ T_{ab\ldots} \end{aligned} \tag{6.4.20}$$

6.5 Differentiation

The derivative of a tensor is supposed to quantify its instantaneous rate of change as we move from point to point on the manifold. In a global Cartesian basis, this is done by measuring the difference between its components at infinitesimally close points, p and p', and then dividing the change by the separation of the points, finally taking the limit as the separation approaches zero. This simple procedure will fail in a general coordinate system because the basis vectors at the two different points, p and p', differ from each other. This means that there is no direct way to compare the components of a vector or tensor at two points. A naïve application of the procedure employed in a Cartesian basis will result in a rate of change that does not have definite transformation properties under general coordinate transformations. This is unsuitable for applications in physics because of the principle of general covariance. Below we will consider two ways to define the "derivative" of a tensor so that the derivative is itself a tensor.

6.5.1 Lie Derivative and Lie Transport

The first derivative we address concerns the change in the components of a tensor field along the flow defined by a vector field.

Consider a one parameter family of coordinate transformations $\xi'^\mu(\lambda,\xi)$ which are such that the $\lambda = 0$ transformation is just the identity, $\xi'(0,\xi) = \xi$. Let the coordinates of point p be ξ_p^μ. Holding ξ_p fixed, $\xi'^\mu(\lambda,\xi_p)$ represents a curve passing through p at $\lambda = 0$. Suppose that we have chosen our one parameter family of transformations so that the curve $\xi'^\mu(\lambda,\xi_p)$ passes through p' at $\delta\lambda$. Let $U^\mu(\lambda,\xi_p)$ be tangent to the curve, $U^\mu(0,\xi_p) = U^\mu(\xi_p)$ is tangent to the curve at p. The (infinitesimally separated) point p' is therefore

$$\xi'^\mu = \xi'^\mu(\delta\lambda,\xi_p) = \xi_p^\mu + \delta\lambda U^\mu(\xi_p) \tag{6.5.1}$$

This is the "active" view of coordinate transformations, where they are used to "push" points along the integral curves of a vector field.

We will be interested in the functional change in the components of a tensor field, $\mathbb{T}$, induced by the coordinate transformation, *i.e.,* we want to compare the original components $T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots}(\xi)$ of $\mathbb{T}$ with the transformed components, $T'^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots}(\xi')$ *at the same point, i.e.,* at the same numerical value of the coordinates, when $\xi = \xi'$. Equivalently, we could compare $T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots}(\xi)$ with $T'^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots}(\xi)$. Therefore, define the Lie derivative of $\mathbb{T}$ along the vector field U in two equivalent ways:

$$[\pounds_U\mathbb{T}]^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots} = \lim_{\delta\lambda\to 0}\frac{1}{\delta\lambda}\left[T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots}(\xi') - T'^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots}(\xi')\right]. \tag{6.5.2}$$

or as

$$[\pounds_U\mathbb{T}]^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots} = \lim_{\delta\lambda\to 0}\frac{1}{\delta\lambda}\left[T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots}(\xi) - T'^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots}(\xi)\right], \tag{6.5.3}$$

It measures the rate of change of the *functional form* of the components of a tensor field along the integral curves of U.

For scalar functions, we see immediately that this is just the directional derivative, for if $\mathbb{T}$ is a scalar function, $f(\xi)$, then

$$f(\xi') - f'(\xi') = f(\xi') - f(\xi) = \delta\lambda U^\mu \partial_\mu f \tag{6.5.4}$$

(to order $\delta\lambda$) because $f'(\xi') = f(\xi)$, therefore

$$\pounds_U f(x) = \lim_{\delta\lambda\to 0} \frac{1}{\delta\lambda}\left[f(\xi) - f'(\xi)\right] = U^\mu \partial_\mu f(\xi). \tag{6.5.5}$$

If $\mathbb{T}$ is a vector field, $V^\mu(\xi)$, then

$$\begin{aligned}
[\pounds_U V]^\mu &= \lim_{\delta\lambda\to 0} \frac{1}{\delta\lambda}\left[V^\mu(\xi') - V'^\mu(\xi')\right] \\
&= \lim_{\delta\lambda\to 0} \frac{1}{\delta\lambda}\left[V^\mu(\xi) + \delta\lambda U^\kappa \partial_\kappa V^\mu(\xi) - \frac{\partial \xi'^\mu}{\partial \xi^\kappa} V^\kappa(\xi)\right] \\
&= \lim_{\delta\lambda\to 0} \frac{1}{\delta\lambda}\left[V^\mu(\xi) + \delta\lambda U^\kappa \partial_\kappa V^\mu(\xi) - (\delta^\mu_\kappa + \delta\lambda \partial_\kappa U^\mu) V^\kappa(\xi)\right] \\
&= U^\kappa \partial_\kappa V^\mu - V^\kappa \partial_\kappa U^\mu \equiv -[\pounds_V U]^\mu
\end{aligned} \tag{6.5.6}$$

and if $\mathbb{T}$ is a co-vector field, $W_\mu(\xi)$

$$\begin{aligned}
[\pounds_U W]_\mu &= \lim_{\delta\lambda\to 0} \frac{1}{\delta\lambda}\left[W_\mu(\xi') - W'_\mu(\xi')\right] \\
&= \lim_{\delta\lambda\to 0} \frac{1}{\delta\lambda}\left[W_\mu(\xi) + \delta\lambda U^\kappa \partial_\kappa W_\mu - \frac{\partial \xi^\kappa}{\partial \xi'^\mu} W_\kappa(\xi)\right] \\
&= \lim_{\delta\lambda\to 0} \frac{1}{\delta\lambda}\left[W_\mu(\xi) + \delta\lambda U^\kappa \partial_\kappa W_\mu - (\delta^\kappa_\mu - \delta\lambda \partial_\mu U^\kappa) W_\kappa(\xi)\right] \\
&= U^\kappa \partial_\kappa W_\mu + W_\kappa \partial_\mu U^\kappa
\end{aligned} \tag{6.5.7}$$

and so on for tensors of higher rank.[5] If $\pounds_U \mathbb{T} = 0$, then $\mathbb{T}$ does not change its functional form as we move along the integral curves of U. In this case, the vector field U is called a

[5] Obtain the Lie derivative of second rank contravariant, covariant and mixed tensors. In general, the Lie derivative of a mixed tensor takes the form

$$\begin{aligned}
[\pounds_U \mathbb{T}]^{\mu_1\mu_2\ldots}{}_{\nu_1\nu_2\ldots} &= U^\sigma \partial_\sigma T^{\mu_1\mu_2\ldots}{}_{\nu_1\nu_2\ldots} - T^{\sigma\mu_2\ldots}{}_{\nu_1\nu_2\ldots} \partial_\sigma U^{\mu_1} - \ldots \\
&\quad + T^{\mu_1\mu_2\ldots}{}_{\sigma\nu_2\ldots} \partial_{\nu_1} U^\sigma + \ldots
\end{aligned}$$

where the ellipsis means that we repeat the terms of each index of the same type.

"symmetry" of $\mathbb{T}$. [6] Note that the Lie derivative of a tensor field $\mathbb{T}$ is again a tensor field and of the same rank as $\mathbb{T}$.

The Lie derivative is linear and satisfies the Leibnitz rule. If a and b are real numbers:[7]

- $\pounds_U(a\mathbb{A} + b\mathbb{B}) = a\pounds_U\mathbb{A} + b\pounds_U\mathbb{B}$
- $\pounds_{aU+bV}\mathbb{T} = a\pounds_U\mathbb{T} + b\pounds_V\mathbb{T}$
- $\pounds_U(\mathbb{A} \otimes \mathbb{B}) = (\pounds_U\mathbb{A}) \otimes \mathbb{B} + \mathbb{A} \otimes (\pounds_U\mathbb{B})$

When $\pounds_U\mathbb{T} = 0$, the tensor $\mathbb{T}$ is said to be "**Lie transported**" along the integral curves of U. If the metric tensor is Lie transported along the integral curves of a vector field, U, then the vector field is called a "**Killing**" field after the mathematician Wilhelm Killing.[8]

6.5.2 Covariant Derivative

The Lie derivative can be thought of as an operator that acts upon a tensor to yield another tensor of the same rank and tells us how the functional form of its components changes along the flow of some vector field. However, when we think of a derivative, we think of the operator (∂_a, say), which has the effect of increasing the rank of the tensor and specifies how it gets transported along a curve. In a global Cartesian basis, if $\mathbb{T}$ is a rank (m, n) tensor (m contravariant indices and n covariant indices) then $\partial\mathbb{T}$ is a tensor of rank $(m, n+1)$. But $\partial\mathbb{T}$ is not a tensor in a general coordinate system, as we will see below, because it compares components in different bases. We would like to obtain a derivative operator, ∇, in general curvilinear coordinates that plays the role of ∂ in Cartesian coordinates, so let us begin with vectors. For this we will need to introduce some additional structure.

Imagine transporting a vector, $\vec{A}$, from some point p to some other point p'. The basis vectors are not necessarily constant during this transport – they would be constant only if the coordinate system is Cartesian. Instead of asking about changes in the components

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[6] More generally, a tensor $\mathbb{T}$ is called **Lie-recurrent** if there exists a non-vanishing scalar field $\lambda(x)$ such that

$$\pounds_U\mathbb{T} = \lambda\mathbb{T}.$$

[7] <u>Problem</u>: Prove these.

[8] <u>Problem</u>: Find the Killing vectors of the sphere of radius r, with metric

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

How many of them are there? A maximally symmetric space is one that possesses the same number of symmetries as Euclidean space. An n dimensional Euclidean space is rotationally and translationally invariant and there are $n(n-1)/2$ rotations and n translations. Therefore a maximally symmetric n dimensional space will have $n(n+1)/2$ symmetries and Killing vectors. Is the two sphere maximally symmetric?

of a vector $\vec{A}$, let's ask instead how the vector as a whole changes as we move from p to p'. To do so it is convenient to work with

$$\vec{A} = A^\mu \vec{e}_{(\mu)} = A^a \vec{u}_{(a)} \tag{6.5.8}$$

$$\delta\vec{A} = (\delta A^\mu)\vec{e}_{(\mu)} + A^\mu(\delta\vec{e}_{(\mu)}) = (\delta A^a)\vec{u}_{(a)} + A^b(\delta\vec{u}_{(b)}) \tag{6.5.9}$$

We see that the change in $\mathbb{A}$ is made up of two contributions, the first coming from a change in its components and the second from a change in the basis as we move from one point to another. This latter contribution is captured by the second term in both expressions on the right above.

Now, since the basis is complete, the change in $\vec{u}_{(\mu)}$ must be a linear combination of the $\vec{u}_{(\mu)}$ themselves and likewise the change in $\vec{u}_{(a)}$ a linear combination of the $\vec{u}_{(a)}$. Then let

$$\delta\vec{e}_{(\mu)} = \vec{e}_{(\nu)}(\delta\Gamma^\nu{}_\mu), \quad \delta\vec{u}_{(b)} = \vec{u}_{(a)}(\delta\omega^a{}_b) \tag{6.5.10}$$

where $\delta\Gamma^\nu{}_\mu(x)$ and $\delta\omega^a{}_b(x)$ will in general be functions of position. Of course, if $\vec{u}_{(a)}$ refers to a global orthogonal tetrad, $\delta\omega^a{}_b = 0$, but, for the moment at least, let us be as general as possible and agree to let the tetrad frame change from point to point (in doing so, we are allowing for space-time to be curved). It should be possible to write the *total* change, $\delta\vec{A}$, as $\delta\vec{A} = (\mathfrak{d}A^\mu)\vec{e}_{(\mu)} = (\mathfrak{d}A^a)\vec{u}_{(a)}$ as well, so we find

$$\mathfrak{d}A^\mu = \delta A^\mu + (\delta\Gamma^\mu{}_\nu)A^\nu \tag{6.5.11}$$

and

$$\mathfrak{d}A^a = \delta A^a + (\delta\omega^a{}_b)A^b \tag{6.5.12}$$

The second term in each expression on the right represents the effect of the changing basis vectors on the components. The derivative corresponding to the infinitesimal changes $\mathfrak{d}$ given above is called the "covariant derivative" of A^μ,

$$\nabla_\lambda A^\mu = \partial_\lambda A^\mu + \Gamma^\mu_{\lambda\nu}A^\nu \tag{6.5.13}$$

and the 3-index object $\Gamma^\mu_{\nu\lambda}$ is called a "**Levi-Civita connection**". We also find from (6.5.12)

$$\nabla_\lambda A^a = \partial_\lambda A^a + \omega^a_{\lambda b}A^b \tag{6.5.14}$$

and the 3-index object $\omega^a_{\lambda b}$ is called the "**spin connection**" (how these expressions generalize to contravariant tensors should be clear).

Now we have to ensure that (6.5.11) and (6.5.12) are compatible, since both refer to the same vector; this will relate the Levi-Civita connection to the spin connection. Thus, starting from (6.5.12),

$$(\nabla_\lambda A^\mu)\vec{e}_{(\mu)} = (\nabla_\lambda A^a)\vec{u}_{(a)} \quad = \quad (\partial_\lambda A^a + \omega^a_{\lambda b}A^b)\vec{u}_{(a)}$$

$$
\begin{aligned}
&= E_a^\mu[\partial_\lambda(e_\nu^a A^\nu) + \omega_{\lambda b}^a e_\nu^b A^\nu]\vec{e}_{(\mu)} \\
&= [\partial_\lambda A^\mu + (E_a^\mu(\partial_\lambda e_\nu^a) + E_a^\mu \omega_{\lambda b}^a e_\nu^b)A^\nu]\vec{e}_{(\mu)} \\
&\equiv (\partial_\lambda A^\mu + \Gamma^\mu_{\lambda\nu}A^\nu)\vec{e}_{(\mu)} \qquad (6.5.15)
\end{aligned}
$$

and comparing the last two expressions above, we find

$$\omega_{\lambda b}^a = e_\mu^a E_b^\nu \Gamma^\mu_{\lambda\nu} - E_b^\nu(\partial_\lambda e_\nu^a). \qquad (6.5.16)$$

which shows that we can determine the spin connection via the the Levi-Civita connection and vierbein. Conversely, if the spin connection and vierbein are known then the Levi-Civita connection is determined. The relation between the two can be put in the more instructive form

$$\partial_\lambda e_\nu^a - \Gamma^\sigma_{\lambda\nu} e_\sigma^a + \omega_{\lambda b}^a e_\nu^b = 0. \qquad (6.5.17)$$

We will call the left hand side the covariant derivative of the vierbein and say that the vierbein is covariantly constant.[9] We will assume that the the Lorentz metric, η^{ab}, is also covariantly constant, which is equivalent to the statement that $\omega_\lambda^{ab} = \omega_{\lambda c}^a \eta^{cb}$ is antisymmetric in the pair (a, b). Often the covariant derivative is represented by a semi-colon and the ordinary derivative by a comma; for example, $\partial_\nu A^\mu \equiv A^\mu{}_{,\nu}$ and $\nabla_\nu A^\mu \equiv A^\mu{}_{;\nu}$. This compact notation can be quite useful in complicated expressions and will be used later.

The Levi-Civita connection measures the rate of change of the coordinate basis as we move from point to point and, under certain conditions, is computed directly from the metric $g_{\mu\nu}$ as we will now see. Consider the change in the metric as we move from x to $x + dx$,

$$
\begin{aligned}
\delta g_{\mu\nu} &= \delta\vec{e}_{(\mu)} \cdot \vec{e}_{(\nu)} + \vec{e}_{(\mu)} \cdot \delta\vec{e}_{(\nu)} = \delta\Gamma^\kappa{}_\mu \vec{e}_{(\kappa)} \cdot \vec{e}_{(\nu)} + \vec{e}_{(\mu)} \cdot \vec{e}_{(\kappa)} \delta\Gamma^\kappa{}_\nu \\
\Rightarrow \quad \partial_\gamma g_{\mu\nu} &= \Gamma^\kappa_{\gamma\mu} g_{\kappa\nu} + \Gamma^\kappa_{\gamma\nu} g_{\mu\kappa}. \qquad (6.5.18)
\end{aligned}
$$

If we take the combination

$$\partial_\gamma g_{\mu\nu} + \partial_\nu g_{\gamma\mu} - \partial_\mu g_{\nu\gamma} = \Gamma^\kappa_{\gamma\mu} g_{\kappa\nu} + \Gamma^\kappa_{\gamma\nu} g_{\mu\kappa} + \Gamma^\kappa_{\nu\gamma} g_{\kappa\mu} + \Gamma^\kappa_{\nu\mu} g_{\gamma\kappa} - \Gamma^\kappa_{\mu\nu} g_{\kappa\gamma} - \Gamma^\kappa_{\mu\gamma} g_{\nu\kappa}$$

[9] Equation (6.5.17) is more directly obtained by considering the change in the coordinate basis vectors,

$$\delta\vec{e}_{(\nu)} = \delta(e_\nu^a \vec{u}_{(a)}) = \delta e_\nu^a \vec{u}_{(a)} + \omega^a{}_b e_\nu^b \vec{u}_{(a)} \equiv \delta\Gamma^\lambda_\nu e_\lambda^a \vec{u}_{(a)},$$

which implies that

$$\partial_\lambda e_\nu^a - \Gamma^\sigma_{\lambda\nu} e_\sigma^a + \omega_{\lambda b}^a e_\nu^b = 0.$$

$$= \Gamma^{\kappa}_{\{\gamma\nu\}} g_{\mu\kappa} + \Gamma^{\kappa}_{[\gamma\mu]} g_{\kappa\nu} + \Gamma^{\kappa}_{[\nu\mu]} g_{\gamma\kappa}, \tag{6.5.19}$$

where we freely use the fact that $g_{\mu\nu}$ is symmetric, and further assume that the Levi-Civita connection is *symmetric* in its lower two indices,

$$\Gamma^{\kappa}_{[\gamma\nu]} = 0, \tag{6.5.20}$$

then

$$\partial_\gamma g_{\mu\nu} + \partial_\nu g_{\gamma\mu} - \partial_\mu g_{\nu\gamma} = 2\Gamma^{\kappa}_{\gamma\nu} g_{\mu\kappa} \tag{6.5.21}$$

and it is easy to see that

$$\Gamma^{\kappa}_{\gamma\nu} g_{\mu\kappa} g^{\mu\rho} = \Gamma^{\kappa}_{\gamma\nu} \delta^{\rho}_{\kappa} = \Gamma^{\rho}_{\gamma\nu} = \frac{1}{2} g^{\rho\mu} \left[\partial_\nu g_{\mu\gamma} + \partial_\gamma g_{\mu\nu} - \partial_\mu g_{\gamma\nu}\right] = \left\{ \begin{array}{c} \rho \\ \gamma\nu \end{array} \right\}. \tag{6.5.22}$$

The expression in braces is also called the **Christoffel symbol**.

A Levi-Civita connection that satisfies (6.5.20) is said to be "**torsion free**" and every torsion free connection can be uniquely related to derivatives of the metric. This is the connection on which Einstein's theory of general relativity is based. However, bear in mind that this is an additional condition that is imposed on Levi-Civita connection. The **torsion** tensor is defined by

$$[\nabla_\mu, \nabla_\nu]\varphi(x) = T^{\lambda}_{\mu\nu} \partial_\lambda \varphi(x) = -\Gamma^{\lambda}_{[\mu\nu]} \partial_\lambda \varphi(x),$$

where $\varphi(x)$ is any scalar function.[10] If the torsion does not vanish, the connection is decomposed into two pieces, one of which is the Christoffel symbol,

$$\Gamma^{\rho}_{\gamma\nu} = \left\{ \begin{array}{c} \rho \\ \gamma\nu \end{array} \right\} + K^{\rho}_{\gamma\nu} \tag{6.5.23}$$

and the other, $K^{\rho}_{\gamma\nu}$, is the **contorsion** tensor. It may be expressed in terms of $T^{\rho}_{\gamma\nu}$, but we will not pursue this here. We will consider only space-times for which the torsion vanishes.

The ordinary derivative, $\partial_\nu A^\mu$, of a contravariant vector does not transform as a (mixed) tensor but the covariant derivative, $\nabla_\nu A^\mu$, does. The fact that the covariant derivative transforms as a tensor is of great importance because the laws of physics should not depend on one's choice of coordinates. This means that they should "look the same" in any system, which is possible only if the two sides of any dynamical equation transform in the same manner, *i.e.,* either as scalars, vectors or tensors under transformations

[10] Problem: Sometimes the Torsion is defined as

$$T^{a}_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^{a}_{\mu c} e^c_\nu - \omega^{a}_{\nu c} e^c_\mu.$$

. Use (6.5.17) to show that this is equivalent to the definition given above.

between coordinate systems. Thus, covariant derivatives and not ordinary derivatives are more meaningful in physics.

First let's see that $\partial_\nu A^\mu$ is not a tensor. For the sake of simplicity consider a flat space-time (for which a global Cartesian basis is available, therefore $\omega^a_{\lambda b} = 0$). Then (6.5.16) gives

$$\Gamma^\mu_{\nu\lambda} = (\partial_\nu \vec{e}_\lambda) \cdot \vec{E}^\mu = \frac{\partial^2 x^a}{\partial \xi^\nu \partial \xi^\lambda} \frac{\partial \xi^\mu}{\partial x^a}, \tag{6.5.24}$$

so $\Gamma^\mu_{\lambda\nu}$ is manifestly symmetric in (ν, λ). We have

$$\frac{\partial A'^\mu}{\partial \xi'^\nu} = \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial}{\partial \xi^\lambda} \left(\frac{\partial \xi'^\mu}{\partial \xi^\kappa} A^\kappa \right) = \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial \xi'^\mu}{\partial \xi^\kappa} \frac{\partial A^\kappa}{\partial \xi^\lambda} + \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial^2 \xi'^\mu}{\partial \xi^\lambda \xi^\kappa} A^\kappa \tag{6.5.25}$$

The first term on the r.h.s. corresponds to the tensor transformation, but the second term spoils the transformation properties of $\partial_\nu A^\mu$. Let us then examine the transformation properties of $\nabla_\nu A^\mu$:

$$\begin{aligned} \nabla'_\nu A'^\mu &= \partial'_\nu A'^\mu + \Gamma'^\mu_{\nu\kappa} A'^\kappa \\ &= \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial \xi'^\mu}{\partial \xi^\gamma} \frac{\partial A^\gamma}{\partial \xi^\lambda} + \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial^2 \xi'^\mu}{\partial \xi^\lambda \xi^\gamma} A^\gamma + \frac{\partial \xi'^\kappa}{\partial \xi^\gamma} \Gamma'^\mu_{\nu\kappa} A^\gamma \end{aligned} \tag{6.5.26}$$

If we can show that

$$\frac{\partial \xi'^\kappa}{\partial \xi^\gamma} \Gamma'^\mu_{\nu\kappa} = \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial \xi'^\mu}{\partial \xi^\sigma} \Gamma^\sigma_{\lambda\gamma} - \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial^2 \xi'^\mu}{\partial \xi^\lambda \xi^\gamma} \tag{6.5.27}$$

then we will have

$$\nabla'_\nu A'^\mu = \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial \xi'^\mu}{\partial \xi^\sigma} \left[\frac{\partial A^\sigma}{\partial \xi^\lambda} + \Gamma^\sigma_{\lambda\gamma} A^\gamma \right] = \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial \xi'^\mu}{\partial \xi^\sigma} \nabla_\lambda A^\sigma = (\Lambda^{-1})^\lambda{}_\nu \Lambda^\mu{}_\sigma \nabla_\lambda A^\sigma \tag{6.5.28}$$

and we will have accomplished the task of showing that $\nabla_\nu A^\mu$ is a tensor. It is not so difficult to show (6.5.27). First put it in the form

$$\Gamma'^\mu_{\nu\kappa} = \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial \xi'^\mu}{\partial \xi^\sigma} \frac{\partial \xi^\gamma}{\partial \xi'^\kappa} \Gamma^\sigma_{\lambda\gamma} - \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial^2 \xi'^\mu}{\partial \xi^\lambda \xi^\gamma} \frac{\partial \xi^\gamma}{\partial \xi'^\kappa} \tag{6.5.29}$$

and write

$$\Gamma'^\mu_{\nu\kappa} = \partial'_\nu \Gamma'^\mu_\kappa = (\partial'_\nu \vec{e}'_\kappa) \cdot \vec{E}'^\mu = -\vec{e}'_\kappa (\partial'_\nu \vec{E}'^\mu) \tag{6.5.30}$$

where we have used $\Gamma^\mu_{\nu\kappa} = (\partial_\nu \vec{e}_\kappa) \cdot \vec{E}^\mu$ as, according to (6.5.24), is appropriate when the spin connection vanishes. Then

$$\Gamma'^\mu_{\nu\kappa} = -\frac{\partial \xi^\gamma}{\partial \xi'^\kappa} \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \vec{e}_\gamma \cdot \frac{\partial}{\partial \xi^\lambda} \left(\frac{\partial \xi'^\mu}{\partial \xi^\sigma} \vec{E}^\sigma \right)$$

$$\begin{aligned} &= -\frac{\partial\xi^\gamma}{\partial\xi'^\kappa}\frac{\partial\xi^\lambda}{\partial\xi'^\nu}\frac{\partial\xi'^\mu}{\partial\xi^\sigma}\vec{e}_\gamma\cdot(\partial_\lambda\vec{E}^\sigma) - \frac{\partial\xi^\gamma}{\partial\xi'^\kappa}\frac{\partial\xi^\lambda}{\partial\xi'^\nu}\frac{\partial^2\xi'^\mu}{\partial\xi^\lambda\partial\xi^\sigma}\vec{e}_\gamma\cdot\vec{E}^\sigma \\ &= \frac{\partial\xi^\gamma}{\partial\xi'^\kappa}\frac{\partial\xi^\lambda}{\partial\xi'^\nu}\frac{\partial\xi'^\mu}{\partial\xi^\sigma}\Gamma^\sigma_{\lambda\gamma} - \frac{\partial\xi^\lambda}{\partial\xi'^\nu}\frac{\partial^2\xi'^\mu}{\partial\xi^\lambda\partial\xi^\gamma}\frac{\partial\xi^\gamma}{\partial\xi'^\kappa} \end{aligned} \tag{6.5.31}$$

which is the desired result. With a little more effort (left as an exercise) we can generalize the discussion to the case with a non-vanishing spin connection.[11] Again, notice that without the second term the above would correspond to a tensor transformation, but the second term spoils the transformation properties. In fact it is precisely because of the presence of the second term that $\nabla_\nu A^\mu$ transforms as a tensor. Note also that if the unprimed coordinates were Cartesian, $(\sigma,\lambda,\gamma)\equiv(a,b,c)$, then $\Gamma^a_{bc}\equiv 0$ and

$$\Gamma^\mu_{\nu\kappa} = -\frac{\partial x^b}{\partial\xi^\nu}\frac{\partial^2\xi^\mu}{\partial x^b\partial x^c}\frac{\partial x^c}{\partial\xi^\kappa} = -\vec{e}_\nu\cdot(\partial_\kappa\vec{E}^\mu) \tag{6.5.32}$$

which is equivalent to our starting point (6.5.24) because $\vec{e}_\nu\cdot\vec{E}^\kappa = \delta^\kappa_\nu$.

In the *global* Cartesian basis we are considering, the derivative of a vector is just $\partial_a A^b$. If we now transform to the curvilinear coordinates,

$$\partial_a A^b = \frac{\partial\xi^\mu}{\partial x^a}\frac{\partial}{\partial\xi^\mu}(A^\nu e^b_\nu) = E^\mu_a(\partial_\mu A^\nu)e^b_\nu + E^\mu_a A^\nu(\partial_\mu e^b_\nu) \tag{6.5.33}$$

so that

$$\begin{aligned} e^a_\sigma E^\lambda_b\partial_a A^b &= e^a_\sigma E^\lambda_b E^\mu_a(\partial_\mu A^\nu)e^b_\nu + e^a_\sigma E^\lambda_b E^\mu_a A^\nu(\partial_\mu e^b_\nu) \\ &= \partial_\sigma A^\lambda + E^\lambda_b A^\nu(\partial_\sigma e^b_\nu) = \partial_\sigma A^\lambda + \Gamma^\lambda_{\sigma\nu}A^\nu \end{aligned} \tag{6.5.34}$$

If we think of $\partial_a A^b$ as the components of a (mixed) tensor in the LLF then, in a general coordinate system, its components should be given by $e^a_\sigma E^\lambda_b\partial_a A^b$. The above equation shows that its components in the general coordinate basis are given by the components of the covariant derivative.[12] In other words, ordinary derivatives of the LLF components of vectors must be replaced by covariant derivatives in general coordinate systems.

It should be clear that the covariant derivative of a tensor copies the covariant derivative of the vector. Setting,

$$\mathbb{T} = T^{\mu\nu\cdots}\vec{e}_{(\mu)}\otimes\vec{e}_{(\nu)}\cdots \tag{6.5.35}$$

[11] Problem: Generalize this to the case when the spin connection does not vanish. Treat $\widehat{\omega}_\lambda$ as a coordinate vector *i.e.,*

$$\omega'^a_{\lambda b} = \frac{\partial\xi^\sigma}{\partial\xi'^\lambda}\omega^a_{\sigma b}$$

and show that (6.5.31) is valid in this case as well.

[12] Problem: Modify this argument to include a non-vanishing spin connection. Employ (6.5.17).

we get

$$\delta\mathbb{T} = \delta T^{\mu\nu\cdots}\vec{e}_{(\mu)} \otimes \vec{e}_{(\nu)} ... + T^{\mu\nu\cdots}(\delta\vec{e}_{(\mu)}) \otimes \vec{e}_{(\nu)} ... + T^{\mu\nu\cdots}\vec{e}_{(\mu)} \otimes (\delta\vec{e}_{(\nu)}) ... + ... \tag{6.5.36}$$

from which it follows that

$$\nabla_\gamma T^{\mu\nu...} = \partial_\gamma T^{\mu\nu...} + \Gamma^\mu_{\gamma\lambda} T^{\lambda\nu...} + \Gamma^\nu_{\gamma\lambda} T^{\mu\lambda...} + ... \tag{6.5.37}$$

We have defined the covariant derivatives of a contravariant vector. How about the covariant derivative of a covariant vector? For a dual vector, ω, we should find

$$\delta\omega = (\delta\omega_\mu)\vec{E}^{(\mu)} + \omega_\mu(\delta\vec{E}^{(\mu)}) \tag{6.5.38}$$

and we want to know what $\delta\vec{E}^{(\mu)}$ is. Use the fact that

$$\vec{e}_{(\mu)} \cdot \vec{E}^{(\nu)} = \delta^\nu_\mu \Rightarrow (\delta\vec{e}_{(\mu)}) \cdot \vec{E}^{(\nu)} + \vec{e}_{(\mu)} \cdot (\delta\vec{E}^{(\nu)}) = 0 \tag{6.5.39}$$

or

$$\vec{e}_{(\mu)} \cdot (\delta\vec{E}^{(\nu)}) = -(\delta\vec{e}_{(\mu)}) \cdot \vec{E}^{(\mu)} = -(\delta\Gamma^\kappa{}_\mu)\vec{e}_{(\kappa)} \cdot \vec{E}^{(\nu)} \tag{6.5.40}$$

so that, writing it out in component form and multiplying the l.h.s. by $E^{(\mu)}_b$ gives

$$\delta\vec{E}^{(\nu)} = -(\delta\Gamma^\nu{}_\mu)\vec{E}^{(\mu)} \tag{6.5.41}$$

which should be compared with (6.5.10) for the variation of $\vec{e}_{(\mu)}$. Therefore

$$\delta\vec{\omega} = (\mathfrak{d}\omega_\mu)\vec{E}^{(\mu)} = (\delta\omega_\mu)\vec{E}^{(\mu)} - \omega_\mu(\delta\Gamma^\mu{}_\kappa)\vec{E}^{(\kappa)} \tag{6.5.42}$$

and we could write the covariant derivative of the covector, A_μ

$$\nabla_\nu\omega_\mu = \partial_\nu\omega_\mu - \Gamma^\kappa_{\nu\mu}\omega_\kappa \tag{6.5.43}$$

and of a covariant tensor, $T_{\mu\nu\kappa...}$

$$\nabla_\gamma T_{\mu\nu...} = \partial_\gamma T_{\mu\nu...} - \Gamma^\lambda_{\gamma\mu} T_{\lambda\nu...} - \Gamma^\lambda_{\gamma\nu} T_{\mu\lambda...} + ... \tag{6.5.44}$$

in complete analogy with the covariant derivative of contravectors and tensors. In particular we see that

$$\nabla_\gamma g_{\mu\nu} = \partial_\gamma g_{\mu\nu} - \Gamma^\kappa_{\gamma\mu} g_{\kappa\nu} - \Gamma^\kappa_{\gamma\nu} g_{\mu\kappa} \equiv \nabla_\gamma g^{\mu\nu} = 0 \tag{6.5.45}$$

by (6.5.18). This is called the "**metricity**" property.[13] The covariant derivative is linear and satisfies the Leibniz rule:[14]

- $\nabla(a\mathbb{A} + b\mathbb{B}) = a\nabla\mathbb{A} + b\nabla\mathbb{B}$
- $\nabla(\mathbb{A} \otimes \mathbb{B}) = (\nabla\mathbb{A}) \otimes \mathbb{B} + \mathbb{A} \otimes (\nabla\mathbb{B})$

[13] Using the Lie derivative of the metric (a rank two co-tensor) show that if U is a symmetry of the metric then it must satisfy

$$\nabla_{(\mu}U_{\nu)} = \nabla_\mu U_\nu + \nabla_\nu U_\mu = 0$$

The symmetry vectors of the metric are called Killing vectors. In Minkowski space there are 10 of them corresponding to the 10 generators of the Poincaré group: translations, spatial rotations and boosts.

[14] Problem: Prove these.

6.5.3 Parallel Transport

Having defined the covariant derivative operator, in a general coordinate system, we may now define the *absolute derivative* (or *directional* derivative, or *total* derivative, or *intrinsic* derivative) of a tensor, $\mathbb{T}$, along the integral curves of any vector field, U, as the projection of its covariant derivative on U, *i.e.,*

$$D_U\mathbb{T} = (U \cdot \nabla)\mathbb{T}. \tag{6.5.46}$$

The absolute derivative measures the total rate of change of $\mathbb{T}$ along any integral curve of U and is a tensor of the same rank as $\mathbb{T}$ itself. It is satisfies both linearity and the Leibnitz rule.

A tensor $\mathbb{T}$ is said to be "**parallel transported**" along the integral curves of a vector field U if and only if

$$D_U\mathbb{T} = f(\lambda)\mathbb{T} \tag{6.5.47}$$

along the curves. Let us consider what parallel transport means geometrically. Let $\xi^\mu(\lambda)$ be an integral curve of U and let $\vec{A}$ be a vector defined at any point, P, on the curve, say at $\lambda = 0$. To "parallel transport" $\vec{A}$ along the curve is to define a vector field along the curve in such a way that at every point on the curve $\vec{A}$ remains parallel to itself. This is only possible if $\vec{A}$, evaluated at neighboring points, changes by some multiple of itself along the curve, *i.e.,*

$$\delta\vec{A} = \delta g(\lambda)\vec{A}, \tag{6.5.48}$$

where $g(\lambda)$ is an arbitrary function of λ. If, in addition, we wish to keep the magnitude of $\vec{A}$ unchanged as well then $\delta g(\lambda) = 0$. Bearing in mind that the changing basis vectors must also be accounted for, we write this in component form as

$$\mathfrak{d}A^\mu = \delta\xi^\alpha\nabla_\alpha A^\mu = \delta g(\lambda)A^\mu \tag{6.5.49}$$

or

$$\frac{d\xi^\alpha}{d\lambda}\nabla_\alpha A^\mu = (U \cdot \nabla)A^\mu = g'(\lambda)A^\mu \tag{6.5.50}$$

which is (6.5.47) for vectors with $f(\lambda) = g'(\lambda)$.

For any vector, $\vec{A}$, we have

$$(U \cdot \nabla)A^\mu = U^\sigma(\partial_\sigma A^\mu + \Gamma^\mu_{\sigma\kappa}A^\kappa) = \frac{dA^\mu}{d\lambda} + \Gamma^\mu_{\sigma\kappa}U^\sigma A^\kappa = f(\lambda)A^\mu, \tag{6.5.51}$$

so, in particular, if a tangent vector is parallel transported along its integral curves, then

$$\frac{dU^\mu}{d\lambda} + \Gamma^\mu_{\sigma\kappa}U^\sigma U^\kappa = \frac{d^2\xi^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\kappa}\frac{d\xi^\sigma}{d\lambda}\frac{d\xi^\kappa}{d\lambda} = f(\lambda)\frac{d\xi^\mu}{d\lambda} \tag{6.5.52}$$

gives a very special second order equation for the integral curves. In the next chapter we will see that this is the **geodesic** equation (geodesics are paths of shortest distance between points) when $g(\lambda) = \ln(ds/d\lambda)$, where s represents the proper distance. For example, if we identify λ with the proper distance, s, itself,[15] then $f(\lambda) = 0$. In the LLF, where the connections vanish, we would find simply that

$$\frac{d^2x^a}{ds^2} = 0, \tag{6.5.53}$$

This will be recognized as the equation of a "straight line" in flat space. We can now give a more general meaning to the notion of "straight": a "straight path" is one along which its tangent vector is parallel transported.

6.5.4 The Divergence and Laplacian

Consider divergence of a vector, which can be defined as

$$\nabla_\mu A^\mu = \partial_\mu A^\mu + \Gamma^\mu_{\mu\lambda} A^\lambda \tag{6.5.54}$$

But

$$\Gamma^\mu_{\mu\kappa} = \frac{1}{2} g^{\mu\rho} [\partial_\kappa g_{\rho\mu} + \partial_\mu g_{\rho\kappa} - \partial_\rho g_{\mu\kappa}] \tag{6.5.55}$$

and interchanging $(\mu\rho)$ in the middle term shows that it cancels the last, so

$$\Gamma^\mu_{\mu\kappa} = \frac{1}{2} g^{\mu\rho} \partial_\kappa g_{\rho\mu} \tag{6.5.56}$$

This expression may be further simplified: let g be the determinant of $g_{\mu\nu}$, then

$$\ln g = \text{tr} \ln \hat{g} \to \delta \ln g = \frac{\delta g}{g} = \text{tr} \hat{g}^{-1} \delta \hat{g} = g^{\mu\rho} \delta g_{\mu\rho} \tag{6.5.57}$$

and therefore

$$\frac{1}{g} \partial_\kappa g = g^{\mu\rho} \partial_\kappa g_{\mu\rho}, \quad \Gamma^\mu_{\mu\kappa} = \partial_\kappa \ln \sqrt{-g}, \tag{6.5.58}$$

which gives

$$\nabla_\mu A^\mu = \partial_\mu A^\mu + \partial_\kappa \ln \sqrt{-g} A^\kappa = \frac{1}{\sqrt{-g}} \partial_\kappa \sqrt{-g} A^\kappa \tag{6.5.59}$$

A similar result holds for any *antisymmetric* tensor (only!), $A^{\mu\nu}$,

$$\nabla_\mu A^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} A^{\mu\nu}) \tag{6.5.60}$$

[15] More generally, if s is a linear function of λ; such parameters are called "affine".

and this can be shown in the same way as the previous result.[16] These formulas can be quite useful while computing divergences.

Another important operation on vectors and tensors in physics is the Laplacian. It is an invariant under coordinate transformations, being defined in an arbitrary system of coordinates as $\Box_x = \nabla_\mu \nabla^\mu$. Because it involves the covariant derivative its action will depend on whether it operates on a scalar, a vector or a tensor. Consider its operation on a scalar function, ϕ (remember that $\nabla^\mu \phi = \partial^\mu \phi$ is a vector)

$$\Box_x \phi = \nabla_\mu \nabla^\mu \phi = \partial_\mu \nabla^\mu \phi + \Gamma^\mu_{\mu\kappa} \nabla^\kappa \phi = \partial_\mu g^{\mu\nu} \partial_\nu \phi + \Gamma^\mu_{\mu\kappa} g^{\kappa\nu} \partial_\nu \phi \tag{6.5.61}$$

where we have used the fact that the covariant derivative operating on a scalar function is just the partial derivative. Inserting (6.5.58) above shows that

$$\Box_x \phi = \partial_\mu (g^{\mu\nu} \partial_\nu \phi) + g^{\mu\nu} (\partial_\mu \ln \sqrt{-g}) \partial_\nu \phi = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \tag{6.5.62}$$

This is a very compact formula. Life is not so easy if the Laplacian, $\Box_x$, operates on a vector (worse, on a tensor), instead of a scalar. Then we have

$$\begin{aligned} \Box_x A^\mu &= \nabla_\nu \nabla^\nu A^\mu = g^{\nu\kappa} \nabla_\nu \nabla_\kappa A^\mu = g^{\nu\kappa} [\partial_\nu \nabla_\kappa A^\mu - \Gamma^\lambda_{\nu\kappa} \nabla_\lambda A^\mu + \Gamma^\mu_{\nu\lambda} \nabla_\kappa A^\lambda] \\ &= g^{\nu\kappa} [\partial_\nu (\partial_\kappa A^\mu + \Gamma^\mu_{\kappa\lambda} A^\lambda) - \Gamma^\lambda_{\nu\kappa} (\partial_\lambda A^\mu + \Gamma^\mu_{\lambda\gamma} A^\gamma) \\ &\qquad + \Gamma^\mu_{\nu\lambda} (\partial_\kappa A^\lambda + \Gamma^\lambda_{\kappa\gamma} A^\gamma)] \end{aligned} \tag{6.5.63}$$

which is certainly more complicated. Let's see how this works through some common examples. Only the results will be given, the details are left to the reader.

6.6 Flatland Examples

6.6.1 Orthogonal Coordinates

A coordinate system, $\xi^\mu(x)$ is considered to be "orthogonal" if all of the coordinate surfaces, defined by ξ^μ =const., intersect at right angles. Cartesian coordinates, for example, form an orthogonal system. Mathematically, orthogonal systems are simpler to work with. For example, differential equations of interest in physics often can be solved by the method of separation of variables if the coordinate system respecting the symmetries of the physical system is orthogonal. In the following examples we will only consider coordinate transformations that do not involve time, *i.e.,* we consider transformations of the form

[16] Problem: Prove it.

$t \to t' = t$ and $\xi^i = \xi^i(\vec{x})$ and suppose that the $\xi^i(\vec{x})$ are all invertible; for a complete set of Cartesian basis vectors, $\vec{u}_{(a)}$, take

$$\vec{u}_{(t)} = (1,0,0,0), \quad \vec{u}_{(x)} = (0,1,0,0), \quad \vec{u}_{(y)} = (0,0,1,0), \quad \vec{u}_{(z)} = (0,0,0,1), \tag{6.6.1}$$

then the vierbein is

$$\begin{aligned} \vec{e}_t &= (\frac{\partial t}{\partial t}, \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}) = (1,0,0,0) \\ \vec{e}_i &= (0, \frac{\partial x}{\partial \xi^i}, \frac{\partial y}{\partial \xi^i}, \frac{\partial z}{\partial \xi^i}) \end{aligned} \tag{6.6.2}$$

and orthogonality is defined by $\vec{e}_i \cdot \vec{e}_j = h_i^2(\xi)\delta_{ij}$. They serve as a complete set of coordinate basis vectors, $\vec{u}_{(\mu)} = \vec{e}_\mu$. Alternatively, in terms of the inverse vierbein

$$\begin{aligned} \vec{E}^t &= (\frac{\partial t}{\partial t}, \frac{\partial t}{\partial x}, \frac{\partial t}{\partial y}, \frac{\partial t}{\partial z}) = (1,0,0,0) \\ \vec{E}^i &= (0, \frac{\partial \xi^i}{\partial x}, \frac{\partial \xi^i}{\partial y}, \frac{\partial \xi^i}{\partial z}), \end{aligned} \tag{6.6.3}$$

the orthogonality condition will take the form $\vec{E}^i \cdot \vec{E}^j = h_i^{-2}(\xi)\delta_{ij}$. They serve as the dual coordinate basis vectors, $\vec{\theta}^{(\mu)} = \vec{E}^\mu$. The metric tensor is therefore diagonal

$$g_{\mu\nu} = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & h_1^2 & 0 & 0 \\ 0 & 0 & h_2^2 & 0 \\ 0 & 0 & 0 & h_3^2 \end{bmatrix} \tag{6.6.4}$$

and the distance function is given explicitly by

$$ds^2 = c^2 dt^2 - g_{ij} d\xi^i d\xi^j = c^2 dt^2 - (h_i d\xi^i)^2. \tag{6.6.5}$$

Notice that it has no off-diagonal components. The functions $h_i(\xi)$ are called "scaling" functions and they can be used directly to compute various differential operators once the Christoffel symbols are determined from $\Gamma^\mu_{\nu\kappa} = (\partial_\nu \vec{e}_\kappa) \cdot \vec{E}^\mu$ or

$$\Gamma^i_{jk} = \frac{\partial^2 x^a}{\partial \xi^j \partial \xi^k} \frac{\partial \xi^i}{\partial x^a}. \tag{6.6.6}$$

(It should be clear that only the spatial components of the Christoffel symbols will be non-vanishing because we have chosen a static (time-independent) frame and because the time-time component of the metric is constant.) Let's look at some common examples of such coordinate systems.

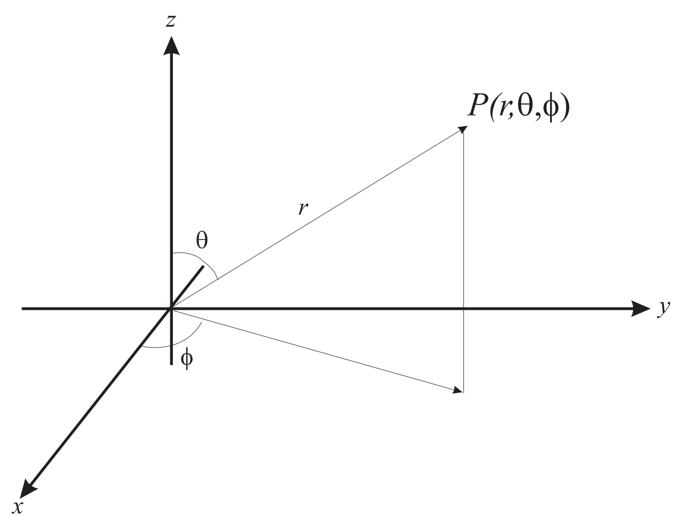

Figure 6.1: Spherical coordinates

Spherical Coordinates

Take the following coordinate functions: $\xi^\mu = (t, r, \theta, \phi)$ where

$$\begin{aligned} t &= t \\ r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \varphi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned} \tag{6.6.7}$$

and the inverse transformations: $x^a = x^a(\xi)$

$$\begin{aligned} t &= t \\ x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned} \tag{6.6.8}$$

Let's compute the vierbein

$$\begin{aligned} \vec{e}_t &= \left(\frac{\partial t}{\partial t}, \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right) = (1,0,0,0) = \widehat{t} \\ \vec{e}_r &= \left(\frac{\partial t}{\partial r}, \frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r}\right) = (0, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \widehat{r} \end{aligned}$$

$$
\begin{aligned}
\vec{e}_\theta &= (\frac{\partial t}{\partial\theta}, \frac{\partial x}{\partial\theta}, \frac{\partial y}{\partial\theta}, \frac{\partial z}{\partial\theta}) = r(0, \cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) = r\ \widehat{\theta} \\
\vec{e}_\varphi &= (\frac{\partial t}{\partial\varphi}, \frac{\partial x}{\partial\varphi}, \frac{\partial y}{\partial\varphi}, \frac{\partial z}{\partial\varphi}) = r(0, -\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0) = r\sin\theta\ \widehat{\varphi} \qquad (6.6.9)
\end{aligned}
$$

and its inverse

$$
\begin{aligned}
\vec{E}^t &= (\frac{\partial t}{\partial t}, \frac{\partial t}{\partial x}, \frac{\partial t}{\partial y}, \frac{\partial t}{\partial z}) = (1,0,0,0) = \widehat{t} \\
\vec{E}^r &= (\frac{\partial r}{\partial t}, \frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}) = (0, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \widehat{r} \\
\vec{E}^\theta &= (\frac{\partial \theta}{\partial t}, \frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}) = \frac{1}{r}(0, \cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) = \frac{\widehat{\theta}}{r} \\
\vec{E}^\varphi &= (\frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}) = \frac{1}{r}(0, -\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0) = \frac{\widehat{\varphi}}{r\sin\theta} \qquad (6.6.10)
\end{aligned}
$$

It is easy to check that $\vec{e}_\mu \cdot \vec{E}^\nu = \delta^\nu_\mu$ and that $e^a_\mu E^\mu_b = \delta^a_b$. Now compute the inner products to get the metric function: $g_{tt} = -1$, $g_{rr} = 1$, $g_{\theta\theta} = r^2$ and $g_{\varphi\varphi} = r^2\sin^2\theta$ (all other components vanish). In matrix notation,

$$
g_{\mu\nu} = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix} \qquad (6.6.11)
$$

and the distance function is given explicitly by,

$$
ds^2 = c^2dt^2 - (dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2) \qquad (6.6.12)
$$

Next compute the connections using either $\Gamma^\mu_{\nu\kappa} = (\partial_\nu \vec{e}_\kappa)\cdot\vec{E}^\mu$ or (6.5.22) to get the non-vanishing components

$$
\begin{aligned}
&\Gamma^r_{\theta\theta} = -r, \qquad \Gamma^r_{\varphi\varphi} = -r\sin^2\theta \\
&\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r} \\
&\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta, \qquad \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \cot\theta \qquad (6.6.13)
\end{aligned}
$$

(all others vanish).[17]

[17]Problem: Show that the metric of the three dimensional line element:

$$
ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)
$$

admits three independent Killing vectors.

Care must be taken in translating the results from working in the basis $\{\vec{e}_\mu\}$ (or $\{\vec{E}^\mu\}$) and the basis $\{\widehat{t},\widehat{r},\widehat{\theta},\widehat{\varphi}\}$ usually found in the literature. For example, the gradient of a scalar function, f has components $\partial_\mu f$ in the basis given by $\{\vec{E}^\mu\}$. Therefore, written in terms of the usual basis, this becomes

$$(\partial_\mu f)\vec{E}^\mu = (\partial_t f)\widehat{t} + (\partial_r f)\widehat{r} + \frac{1}{r}(\partial_\theta f)\widehat{\theta} + \frac{1}{r\sin\theta}(\partial_\varphi f)\widehat{\varphi} \tag{6.6.14}$$

Likewise, the divergence of a vector $U = U^\mu \vec{e}_\mu$ is given as

$$\nabla\cdot U = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}\; U^\mu = \partial_t U^t + \frac{1}{r^2}\partial_r(r^2U^r) + \frac{1}{\sin\theta}\partial_\theta(\sin\theta U^\theta) + \partial_\varphi U^\varphi. \tag{6.6.15}$$

However, if we are given the vector U in the basis $\{\widehat{t},\widehat{r},\widehat{\theta},\widehat{\varphi}\}$ as

$$U = u^t\widehat{t} + u^r\widehat{r} + u^\theta\widehat{\theta} + u^\varphi\widehat{\varphi} \tag{6.6.16}$$

then

$$U^t = u^t,\;\; U^r = u^r,\;\; U^\theta = \frac{u^\theta}{r},\;\; U^\varphi = \frac{u^\varphi}{r\sin\theta} \tag{6.6.17}$$

and

$$\nabla\cdot U = \partial_t u^t + \frac{1}{r^2}\partial_r(r^2u^r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta u^\theta) + \frac{1}{r\sin\theta}\partial_\varphi u^\varphi \tag{6.6.18}$$

which is the standard expression found in the literature. Henceforth we will take for granted that our components are always given in the vierbein basis.

What is the action of of the Laplacian, $\Box_x$, on a scalar function? Directly applying the expression in (6.5.62) we find

$$\Box_x\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = -\frac{1}{c^2}\partial_t^2\phi + \frac{1}{r^2}\partial_r(r^2\partial_r\phi) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\phi) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2\phi \tag{6.6.19}$$

the spatial part of which will be recognized as the standard result from ordinary vector analysis. Its action on vectors is quite a bit more complicated but can be written out,

$$\begin{aligned}
\Box_x A^0 &= \left[-\frac{1}{c^2}\partial_t^2 A^0 + \frac{1}{r^2}\partial_r(r^2\partial_r A^0) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta A^0) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 A^0\right]\\
\Box_x A^r &= \left[-\frac{1}{c^2}\partial_t^2 A^r + \frac{1}{r^2}\partial_r(r^2\partial_r A^r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta A^r) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 A^r\right.\\
&\qquad\left. -\frac{2}{r^2}(A^r + r\cot\theta A^\theta + r\partial_\theta A^\theta + r\partial_\varphi A^\varphi)\right]\\
\Box_x A^\theta &= \left[-\frac{1}{c^2}\partial_t^2 A^\theta + \frac{1}{r^4}\partial_r(r^4\partial_r A^\theta) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta A^\theta) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 A^\theta\right.
\end{aligned}$$

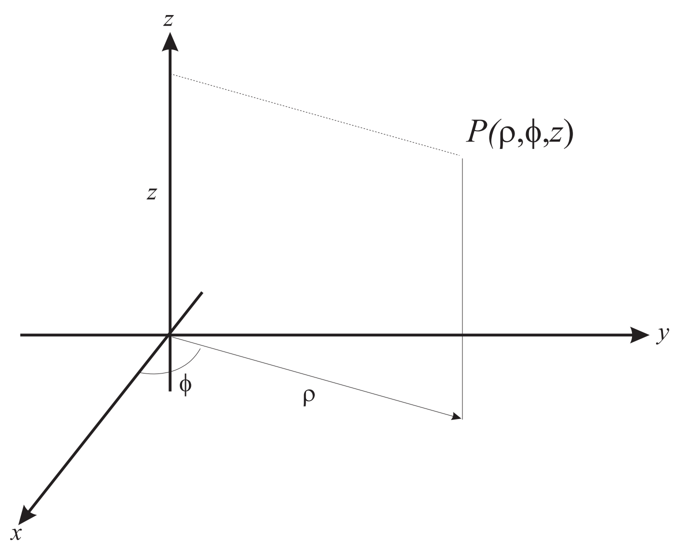

Figure 6.2: Cylindrical coordinates

$$+\frac{2}{r^3}(\partial_\theta A^r - \frac{1}{2}r\cos 2\theta A^\theta - r\cot\theta\partial_\phi A^\varphi)\Big]$$

$$\Box_x A^\varphi = \Big[-\frac{1}{c^2}\partial_t^2 A^\varphi + \frac{1}{r^4}\partial_r(r^4\partial_r A^\varphi) + \frac{1}{r^2\sin^3\theta}\partial_\theta(\sin^3\theta\partial_\theta A^\varphi) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 A^\varphi$$
$$+\frac{2}{r^3\sin^3\theta}(\sin\theta\partial_\varphi A^r + r\cos\theta\partial_\varphi A^\theta)\Big] \qquad (6.6.20)$$

We see that the Laplacian acts on the time component, A^0, of A^μ, just exactly as it does on a scalar. This is because the coordinate transformation was purely spatial. On the other hand, its action on the space components mixes them.

Cylindrical coordinates

Take the following coordinate functions: $\xi^\mu = (t, \rho, \varphi, z)$ where

$$\begin{aligned} t &= t \\ \rho &= \sqrt{x^2+y^2} \\ \varphi &= \tan^{-1}\left(\frac{y}{x}\right) \\ z &= z \end{aligned} \qquad (6.6.21)$$

and the inverse transformations: $x^a = x^a(\xi)$

$$t = t$$

$$\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \\ z &= z \end{aligned} \tag{6.6.22}$$

Let's compute the vierbein

$$\begin{aligned} \vec{e}_t &= (\frac{\partial t}{\partial t}, \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}) = (1,0,0,0) \\ \vec{e}_\rho &= (\frac{\partial t}{\partial \rho}, \frac{\partial x}{\partial \rho}, \frac{\partial y}{\partial \rho}, \frac{\partial z}{\partial \rho}) = (0,\cos\varphi,\sin\varphi,0) \\ \vec{e}_\varphi &= (\frac{\partial t}{\partial \varphi}, \frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial z}{\partial \varphi}) = \rho(0,-\sin\varphi,\cos\varphi,0) \\ \vec{e}_z &= (\frac{\partial t}{\partial z}, \frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \frac{\partial z}{\partial z}) = (0,0,0,1) \end{aligned} \tag{6.6.23}$$

and its inverse

$$\begin{aligned} \vec{E}^t &= (\frac{\partial t}{\partial t}, \frac{\partial t}{\partial x}, \frac{\partial t}{\partial y}, \frac{\partial t}{\partial z}) = (1,0,0,0) \\ \vec{E}^\rho &= (\frac{\partial \rho}{\partial t}, \frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial y}, \frac{\partial \rho}{\partial z}) = (0,\cos\varphi,\sin\varphi,0) \\ \vec{E}^\varphi &= (\frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}) = \frac{1}{\rho}(0,-\sin\varphi,\cos\varphi,0) \\ \vec{E}^z &= (\frac{\partial z}{\partial t}, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial z}{\partial z}) = (0,0,0,1) \end{aligned} \tag{6.6.24}$$

Again, it's easy to check that $\vec{e}_\mu \cdot \vec{E}^\nu = \delta^\nu_\mu$ and that $e^a_\mu E^\mu_b = \delta^a_b$. Now compute the inner products to get the metric function: $g_{tt} = -1$, $g_{\rho\rho} = 1$, $g_{\varphi\varphi} = \rho^2$ and $g_{zz} = 1$ (all other components vanish). In matrix notation,

$$g_{\mu\nu} = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.6.25}$$

and the distance function is given explicitly by,

$$ds^2 = c^2dt^2 - (d\rho^2 + \rho^2 d\varphi^2 + dz^2) \tag{6.6.26}$$

The non-vanishing components of the connections, obtained by using either $\Gamma^\mu_{\nu\kappa} = (\partial_\nu \vec{e}_\kappa)\cdot \vec{E}^\mu$ or (6.5.22) are just

$$\Gamma^\rho_{\varphi\varphi} = -\rho,$$

$$\Gamma^{\varphi}_{\rho\varphi} = \Gamma^{\varphi}_{\varphi\rho} = \frac{1}{\rho} \tag{6.6.27}$$

(all others vanish)[18], while the action of the Laplacian, $\Box_x$, on a scalar function is

$$\Box_x\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = -\frac{1}{c^2}\partial_t^2\phi + \frac{1}{\rho}\partial_\rho(\rho\partial_\rho\phi) + \frac{1}{\rho^2}\partial_\varphi^2\phi + \partial_z^2\phi \tag{6.6.28}$$

the spatial part of which being, as before, the standard result from ordinary vector analysis. Its action on vectors can be written out and we leave this as a straightforward exercise.[19]

6.6.2 Rindler Coordinates

The formalism we developed applies just as well when the transformations mix the space and time coordinates. As an example, consider the two dimensional Rindler space of the previous chapter. The coordinate functions we take will be $\xi^\mu = (\eta, \xi)$ where

$$\begin{aligned} \eta &= \frac{c}{a}\tanh^{-1}\frac{ct}{x} \\ \xi &= \frac{c^2}{2a}\ln\left[\frac{a^2}{c^4}(x^2 - c^2t^2)\right] \end{aligned} \tag{6.6.29}$$

and the inverse transformations: $x^a = x^a(\xi)$

$$\begin{aligned} t &= \frac{c}{a}e^{a\xi/c^2}\sinh\frac{a\eta}{c} \\ x &= \frac{c^2}{a}e^{a\xi/c^2}\cosh\frac{a\eta}{c} \end{aligned} \tag{6.6.30}$$

Let's compute the vierbein[20]

$$\begin{aligned} \vec{e}_\eta &= (\frac{\partial t}{\partial\eta}, \frac{\partial x}{\partial\eta}) = e^{a\xi/c^2}(\cosh\frac{a\eta}{c}, c\sinh\frac{a\eta}{c}) \\ \vec{e}_\xi &= (\frac{\partial t}{\partial\xi}, \frac{\partial x}{\partial\xi}) = e^{a\xi/c^2}(\frac{1}{c}\sinh\frac{a\eta}{c}, \cosh\frac{a\eta}{c}) \end{aligned} \tag{6.6.31}$$

and its inverse

$$\vec{E}^\eta = (\frac{\partial\eta}{\partial t}, \frac{\partial\eta}{\partial x}) = e^{-a\xi/c^2}(\cosh\frac{a\eta}{c}, -\frac{1}{c}\sinh\frac{a\eta}{c})$$

[18] Problem: How many Killing vectors does the three dimensional cylindrical metric admit? Determine them.

[19] Problem: Write out $\Box_x A^\mu$ for each component of A^μ in cylindrical coordinates.

[20] Problem: Is the Rindler system orthogonal?

$$\vec{E}^\xi = (\frac{\partial\xi}{\partial t}, \frac{\partial\xi}{\partial x}) = e^{-a\xi/c^2}(-c\sinh\frac{a\eta}{c}, \cosh\frac{a\eta}{c}) \tag{6.6.32}$$

Again, it's easy to check that $\vec{e}_\mu \cdot \vec{E}^\nu = \delta^\nu_\mu$ and that $e^a_\mu E^\mu_b = \delta^a_b$. Now compute the inner products to get the metric function: $g_{tt} = -c^2 e^{2a\xi/c^2}$, $g_{\xi\xi} = e^{2a\xi/c^2}$, (all other components vanish). In matrix notation,

$$g_{\mu\nu} = \begin{bmatrix} -c^2 e^{2a\xi/c^2} & 0 \\ 0 & e^{2a\xi/c^2} \end{bmatrix} \tag{6.6.33}$$

and the distance function is given explicitly by,[21]

$$ds^2 = e^{2a\xi/c^2}(c^2 d\eta^2 - d\xi^2) \tag{6.6.34}$$

The non-vanishing components of the connections are obtained by using either $\Gamma^\mu_{\nu\kappa} = (\partial_\nu \vec{e}_\kappa) \cdot \vec{E}^\mu$ or (6.5.22),

$$\begin{aligned} \Gamma^\eta_{\eta\xi} &= \frac{a}{c^2}, \\ \Gamma^\xi_{\eta\eta} &= a \\ \Gamma^\xi_{\xi\xi} &= \frac{a}{c^2} \end{aligned} \tag{6.6.35}$$

(all others vanish), while the action of the Laplacian, $\Box_x$, on a scalar function is[22]

$$\Box_x \phi = -\frac{e^{-2a\xi/c^2}}{c^2}(\partial^2_\eta \varphi - c^2 \partial^2_\xi \varphi) \tag{6.6.36}$$

6.7 Integration

We have spent much time discussing differentiation. Let us now summarize some important aspects of integration. We will focus on defining volume integrations, integrations over hypersurfaces and general forms of the integral theorems.

6.7.1 The Levi-Civita tensor

It is useful to develop a generalization of the Levi-Civita tensor (density), which is adapted to general coordinates in four dimensions. To do so, we will make use of the permutation

[21] Problem: Determine the single Killing vector of the Rindler metric.

[22] Problem: Write out $\Box_x A^\mu$ for each component of A^μ in Rindler coordinates.

symbol in (1.3.50) and recall that this is the object one uses in the definition of the determinant, M, of a four dimensional matrix, $\widehat{M}$,

$$M = [\alpha\beta\mu\nu] M_{\alpha 0} M_{\beta 1} M_{\mu 2} M_{\nu 3} \tag{6.7.1}$$

The permutation symbol, by itself, is only a tensor if the coordinate system employed is Cartesian. In a general coordinate system,

$$\epsilon_{\alpha\beta\mu\nu} = \sqrt{-g}\ [\alpha\beta\mu\nu] \tag{6.7.2}$$

is what transforms as a tensor called the Levi-Civita tensor. The proof goes as follows: the quantity

$$[\alpha\beta\mu\nu] \frac{\partial\xi^\alpha}{\partial\xi'^\gamma} \frac{\partial\xi^\beta}{\partial\xi'^\delta} \frac{\partial\xi^\mu}{\partial\xi'^\lambda} \frac{\partial\xi^\nu}{\partial\xi'^\sigma} \tag{6.7.3}$$

is completely antisymmetric in $\{\gamma, \delta, \lambda, \sigma\}$, so it must be proportional to $[\gamma\delta\lambda\sigma]$. We say that

$$[\alpha\beta\mu\nu] \frac{\partial\xi^\alpha}{\partial\xi'^\gamma} \frac{\partial\xi^\beta}{\partial\xi'^\delta} \frac{\partial\xi^\mu}{\partial\xi'^\lambda} \frac{\partial\xi^\nu}{\partial\xi'^\sigma} = \lambda(\xi)\ [\gamma\delta\lambda\sigma], \tag{6.7.4}$$

where the proportionality factor, λ, could depend on ξ. Take $[\gamma\delta\lambda\sigma] = [0123]$, then

$$[\alpha\beta\mu\nu] \frac{\partial\xi^\alpha}{\partial\xi'^0} \frac{\partial\xi^\beta}{\partial\xi'^1} \frac{\partial\xi^\mu}{\partial\xi'^2} \frac{\partial\xi^\nu}{\partial\xi'^3} = \lambda = \left\| \frac{\partial\xi}{\partial\xi'} \right\| \tag{6.7.5}$$

Now if x^a represents a Cartesian system, in terms of which the coordinates ξ^μ and ξ'^μ are defined via invertible functions $\xi^\mu = \xi^\mu(x)$ and $\xi'^\mu = \xi'^\mu(x)$, the last determinant can be written as

$$\left\| \frac{\partial\xi}{\partial\xi'} \right\| = \left\| \frac{\partial\xi}{\partial x} \frac{\partial x}{\partial\xi'} \right\| = \left\| \frac{\partial\xi}{\partial x} \right\| \left\| \frac{\partial x}{\partial\xi'} \right\|. \tag{6.7.6}$$

Notice that under the coordinate transformation that took $x^a \to \xi^\mu$, the metric also underwent a transformation

$$\eta_{ab} \to g_{\mu\nu} = \eta_{ab} \frac{\partial x^a}{\partial\xi^\mu} \frac{\partial x^a}{\partial\xi^\mu} = \eta_{ab} e^a_\mu e^b_\mu \tag{6.7.7}$$

It follows, upon taking determinants, that

$$g = -c^2 \left\| \frac{\partial x}{\partial\xi} \right\|^2 \tag{6.7.8}$$

where we have used the fact that the determinant of the Lorentz metric is just $-c^2$. It follows that

$$\left\| \frac{\partial x}{\partial\xi} \right\| = \frac{1}{c}\sqrt{-g} \tag{6.7.9}$$

and from here we find

$$\lambda = \sqrt{\frac{g'}{g}}. \tag{6.7.10}$$

Therefore we see that $\epsilon_{\alpha\beta\mu\nu}$ of (6.7.2) transforms as a $(0,4)$ tensor,

$$\sqrt{-g}\ [\alpha\beta\mu\nu]\frac{\partial \xi^\alpha}{\partial \xi'^\gamma}\frac{\partial \xi^\beta}{\partial \xi'^\delta}\frac{\partial \xi^\mu}{\partial \xi'^\lambda}\frac{\partial \xi^\nu}{\partial \xi'^\sigma} = \sqrt{-g'}\ [\gamma\delta\lambda\sigma]. \tag{6.7.11}$$

It is straightforward now to argue in the same way that

$$\epsilon^{\alpha\beta\mu\nu} = \frac{1}{\sqrt{-g}}[\alpha\beta\mu\nu] \tag{6.7.12}$$

transforms as a $(4,0)$ tensor.[23] Any object that can be written as

$$\mathfrak{T} = (-g)^{\Delta/2}\ \mathbb{T}, \tag{6.7.13}$$

where $\mathbb{T}$ is a rank (m,n) tensor, is called a tensor **density** of weight Δ. Hence, the permutation symbol is revealed either as a covariant tensor density of weight -1 or a contravariant tensor density of weight $+1$.

6.7.2 The four dimensional Volume element

The invariant volume integral is always written as

$$\int dV = \frac{1}{c}\int (\epsilon_{abcd}e_1^a e_2^b e_3^c e_4^d) d\xi^1 d\xi^2 d\xi^3 d\xi^4 = \frac{1}{c}\int\ d^4\xi\sqrt{-g} \tag{6.7.14}$$

To see that it is indeed invariant under coordinate transformations, let $\xi \to \xi'$ and use (6.7.10) to find

$$d^4\xi = d^4\xi' \left\| \frac{\partial \xi}{\partial \xi'} \right\| = d^4\xi' \sqrt{\frac{g'}{g}}, \tag{6.7.15}$$

that is,

$$d^4\xi\sqrt{-g} = d^4\xi'\sqrt{-g'}. \tag{6.7.16}$$

Simple examples are (i) spherical coordinates, in which

$$\frac{1}{c}\int d^4\xi\sqrt{-g} = \int dt\ \int dr\ r^2 \int_0^\pi d\theta\ \sin\theta \int_0^{2\pi} d\varphi, \tag{6.7.17}$$

[23] Problem: Follow the same proof to obtain this result.

(ii) cylindrical coordinates:

$$\frac{1}{c}\int d^4\xi\sqrt{-g} = \int dt \int dz \int d\rho\ \rho^2 \int_0^{2\pi} d\varphi \tag{6.7.18}$$

and (iii) the Rindler frame,

$$\frac{1}{c}\int d^2\xi\sqrt{-g} = \int d\eta \int d\xi\ e^{a\xi/c^2}. \tag{6.7.19}$$

However, (6.7.14) has general applicability.

6.7.3 Three dimensional Hypersurfaces

Any hypersurface, Σ, of dimension one less than then dimension of space-time, (called a hypersurface of **codimension** one) can be specified by a constraint of the form

$$\Phi(\xi) = \text{constant} \tag{6.7.20}$$

The normal to this hypersurface is given by the vector $n_\mu \sim \partial_\mu\Phi$.

Spacelike or Timelike hypersurfaces

If n_μ is not null, we will take it to be the *unit* normal, *i.e.,* we let

$$n_\mu = \epsilon N\partial_\mu\Phi, \quad n^\mu n_\mu = \epsilon \tag{6.7.21}$$

where $N > 0$ is a normalization and $\epsilon = \pm 1$ is positive if the normal is spacelike and negative if it is time-like. The unit normal has been defined so that $n\cdot\partial\Phi > 0$, *i.e.,* n_μ always points in the direction of increasing Φ. Hypersurfaces with time-like normals will be called spacelike and, vice-versa, hypersurfaces with spacelike normals will be called time-like. Null surfaces are more subtle because ϵ vanishes. Confining our attention, for the present, to hypersurfaces that are either spacelike or time-like, the normalization, N, will be given by

$$N^{-2} = \epsilon g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi, \quad \epsilon = \pm 1 \tag{6.7.22}$$

and

$$h_{\mu\nu} = g_{\mu\nu} - \epsilon\ n_\mu n_\nu \tag{6.7.23}$$

projects any vector onto the hypsersurface because the component perpendicular to it is annihilated by $h_{\mu\nu}$. In principle, the constraint $\Phi(\xi) = $ const. could be solved (as we have done in many examples before) by introducing parametric equations of the form

$$\xi^\mu = \xi^\mu(y^a) \tag{6.7.24}$$

where y^a are coordinates "on" (intrinsic to) the hypersurface.[24] Consider, then, a transformation from the original coordinates, ξ^μ, to $\{\Phi, y^a\}$ (although Φ is placed in the first slot here, we do not imply that Φ is a time-like coordinate, merely that we are singling it out as being the coordinate orthogonal to our hypersurface). In the new coordinate system,

$$\begin{aligned} n'_\Phi &= \partial_\Phi \xi^\alpha n_\alpha = \epsilon N \frac{\partial \xi^\alpha}{\partial \Phi}\frac{\partial \Phi}{\partial \xi^\alpha} = \epsilon N \\ n'_a &= \partial_a \xi^\alpha n_\alpha = \epsilon N \frac{\partial \xi^\alpha}{\partial y^a}\frac{\partial \Phi}{\partial \xi^\alpha} = 0 \end{aligned} \tag{6.7.25}$$

where we have used the usual transformation properties of covariant vectors. Define $Q^a_\alpha = \partial_\alpha y^a$ and $q^\alpha_a = \partial_a \xi^\alpha$, then $Q^a_\alpha q^\alpha_b = \delta^a_b$ is the completeness relation. The second equation in (6.7.25) shows that $q^\alpha_a n_\alpha = 0$, so $\vec{q}_{(a)} = q^\alpha_a \vec{u}_{(\alpha)}$ is a coordinate basis in Σ and $\vec{Q}^{(a)} = Q^a_\alpha \vec{\theta}^{(\alpha)}$ will be its dual basis. The distance between two points in Σ is

$$ds^2_\Sigma = -g_{\mu\nu} d\xi^\mu d\xi^\nu = -g_{\mu\nu} q^\mu_a q^\nu_b dy^a dy^b \stackrel{\text{def}}{=} -\gamma_{ab} dy^a dy^b. \tag{6.7.26}$$

$\gamma_{ab} = g_{\mu\nu} q^\mu_a q^\nu_b = h_{\mu\nu} q^\mu_a q^\nu_b$ is called the **induced metric** on Σ. An explicit form for the metric $g_{\mu\nu}$ in the system $\{\Phi, y^a\}$ may be given as follows: under the coordinate transformation from $\xi^\mu \to \{\Phi, y^a\}$, the contravariant components of the metric transform as

$$\begin{aligned} g'^{\Phi\Phi} &= g^{\mu\nu}\partial_\mu \Phi \partial_\nu \Phi = \frac{\epsilon}{N^2} \\ g'^{\Phi a} &= g^{\mu\nu}\partial_\mu \Phi \partial_\nu y^a = \frac{\epsilon}{N} n \cdot \partial y^a \stackrel{\text{def}}{=} \frac{N^a}{N^2} \\ g'^{ab} &= g^{\mu\nu}\partial_\mu y^a \partial_\nu y^b \stackrel{\text{def}}{=} \Sigma^{ab}. \end{aligned} \tag{6.7.27}$$

In doing so we have introduced three new functions, *viz.*, $N^a = \epsilon N n^\mu Q^a_\mu$ in the second line above; the metric has been decomposed into the six independent components of Σ^{ab}, the normalization function N and the three functions N^a. In matrix form we may write

$$g'^{\alpha\beta} = \begin{pmatrix} \frac{\epsilon}{N^2} & \frac{N^b}{N^2} \\ & \\ \frac{N^a}{N^2} & \Sigma^{ab} \end{pmatrix}. \tag{6.7.28}$$

Take the covariant components of the metric to be of the form

$$g'_{\alpha\beta} = \begin{pmatrix} A & B_b \\ & \\ B_a & \gamma_{ab} \end{pmatrix} \tag{6.7.29}$$

[24] Consider the sphere in three dimensions, defined by the constraint $\sum_i x_i^2 = r^2$ and parametrize it in the usual way: $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$ and $z = r\cos\theta$. The "latitude", θ, and "longitude", φ, are intrinsic coordinates on the sphere.

(its spatial components follow from the definition of the induced metric) and require that $g'^{\alpha\beta}g'_{\beta\gamma} = \delta^\alpha_\gamma$. We find the conditions,

$$\begin{aligned}
&\epsilon A + N^b B_b = N^2 \\
&\epsilon B_c + N_c = 0 \\
&N^a A + N^2 \Sigma^{ab} B_b = 0 \\
&N^a B_c + N^2 \Sigma^{ab}\gamma_{bc} = N^2 \delta^a_c
\end{aligned} \tag{6.7.30}$$

where we define $N_a = \gamma_{ab}N^b$. By the second equation, $B_a = -\epsilon N_a$ and therefore, by the first, $A = \epsilon N^2 + N^b N_b$. The last equation then yields

$$\Sigma^{ab} = \gamma^{ab} + \epsilon \frac{N^a N^b}{N^2} \tag{6.7.31}$$

where γ^{ab} is the inverse of the induced metric, defined in the usual way as $\gamma^{ab}\gamma_{bc} = \delta^a_c$. Returning to (6.7.27),

$$\Sigma^{ab} = g^{\mu\nu} Q^a_\mu Q^b_\nu = \gamma^{ab} + \epsilon \frac{N^a N^b}{N^2} \;\Rightarrow\; \gamma^{ab} = h^{\mu\nu} Q^a_\mu Q^b_\nu \tag{6.7.32}$$

gives a compact form for γ^{ab}.

We define the directed hypersurface integration measure to be

$$\int d\Sigma_\mu = \int \epsilon_{\mu\alpha\beta\gamma} q^\alpha_1 q^\beta_2 q^\gamma_3 dy^1 dy^2 dy^3. \tag{6.7.33}$$

The quantity $\chi_\mu = \epsilon_{\mu\alpha\beta\gamma} q^\alpha_1 q^\beta_2 q^\gamma_3$ must be proportional to n_μ, because $q^\mu_b \chi_\mu = 0$ by the antisymmetry of the Levi-Civita tensor. Therefore, we set $\chi_\mu = \lambda n_\mu$, which means that

$$\epsilon\lambda = \chi_\mu n^\mu = \sqrt{-g}[\mu\alpha\beta\gamma] n^\mu q^\alpha_1 q^\beta_2 q^\gamma_3. \tag{6.7.34}$$

The inner product is simplest to evaluate in the $\{\Phi, y^a\}$ system; here $q^\alpha_1 = \delta^\alpha_1$ etc., and we find

$$\epsilon\lambda = \sqrt{-g} n^\Phi = \sqrt{-g} g^{\Phi\Phi} \epsilon N = \frac{1}{N}\sqrt{-g} = \sqrt{|\gamma|}, \tag{6.7.35}$$

from which it follows that

$$\int d\Sigma_\mu = \epsilon \int d^3y \sqrt{|\gamma|}\; n_\mu. \tag{6.7.36}$$

The metric γ_{ab} will be of indefinite signature if the hypersurface is time-like and of Euclidean signature if the hypersurface is spacelike; that is why we have introduced the absolute value of the determinant under the square root.

Null Hypersurfaces

While one cannot define a *unit* normal for a null hypersurface, $n_\mu = -\Phi_{,\mu}$ is in fact the normal vector. Because it is null, the vector is orthogonal to itself and *also tangent to the null hypersurface*. First, let us show that n_μ is parallel transported along its integral curves on the hypersurface. Let λ represent the parameter along the integral curves of n^μ, *i.e.*, $\partial\xi^\mu/\partial\lambda = n^\mu$ and consider

$$\frac{Dn_\mu}{D\lambda} = (n\cdot\nabla)n_\mu. \tag{6.7.37}$$

However,

$$(n\cdot\nabla)n_\mu = -n^\lambda\Phi_{,\mu;\lambda} = -n^\lambda\Phi_{,\lambda;\mu} = -n^\lambda\nabla_\mu n_\lambda = -\frac{1}{2}\nabla_\mu n^2 \tag{6.7.38}$$

(assuming a torsion free connection). But, because $n^2 = 0$ on the surface, we could set $n^2 = -2\kappa(\xi)\Phi(\xi)$ (assuming $\Phi(\xi) = 0$ descibes the constraint surface) for some scalar function, κ. Then $(n\cdot\nabla)n_\mu$ must be proportional to n_μ,

$$(n\cdot\nabla)n_\mu = \kappa n_\mu \tag{6.7.39}$$

on $\Phi(\xi) = 0$. Thus n_μ is parallel transported along its integral curves on the constraint surface,

$$\frac{Dn_\mu}{D\lambda} = (n\cdot\nabla)n_\mu = \kappa n_\mu. \tag{6.7.40}$$

The parameter λ is not affine in general, if it is then $\kappa(\xi) = 0$. A convenient choice of coordinates on the null hypersurface is to pick one of them, say y^1, to be λ and the remaining two to span the space transverse to n^μ. In that case, $q_1^\mu = \partial\xi^\mu/\partial\lambda = n^\mu$. In follows that $\gamma_{11} = 0 = \gamma_{1A}$, $A \in \{2,3\}$. The first is because n^μ is null, the second because $\gamma_{1A} = g_{\mu\nu}n^\mu(\partial\xi^\nu/\partial y^A)$ and $\partial\xi^\nu/\partial y^A$ is orthogonal to n^μ. The induced metric,

$$ds_\Sigma^2 = \sigma_{AB}dy^A dy^B, \tag{6.7.41}$$

where

$$\sigma_{AB} = g_{\mu\nu}\left(\frac{\partial\xi^\mu}{\partial y^A}\right)\left(\frac{\partial\xi^\nu}{\partial y^B}\right), \tag{6.7.42}$$

is therefore two dimensional.[25]

As before, we define the directed hypersurface volume element by

$$d\Sigma_\mu = \epsilon_{\mu\nu\alpha\beta}q_1^\nu q_2^\alpha q_3^\beta \tag{6.7.43}$$

[25] To project onto the hypersurface, one must introduce an auxilliary null vector field, n_μ, such that $n\cdot k = -1$. This choice is not unique, but with its help we can define

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu k_\nu + k_\mu n_\nu,$$

which clearly projects transverse to both k and n, so it is a two dimensional projector, $h^\alpha{}_\alpha = 2$.

and evaluate it explicitly in the coordinate system $\{\Phi, \lambda, y^1, y^2\}$. In this coordinate system,

$$\begin{aligned} g'^{\Phi\Phi} &= g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi = 0 \\ g'^{\Phi\lambda} &= g^{\mu\nu}\partial_\mu\Phi\partial_\nu\lambda = -n\cdot\partial\lambda = -1 \\ g'^{\Phi A} &= g^{\mu\nu}\partial_\mu\Phi\partial_\nu y^A = -n\cdot\partial y^A = 0 \\ g'^{\lambda\lambda} &= g^{\mu\nu}\partial_\mu\lambda\partial_\nu\lambda \stackrel{\text{def}}{=} N^\lambda \\ g'^{\lambda A} &= g^{\mu\nu}\partial_\mu\lambda\partial_\nu y^A \stackrel{\text{def}}{=} N^A \\ g'^{AB} &= g^{\mu\nu}\partial_\mu y^A\partial_\nu y^B \stackrel{\text{def}}{=} \Sigma^{AB}. \end{aligned} \tag{6.7.44}$$

that is,

$$g'^{\alpha\beta} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & N^\lambda & N^B \\ 0 & N^A & \Sigma^{AB} \end{pmatrix} \tag{6.7.45}$$

and it follows that the non-vanishing components of the normal in this system are

$$n_\Phi = -1, \quad n^\lambda = 1. \tag{6.7.46}$$

Making the same arguments as before we write

$$fn_\mu = \epsilon_{\mu\nu\alpha\beta}n^\nu q_1^\alpha q_2^\beta \tag{6.7.47}$$

but, because n_μ is null, we cannot find f as we did before. Instead, introduce an auxilliary null vector k^μ, such that $n\cdot k = -1$. Then

$$f = -\epsilon_{\mu\nu\alpha\beta}k^\mu n^\nu q_1^\alpha q_2^\beta = -\sqrt{-g}[\mu\nu\alpha\beta]k^\mu n^\nu q_1^\alpha q_2^\beta \tag{6.7.48}$$

In our preferred coordinates, a suitable choice for the auxilliary vector would be to take the only non-vanishing components of k to be $k_\lambda = -1$. We see that $f = \sqrt{-g}$. From (6.7.45), then $-g = \sigma$, where σ_{AB} is the induced metric on the null hypersurface. Therefore, we have

$$\int d\Sigma_\mu = \int_\Sigma d\lambda d^2y\sqrt{\sigma}n_\mu, \tag{6.7.49}$$

which is similar to (6.7.36).

A simple example should serve to illustrate the above constructions. Consider the following constraints surface on four dimensional Minkowski space,

$$\Phi(t,x) = c^2t^2 - \sum_i x^{i2} = \begin{cases} -a^2 \\ +a^2 \end{cases} \tag{6.7.50}$$

If $a \neq 0$, the normal to the hypersurface described by this constraint is evidently

$$n_\mu = \epsilon N \partial_\mu \Phi = \frac{\epsilon}{|a|}(c^2 t, -x^1, -x^2, -x^3) \tag{6.7.51}$$

which is spacelike in the first case and timelike in the second. Let us consider the timelike hypersurface, given by

$$\Phi(t, x) = c^2 t^2 - \sum_i x^{i2} = -a^2 \tag{6.7.52}$$

and choose, of many possible parametrizations, the following:

$$ct = a \sinh c\eta/a, \quad \vec{x} = a \cosh c\eta/a(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \tag{6.7.53}$$

so that the hypersurface coordinates are $\{\eta, \theta, \varphi\}$. Then

$$q^\mu{}_a = \begin{pmatrix} \cosh(c\eta/a) & 0 & 0 \\ c\sinh(c\eta/a)\sin\theta\cos\varphi & a\cosh(c\eta/a)\cos\theta\cos\varphi & -a\cosh(c\eta/a)\sin\theta\sin\varphi \\ c\sinh(c\eta/a)\sin\theta\sin\varphi & a\cosh(c\eta/a)\cos\theta\sin\varphi & a\cosh(c\eta/a)\sin\theta\cos\varphi \\ c\sinh(c\eta/a)\cos\theta & -a\cosh(c\eta/a)\sin\theta & 0 \end{pmatrix} \tag{6.7.54}$$

and we find the induced metric on the three dimensional hypersurface,

$$\gamma_{ab} = \eta_{\mu\nu} q^\mu{}_a q^\mu{}_b = \begin{pmatrix} -c^2 & 0 & 0 \\ 0 & a^2\cosh^2(c\eta/a) & 0 \\ 0 & 0 & a^2\cosh^2(c\eta/a)\sin^2\theta \end{pmatrix} \tag{6.7.55}$$

The hypersurface line element,

$$ds_\Sigma^2 = c^2 d\eta^2 - \cosh^2(c\eta/a) d\Omega^2 \tag{6.7.56}$$

represents a three dimensional maximally symmetric space called "de Sitter space", in global (isotropic) coordinates. An alternative parametrization of the same constraint:

$$ct = \sqrt{a^2 - r^2}\sinh\eta/a, \quad x^1 = \sqrt{a^2 - r^2}\cosh\eta/a, \quad x^2 = r\sin\theta, \; x^3 = r\cos\theta \tag{6.7.57}$$

($r < a$) uses the hypersurface coordinates $\{\eta, r, \theta\}$ and gives the line element

$$ds_\Sigma^2 = c^2\left(1 - \frac{r^2}{a^2}\right)d\eta^2 - \left(1 - \frac{r^2}{a^2}\right)^{-1} dr^2 - r^2 d\theta^2, \tag{6.7.58}$$

which is also de Sitter space, this time in static coordinates.

If $a = 0$, we have a null surface. In this case, $n_\mu = (-c^2 t, x^1, x^2, x^3)$ is orthogonal to the surface but cannot be normalized. Note that

$$\frac{\partial x^\mu}{\partial \lambda} = n^\mu \;\Rightarrow\; t = \frac{f(y^A)}{c} e^\lambda, \tag{6.7.59}$$

where $f(y^A)$ is any function of the transverse coordinates, which we may take to be a constant (say a). This suggests the following global parametrization of the constraint:

$$t = \frac{a}{c} e^\lambda, \quad \vec{x} = a e^\lambda (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \tag{6.7.60}$$

and gives

$$q^\mu{}_a = a e^\lambda \begin{pmatrix} \frac{1}{c} & 0 & 0 \\ \sin\theta\sin\varphi & \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}, \tag{6.7.61}$$

whence

$$\sigma_{AB} = \eta_{\mu\nu} q^\mu_A q^\nu_B = a^2 e^{2\lambda} \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}. \tag{6.7.62}$$

Other parametrizations are possible.[26]

6.7.4 Gauss' Theorem

We may now state Gauss' theorem.

- Let M be any region of space-time bounded by a closed hypersurface ∂M; then for any differentiable vector field V^α defined in M,

$$\int_M d^4x \sqrt{-g} V^\alpha{}_{;\alpha} = \oint_{\partial M} d\Sigma_\alpha V^\alpha \tag{6.7.63}$$

where n_α is the *outward* normal to the boundary, ∂M of M.

---

[26] Problem: For example, take

$$t = \frac{a}{c} e^\lambda, \quad \vec{x} = e^\lambda \left(\sqrt{a^2 - r^2}, r\sin\theta, r\cos\theta \right)$$

$(r < a)$ and show that

$$\sigma_{AB} = e^{2\lambda} \begin{pmatrix} \left(1 - \frac{r^2}{a^2}\right)^{-1} & 0 \\ 0 & r^2 \end{pmatrix}$$

i.e.,

$$ds^2_\Sigma = e^{2\lambda} \left(\frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2 d\theta^2 \right).$$

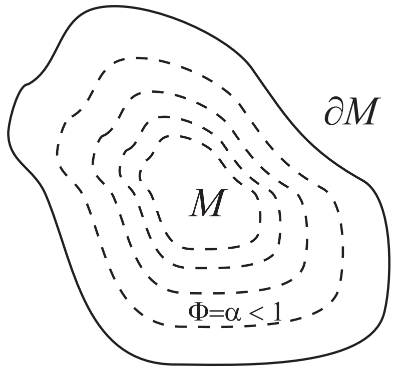

Figure 6.3: The volume M is foliated by closed surfaces $\Phi = \alpha < 1$, with $\Phi = 1$ being the boundary, ∂M.

To prove this theorem, let's consider the boundary of the region M to be given by the constraint $\Phi = 1$. Now $\Phi = \alpha$ defines a hypersurface, $\mathcal{H}_\alpha$, of codimension one for every α. Suppose Φ varies from zero to one (more generally, *any* closed interval in $\mathbb{R}$) in such a way that the union of the set of hypersurfaces, $\mathcal{H}_\alpha$, is M, *i.e.*,

$$M = \bigcup_{\alpha\in[0,1]} \mathcal{H}_\alpha. \tag{6.7.64}$$

We say that the hypersurfaces $\mathcal{H}_\alpha$ "foliate" our volume M, with $\Phi = 1$ being the outermost layer of the foliation and coinciding with the boundary, ∂M, of M, as shown in figure 6.3. The coordinates on the closed hypersurface, y^a, will all be compact coordinates because the hypersurfaces $\Phi = \alpha$ are all closed. Consider the volume integral in the coordinates $\xi := \{\Phi, y^a\}$,

$$\begin{aligned}
\int_M d^4\xi\sqrt{-g}\; V^\alpha{}_{;\alpha} &= \int d^4\xi\partial_\alpha(\sqrt{-g}V^\alpha) \\
&= \int_0^1 d\Phi \oint d^3y\partial_\Phi(\sqrt{-g}V^\Phi) + \int_0^1 d\Phi \oint d^3y\partial_a(\sqrt{-g}V^a) \\
&= \int_0^1 d\Phi\frac{d}{d\Phi}\oint d^3y\sqrt{-g}V^\Phi \\
&= \oint d^3y\sqrt{-g}V^\Phi\Big|_0^1 = \oint_{\partial M} d^3y\sqrt{-g}V^\Phi \\
&= \oint_{\partial M} d^3yN\sqrt{|\gamma|}V^\Phi = \epsilon\oint_{\partial M} d^3y\sqrt{|\gamma|}\; n_\alpha V^\alpha \\
&= \oint_{\partial M} d\Sigma_\alpha V^\alpha.
\end{aligned} \tag{6.7.65}$$

The second line in the proof follows simply by applying the divergence formula of (6.5.59). The second integral in this line is vanishing because the coordinates y^a are all compact. The rest is a straightforward application of the fundamental theorem of calculus when the hypersurface at $\Phi = 0$ is taken to have zero volume.

6.7.5 Two dimensional Hypersurfaces

A two dimensional surface, S, (or, more generally, a hypersurface of codimension two) can be defined in a similar manner, by imagining it to be embedded in a three dimensional hypersurface. As before, let S be specified by a constraint of the form $\Psi(y^a) = \text{constant}$ and let $r_a \sim \partial_a \Psi$ be the unit normal to S. The projector onto S is $h_{ab} = \gamma_{ab} - \epsilon' r_a r_b$, where $\epsilon' = \mp 1$, depending on whether r_a is timelike or spacelike. Let $y^a = y^a(\theta^A)$ solve the constraint so that θ^A are coordinates intrinsic to S and define $P^A_a = \partial_a \theta^A$, $p^a_A = \partial_A y^a$ as before. Then

$$\sigma_{AB} = h_{ab} p^a_A p^b_B \tag{6.7.66}$$

will be the induced metric on S with inverse $\sigma^{AB} = h^{ab} P^A_a P^B_b$. Now S admits two mutually perpendicular normals, because $r^\mu = q^\mu_a r^a = q^\mu_a \gamma^{ab} r_b$ satisfies $r^\mu n_\mu = 0$. We define the integration measure in S by

$$\int dS_{\mu\nu} = \frac{1}{2}\epsilon\epsilon' \int_S d^2\theta \sqrt{|\sigma|}\; n_{[\mu} r_{\nu]} \tag{6.7.67}$$

where σ is the determinant of the induced metric on S, and ϵ (ϵ') are defined according to whether n_μ (r_μ) are timelike or spacelike respectively. If one of them is timelike then the other must be spacelike and $\epsilon\epsilon' = -1$. However, both ϵ and ϵ' may be spacelike, in which case $\epsilon\epsilon' = +1$.

6.7.6 Stokes' Theorem

Stokes theorem concerns the integration of an anisymmetric tensor field on a surface of codimension two.

- Let R be any region in a codimension one hypersurface, Σ, bounded by a closed (codimension two) hypersurface S and let $B^{\alpha\beta}$ be any antisymmetric tensor field in Σ, then

$$\int_R d\Sigma_\alpha \nabla_\beta B^{\alpha\beta} = \oint_S dS_{\alpha\beta} B^{\alpha\beta} \tag{6.7.68}$$

The proof follows the same pattern as the proof of Gauss' theorem.[27] We consider the hypersurface integral in the coordinates $\{\Phi, \Psi, \theta^A\}$ and employ the expression for the

[27] This is to be expected as they are both particular cases of one theorem, the Gauss-Stokes theorem, which is easiest to prove using differential forms.

divergence of an antisymmetric tensor. We imagine, as before, that the boundary of Σ is given by the constraint $\Psi = 1$ and let Ψ vary from zero to unity so that surfaces of constant Ψ foliate the hypersurface with $\Psi = 1$ being the outermost layer of the foliation and coinciding with S. The coordinates θ^A are all compact because S is closed. Then

$$\begin{aligned}\int_R d\Sigma_\alpha \nabla_\beta B^{\alpha\beta} &= \epsilon \int d^3y\sqrt{|\gamma|}n_\alpha \frac{1}{\sqrt{|g|}}\partial_\beta\left(\sqrt{|g|}B^{\alpha\beta}\right) = \int d^3y\ \partial_a\left(\sqrt{|g|}B^{\Phi a}\right)\\ &= \int d^2\theta \int_0^1 d\Psi \partial_\Psi\left(\sqrt{|g|}B^{\Phi\Psi}\right) + \int_0^1 d\Psi \int d^2\theta\partial_A\left(\sqrt{|g|}B^{\Phi A}\right)\\ &= \oint d^2\theta\left(\sqrt{|g|}B^{\Phi\Psi}\right)_0^1\\ &= \oint_S d^2\theta N N'\sqrt{|\sigma|}B^{\Phi\Psi} = \frac{1}{2}\epsilon\epsilon' \oint_S d^2\theta\sqrt{|\sigma|}\ n_{[\mu}r_{\nu]}B^{\mu\nu}\\ &= \int_S dS_{\mu\nu}B^{\mu\nu}\end{aligned} \tag{6.7.69}$$

where we used $n_\Phi = \epsilon N$, $r_\Psi = \epsilon' N'$ and took the surface with $\Psi = 0$ to have zero area.

6.8 Riemann Curvature

The quantity that distinguishes between "flat" and "curved" space-times is the Riemann curvature. As in the case of gauge theories, the covariant derivative does not commute with itself except when it acts on scalars (assuming a torsion free connection). Indeed, if we consider the action of its commutator on a vector we find an expression of the form

$$[\nabla_\mu, \nabla_\nu]A^\alpha = R^\alpha{}_{\beta\mu\nu}A^\beta \tag{6.8.1}$$

where (using commas for partial derivatives)

$$R^\alpha{}_{\beta\mu\nu} \stackrel{\text{def}}{=} \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\eta_{\beta\nu}\Gamma^\alpha_{\eta\mu} - \Gamma^\eta_{\beta\mu}\Gamma^\alpha_{\eta\nu} \tag{6.8.2}$$

is called the **Riemann** or **curvature** tensor. It transforms as a mixed tensor of rank four and contains second order derivatives of the metric. It can also be shown that

$$[\nabla_\mu, \nabla_\nu]A_\alpha = R_\alpha{}^\beta{}_{\mu\nu}A_\beta = -R^\beta{}_{\alpha\mu\nu}A_\beta \tag{6.8.3}$$

and, more generally, that the action of the commutator on a general mixed tensor follows the same pattern:

$$[\nabla_\mu, \nabla_\nu]T^{\alpha_1\ldots\alpha_n}{}_{\beta_1\ldots\beta_m} = R^{\alpha_1}{}_{\sigma_1\mu\nu}T^{\sigma_1\alpha_2\ldots\alpha_n}{}_{\beta_1\ldots\beta_m}\ldots - R^{\sigma_1}{}_{\beta_1\mu\nu}T^{\alpha_1\ldots\alpha_n}{}_{\sigma_1\beta_2\ldots\beta_m}, \tag{6.8.4}$$

with one Riemann tensor appearing for each index (upper and lower) of $\mathbb{T}$ exactly as in (6.8.1) and (6.8.3). Let us now see that the Riemann tensor depends only on the spin connection and its derivatives. Consider

$$[\nabla_\mu, \nabla_\nu]A^\alpha = [\nabla_\mu, \nabla_\nu]E^\alpha_a A^a = E^\alpha_a [\nabla_\mu, \nabla_\nu]A^a \tag{6.8.5}$$

where we used the fact that the vierbein (and its inverse) are covariantly conserved. Expanding the last expression,

$$[\nabla_\mu, \nabla_\nu]A^\alpha = E^\alpha_a e^b_\beta \left(\partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b} + \omega^a_{\mu c}\omega^c_{\nu b} - \omega^a_{\nu c}\omega^c_{\mu b}\right) A^\beta \tag{6.8.6}$$

where we assume a torsion free connection. Comparing this with our definition of the Riemann tensor in (6.8.2),

$$R^\alpha{}_{\beta\mu\nu} = E^\alpha_a e^b_\beta \left(\partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b} + \omega^a_{\mu c}\omega^c_{\nu b} - \omega^a_{\nu c}\omega^c_{\mu b}\right) = E^\alpha_a e^b_\beta R^a{}_{b\mu\nu} \tag{6.8.7}$$

It follows that if a global Cartesian frame is available ($\omega^a_{\mu b} = 0$) then the Riemann tensor must vanish for, as a tensor, if it vanishes identically in one frame then it must also vanish in every other frame.

6.8.1 Algebraic Symmetries and Bianchi Identities

The Riemann tensor possesses some algebraic symmetries that are worth taking note of because they greatly reduce the number of *independent* components of this tensor. These identities can be verified by direct computation: $R_{\alpha\beta\mu\nu} = g_{\alpha\eta}R^\eta{}_{\beta\mu\nu}$:

- $R_{\alpha\beta\mu\nu}$ is antisymmetric in (μ, ν), *i.e.*, $R_{\alpha\beta\{\mu\nu\}} = 0$,
- $R_{\alpha\beta\mu\nu}$ is antisymmetric in (α, β) *i.e.*, $R_{\{\alpha\beta\}\mu\nu} = 0$,
- $R_{\alpha\{\beta\mu\nu\}} = 0$ and
- $R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$.

The first of these follows from the definition of the Riemann curvature. The second is a consequence of the metricity condition,

$$[\nabla_\mu, \nabla_\nu]g_{\alpha\beta} = 0 = R^\sigma{}_{\alpha\mu\nu}g_{\sigma\beta} + R^\sigma{}_{\beta\mu\nu}g_{\alpha\sigma}. \tag{6.8.8}$$

The third is a Bianchi identity, which we will derive below, and the last is not independent but can be obtained using the first three.

The Bianchi identities for the Riemann tensor follow from the Jacobi identity

$$\{[[\nabla_\mu, \nabla_\nu], \nabla_\lambda] + [[\nabla_\lambda, \nabla_\mu], \nabla_\nu] + [[\nabla_\nu, \nabla_\lambda], \nabla_\mu]\}T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_m} = 0 \tag{6.8.9}$$

as did the Bianchi identities for the Maxwell tensor. Consider the first term when $\mathbb{T}$ is a vector A_α,

$$\begin{aligned}
[[\nabla_\mu, \nabla_\nu], \nabla_\lambda] A_\alpha &= [\nabla_\mu, \nabla_\nu](\nabla_\lambda A_\alpha) - \nabla_\lambda([\nabla_\mu, \nabla_\nu] A_\alpha) \\
&= R^\sigma{}_{\lambda\mu\nu} \nabla_\sigma A_\alpha + R^\sigma{}_{\alpha\mu\nu} \nabla_\lambda A_\sigma - \nabla_\lambda (R^\sigma{}_{\alpha\mu\nu} A_\sigma) \\
&= +R^\sigma{}_{\lambda\mu\nu} \nabla_\sigma A_\alpha - \nabla_\lambda (R^\sigma{}_{\alpha\mu\nu}) A_\sigma.
\end{aligned} \tag{6.8.10}$$

Now add all terms in (6.8.9) and get (using semi-colons for covariant derivatives)

$$(R^\sigma{}_{\lambda\mu\nu} + R^\sigma{}_{\nu\lambda\mu} + R^\sigma{}_{\mu\nu\lambda}) A_{\alpha;\sigma} - (R^\sigma{}_{\alpha\mu\nu;\lambda} + R^\sigma{}_{\alpha\lambda\mu;\nu} + R^\sigma{}_{\alpha\nu\lambda;\mu}) A_\sigma = 0 \tag{6.8.11}$$

but, since A_α is arbitrary, this is only possible if each term vanishes; the first gives just the third of the identities listed above and the last gives a new (differential) identity

- $R^\sigma{}_{\alpha\{\mu\nu;\lambda\}} = 0$.

The five bulleted identities above are worth remembering.

6.8.2 Independent Components

Let us now count the number of independent components of the Riemann tensor. The first and second identities say that the first and second pairs of indices each satisfy $n(n+1)/2$ conditions (in n dimensions, 10 in four dimensions) and so, with these two conditions we should have only

$$\left[n^2 - \frac{n(n+1)}{2}\right]^2 = \left[\frac{n(n-1)}{2}\right]^2 \tag{6.8.12}$$

independent components. However, the antisymmetry implied by the third identity is an additional set of $n^2(n-1)(n-2)/6$ conditions and so the number of independent components will be

$$\left[\frac{n(n-1)}{2}\right]^2 - \frac{1}{6} n^2 (n-1)(n-2) = \frac{1}{12} n^2 (n^2 - 1) \tag{6.8.13}$$

independent components, *i.e.,* 20 in four dimensions. This is far greater than the $n(n+1)/2$ independent components (10 in four dimensions) of the metric tensor which, in Einstein's theory, describes the gravitational field and for which we will eventually seek dynamical equations. Therefore the full Riemann tensor cannot be required for a complete set of equations governing the gravitational field. Instead, the following "contraction" of $R^\alpha{}_{\beta\mu\nu}$,

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\eta_{\mu\nu} \Gamma^\alpha_{\eta\alpha} - \Gamma^\eta_{\mu\alpha} \Gamma^\alpha_{\eta\nu}, \tag{6.8.14}$$

called the **Ricci tensor** is symmetric in its two indices (one can show this by direct computation, using the listed properties of the Riemann tensor) and therefore has precisely the required $n(n+1)/2$ independent components. Another contraction gives the **scalar curvature** or **Ricci scalar**,

$$R = g^{\mu\nu} R_{\mu\nu}, \tag{6.8.15}$$

Both the scalar curvature and Ricci tensor will play important roles in the gravitational field equations.

6.9 Curved Space Examples

To accustom ourselves to working with the connections and curvature tensors, let us take a few simple examples of the non-trivial, curved space-times that we will encounter in future chapters.

6.9.1 Homogeneous and Isotropic Metrics

If the universe is assumed to be homogeneous and isotropic, then one can show that space-time may be described by a metric of the form

$$ds^2 = c^2 dt^2 - a^2(t)\left[d\chi^2 + f^2(\chi) d\Omega^2\right] \tag{6.9.1}$$

where $a(t)$ is a function of time, $d\Omega$ is the two dimensional solid angle and

$$f(\chi) = \begin{cases} \sin\chi & \chi \in [0, \pi/2) \\ \chi & \chi \in [0, \infty) \\ \sinh\chi & \chi \in [0, \infty) \end{cases} \tag{6.9.2}$$

One should think of $a(t)$ as a time dependent "scale factor", whose effect is to scale the spatial metric in a time dependent way. With $f(\chi) = \sin\chi$ the spatial metric itself represents a three sphere, S^3, of unit radius. Likewise, with $f(\chi) = \chi$ the spatial metric is that of $\mathbb{R}^3$ (in spherical coordinates) and, with $f(\chi) = \sinh\chi$ the spatial metric is hyperbolic. These different cases describe the "closed", "flat" and "open" universes respectively.

The unique torsion free connection has non-vanishing components

$$\begin{aligned}
&\Gamma^t_{\chi\chi} = \frac{1}{c^2} a\dot{a}, \quad \Gamma^t_{\theta\theta} = \frac{1}{c^2} a\dot{a} f^2, \quad \Gamma^t_{\theta\theta} = \frac{1}{c^2} a\dot{a} f^2 \sin^2\theta \\
&\Gamma^\chi_{t\chi} = \Gamma^\chi_{\chi t} = \frac{\dot{a}}{a}, \quad \Gamma^\chi_{\theta\theta} = -ff', \quad \Gamma^\chi_{\phi\phi} = -ff'\sin^2\theta \\
&\Gamma^\theta_{t\theta} = \Gamma^\theta_{\theta t} = \frac{\dot{a}}{a}, \quad \Gamma^\theta_{\chi\theta} = \Gamma^\theta_{\theta\chi} = \frac{f'}{f}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \\
&\Gamma^\phi_{t\phi} = \Gamma^\phi_{\phi t} = \frac{\dot{a}}{a}, \quad \Gamma^\phi_{\chi\phi} = \Gamma^\phi_{\phi\chi} = \frac{f'}{f}, \quad \Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta
\end{aligned} \tag{6.9.3}$$

The Riemann tensor, Ricci tensor and scalar curvature can be evaluated from (6.8.2), (6.8.14) and (6.8.15) respectively; in particular the scalar curvature,

$$R = \frac{1}{c^2 f^2 a^2}\left[2c^2(1-f'^2) + 6f^2(\dot{a}^2 + a\ddot{a}) - 4c^2 f f''\right] \tag{6.9.4}$$

is non-vanishing. Even if $a(t) = a$ (constant) we find

$$R = \begin{cases} +\frac{6}{a^2} & f(\chi) = \sin\chi \\ 0 & f(\chi) = \chi \\ -\frac{6}{a^2} & f(\chi) = \sinh\chi \end{cases} \tag{6.9.5}$$

showing that only if $f(\chi) = \chi$ is the space-time of zero curvature. This is, of course, just Minkowski space in spherical coordinates and χ is the radial coordinate. In this case one also finds that all components of the Riemann and Ricci tensor vanish.

An equivalent approach is via the vierbeins and spin connections. There is no unique choice for e^a_μ because it depends on how we choose to define the LLF at each point. One choice that generalizes the flat space vierbeins for spherical coordinates is

$$\widetilde{e}^a_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(t)\sin\theta\cos\phi & a(t)f(\chi)\cos\theta\cos\phi & -a(t)f(\chi)\sin\theta\sin\phi \\ 0 & a(t)\sin\theta\sin\phi & a(t)f(\chi)\cos\theta\sin\phi & a(t)f(\chi)\sin\theta\cos\phi \\ 0 & a(t)\cos\theta & -a(t)f(\chi)\sin\theta & 0 \end{pmatrix}. \tag{6.9.6}$$

It is easy to see that it satisfies (6.2.9). However, a far simpler choice that also satisfies (6.2.9) is

$$e^a_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(t) & 0 & 0 \\ 0 & 0 & a(t)f(\chi) & 0 \\ 0 & 0 & 0 & a(t)f(\chi)\sin\theta \end{pmatrix}. \tag{6.9.7}$$

With this choice, together with the unique torsion free connection obtained above, the spin connection is obtained as,

$$\begin{aligned} &\omega^0_{r1} = \frac{\dot{a}}{c^2}, \quad \omega^0_{\theta 2} = \frac{1}{c^2} f\dot{a}, \quad \omega^0_{\phi 3} = \frac{1}{c^2} f\dot{a}\sin\theta \\ &\omega^1_{r0} = \dot{a}, \quad \omega^1_{\theta 2} = -f', \quad \omega^1_{\phi 3} = -f'\sin\theta \\ &\omega^2_{\theta 0} = f\dot{a}, \quad \omega^2_{\theta 1} = f', \quad \omega^2_{\phi 3} = -\cos\theta \\ &\omega^3_{\phi 0} = f\dot{a}\sin\theta, \quad \omega^3_{\phi 1} = f'\sin\theta, \ \omega^3_{\phi 2} = \cos\theta, \end{aligned} \tag{6.9.8}$$

using (6.5.16).[28]

[28]Problem: What rotation matrix transforms the vierbein in (6.9.6) into the one in (6.9.7)? Verify the spin connections in (6.9.8), compute the components of $R^\alpha{}_{\beta\mu\nu}$ and show that it is identical to the Riemann tensor obtained directly from the Christofel symbols.

6.9.2 Static, Spherically Symmetric Metrics

The most general static, spherically symmetric metric can be written in terms of two non-negative functions, $A(r)$ and $B(r)$ as

$$ds^2 = c^2 A(r) dt^2 - B(r) dr^2 - r^2 d\Omega^2. \tag{6.9.9}$$

The unique, torsion free connection associated with this metric is

$$\begin{aligned}
&\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{A'}{2A}, \\
&\Gamma^r_{tt} = \frac{c^2 A'}{2B}, \quad \Gamma^r_{rr} = \frac{B'}{2B}, \quad \Gamma^r_{\theta\theta} = -\frac{r}{B}, \quad \Gamma^r_{\phi\phi} = -\frac{r}{B}\sin^2\theta \\
&\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \\
&\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta
\end{aligned} \tag{6.9.10}$$

from which follow the components of the Riemann tensor, the Ricci tensor and the scalar curvature. They are non vanishing for general functions, A and B.[29]

The simplest vierbein that is compatible with the static, spherical metric is

$$e^a_\mu = \begin{pmatrix} \sqrt{A} & 0 & 0 & 0 \\ 0 & \sqrt{B} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix} \tag{6.9.11}$$

and, in conjunction with the Christoffel symbols, yields the following spin connection

$$\begin{aligned}
&\omega^0_{t1} = \frac{A'}{2\sqrt{AB}}, \\
&\omega^1_{t0} = \frac{c^2 A'}{2\sqrt{AB}}, \quad \omega^1_{\theta 2} = -\frac{1}{\sqrt{B}}, \quad \omega^1_{\phi 3} = -\frac{\sin\theta}{\sqrt{B}}, \\
&\omega^2_{\theta 1} = \frac{1}{\sqrt{B}}, \quad \omega^2_{\phi 3} = -\cos\theta, \\
&\omega^3_{\phi 1} = \frac{\sin\theta}{\sqrt{B}}, \quad \omega^3_{\phi 2} = \cos\theta
\end{aligned} \tag{6.9.12}$$

As always, the vierbein and spin connection provide an alternate route to the curvature tensor and its contractions.

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[29] Problem: Determine the components of the Riemann tensor, the Ricci tensor and the scalar curvature of the general static, spherically symmetric space-time.

Chapter 7

The Gravitational Field

Once the principle of covariance was accepted, Einstein was faced with the following dilemma: the equations of Newtonian gravity are not Lorentz but Galilei covariant, and therefore imply that the gravitational force acts instantaneously at all points of space propagating, as it were, at infinite speed. This is unacceptable in the face of special relativity as no information is allowed to travel faster than the speed of light and it becomes clear that gravity must be described by a Lorentz covariant field theory that reduces, in the limit as the speed of light approaches infinity, to the Newtonian theory.

This was no easy task because gravity couples to the mass according to Newton and according to special relativity mass is only a form of energy. Therefore in a relativistic theory, gravity must couple not just to mass but to all forms of *energy*. However, as a dynamical field, the gravitational field would carry its own energy and momentum and so it would have to couple also to itself. Thus a relativistic theory of gravity would have to be nonlinear. Non-linearity eliminates the possibility of superposition; as a consequence, the field of two masses would not be the sum of the fields of the individual masses but would be modified by the gravitational interaction between them.

7.1 The Equivalence Principle

Newton assumed, without further explanation, that the "gravitational masses", by which is meant the masses that appear in his force law,

$$\vec{F} = -G\frac{m_1 m_2}{r_{12}^2}\,\widehat{r}_{12}, \tag{7.1.1}$$

are the the same as the "inertial masses", by which is meant the masses that define the momenta of the particles,

$$\vec{p}_1 = m_1\vec{v}_1, \quad \vec{p}_2 = m_2\vec{v}_2. \tag{7.1.2}$$

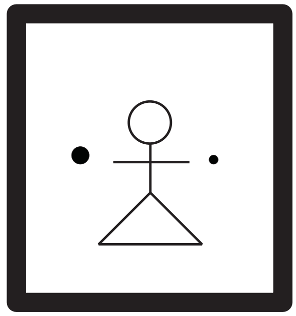

Figure 7.1: Everything floats. Is the spaceship free or freely falling?

As a consequence,

- *all particles satisfying the same initial conditions would follow the same path in a gravitational field and this path would be independent of any characteristic property of the particles, including the rest mass.*

This is the **weak equivalence principle** and dates back to Galileo, whose famous experiment at Pisa was meant to demonstrate it.

A consequence of the universality of the motion of particles is that, at least *locally*, it should always be possible to find a reference frame that makes the effects of gravity disappear. To understand this rather subtle point, Einstein proposed the following "gedanken" ("thought") experiment. A thought experiment is an "experiment" that is not performed in a laboratory but in the experimenter's mind: it consists of analyzing the consequences of an hypothesis in an imagined, idealized experimental setup that may itself not be realizable in practice, but is, in principle. Its goal is to rigorously pursue all the logical consequences of any hypothesis. In this spirit, consider an experimentalist situated in a closed spaceship, with no access to the universe outside. His task is to determine if he is in a gravitational field.

Now imagine that the spaceship is floating freely in deep space, far from the gravitational effects of stars and planets. In the absence of any forces, our astronaut scientist will simply float in his spaceship and if he releases two stones of unequal mass they too will float along with him. The "weightlessness" of all he is able to observe will convince him that a gravitational field is absent. However, there is another possibility: he could conclude, by the weak equivalence principle, that he is instead freely falling in an uniform, external gravitational field. In fact he has no way of distinguishing between these two possibilities because there is no experiment he could perform within the confines of his spaceship that would distinguish between them.

Again, suppose that some alien power grabbed the spaceship and began dragging it

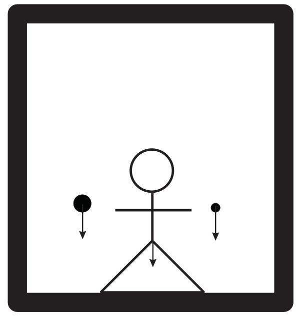

Figure 7.2: Everything "falls": Is the spaceship being dragged "up" or is it stationary in an external gravitational field?

through space with a constant acceleration. Our experimentalist first finds that he is pressed against the trailing end of the spaceship; then he releases his stones of unequal mass and finds that they are both accelerated toward that end of the spaceship at the same rate. He *could* conclude that his spaceship is being accelerated by some alien power, but he could also conclude that he is stationary in an uniform, external gravitational field. Again, no experiment performed inside the spaceship would be able to discriminate between the two possibilities.

One conclusion that stands out is that in both cases our experimentalist is unable to distinguish between the presence of an external gravitational field and some inertial effect. Forces whose effects are the same for all bodies are well known: familiar examples are the "fictitious" forces which depend only on the reference frame of the observer, *viz.,* the centripetal force and the Coriolis force. Our thought experiment therefore suggests that gravitation must now be included in this class.

There is a subtlety, however: we have assumed that the field is uniform inside the spaceship in which the experiment is being performed. This cannot hold in a general external gravitational field because some degree of non-uniformity will be present in any gravitational field due to a realistic mass distribution, owing to both its shape and the distribution of matter within it (for example, the gravitational field due to the earth is not constant but spherically symmetric and points toward a single point – the center of the earth). Thus the falling objects, our experimentalist included, will follow paths that focus at the earth's center (see figure 7.3). Moreover, a sensitive enough experiment would distinguish between the magnitude of the acceleration at different points within the spaceship. The changing mangitude and direction of the acceleration at different locations provides a way to distinguish between the presence of an external gravitational field and kinematical acceleration through space. To overcome this difficulty, we can imagine that the spaceship is very small, occupying, in the limit, an infinitesimal volume

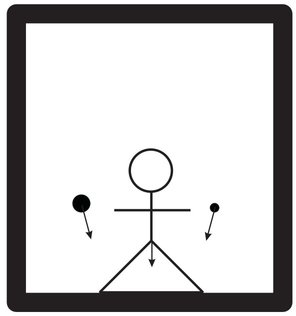

Figure 7.3: In a non-uniform field, trajectories will not generally be parallel.

in which all non-uniformity of the field can be ignored. Thus while the effects of an external gravitational field cannot be completely eliminated or mimicked kinematically on a large scale, it should be possible to do so *locally*. This leads us to a stronger form of the equivalence principle:

- *the effects of a gravitational field can be eliminated or mimicked locally by an appropriate choice of reference frame.*

But if the effects of the gravitational field can be eliminated locally by an appropriate choice of reference frame then we can make an even stronger statement that would agree with the special theory of relativity. This is **Einstein's equivalence principle**, or the **strong form** of the equivalence principle, which may be stated as:

- *it is always possible to find a reference frame in which all the laws of physics reduce locally to those of the Special Theory of Relativity.*

In this frame the motion of a free particle would be given by the usual equation

$$\frac{d^2x^a}{ds^2} = 0, \tag{7.1.3}$$

but, because the motion of otherwise free particles in a gravitational field is also described by a second order differential equation in which no characteristic of the particles appears, Einstein proposed that free particle motion in a gravitational field should be interpreted as geodesic motion. Since the trajectories are no longer "straight" world lines in the presence of a gravitational field, he concluded that the effect of gravity is to deform the space-time metric of special relativity.

This is a bold step if we take into account the pivotal role played by inertial frames in the special theory of relativity and their intimate connection with the Minkowski metric and Cartesian frames that extend over all of space-time. By allowing deformations of the Minkowski metric we are saying that whereas inertial can be defined *locally* according to the strong form of the equivalence principle, *global* inertial frames do not exist. This is very similar to the idea in Riemannian geometry that a smooth enough curved space can always be considered "flat" in a small enough neighborhood of every point and it led Einstein to conjecture that gravity in fact manifests itself as the curvature of space-time. If we take this seriously, then the strong form of the equivalence principle suggests that the gravitational field must be represented by a general metric, $g_{\mu\nu}(\xi)$, in a curvilinear coordinate system, ξ^μ, that is not necessarily related to a global Cartesian system by a coordinate transformation, but is always related to one in small enough neighborhoods of points.

But what does the Einstein equivalence principle imply for the other laws of physics? Since it declares that there is a reference frame at every point in which the laws of physics take the form that they do in special relativity, we again begin in a local Minkowski frame. In this frame, Maxwell's laws of electromagnetism, for example, would have the form

$$F_{ab} = \partial_a A_b - \partial_b A_a,$$

$$\partial_a F^{ab} = -j^b, \quad \partial_a {}^*F^{ab} = 0. \tag{7.1.4}$$

In a small neighborhood of the point we can transform these equations to a general coordinate system

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu,$$

$$\nabla_\mu F^{\mu\nu} = -j^\nu, \quad \nabla_\mu {}^*F^{\mu\nu} = 0 \tag{7.1.5}$$

by making a coordinate transformation. Since the transformation properties of the left and right hand sides are the same under general coordinate transformations, we simply declare that these are Maxwell's equations in *any* coordinate system and for *any* metric. This procedure is often referred to as the principle of **general covariance**:

- *the laws of physics must preserve their form under general coordinate transformations and should reduce to the laws of Special Relativity in a space-time that is described by the Minkowski metric*

By virtue of this principle, if the laws of physics in an inertial (Minkowski) frame are known then they are also known in the presence of a gravitational field. The principle of general covariance combines the Einstein equivalence principle and the requirement that the laws of physics should be independent of the coordinate system.

The principle of general covariance offers guidance but not uniqueness in the construction of generally covariant equations. For example, an ambiguity appears when we want to apply its prescription to quantities involving second order or higher than second order derivatives, for then there is no unique ordering of the derivatives in flat space but in curved space the difference between different orderings will be depend on the curvature. For example, there is no difference in flat space between $X^c_{ab} = \partial_a \partial_b A^c$ and and $X^c_{ba} = \partial_b \partial_a A^c$. In curved space, however,

$$X^\gamma_{\mu\nu} - X^\gamma_{\nu\mu} = R^\gamma{}_{\beta\mu\nu} A^\beta \tag{7.1.6}$$

In particluar, if we define the scalar in flat space by $X = B^a X^c_{ac}$, then $B^a X^c_{[ac]} = 0$, but in curved space

$$B^\mu X^\gamma_{[\mu\gamma]} = -R_{\alpha\beta} A^\alpha B^\beta. \tag{7.1.7}$$

Because the principle of general covariance does not provide any preference for the ordering of derivatives, it cannot determine the status in the theory of additional curvature dependent terms that may arise by different choices.

7.2 Geodesic Motion

If we accept the conjecture that particle motion in a gravitational field is described by the geodesics of a curved space metric then the equations of motion should be obtained by extremizing the proper distance between two points

$$S = -mc \int_1^2 ds = -mc \int_1^2 d\lambda \sqrt{-g_{\mu\nu}(\xi) \frac{d\xi^\mu}{d\lambda} \frac{d\xi^\nu}{d\lambda}}, \tag{7.2.1}$$

where λ is an arbitrary parameter (the action is reparameterization invariant) and

$$\frac{ds}{d\lambda} = L = \sqrt{-g_{\mu\nu} \frac{d\xi^\mu}{d\lambda} \frac{d\xi^\nu}{d\lambda}}, \tag{7.2.2}$$

Setting $U^\mu_{(\lambda)} = d\xi^\mu / d\lambda$, we apply Euler's equations to get[1]

$$\frac{dU^\mu_{(\lambda)}}{d\lambda} + \Gamma^\mu_{\alpha\beta} U^\alpha_{(\lambda)} U^\beta_{(\lambda)} = f(\lambda) U^\mu_{(\lambda)} \tag{7.2.3}$$

with $f(\lambda) = d \ln L / d\lambda$. These are just the equations of parallel transport in (6.5.52), with a specific function, $f(\lambda)$, which can be made to vanish by choosing λ so that L is a positive constant,

$$\frac{d\lambda}{ds} = a \quad \Rightarrow \quad \lambda = as + b \tag{7.2.4}$$

[1] Problem: Derive the geodesic equation starting from the action in (7.2.1) and applying Euler's equations,

$$\frac{d}{d\lambda} \left(\frac{\partial L}{\partial U^\mu_{(\lambda)}} \right) - \frac{\partial L}{\partial \xi^\mu} = 0.$$

and then the geodesic equation reads

$$\frac{d^2\xi^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{d\xi^\alpha}{d\lambda}\frac{d\xi^\beta}{d\lambda} = 0. \tag{7.2.5}$$

The geodesic equation has this form only for parameters that are related to the proper distance by linear transformations. Parameters for which the geodesic equation has the form in (7.2.5) are called **affine** parameters. An alternate and sometimes more useful form for the geodesic equation is

$$(U_{(\lambda)} \cdot \nabla)U^\mu_{(\lambda)} = f(\lambda)U^\mu_{(\lambda)}.$$

Although derived for time-like paths, (7.2.5) should also be valid for null paths. In the latter case λ cannot be linearly related to s, since $ds = 0$ for null trajectories. However, affine parameters do exist for null geodesics. To see this, begin with the general form in (6.5.52) and ask if, for *any* λ, it is possible to find a transformation $\lambda \to \lambda'(\lambda)$ which is such that (7.2.5) is recovered. From (6.5.52) we find

$$\left(\frac{d\lambda'}{d\lambda}\right)^2 \left[\frac{d^2\xi^\mu}{d\lambda'^2} + \Gamma^\mu_{\alpha\beta}\frac{d\xi^\alpha}{d\lambda'}\frac{d\xi^\beta}{d\lambda'}\right] = \left[f(\lambda)\frac{d\lambda'}{d\lambda} - \frac{d^2\lambda'}{d\lambda^2}\right]\frac{d\xi^\mu}{d\lambda'} \tag{7.2.6}$$

Evidently, what we want is

$$f(\lambda)\frac{d\lambda'}{d\lambda} - \frac{d^2\lambda'}{d\lambda^2} = 0 \tag{7.2.7}$$

assuming that $d\lambda'/d\lambda \neq 0$. The solution is

$$\lambda'(\lambda) = a\int^\lambda d\lambda'' e^{\int^{\lambda''} d\lambda''' f(\lambda''')} + b \tag{7.2.8}$$

but the resulting equation for null geodesics cannot be thought of as a limit of the geodesic equation for time-like trajectories. Remember also that the four velocities are subject to the constraints $U^\mu_{(\lambda)}U_{(\lambda)\mu} = -(ds/d\lambda)^2$ for time like geodesics and $U^\mu_{(\lambda)}U_{(\lambda)\mu} = 0$ for null geodesics.

The existence of Killing vectors implies that there will be locally conserved quantities characterizing the geodesics: let ϵ^μ be a Killing vector of the metric, then $\epsilon \cdot U$ is conserved along the geodesic. This follows because, if λ is an affine parameter,

$$\frac{D}{D\lambda}(\epsilon \cdot U) = U \cdot \nabla(\epsilon \cdot U) = U^\alpha U^\beta \nabla_\alpha \epsilon_\beta = \frac{1}{2}U^\alpha U^\beta \nabla_{(\alpha}\epsilon_{\beta)} = 0 \tag{7.2.9}$$

by the Killing equation.

7.3 The Einstein Equations

Let's consider the spatial acceleration of a slowly moving "test particle" in a weak, static gravitational field. A "test particle" is an idealized particle whose energy is small enough that it does not significantly alter the gravitational field in its neighborhood. A weak gravitational field should be given by a small deformation of the Minkowski metric, so we set

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{7.3.1}$$

where $|h_{\mu\nu}| \ll |\eta_{\mu\nu}|$.[2] If the metric is static then $h_{\mu\nu,t} = 0$ and we can also set $h_{ti} = 0$ because, by time reversal invariance, terms of the form $h_{ti}dtdx^i$ in the expression for the proper distance should vanish identically. As far as the test particle is concerned, by slowly moving we will mean that

$$\left|\frac{dx^i}{ds}\right| \ll c\frac{dt}{ds} \tag{7.3.2}$$

so the geodesic equation for such a particle would be

$$\frac{d^2x^\mu}{ds^2} \approx -\Gamma^\mu_{tt}\left(\frac{dt}{ds}\right)^2 \tag{7.3.3}$$

Now to first order in $h_{\mu\nu}$ we find

$$\Gamma^\mu_{tt} \approx -\frac{1}{2}\eta^{\mu\nu}h_{tt,\nu} \tag{7.3.4}$$

and therefore

$$\frac{d^2t}{ds^2} \approx 0, \quad \text{and} \quad \frac{d^2x_i}{ds^2} \approx \frac{1}{2}h_{tt,i}\left(\frac{dt}{ds}\right)^2. \tag{7.3.5}$$

The first equation tells us that $t \approx as + b$, where a, b are constants. Without any loss of generality take $a = 1/c$ and $b = 0$ then the second equation is

$$\frac{d^2x_i}{dt^2} = \frac{1}{2}h_{tt,i} \tag{7.3.6}$$

which has the same form as the Newtonian force law,

$$\frac{d^2\vec{r}}{dt^2} = \vec{g} = -\vec{\nabla}\phi, \tag{7.3.7}$$

and leads us to interpret $\phi = -h_{tt}/2$ as the Newtonian potential of the gravitational field.

[2] Problem: The inverse metric will be $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, where $h^{\mu\nu}$ is obtained by raising the indices of $h_{\mu\nu}$ using the inverse Minkowski metric, $\eta^{\mu\nu}$. Show that the negative sign is necessary to ensure that $g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu$ up to $\mathcal{O}(h)$.

It would therefore appear that the Newtonian theory is the non-relativistic, static, weak field limit of a more general theory in which gravity is described by the geometry of a curved space-time. It remains to determine the equations that govern the metric, given any distribution of matter in space-time; in other words we would like to know how the Newtonian Poisson equation

$$\vec{\nabla}^2\phi(\vec{r}) = 4\pi G\rho(\vec{r}), \tag{7.3.8}$$

is generalized in such a theory. We will seek equations that are similar in structure to this, in the sense that on the left it should have terms involving second derivatives of the gravitational potentials, which our examination of geodesics has indicated must be related to the metric coefficients, and, on the right, terms involving the matter distribution. Moreover, not one but $n(n+1)/2$ such equations are needed, one for each of the $n(n+1)/2$ components of the metric tensor in n dimensions and they should obey the principle of general covariance, *i.e.,* they should relate tensors to tensors. We must therefore relate a symmetric, second rank tensor involving non-vanishing second order derivatives of the metric (we must allow for a curved space-time) to an algebraically similar tensor involving energy and momentum. Finally, the $n(n+1)/2$ equations should reduce to (7.3.8) in the non-relativistic, static, weak field limit.[3], [4]

From the discussion above it should have become clear that the equations we seek will relate second rank tensors as follows:

Riemann / Contractions = const. × Energy/Momentum of matter distribution. (7.3.9)

The simplest candidate for the left hand side is

$$R_{\mu\nu} + Ag_{\mu\nu}(R - \Lambda), \tag{7.3.10}$$

where A is a dimensionless constant and Λ has dimension l^{-2}. We also know that the stress energy tensor is a second-rank tensor that represents the energy and momentum carried by the matter fields, so the equations governing the gravitational field should have the form

$$R_{\mu\nu} + Ag_{\mu\nu}(R - \Lambda) = \frac{DG}{c^4}T_{\mu\nu} \tag{7.3.11}$$

[3] Problem: Obtain (7.3.6) by expanding the particle action to second order, then directly applying Euler's equations.

[4] Problem: Consider a space-time that is not static but stationary. In this case, we can still take $h_{\mu\nu,t} = 0$ but we cannot assume that $h_{ti} = 0$ because time reversal is not a symmetry of such space-times. Calling $A_i(\vec{r}) = h_{ti}(\vec{r})/2$, repeat the calculation in the previous problem to find the lowest order particle Lagrangian

$$\mathcal{L} = m\left[\frac{1}{2}\vec{v}^2 - \phi(\vec{r}) + \vec{v}\cdot\vec{A}\right],$$

then directly determine the modification of Newton's force law by $\vec{A}$. Of which well-known interaction is the addtitional term reminiscent? It is called the "gravitomagnetic" term.

where we have introduced another dimensionless constant D on the right hand side and G/c^4 makes the two sides agree dimensionally. We must now determine the constants A, Λ and D. Since the stress-energy tensor is conserved, the equation above can be consistent only if the left hand side is also conserved, *i.e.,*

$$\nabla^\mu R_{\mu\nu} + AR_{,\nu} = 0, \tag{7.3.12}$$

but by contracting the Bianchi identities in (6.8.9) we determine

$$2\nabla^\mu R_{\mu\nu} - R_{,\nu} = 0, \tag{7.3.13}$$

so take $A = -1/2$ and it remains only to find Λ and D in

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(R - \Lambda) = \frac{DG}{c^4} T_{\mu\nu} \tag{7.3.14}$$

For this, we return to the weak field limit of the left hand side of (7.3.14). Setting $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we expand to first order in $h_{\mu\nu}$, so

$$\Gamma^\alpha_{\mu\nu} \approx \frac{1}{2}\eta^{\alpha\beta}\left[h_{\mu\beta,\nu} + h_{\beta\nu,\mu} - h_{\mu\nu,\beta}\right] \tag{7.3.15}$$

and

$$\begin{aligned} R_{\mu\nu} &\approx \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} \\ &= \frac{1}{2}\eta^{\alpha\beta}\left[h_{\mu\beta,\alpha\nu} + h_{\nu\beta,\mu\alpha} - h_{\mu\nu,\alpha\beta} - h_{\mu\beta,\alpha\nu} - h_{\alpha\beta,\mu\nu} + h_{\mu\alpha,\beta\nu}\right] \\ &= -\frac{1}{2}\left[h_{,\mu\nu} - \eta^{\alpha\beta}(h_{\mu\alpha,\beta\nu} + h_{\beta\nu,\mu\alpha}) - \Box h_{\mu\nu}\right]. \end{aligned} \tag{7.3.16}$$

In the *static* approximation, the time-time component of the Ricci tensor is

$$R_{tt} = \frac{1}{2}\Box h_{tt} = \vec{\nabla}^2 \phi(\vec{r}) \tag{7.3.17}$$

and we will now see that this is sufficient to obtain the Newton-Poisson equation for the gravitational potential. By contracting (7.3.14) we get

$$R = -\frac{DG}{c^4} T + 2\Lambda, \tag{7.3.18}$$

therefore (7.3.14) can also be written as

$$R_{\mu\nu} = \frac{DG}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) + \frac{1}{2}\Lambda g_{\mu\nu}. \tag{7.3.19}$$

For the right hand side we use the perfect fluid stress tensor,

$$T_{\mu\nu} = pg_{\mu\nu} + \left(\rho + \frac{p}{c^2}\right) U_\mu U_\nu, \quad T = 3p - \rho c^2, \tag{7.3.20}$$

where ρ is the mass density of the fluid and p is the pressure. But, in the non-relativistic limit $U^i \approx 0$ and we can ignore the pressure (as $c \to \infty$)

$$T_{tt} = \rho c^4, \quad T = -\rho c^2 \tag{7.3.21}$$

and the time-time component of field equations should read

$$\vec{\nabla}^2 \phi(\vec{r}) = \frac{1}{2} DG\rho - \frac{1}{2}\Lambda c^2. \tag{7.3.22}$$

This compares with Newton's equation (7.3.8) if $D = 8\pi$ and $\Lambda = 0$. Thus we arrive at **Einstein's equations** for the gravitational field,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{7.3.23}$$

The left hand side is often called the **Einstein tensor**, $E_{\mu\nu}$. It has the property that it is divergence free, like the stress-energy tensor.

Λ represents a constant, intrinsic energy density of the vacuum and an associated pressure. It can still be included in the equations (and was originally introduced by Einstein to achieve a static universe) provided it is small enough so that it does not significantly alter the predictions of Newton's theory in experiments conducted within our solar system. This is the regime in which Newton's law of gravitation is extremely well tested. Nevertheless, as the energy density of the vacuum, it may be important on larger, even cosmological, distance scales. When it is included it is called the **cosmological constant**, and Einstein's equations *with* a cosmological constant,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - \Lambda) = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{7.3.24}$$

represent the most general, self-consistent, second order equations for the gravitational field in four space-time dimensions. Other possibilities involving higher powers of the Riemann curvature and its contractions in very special combinations exist in higher dimensions. They must all be taken into account in any theory, such as string theory, that aims to describe physics in more than four dimensions.

7.4 The Hilbert Action

Now that the equations governing the gravitational field are known, the next question to ask is whether or not those equations are derivable from an action principle, *i.e.,* as

Euler-Lagrange equations of a particular action. Let us first show that the left hand side of (7.3.23) can be obtained by a functional variation with respect to the metric of the action

$$S_G = \frac{c^4}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\; R, \tag{7.4.1}$$

where R is the curvature scalar, $\mathcal{M}$ represents the space-time manifold and the constant in front has been chosen so that S_G has the dimension of "action". We have returned to using "x" for our coordinates; henceforth they will represent general curvilinear coordinates unless otherwise stated.

Notice that the action involves second order derivatives of $g_{\mu\nu}$ via the curvature scalar, which is unusual: in all the examples we have worked with so far, the action has only involved first derivatives of the fields. However, the gravitational action must be a scalar under coordinate transformations and there is no geometric scalar that involves only first order derivatives of the metric, so we are left with no choice. Ordinarily, the presence of second order derivatives in the action would lead to higher than second order equations of motion *and* require us to impose additional boundary conditions on the variations so that the contributions from the boundary vanish. However, the equations of motion will continue to be second order if the higher derivative action can be shown to split into a bulk (volume) action involving only first order derivatives and a total derivative, or surface, term. Only the boundary conditions then need to be modified so that the surface contribution to the variation vanishes. This is the case with the Einstein-Hilbert action.

To see how this comes about, let us rewrite the action as

$$S_G = \frac{c^4}{32\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\; P^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} \tag{7.4.2}$$

where $P^{\mu\nu\alpha\beta} = g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}$. Noting the symmetries of $P^{\mu\nu\alpha\beta}$, it is antisymmetric in (μ,ν) and in (α,β), we could rewrite the action as

$$\begin{aligned} S_G &= \frac{c^4}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\; P_\mu{}^{\nu\alpha\beta}\left[\Gamma^\mu_{\nu\beta,\alpha} + \Gamma^\eta_{\nu\beta}\Gamma^\mu_{\eta\alpha}\right] \\ &= \frac{c^4}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[P_\mu{}^{\nu\alpha\beta}\Gamma^\eta_{\nu\beta}\Gamma^\mu_{\eta\alpha} - \frac{1}{\sqrt{-g}}\Gamma^\mu_{\nu\beta}\partial_\alpha(\sqrt{-g}P_\mu{}^{\nu\alpha\beta})\right] \\ &\qquad + \frac{c^4}{16\pi G}\int_{\mathcal{M}} d^4x\; \partial_\alpha(\sqrt{-g}P_\mu{}^{\nu\alpha\beta}\Gamma^\mu_{\nu\beta}) \end{aligned} \tag{7.4.3}$$

Now simplify the second term using

$$\frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}P_\mu{}^{\nu\alpha\beta}) = \partial_\alpha(P_\mu{}^{\nu\alpha\beta}) + \partial_\alpha(\ln\sqrt{-g})P_\mu{}^{\nu\alpha\beta} \tag{7.4.4}$$

together with the identities

- $\partial_\alpha \ln \sqrt{-g} = \Gamma^\lambda_{\lambda\alpha}$ and
- $\nabla_\alpha P_\mu{}^{\nu\alpha\beta} = 0 = \partial_\alpha P_\mu{}^{\nu\alpha\beta} - \Gamma^\lambda_{\alpha\mu} P_\lambda{}^{\nu\alpha\beta} + \Gamma^\nu_{\alpha\lambda} P_\mu{}^{\lambda\alpha\beta} + \Gamma^\alpha_{\alpha\lambda} P_\mu{}^{\nu\lambda\beta}$,

and find

$$\frac{1}{\sqrt{-g}} \Gamma^\mu_{\nu\beta} \partial_\alpha (\sqrt{-g} P_\mu{}^{\nu\alpha\beta}) = -2 P_\mu{}^{\nu\alpha\beta} \Gamma^\eta_{\nu\alpha} \Gamma^\mu_{\eta\beta}, \tag{7.4.5}$$

which leads to the compact result $S_G = S_{\text{bulk}} + S_{\text{surf}}$, where

$$S_{\text{bulk}} = \frac{c^4}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} P_\mu{}^{\nu\alpha\beta} \Gamma^\eta_{\nu\alpha} \Gamma^\mu_{\eta\beta} \tag{7.4.6}$$

and

$$S_{\text{surf}} = \frac{c^4}{16\pi G} \int_{\mathcal{M}} d^4x \ \partial_\alpha (\sqrt{-g} P_\mu{}^{\nu\alpha\beta} \Gamma^\mu_{\nu\beta}). \tag{7.4.7}$$

The "bulk" portion of the action contains only first order derivatives of the metric. The boundary or surface portion contains second order derivatives. Neither of the actions is separately coordinate invariant. This was to be expected because, as mentioned in the introduction to this section, no coordinate scalar exists which involves only first derivatives of the metric; so we have learned that the surface term in the Einstein Hilbert action is unavoidable and it is a direct consequence of requiring coordinate invariance together with second order dynamics for the gravitational field. It is, furthermore, a curious result (proved by explicit computation) that

$$\mathfrak{L}_{\text{surf}} = -\frac{1}{2} \partial_\alpha \left(g_{\lambda\kappa} \frac{\partial \mathfrak{L}_{\text{bulk}}}{\partial g_{\lambda\kappa,\alpha}} \right) \tag{7.4.8}$$

where $\mathfrak{L}_{\text{bulk}} = \sqrt{-g} P_\mu{}^{\nu\alpha\beta} \Gamma^\eta_{\nu\beta} \Gamma^\mu_{\eta\alpha}$, so that the gravitational action can be put into the form

$$S_G = \frac{c^4}{16\pi G} \int_{\mathcal{M}} d^4x \left[\mathfrak{L}_{\text{bulk}} - \frac{1}{2} \partial_\alpha \left(g_{\lambda\kappa} \frac{\partial \mathfrak{L}_{\text{bulk}}}{\partial g_{\lambda\kappa,\alpha}} \right) \right] \tag{7.4.9}$$

This is a special form.[5] With it we are guaranteed that the Euler equations will be second order, if the momenta and not the configuration space variables are held fixed on the boundary.

[5] Problem: Consider the particle action

$$S = \int_1^2 \left\{ \mathcal{L}(q, \dot{q}, t) - \frac{d}{dt} \left[q \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \right] \right\} dt.$$

The total derivative term introduces higher order derivatives, but it will be recognized as a mere canonical transformation that interchanges $q \to -P$ and $p \to Q$. Show that the Euler equations are second order and that it is the momentum, p, not the coordinate, q, that is required to be held fixed at the boundary points.

7.4.1 Variation of S_G

Let us proceed with the variation of S_G,

$$\delta_0 S_G = \frac{c^4}{16\pi G}\int d^4x\left[(\delta_0\sqrt{-g})R + \sqrt{-g}(\delta_0 R)\right]. \tag{7.4.10}$$

This involves the variations $\delta\sqrt{-g}$ and δR. The first is easy to determine if we recognize that for any matrix $\widehat{A}$ the identity

$$\ln\det(\widehat{A}) = \mathrm{tr}\ln(\widehat{A}) \tag{7.4.11}$$

implies, varying both sides and using $A = \det\widehat{A}$, that

$$\delta\ln(A) = 2\delta\ln\sqrt{A} = \frac{2}{\sqrt{A}}\delta\sqrt{A} = \mathrm{tr}(\widehat{A}^{-1}\delta\widehat{A}) \tag{7.4.12}$$

and therefore taking $\widehat{A}$ to be the metric, the variation of $\sqrt{-g}$ will be

$$\delta_0\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta_0 g_{\mu\nu} \tag{7.4.13}$$

but if we use the fact that

$$\delta_0(g^{\alpha\sigma}g_{\sigma\beta}) = 0 \Rightarrow \delta_0(g^{\alpha\sigma})g_{\sigma\beta} = -g^{\alpha\sigma}(\delta_0 g_{\sigma\beta}) \tag{7.4.14}$$

then $\delta_0 g_{\mu\nu} = -g_{\mu\alpha}g_{\nu\beta}\delta_0 g^{\alpha\beta}$ and we have the equivalent expression

$$\delta_0\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta_0 g^{\mu\nu}. \tag{7.4.15}$$

for the variation of $\sqrt{-g}$.

The variation of R is a little more delicate. Consider

$$\delta_0 R = \delta_0(g^{\mu\nu}R_{\mu\nu}) = (\delta_0 g^{\mu\nu}R_{\mu\nu}) + g^{\mu\nu}\delta_0 R_{\mu\nu} \tag{7.4.16}$$

and let us now concentrate on the last term. It's more convenient to consider

$$\delta_0 R^\alpha{}_{\beta\mu\nu} = \delta_0\Gamma^\alpha_{\beta\nu,\mu} - \delta_0\Gamma^\alpha_{\beta\mu,\nu} + (\delta_0\Gamma^\eta_{\beta\nu})\Gamma^\alpha_{\eta\mu} + \Gamma^\eta_{\beta\nu}(\delta_0\Gamma^\alpha_{\eta\mu}) - (\delta_0\Gamma^\eta_{\beta\mu})\Gamma^\alpha_{\eta\nu} - \Gamma^\eta_{\beta\mu}(\delta_0\Gamma^\alpha_{\eta\nu}) \tag{7.4.17}$$

instead. Direct computation shows that[6]

$$\delta_0\Gamma^\alpha_{\beta\nu} = \frac{1}{2}g^{\alpha\eta}\left(\delta_0 g_{\eta\beta;\nu} + \delta_0 g_{\eta\nu;\beta} - \delta_0 g_{\beta\nu;\eta}\right), \tag{7.4.18}$$

[6] Problem: Show this.

which is manifestly a type $(1,2)$ tensor. If we take the covariant derivative of $\delta_0\Gamma$ and compare it to our expression for $\delta_0 R^\alpha{}_{\beta\mu\nu}$ we will find the remarkable result that

$$\delta_0 R^\alpha{}_{\beta\mu\nu} = \delta_0\Gamma^\alpha_{\beta\nu;\mu} - \delta_0\Gamma^\alpha_{\beta\mu;\nu} \tag{7.4.19}$$

and therefore

$$\delta_0 R_{\mu\nu} = \delta_0\Gamma^\alpha_{\mu\nu;\alpha} - \delta_0\Gamma^\alpha_{\mu\alpha;\nu}. \tag{7.4.20}$$

Now insert

$$g^{\mu\nu}\delta_0 R_{\mu\nu} = (g^{\mu\nu}\delta_0\Gamma^\alpha_{\mu\nu})_{;\alpha} - (g^{\mu\nu}\delta_0\Gamma^\alpha_{\mu\alpha})_{;\nu} \tag{7.4.21}$$

into (7.4.10) and find

$$\begin{aligned}\delta_0 S_G &= \frac{c^4}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu}\right]\delta_0 g^{\mu\nu} \\ &\quad + \frac{c^4}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[(g^{\mu\nu}\delta_0\Gamma^\alpha_{\mu\nu})_{;\alpha} - (g^{\mu\nu}\delta_0\Gamma^\alpha_{\mu\alpha})_{;\nu}\right].\end{aligned} \tag{7.4.22}$$

The second term above is a boundary term and can be written as

$$\frac{c^4}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[P^{\mu\alpha\nu}{}_\sigma\delta_0\Gamma^\sigma_{\mu\nu}\right]_{;\alpha} \tag{7.4.23}$$

so it follows that

$$\delta_0 S_G = \frac{c^4}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\left\{\left[-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu}\right]\delta_0 g^{\mu\nu} + \nabla_\alpha \delta_0 j^\alpha\right\} \tag{7.4.24}$$

where

$$\delta_0 j^\alpha = P^{\mu\alpha\nu}{}_\sigma\delta_0\Gamma^\sigma_{\mu\nu} = P^{\mu\alpha\sigma\nu}\delta_0 g_{\mu\nu;\sigma} = (g^{\mu\sigma}g^{\alpha\nu} - g^{\mu\nu}g^{\alpha\sigma})\,\delta_0 g_{\mu\nu;\sigma}. \tag{7.4.25}$$

involves derivatives of the variations. Notice, however, that

$$\delta_0 S_{\rm surf} = \frac{c^4}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\ \nabla_\alpha\delta_0 j^\alpha + \frac{c^4}{16\pi G}\int_{\mathcal{M}} d^4x\ \partial_\alpha[\Gamma^\mu_{\nu\beta}\delta_0(\sqrt{-g}P_\mu{}^{\nu\alpha\beta})] \tag{7.4.26}$$

so the variation of the "bulk" action, $S_{\rm bulk} = S_G - S_{\rm surf}$, would involve a surface term which does *not* contain derivatives of the metric variations. Therefore we conclude that:

- if the gravitational action is taken to be the coordinate invariant functional S_G then

$$-\frac{2}{\sqrt{-g}}\frac{\delta S_G}{\delta_0 g^{\mu\nu}} = -\frac{c^4}{8\pi G}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right] \tag{7.4.27}$$

  subject to the condition that $\delta_0 j^\alpha$ vanishes at the boundary of $\mathcal{M}$;

- alternatively, if it is taken to be the non-invariant functional $S_{\rm bulk}$ then

$$-\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm bulk}}{\delta_0 g^{\mu\nu}} = -\frac{c^4}{8\pi G}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right] \tag{7.4.28}$$

  subject only to the condition that $\delta_0 g^{\mu\nu} = 0$ on the boundary of $\mathcal{M}$.

7.4.2 Inclusion of Matter

Now if the left hand side is derived via a variation with respect to the metric then it is reasonable to expect that the right hand side must also be derivable from a similar procedure. Let S_M be an action describing the matter fields, which must be added to the total action giving a combined action of $S_G + S_M$. Then we *define* the stress-energy tensor of the matter fields as the symmetric tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta_0 g^{\mu\nu}}. \tag{7.4.29}$$

How does it compare with the generalization, prompted by the principle of general covariance, of the stress tensors obtained in Chapter 2 from translation invariance in flat space? Consider the examples of a scalar field and a massless vector field. The stress energy tensor for a scalar field in a flat background is given in (2.5.22). Therefore, by the principle of general covariance, it should be generalized to

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + \frac{1}{\sqrt{-g}} g_{\mu\nu} \mathfrak{L}. \tag{7.4.30}$$

One can check explicitly that this is recovered from (7.4.29) by varying the scalar field action (notice the definition of $\mathfrak{L}$ below)

$$S_{\text{scalar}} = \int d^4x\ \mathfrak{L} = -\frac{1}{2} \int d^4x \sqrt{-g} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 2V(\phi) \right]. \tag{7.4.31}$$

Likewise, applying the definition of the stress energy tensor to the action for the massless vector field

$$S_{\text{em}} = \int d^4x\ \mathfrak{L}_{\text{em}} = -\frac{gc}{4} \int d^4x \sqrt{-g} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} \tag{7.4.32}$$

gives precisely

$$T_{\mu\nu} = \left[gc F_\mu{}^\alpha F_{\nu\alpha} + \frac{1}{\sqrt{-g}} g_{\mu\nu} \mathfrak{L}_{\text{em}} \right] \tag{7.4.33}$$

as in (2.7.28). Note that this is the *symmetrized* version of the vector field stress tensor. In fact (7.4.29) will always yield the symmetric generalization of the stress tensors obtained earlier because $g^{\mu\nu}$ is symmetric. If the total action is now taken to be the sum of the gravitational and matter actions, $S = S_G + S_M$, then requiring that S is stationary under variations of the metric will yield precisely Einstein's equations.

For phenomenological applications, it is often useful to consider non-interacting particles. The stress energy tensor for a single free particle may be given as (see the footnote)

$$T_{\mu\nu} = \frac{c^2}{\sqrt{-g}} \frac{p_\mu p_\nu}{E} \delta(\vec{r} - \vec{r}(t)) \tag{7.4.34}$$

where $\delta(\vec{r} - \vec{r}(t))$ is defined by the integral $\int d^3\vec{r}\ \delta(\vec{r} - \vec{r}(t)) = 1$ and $E = p^0c^2 = mc^2(dt/d\tau)$.[7] For an collection of non-interacting particles it follows that

$$T_{\mu\nu} = \sum_n \frac{c^2}{\sqrt{-g}} \frac{p_\mu^{(n)} p_\nu^{(n)}}{E^{(n)}} \delta(\vec{r} - \vec{r}_n(t)) \tag{7.4.35}$$

and in the continuum limit, for an ideal fluid,

$$T_{\mu\nu} = pg_{\mu\nu} + \left(\rho + \frac{p}{c^2}\right) U_\mu U_\nu \tag{7.4.36}$$

where U^μ is the fluid four-velocity, p its pressure and ρ its mass density.[8]

7.5 Symmetries and Conservation Laws

The gravitational action is not scale invariant but it *is* invariant under general coordinate transformations,

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x), \quad g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \tag{7.5.1}$$

and thus

$$\delta x^\mu = \varepsilon^\mu(x), \quad \delta g_{\mu\nu} = -g_{\mu\alpha}\varepsilon^\alpha{}_{,\nu} - g_{\nu\alpha}\varepsilon^\alpha{}_{,\mu} \tag{7.5.2}$$

As the symmetry is local, we want to determine the conserved current and Noether's identities for the theory. Even though the Lagrange density in the case of gravity is not first order as assumed in writing the Noether identities in (2.2.15), their form remains the same provided that we treat the $\mathcal{E}_A$ as the Euler derivative for the higher order theory. Then with

$$G^\mu{}_\alpha = \frac{\delta x^\mu}{\delta \varepsilon^\alpha} = \delta^\mu_\alpha, \quad T^\beta{}_{\mu\nu\alpha} = \frac{\delta g_{\mu\nu}}{\delta(\partial_\beta \varepsilon^\alpha)} = -g_{\mu\alpha}\delta^\beta_\nu - g_{\nu\alpha}\delta^\beta_\mu \tag{7.5.3}$$

we find from (2.2.15), using $\mathcal{E}_A = -\sqrt{-g}E^{\mu\nu}$, that

$$-2\partial_\beta E_\alpha{}^\beta - E_\alpha{}^\beta g^{\lambda\sigma} g_{\lambda\sigma,\beta} + E^{\lambda\sigma} g_{\lambda\sigma,\alpha} = -2\nabla_\beta E_\alpha{}^\beta = 0, \tag{7.5.4}$$

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[7] Write the action for a free particle as

$$S = -mc\int ds = -mc\int dt\sqrt{-g^{\mu\nu}v_\mu v_\nu} = -mc\int d^4x\sqrt{-g^{\mu\nu}v_\mu v_\nu}\ \delta(\vec{r} - \vec{r}(t))$$

and take the functional derivative w.r.t. $g^{\mu\nu}$. We find

$$T_{\mu\nu} = \frac{mc}{\sqrt{-g}} \frac{v_\mu v_\nu}{\sqrt{-g^{\mu\nu}v_\mu v_\nu}} \delta(\vec{r} - \vec{r}(t)) = \frac{m}{\sqrt{-g}} \frac{U_\mu U_\nu}{(dt/d\tau)} \delta(\vec{r} - \vec{r}(t)) = \frac{c^2}{\sqrt{-g}} \frac{p_\mu p_\nu}{E} \delta(\vec{r} - \vec{r}(t)).$$

[8] In flat space and in the rest frame of the fluid, $T^0{}_0 = -\rho c^2$, $T^0{}_i = 0$ and $T^i{}_j = p\delta^i_j$.

which is the Bianchi identity of (7.3.13), so it tells us nothing new. This was also the case for gauge fields.

For consistency we must also show that general coordinate invariance requires that $T_{\mu\nu}$ is divergence free, so consider what coordinate invariance would mean for the matter action. Varying the action would involve two terms: (a) a variation with respect to the metric and (b) a variation with respect to the matter fields. The variation with respect to the matter fields vanishes on shell (*i.e.,* if the equations of motion are satisfied). For the variation with respect to the metric, use $\mathcal{E}_A = \frac{1}{2}\sqrt{-g}T^{\mu\nu}$, according to (7.4.29), in (2.2.15). Because $T^{\mu\nu}$ is symmetric we will find, as we did in the case of $E_{\mu\nu}$, that

$$2\partial_\beta T_\alpha{}^\beta + T_\alpha{}^\beta g^{\lambda\sigma} g_{\lambda\sigma,\beta} - T^{\lambda\sigma} g_{\lambda\sigma,\alpha} = 2\nabla_\beta T_\alpha{}^\beta = 0, \tag{7.5.5}$$

but only on-shell. One should be quite careful in interpreting these results.

Firstly, being *covariantly* divergence free is not the same as being divergence free because of the added contributions from the connections

$$T^{\alpha\beta}{}_{;\beta} = 0 \Rightarrow T^{\alpha\beta}{}_{,\beta} = -\Gamma^\beta_{\beta\sigma} T^{\alpha\sigma} - \Gamma^\alpha_{\beta\sigma} T^{\sigma\beta} \tag{7.5.6}$$

and the right hand side should be viewed as a "gravitational" source term. It encapsulates the exchange of energy and momentum between the matter and the gravitational fields. Secondly, tempting as it may be, it would be wrong to consider $E_{\mu\nu}$ as representing the stress energy tensor of the gravitational field. Although it is symmetric, covariantly divergence free and constructed entirely out of the metric tensor, it contains second derivatives of the metric tensor in contrast with the stress energy tensor of matter, which contains only first derivatives of the fields. As a consequence, $E_{\mu\nu}$ does not vanish in a local inertial frame. On the other hand, we have seen that the gravitational *force* vanishes in such a frame. Therefore, if $E_{\mu\nu}$ were to represent the energy and momentum of the gravitational field then we would arrive at the untenable conclusion that the gravitational field may exert no force while still carrying energy and momentum in a locally inertial frame.

Apart from the Bianchi identities, general coordinate invariance will also lead to a strongly conserved current density, but we cannot use (2.2.14) directly because it is adapted to first order Lagrangians. Instead we use (2.2.8) to write

$$\delta S_G = \int d^4x \, [\partial_\mu(\mathfrak{L}\varepsilon^\mu) + \delta_0 \mathfrak{L}] \tag{7.5.7}$$

where $\mathfrak{L} = \sqrt{-g}R$. We have already seen that

$$\frac{8\pi G}{c^4} \, \delta_0 \mathfrak{L} = -\sqrt{-g} E^{\mu\nu} \delta_0 g_{\mu\nu} + \partial_\mu \sqrt{-g} \delta_0 j^\mu \tag{7.5.8}$$

so a variation of S_G gives

$$\delta S_G = \frac{c^4}{8\pi G} \int d^4x \left[-\sqrt{-g} E^{\mu\nu} \delta_0 g_{\mu\nu} + \partial_\mu \sqrt{-g} \left(R\varepsilon^\mu + \delta_0 j^\mu\right)\right]$$

$$= \frac{c^4}{8\pi G}\int d^4x \left[-\sqrt{-g}E^{\mu\nu}\delta_0 g_{\mu\nu} + \partial_\mu\sqrt{-g}\left(R\varepsilon^\mu + P^{\alpha\mu\beta}{}_\sigma\delta_0\Gamma^\sigma_{\alpha\beta}\right)\right] \quad (7.5.9)$$

Now $\delta_0 g_{\mu\nu}$ represents a functional change in the metric under the diffeomorphism $\xi^\mu \to x'^\mu = x^\mu + \varepsilon^\mu$, therefore

$$\delta_0 g_{\mu\nu} = \delta g_{\mu\nu} - \varepsilon\cdot\partial g_{\mu\nu} = -\nabla_\mu\varepsilon_\nu - \nabla_\nu\varepsilon_\mu = -\pounds_\varepsilon g_{\mu\nu} \quad (7.5.10)$$

and substituting this into the variation of S_G, then integrating the first term by parts, we determine

$$\delta S_G = \frac{c^4}{8\pi G}\int d^4x \left\{-2\sqrt{-g}E^{\mu\nu}{}_{;\nu}\varepsilon_\mu + \partial_\mu\sqrt{-g}\left[2E^{\mu\nu}\varepsilon_\nu + R\varepsilon^\mu + P^{\alpha\mu\beta}{}_\sigma\delta_0\Gamma^\sigma_{\alpha\beta}\right]\right\} \quad (7.5.11)$$

The first term vanishes by the Bianchi identity, showing that diffeomorphism invariance requires the conservation of the current

$$\begin{aligned} J^\mu &= \frac{c^4\sqrt{-g}}{8\pi G}\left[2E^{\mu\nu}\varepsilon_\nu + R\varepsilon^\mu - \left(g^{\alpha\sigma}g^{\mu\beta} - g^{\alpha\beta}g^{\mu\sigma}\right)\nabla_\sigma\nabla_{(\alpha}\varepsilon_{\beta)}\right] \\ &= \frac{c^4\sqrt{-g}}{8\pi G}\left[2R^{\mu\nu}\varepsilon_\nu - P^{\alpha\mu\sigma\beta}\nabla_\sigma\nabla_{(\alpha}\varepsilon_{\beta)}\right] \end{aligned} \quad (7.5.12)$$

where the brackets, $(\ldots)$ mean symmetrization of the indices, as usual. We see right away that if ε^μ is a Killing vector of the metric then

$$J^\mu = \frac{c^4\sqrt{-g}}{4\pi G}R^{\mu\nu}\varepsilon_\nu \quad (7.5.13)$$

For a general diffeomorphism, the current can be simplified even further by reordering the derivatives of ε in a suitable way so as to eliminate the curvature term. We get[9]

$$J^\mu = \frac{c^4\sqrt{-g}}{8\pi G}P^{\mu\alpha\beta\sigma}\nabla_\alpha\nabla_\beta\varepsilon_\sigma, \quad (7.5.14)$$

which can further be written in terms of the antisymmetric tensor

$$J^{\mu\alpha} = \frac{c^4\sqrt{-g}}{8\pi G}P^{\mu\alpha\beta\sigma}\nabla_\beta\varepsilon_\sigma = \frac{c^4\sqrt{-g}}{8\pi G}\ \nabla^{[\mu}\varepsilon^{\alpha]}, \quad (7.5.15)$$

where $[\ldots]$ mean *anti*symmetrization of the indices, as $J^\mu = \nabla_\alpha J^{\mu\alpha}$. As we know from (2.2.16), conserved currents imply conserved charges, but we must first provide a general, covariant proof of this statement.

[9] Problem: Derive the result by first showing that

$$P^{\alpha\mu\sigma\beta}\nabla_\sigma\nabla_{(\alpha}\varepsilon_{\beta)} = \nabla^2\varepsilon^\mu + \nabla_\lambda\nabla^\mu\varepsilon^\lambda - 2\nabla^\mu(\nabla\cdot\varepsilon)$$

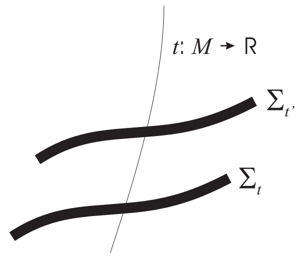

Figure 7.4: The space-time split of $\mathcal{M} = \mathbb{R} \times \Sigma$

To this end, let $\mathcal{M}$ be a space-time that is globally hyperbolic, *i.e.,* of the form $\mathbb{R} \times \Sigma$, where $\mathbb{R}$ is a time-like direction and Σ is a spatial hypersurface. Let $g_{\mu\nu}(X)$ be a metric on this space-time in some coordinates X^μ and let $t(X)$ be a smooth, real-valued function on $\mathcal{M}$ such that there is a differentiable bijection between spacelike surfaces, denoted by Σ_t, of constant $t(X)$. $\mathcal{M}$ may then be described as a sequence of spacelike hypersurfaces (a foliation), which are the level surfaces of $t(X)$ (see figure 7.4). Let a region M of this space-time be bounded by a closed hypersurface ∂M made up of two spacelike hypersurfaces, $\Sigma_\pm$, characterised by constant times, $t_-(X) < t_+(X)$, and a timelike hypersurface, $\mathcal{B}$. Integrating the conservation equation for a (covariantly) conserved current, $\mathcal{J}^\mu = J^\mu/\sqrt{-g}$, and applying Gauss' theorem (6.7.63),

$$0 = \int_M \nabla_\mu \mathcal{J}^\mu = \oint_{\partial M} n_\mu \mathcal{J}^\mu = \int_{\Sigma_+} d\Sigma_+ n^+_\mu j^\mu - \int_{\Sigma_-} d\Sigma_- n^-_\mu \mathcal{J}^\mu + \int_{\mathcal{B}} d\Sigma_{\mathcal{B}}\, b_\mu \mathcal{J}^\mu \quad (7.5.16)$$

where $n^\pm_\mu$ is the timelike, future directed, unit normal to the $\Sigma_\pm$, the negative sign in the second term on the right hand side compensates for the fact that we have chosen to use the *future* directed unit normal on Σ_-, which is *inward* pointing, and b_μ is the outward, spacelike unit normal to $\mathcal{B}$. If we define

$$Q = \int_{\Sigma_t} d\Sigma_t\, n_\mu \mathcal{J}^\mu \quad (7.5.17)$$

where Σ is any spacelike hypersurface then,

$$Q_+ - Q_- = -\int_{\mathcal{B}} d\Sigma_{\mathcal{B}}\, b_\mu \mathcal{J}^\mu \quad (7.5.18)$$

Then, using $[\nabla^\mu, \nabla_\alpha]\varepsilon^\alpha = -R^{\mu\lambda}\varepsilon_\lambda$, simplify the last term above to get

$$P^{\alpha\mu\sigma\beta}\nabla_\sigma\nabla_{(\alpha}\varepsilon_{\beta)} = \nabla^2\varepsilon^\mu - \nabla_\lambda\nabla^\mu\varepsilon^\lambda + 2R^{\mu\lambda}\varepsilon_\lambda$$

and insert this into the expression for J^μ.

This is the appropriate generalization of the statement in (2.2.16). The integral on the right represents the flux of $\mathcal{J}^\mu$ through the timelike portion of the boundary, $\mathcal{B}$. If this flux vanishes then, because $t_\pm(X)$ can be chosen arbitrarily, it follows that Q is conserved. Using the fact that our current is expressible as the derivative of an antisymmetric tensor, we have

$$Q = \int_{\Sigma_t} d\Sigma_\mu \mathcal{J}^\mu = \int_{\Sigma_t} d\Sigma_\mu \nabla_\nu \mathcal{J}^{\mu\nu} = \frac{c^4}{8\pi G} \oint_S dS_{\mu\nu} \nabla^{[\mu} \varepsilon^{\nu]}, \tag{7.5.19}$$

where we have applied Stokes' theorem (6.7.68) and S is the two dimensional boundary of the spatial hypersurface Σ_t. When ε^μ is a Killing vector, the integral on the right is called a **Komar** integral. When the spacetime is asymptotically flat, the integral over the two sphere at infinity is the total mass energy as measured at infinity if ε^μ is the timelike Killing vector and the angular momentum measured at infinity if ε^μ is the azimuthal Killing vector.

7.6 Energy Conditions

Before we can look for solutions of the Einstein equations, we should ensure that the sources of the gravitational field obey some general physical requirements. These are intended to exclude matter that would lead to "unphysical" solutions of the Einstein equations. One or more of these requirements are generally met by most forms of normal matter and non-gravitational fields. There are four "energy conditions", defined in terms of unit time-like vector fields, n^μ ($n^2 = -1$), and light-like vectors fields, l^μ ($l^2 = 0$). Time-like vector fields represent tangent lines to time-like trajectories and define the worldlines of (possibily accelerated) time-like observers. Light-like vector fields represent the trajectories of massless particles. All the energy conditions are defined locally.

- **Weak Energy Condition:** For every future directed time like vector field, n^μ,

$$T_{\mu\nu} n^\mu n^\nu \geq 0. \tag{7.6.1}$$

  It states that the energy density of the matter distribution measured by every time-like observer, with four velocity n^μ, must be non-negative. Using Einstein's equation we can think of it as a statement about the geometry of space-time. The condition directly translates into the requirement that $R_{\mu\nu} n^\mu n^\nu + \frac{1}{2} R \geq 0$.

- **Null Energy Condition:** For every future directed null vector field, l^μ,

$$T_{\mu\nu} l^\mu l^\nu \geq 0. \tag{7.6.2}$$

  This is essentially the weak energy condition applied to null vector fields. Since $l^2 = 0$ it follows from Einstein's equations that the null condition requires $R_{\mu\nu} l^\mu l^\nu \geq 0$ for any future directed null vector field.

- **Strong energy Condition:** For every future directed time-like vector field, n^μ,

$$\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) n^\mu n^\nu \geq 0. \tag{7.6.3}$$

The strong energy condition can also be put in a geometric form. Einstein's equations imply that

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

so the statement is that $R_{\mu\nu}n^\mu n^\nu \geq 0$ for any future directed, time-like vector field.

- **Dominant Energy Condition:** In addition to the weak energy condition, the current vector field

$$j_\mu = -T_{\mu\nu}k^\nu \tag{7.6.4}$$

is always future directed and null or time-like, provided that k^μ is future directed and either null or time-like. This condition requires matter to *always* flow in future directed, time-like or null directions.

Often these conditions are not required to hold at every space-time point but only on the average; for example, the *averaged* null energy condition would read

$$\int T_{\mu\nu}l^\mu l^\nu d\lambda \geq 0 \tag{7.6.5}$$

where λ is the affine parameter along the integral curves of l^μ. This allows us to include stress energy tensors of quantum fields as sources for the gravitational field, although there is some argument about whether or not this is very meaningful.

As an example, consider the case of the stress tensor for the time-like ideal fluid,

$$T_{\mu\nu} = pg_{\mu\nu} + \left(\rho + \frac{p}{c^2}\right)U_\mu U_\nu \tag{7.6.6}$$

where U^μ is the time-like four velocity vector of the fluid elements. Since all the energy conditions are scalar we can choose a particular frame in which to evaluate them, the most convenient being the **comoving frame** of the fluid, *i.e,* the one in which $U^0 = 1$ and $U^i = 0$.

- The weak energy condition will be

$$-p + \left(\rho + \frac{p}{c^2}\right)(n \cdot U)^2 = -p + \left(\rho + \frac{p}{c^2}\right)n_0^2 \geq 0 \tag{7.6.7}$$

since $n^2 = -1$ and where we have set $(n \cdot U)^2 = n_0^2$, which is positive but otherwise arbitrary. If we take $n^\mu = U^\mu/c$ then we find that $\rho \geq 0$. But since n_0^2 can be made arbitrarily large, we must also require $\rho c^2 + p \geq 0$. This second condition restricts how negative the pressure can get, but places no other restriction on its magnitude.

- The null energy condition reads

$$\left(\rho + \frac{p}{c^2}\right)(l \cdot U)^2 \geq 0 \Rightarrow \rho c^2 + p \geq 0. \tag{7.6.8}$$

  since $l^2 = 0$.

- Applying the strong energy condition and taking $n^\mu = U^\mu/c$ gives $\rho c^2 + 3p \geq 0$ and we must also have $\rho c^2 + p \geq 0$ because n_0 is arbitrary. Again, there is no restriction on the magnitude of the pressures.

- Finally, the dominant energy condition requires, besides the weak energy conditions, that $-\rho c^2 \leq p \leq \rho c^2$ and thus restricts the magnitude of the pressure as well.

In the general case, determining the implications of the energy conditions for *any* stress tensor is essentially an algebraic problem and can always be formulated in terms of the eigenvalues of the stress energy tensor.

7.7 Geodesic Congruences

The behavior of both time-like and null geodesics is an essential tool in analyzing the space-time produced by a matter distribution.

7.7.1 Time-like Congruences

We will begin with a look at time-like geodesics. In fact, we will look at time-like geodesic **congruences** in regions of space-time, by which we mean entire one parameter families of time-like geodeiscs that do not intersect anywhere in the region. Therefore, only one curve of a geodesic congruence passes through any given event in our region. Our goal is to determine how the congruence "evolves" over time, which we will make precise below.

Let $x^\mu(\lambda, \sigma)$ represent the geodesic congruence, shown in figure 7.5, where λ is an affine parameter and σ labels the geodesics. Holding σ fixed and varying λ yields the geodesic curve labeled by σ, but holding λ and varying σ yields a curve labeled by λ and parameterized by σ. Thus $U^\mu(\lambda, \sigma) = \partial x^\mu(\lambda, \sigma)/\partial\lambda$ would represent the tangent vector field to the geodesic curve labeled by σ. Because each $x^\mu(\lambda, \sigma)$ is a solution of the geodesic equation, the acceleration vector vanishes,

$$a^\mu = (U \cdot \nabla)U^\mu = 0. \tag{7.7.1}$$

In analogy with the four velocity, let us define the vector field $V^\mu(\lambda, \sigma) = \partial x^\mu(\lambda, \sigma)/\partial\sigma$. $V^\mu(\lambda, \sigma)$ represents the tangent vector field to the second family of curves mentioned

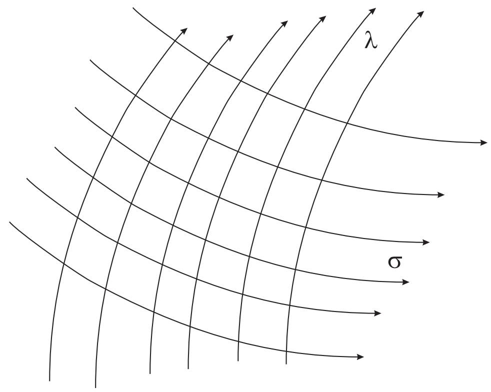

Figure 7.5: The congruence $x^\mu(\lambda, \sigma)$

above that were labeled by λ and parameterized by σ. We want to determine some of its properties, so first consider the fact that integrability requires,[10]

$$\frac{\partial V^\mu}{\partial \lambda} = \frac{\partial U^\mu}{\partial \sigma} \;\Rightarrow\; (U \cdot \nabla)V^\mu = (V \cdot \nabla)U^\mu. \tag{7.7.2}$$

We can combine this with the geodesic equation to prove that $\partial(V \cdot U)/\partial\lambda = 0$ as follows

$$\frac{\partial}{\partial \lambda}(V \cdot U) = (U \cdot \nabla)(V \cdot U) = [(U \cdot \nabla)V] \cdot U + [(U \cdot \nabla)U] \cdot V. \tag{7.7.3}$$

The last term vanishes by the geodesic equation and using (7.7.2) we have

$$\frac{\partial}{\partial \lambda}(V \cdot U) = [(V \cdot \nabla)U] \cdot U \tag{7.7.4}$$

Now because $U^2 = -1$ for time-like geodesics it follows that $U^\mu(\nabla_\alpha U_\mu) = 0$, therefore

$$\frac{\partial}{\partial \lambda}(V \cdot U) = V^\alpha(\nabla_\alpha U^\mu)U_\mu = 0 \tag{7.7.5}$$

(as promised), which says that $V \cdot U$ is a constant in the affine parameter. Now we recall that the affine parameter is not uniquely defined, being defined up to linear transformations. Thus we could shift the affine parameter by a constant that depends on the geodesic label, $\lambda \to \lambda'(\sigma) = \lambda + \mu(\sigma)$, for any arbitrary function of σ. Under such a transformation,

$$V^\mu \to V'^\mu = V^\mu + \frac{d\mu}{d\sigma}U^\mu \tag{7.7.6}$$

[10] Alternatively,

$$\pounds_U V^\mu = 0 = \pounds_V U^\mu$$

Now $V' \cdot U = V \cdot U - c^2 d\mu(\sigma)/d\sigma$. One could therefore choose $\mu(\sigma)$ so that $V' \cdot U = 0$ at some point on each geodesic. Then by (7.7.5) the vector field V' would be *everywhere* orthogonal to each geodesic in the congruence. The vector field V (or V') can be thought of as characterizing the deviation between neighboring geodesics and is called the **geodesic deviation** vector.

Expansion, Shear and Rotation

Given any time-like vector field, say U^μ, one defines a "projector" transverse to U^μ by

$$h^\mu{}_\nu = \delta^\mu{}_\nu + U^\mu U_\nu \tag{7.7.7}$$

To see that $h^\mu{}_\nu$ projects transverse to U^μ, notice that $h^\mu{}_\nu U^\nu = 0 = U_\mu h^\mu{}_\nu$ because $U^2 = -1$ and therefore, for any vector A^μ, if $A^\mu_\perp = h^\mu{}_\nu A^\nu$ then $U \cdot A_\perp = 0$. $h_{\mu\nu}$ satisfies $h^\alpha{}_\alpha = 3$ and therefore projects onto a three dimensional (spatial) hypersurface, orthogonal to U. The tensor

$$B_{\mu\nu} = U_{\mu;\nu}, \tag{7.7.8}$$

is also transverse because $B_{\mu\nu}U^\nu = U^\nu U_{\mu;\nu} = (U \cdot \nabla)U_\mu = 0$ by the geodesic equation and $U^\mu B_{\mu\nu} = U^\mu U_{\mu;\nu} = \frac{1}{2}\nabla_\nu U^2 = 0$.[11] Furthermore, by (7.7.2),

$$\frac{\partial V^\mu}{\partial \lambda} = (U \cdot \nabla)V^\mu = (V \cdot \nabla)U^\mu = B^\mu{}_\nu V^\nu, \tag{7.7.9}$$

so $B_{\mu\nu}$ governs the evolution of the deviation vector. It is convenient to separate $\widehat{B}$ into a trace part, a trace free symmetric part and an antisymmetric part as follows,

$$B_{\mu\nu} = \frac{1}{3}\Theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}, \tag{7.7.10}$$

where $\Theta = B^\alpha{}_\alpha$ (hence the factor of three) is called the **expansion scalar**, the symmetric tensor $\sigma_{\mu\nu} = \frac{1}{2}B_{(\mu\nu)} - \frac{1}{3}\Theta h_{\mu\nu}$ is called the **shear tensor** and the antisymmetric part, $\omega_{\mu\nu} = \frac{1}{2}B_{[\mu\nu]}$ is called the **vorticity** or **rotation tensor**. The expansion represents the rate of change of the volume of an infinitesimal sphere of geodesics with respect to the affine parameter of the geodesic passing through its center (when $\Theta > 0$ the geodesic congruence is diverging and when $\Theta < 0$ it is converging). The shear represents the rate at which the original spherical volume gets distorted and the rotation describes the tendency of the original sphere to rotate during its evolution, causing the geodesics to twist about one another.

Let us examine the evolution of $B_{\mu\nu}$,

$$\frac{\partial}{\partial \lambda}B_{\mu\nu} = U^\alpha B_{\mu\nu;\alpha} = U_{\mu;\nu\alpha}U^\alpha \tag{7.7.11}$$

[11] Problem: Show this explicitly.

Rewrite the right hand side as follows:

$$\begin{aligned}\frac{\partial}{\partial\lambda}B_{\mu\nu} &= [U_{\mu;[\nu\alpha]}+U_{\mu;\alpha\nu}]U^\alpha = U_{\mu;\alpha\nu}U^\alpha - R^\gamma{}_{\mu\alpha\nu}U_\gamma U^\alpha \\ &= (U_{\mu;\alpha}U^\alpha)_{;\nu} - U_{\mu;\alpha}U^\alpha{}_{;\nu} - R^\gamma{}_{\mu\alpha\nu}U_\gamma U^\alpha\end{aligned} \tag{7.7.12}$$

The first term on the right vanishes by the geodesic equation, so

$$\frac{\partial}{\partial\lambda}B_{\mu\nu} = -B_{\mu\alpha}B^\alpha{}_\nu - R^\gamma{}_{\mu\alpha\nu}U_\gamma U^\alpha. \tag{7.7.13}$$

This allows us to obtain the **geodesic deviation** equation by taking a derivative of (7.7.9),

$$\frac{\partial^2 V^\mu}{\partial\lambda^2} = -R^\mu{}_{\alpha\beta\gamma}U^\alpha U^\gamma V^\beta, \tag{7.7.14}$$

which shows that it is the Riemann curvature that is responsible for the relative acceleration between neighboring geodesics. This relative acceleration is sometimes referred to as the **tidal acceleration**.

Hypersurface Orthogonal Vector Fields

Any vector field, ξ^μ, identifies a congruence of world lines to which the field is tangent at every point. When are we guaranteed that there exists a family of hypersurfaces to which these world lines are everywhere perpendicular? If there exists such a family of hypersurfaces, $\Psi(x) =$ const., then the vector field is said to be **hypersurface orthogonal**. In that case,

$$\xi_\mu(x) = N(x)\Psi_{,\mu}(x) \tag{7.7.15}$$

where $N(x)$ is some function. It follows that

$$\xi_{\mu;\nu} = N_{,\nu}\Psi_{,\mu} + N\Psi_{;\mu\nu} \tag{7.7.16}$$

and therefore

$$\xi_{[\mu;\nu]} = N_{[,\nu}\Psi_{,\mu]} = \frac{1}{N}N_{[,\nu}\xi_{\mu]} = \frac{1}{N}[N_{,\nu}\xi_\mu - N_{,\mu}\xi_\nu] \tag{7.7.17}$$

or

$$\epsilon^{\alpha\mu\nu\beta}\xi_{[\mu;\nu]}\xi_\beta = 0 \tag{7.7.18}$$

by the antisymmetry of the Levi-Civita tensor. The vector field

$$\omega^\alpha = \frac{1}{2}\epsilon^{\alpha\mu\nu\beta}\xi_{[\mu;\nu]}\xi_\beta \tag{7.7.19}$$

is called the **rotation** of the congruence of world lines defined by ξ^μ. We have shown that a vanishing rotation is necessary for a vector field to be hypersurface orthogonal, but it turns out that it is also sufficient, *i.e.,* if the rotation vanishes then *locally* there are functions $N(x)$ and $\Psi(x)$ such that $\xi_\mu = N(x)\partial_\mu \Psi(x)$. This is more difficult to prove (and we will not do so here). The result is called **Frobenius' theorem**: *a necessary and sufficient condition for a vector field to be hypersurface orthogonal is that its rotation vanishes.*

A slightly simpler condition is obtained when Frobenius' theorem is applied to a geodesic congruence, U^μ. In this case, because U^μ is hypersurface orthogonal, *i.e.,*

$$U_{[\mu;\nu]} = \frac{1}{N}[N_{,\nu}U_\mu - N_{,\mu}U_\nu], \tag{7.7.20}$$

and $B_{\mu\nu}$ is transverse, *i.e.,* $U_{\mu;\nu}U^\nu = 0 = U_{\mu;\nu}U^\mu$, it follows that $U_{[\mu;\nu]}U^\nu = 0$ or

$$[(U \cdot \nabla)N]U_\mu + c^2 N_{,\mu} \;\Rightarrow\; N_{,\mu} = -\frac{1}{c^2}[(U \cdot \nabla)N]U_\mu. \tag{7.7.21}$$

Inserting this result back into (7.7.20) shows that

$$\frac{1}{2}B_{[\mu\nu]} = \omega_{\mu\nu} = \frac{1}{2c^2 N}[-[(U \cdot \nabla)N]U_\nu U_\mu + [(U \cdot \nabla)N]U_\mu U_\nu] = 0. \tag{7.7.22}$$

i.e., the rotation tensor vanishes. The converse is also true: if the rotation tensor vanishes then the geodesic congruence is hypersurface orthogonal.

Evolution of the Expansion, Shear and Rotation Tensors

Taking the trace of (7.7.13) gives the **Raychoudhuri** equation,

$$\begin{aligned} \frac{\partial \Theta}{\partial \lambda} &= -B_{\mu\nu}B^{\nu\mu} - R_{\mu\nu}U^\mu U^\nu \\ &= -\frac{1}{3}\Theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}U^\mu U^\nu \end{aligned} \tag{7.7.23}$$

The first two terms are always *negative* because the tensor $\sigma^{\mu\nu}$ is transverse to a time-like congruence. If the strong energy condition holds, then the last term is negative as well. Therefore, the divergence of a hypersurface orthogonal congruence (rotation free, $\widehat{\omega} = 0$) will always decrease if the strong energy condition holds,

$$\frac{\partial \Theta}{\partial \lambda} = -\frac{1}{3}\Theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} - R_{\mu\nu}U^\mu U^\nu \le 0. \tag{7.7.24}$$

This is the **focusing theorem**. In particular,

$$\frac{\partial \Theta}{\partial \lambda} \le -\frac{1}{3}\Theta^2 \;\Rightarrow\; \Theta^{-1}(\lambda) \ge \Theta_i^{-1} + \frac{1}{3}(\lambda - \lambda_i) \tag{7.7.25}$$

means that if $\Theta_i < 0$ then $\Theta^{-1} \to 0^-$ or $\Theta \to -\infty$ for some value of the affine parameter. This happens when the geodesics converge to a point called a **caustic**. A caustic is a singularity of the congruence. At this point the decomposition (7.7.10) breaks down and the expansion, shear and rotation are meaningless.

The evolution of the rotation tensor also follows directly from (7.7.13) because

$$\frac{\partial}{\partial\lambda}\omega_{\mu\nu} = \frac{1}{2}\frac{\partial}{\partial\lambda}B_{[\mu\nu]} = -\frac{2}{3}\Theta\ \omega_{\mu\nu} - \sigma_{\mu\alpha}\omega^{\alpha}{}_{\nu} + \sigma_{\nu\alpha}\omega^{\alpha}{}_{\mu} \tag{7.7.26}$$

where we exploited the third of the algebraic symmetries of the Riemann tensor listed in the previous chapter. Similarly, the evolution of the shear tensor can be obained as

$$\frac{\partial}{\partial\lambda}\sigma_{\mu\nu} = -\frac{2}{3}\Theta\sigma_{\mu\nu} - \sigma_{\mu\alpha}\sigma^{\alpha}{}_{\nu} - \omega_{\mu\alpha}\omega^{\alpha}{}_{\nu} - R_{\alpha\mu\beta\nu}U^{\alpha}U^{\beta} + \frac{1}{3}\left(\sigma^2 - \omega^2 + R_{\alpha\beta}U^{\alpha}U^{\beta}\right)h_{\mu\nu}, \tag{7.7.27}$$

where $\sigma^2 = \sigma_{\alpha\beta}\sigma^{\alpha\beta}$ and $\omega^2 = \omega_{\alpha\beta}\omega^{\alpha\beta}$. We leave this as an exercise.[12]

7.7.2 Null Congruences

The analysis of null geodesic congruences has much in common with the anaylsis of timelike geodesics, but there are also several complications, all of which are brought about by the fact that $K^2 = 0$, where $K^\mu = \partial x^\mu/\partial\lambda$. Thus we could define the geodesic deviation, V^μ in the same way as before and we would obtain $\pounds_V K^\mu = \pounds_K V^\mu$ as before. The acceleration of the geodesics is zero by definition, *i.e.,* $a^\mu = (K \cdot \nabla)K^\mu = 0$ and it is straightforward to show that $\partial(V \cdot K)/\partial\lambda = 0$, this time because $K^2 = 0$.

The dificulties arise from the fact that, because K^μ is a null vector field, $\delta^\mu{}_\nu + K^\mu K_\nu$ does not not annihilate K^μ and so it does not project transverse to K^μ. This is because the hypersurface orthogonal to any null vector is two and not three dimensional.[13] To correct this, we must introduce an auxilliary null vector field N^μ, which is such that $K \cdot N = -1$. We do this by first picking any spacelike hypersurface, Σ, and a null vector field, N^μ, defined on Σ, which satisfies $K \cdot N = -1$. If we extend N off Σ by parallel transport along geodesics, $(K \cdot \nabla)N^\mu = 0$, then the two conditions $N^2 = 0$ and $N \cdot K = -1$ are guaranteed to hold. While this can always be done, the conditions do *not* uniquely determine the null vector N^μ. However, once a particular N^μ is chosen on Σ the projector

$$h^\mu{}_\nu = \delta^\mu{}_\nu + K^\mu N_\nu + N^\mu K_\nu \tag{7.7.28}$$

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[12] Problem: Derive the evolution of the rotation and shear from (7.7.13).

[13] To see this, consider flat space in null coordinates, $u_\pm = ct \pm x$. The line element is

$$ds^2 = du_+ du_- - dy^2 - dz^2.$$

If we let K be tangent to the curves $u_\pm$ = const., then the transverse line element is given by $ds_\perp^2 = -dy^2 - dz^2$, which is two dimensional.

does indeed project transverse to K^μ. It also projects transverse to N^μ and therefore $h_{\mu\nu}$ is a two dimensional projector, satisfying $h^\alpha{}_\alpha = 2$. Furthermore, because both K and N are parallel transported along geodesics, it follows that so is $h^\mu{}_\nu$, *i.e.,* $(K \cdot \nabla) h^\mu{}_\nu = 0$.

Expansion, Shear and Twist

Taking our cue from the treatment of time-like geodesics, we define the tensor

$$B_{\mu\nu} = K_{\mu;\nu}. \tag{7.7.29}$$

It follows that

$$\frac{\partial V^\mu}{\partial \lambda} = B^\mu{}_\nu \xi^\nu, \tag{7.7.30}$$

but $B_{\mu\nu}$ is orthogonal only to K^μ, not to N^μ, so it is not transverse. Therefore, we use the projector and define the transverse tensor

$$\widetilde{B}_{\alpha\beta} = B_{\mu\nu} h^\mu{}_\alpha h^\nu{}_\beta. \tag{7.7.31}$$

Expanding $\widetilde{B}$ as before into a trace part, a symmetric, traceless part and an antisymmetric part,

$$\widetilde{B}_{\mu\nu} = \frac{1}{2}\Theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} \tag{7.7.32}$$

defines $\Theta = \widetilde{B}^\alpha{}_\alpha$, $\sigma_{\mu\nu} = \frac{1}{2}[\widetilde{B}_{(\mu\nu)} - \Theta h_{\mu\nu}]$ and $\omega_{\mu\nu} = \frac{1}{2}\widetilde{B}_{[\mu\nu]}$, which are the expansion, shear and twist of the null congruence, respectively. Note that the geodesic equation and the null condition together ensure that $\widetilde{B}^\alpha{}_\alpha = B^\mu{}_\mu$.

The argument, given for timelike geodesics, to show that the deviation vector can always be chosen to be transverse, fails in the case of the null geodesic congruence because $K^2 = 0$. Thus, even though $V \cdot K$ is a constant along each geodesic, it is not guaranteed to vanish everywhere by a suitable choice of parameterization. In general, the deviation vector could be expanded as

$$V^\mu = aK^\mu + bN^\mu + \widetilde{V}^\mu, \tag{7.7.33}$$

where $K \cdot \widetilde{V} = 0 = N \cdot \widetilde{V}$ and we are only guaranteed that $b = -V \cdot K$ is constant along each geodesic. Because $h_{\mu\nu}$ is parallel transported along geodesics, the evolution of the transverse part of the deviation vector, $\widetilde{V}^\mu = h^\mu{}_\alpha V^\alpha$, follows directly from (7.7.30) as

$$\frac{\partial \widetilde{V}^\mu}{\partial \lambda} = h^\mu{}_\alpha B^\alpha{}_\beta V^\beta. \tag{7.7.34}$$

Now we have

$$\frac{\partial \widetilde{V}^\mu}{\partial \lambda} = h^\mu{}_\alpha B^\alpha{}_\beta (aK^\beta + bN^\beta + \widetilde{V}^\beta) = h^\mu{}_\alpha B^\alpha{}_\beta (bN^\beta + \widetilde{V}^\beta), \tag{7.7.35}$$

where the last equation follows because $B^\alpha{}_\beta K^\beta = 0$. Thus deviation vectors for which $b = -K \cdot V = 0$ satisfy

$$\frac{\partial \widetilde{V}^\mu}{\partial \lambda} = h^\mu{}_\alpha B^\alpha{}_\beta \widetilde{V}^\beta = h^\mu{}_\alpha B^\alpha{}_\beta h^\beta{}_\gamma \underbrace{h^\gamma{}_\delta V^\delta} = \underbrace{h^\mu{}_\alpha h^\beta{}_\gamma B^\alpha{}_\beta} \widetilde{V}^\gamma = \widetilde{B}^\mu{}_\gamma \widetilde{V}^\gamma, \tag{7.7.36}$$

where we have used the definition of the transverse tensor $\widetilde{B}_{\mu\nu}$ in (7.7.31).

The evolution equation for $B_{\mu\nu}$ is obtained in precisely the same way as (7.7.13) with the same result,

$$\frac{\partial}{\partial \lambda} B_{\mu\nu} = -B_{\mu\alpha} B^\alpha{}_\nu - R^\gamma{}_{\mu\alpha\nu} K_\gamma K^\alpha, \tag{7.7.37}$$

which allows us to find the null geodesic deviation equation,

$$\frac{\partial^2 V^\mu}{\partial \lambda^2} = -R^\mu{}_{\alpha\beta\gamma} K^\alpha K^\gamma \xi^\beta \tag{7.7.38}$$

and, once again, it is the Riemann curvature that is responsible for the relative acceleration between neighboring (null) geodesics.[14]

Frobenius' Theorem

The congruence K^μ is hypersurface orthogonal if and only if $\omega_{\mu\nu} = 0$. As before, we will only prove that if the congruence is hypersurface orthogonal then $\omega_{\mu\nu} = 0$. Thus, suppose that the congruence is orthogonal to the family of (null) hypersurfaces given by $\Psi(x) =$ const., then it follows that $K_\alpha \propto \Psi_{,\alpha}$. However, then K^μ is at once both orthogonal and parallel to the hypersurfaces, since $K^\alpha \Psi_{,\alpha} = 0$, and therefore K^μ is also tangent to the hypersurfaces and lives within them. They are called the **null generators** of the hypersurfaces $\Psi(x) =$ const.

Now we may set $K_\mu = f\Psi_{,\mu}$ as before, but now $f(x)$ is any arbitrary function. Then

$$B_{\mu\nu} = f_{,\nu}\Psi_{,\mu} + f\Psi_{;\mu\nu} = \frac{f_{,\nu}}{f} K_\mu + \Psi_{;\mu\nu} \tag{7.7.39}$$

and, if we assume that space-time is torsion free, it follows that

$$B_{[\mu\nu]} = \frac{1}{f}\left(f_{,\nu} K_\mu - f_{,\mu} K_\nu\right). \tag{7.7.40}$$

Therefore after projecting in the transverse direction we will find that $\widetilde{B}_{[\mu\nu]} = 2\omega_{\mu\nu} \equiv 0$.

[14] Problem: Determine the evolution equation for $\widetilde{B}_{\mu\nu}$ and the geodesic deviation equation for the transverse part of V^μ.

7.8 Hamiltonian Description

The Hamiltonian dynamics of particles and fields relies heavily upon the splitting of space-time into space and time. In other words, we must consider space-times, $\mathcal{M}$, that are globally $\mathbb{R} \times \Sigma_3$ where $\mathbb{R}$ represents a time-like direction and Σ a spatial hypersurface. We therefore return to the foliation of $\mathcal{M}$ by spatial hypersurfaces as described in section 7.5. This approach, developed by Arnowitt, Deser and Misner, allows the problem of finding the geometry of $\mathcal{M}$ to be treated as an initial value problem in which the initial data are prescribed on an "initial" hypersurface and then evolved according to the field equations.

7.8.1 Arnowitt-Deser-Misner (ADM) metric

The treatment here will closely resemble the discussion of three dimensional hypersurfaces in the previous chapter. Because Σ_t is a surface of constant t, $\partial_\mu t$ is everywhere perpendicular to Σ_t. Assuming that $\partial_\mu t$ is future directed and nowhere lightlike, let

$$N^{-2} = -g^{\mu\nu}\partial_\mu t \partial_\nu t > 0 \tag{7.8.1}$$

be its norm and let n_μ be the future directed, unit normal to Σ_t,

$$n_\mu = -N\partial_\mu t. \tag{7.8.2}$$

Now let us transform from the original system of coordinates, X^μ, to a new system, $X'^\mu = \{t, x^i\}$. By the transformation laws,

$$n'^0 = n^\mu \partial_\mu t = \frac{1}{N} \tag{7.8.3}$$

and we can define three new functions $N^i(X')$ such that

$$n'^i = n^\mu \partial_\mu x^i = -\frac{N^i}{N} \tag{7.8.4}$$

The metric on $\mathcal{M}$ in the coordinates X' captures the geometric structure of the factorizable manifold $\mathcal{M}$; by a straightforward calculation, we arrive at

$$\begin{aligned}
g'^{00} &= g^{\mu\nu}\partial_\mu t \partial_\nu t = -\frac{1}{N^2} \\
g'^{0i} = g'^{i0} &= g^{\mu\nu}\partial_\mu t \partial_\nu x^i = -\frac{1}{N} n^\mu \partial_\mu x^i = \frac{N^i}{N^2} \\
g'^{ij} &= g^{\mu\nu}\partial_\mu x^i \partial_\nu x^j = h^{\mu\nu}\partial_\mu x^i \partial_\nu x^j - \frac{N^i N^j}{N^2},
\end{aligned} \tag{7.8.5}$$

where we have defined the "projection" $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$, which annihilates any vector in the direction of n^μ. Since the first term on the right hand side of the last equation "lives" only on Σ_t, we let

$$\gamma^{ij} = h^{\mu\nu} \partial_\mu x^i \partial_\nu x^j \tag{7.8.6}$$

and write the (inverse) metric as

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & N^j/N^2 \\ \\ N^i/N^2 & \gamma^{ij} - N^i N^j/N^2 \end{pmatrix}, \tag{7.8.7}$$

where we have dropped the primes. Finally, *defining* γ_{ij} to be the inverse of γ^{ij}, *i.e.,* $\gamma^{ik}\gamma_{kj} = \delta^i_j$, we have a canonical form for the metric on $\mathcal{M}$:

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \gamma_{lm} N^l N^m & N_j \\ \\ N_i & \gamma_{ij} \end{pmatrix} \tag{7.8.8}$$

where $N_i = \gamma_{ij} N^j$. The matrix γ_{ij} is the induced metric on Σ_t, or the **first fundamental form**. It defines the distance between infinitesimally separated points on Σ_t. When it is given as (7.8.8), the metric is said to be in the **Arnowitt-Deser-Misner** (or ADM, for short) form. The function N is called the **shift** and N^i are called the **lapse** functions. The ADM distance between two points on $\mathcal{M}$ will be

$$ds^2 = -g_{\mu\nu} dx^\mu dx^\nu = N^2 dt^2 - \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{7.8.9}$$

and the 10 functions comprising the shift, the lapse and the spatial metric are to be recovered from Einstein's equations. The volume element, written in terms of these 10 functions becomes

$$d^4x\sqrt{-g} = d^4x N\sqrt{\gamma} \tag{7.8.10}$$

where γ represents the determinant of the induced metric.

The ADM form of the metric is designed to turn the problem of finding solutions to Einstein's equations into a Cauchy problem, *i.e.,* into the problem of finding solutions subject to initial data specified on some initial spatial hypersurface. This provides a clean way to classify the initial data while also allowing us to formulate the concept of causality in general relativity. Our aim in this section is more limited. We only seek a Hamiltonian formulation of the theory.

7.8.2 Hamiltonian of a scalar field

As a preparatory exercise, let us first apply this general metric to determine the Hamiltonian of, say, a real scalar field. The procedure is similarly carried out for all other fields.

Consider a (real) scalar field minimally coupled to the gravitational field so that its action is

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}\,[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 2V(\phi)] \tag{7.8.11}$$

and re-express it, using the ADM decomposition in the previous subsection,

$$S = -\frac{1}{2}\int dt\int d^3xN\sqrt{\gamma}\left[-\frac{1}{N^2}\dot{\phi}^2 + \frac{2N^i\phi_{,i}}{N^2}\dot{\phi} - \frac{(N^i\phi_{,i})^2}{N^2} + \gamma^{ij}\phi_{,i}\phi_{,j} + 2V(\phi)\right] \tag{7.8.12}$$

The Lagrangian density for the field is

$$\mathfrak{L} = \frac{1}{2}N\sqrt{\gamma}\left[\frac{1}{N^2}\dot{\phi}^2 - \frac{2N^i\phi_{,i}}{N^2}\dot{\phi} + \frac{(N^i\phi_{,i})^2}{N^2} - \gamma^{ij}\phi_{,i}\phi_{,j} - 2V(\phi)\right] \tag{7.8.13}$$

and the momentum conjugate to the field is

$$\pi = \frac{\partial\mathfrak{L}}{\partial\dot{\phi}} = \frac{\sqrt{\gamma}}{N}\left[\dot{\phi} - N^i\phi_{,i}\right]. \tag{7.8.14}$$

Solving for the field velocities,

$$\dot{\phi} = \frac{N\pi}{\sqrt{\gamma}} + N^i\phi_{,i}, \tag{7.8.15}$$

and performing the Legendre transfrmation in the usual way, gives the Hamiltonian of the scalar field,

$$\begin{aligned} H &= \int d^3x\left[\pi\dot{\phi} - \mathfrak{L}\right] \\ &= \int d^3xN\sqrt{\gamma}\left[\frac{\pi^2}{2\gamma} + \frac{(N^i\phi_{,i})\pi}{N\sqrt{\gamma}} + \frac{1}{2}\left\{\gamma^{ij}\phi_{,i}\phi_{,j} + 2V(\phi)\right\}\right]. \end{aligned} \tag{7.8.16}$$

Now if ξ^μ is a Killing vector then the four momentum, $p^\mu = \xi^\mu T^\mu{}_\nu$, is (covariantly) conserved ($\nabla_\mu p^\mu = 0$), so integrating p^μ over the spatial hypersurface, Σ, yields the conserved quantity,

$$Q_\xi = \int_\Sigma d^3x\sqrt{\gamma}n_\mu(\xi^\nu T^\mu{}_\nu). \tag{7.8.17}$$

In a stationary spacetime, taking ξ^μ to be the time-like Killing vector $\xi^\mu = (1, 0, 0, 0)$, we find that $Q_\xi \equiv H$.[15]

[15] Problem: Work this out using the unit normal, $n_\mu = (-N, 0, 0, 0)$, the ADM metric in (7.8.9) and the stress tensor for a real scalar field in (7.4.30).

7.8.3 Extrinsic and Intrinsic curvatures of a hypersurface

The metric and the unit normal together will define a projector onto the spatial hypersurfaces,

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{7.8.18}$$

To see that this is indeed a projector consider that acting on any vector parallel to n_μ it must yield exactly zero, since $h_{\mu\nu}n^\nu = 0$, and it also satisfies

$$h_{\mu\lambda}h^\lambda{}_\nu = (g_{\mu\lambda} + n_\mu n_\lambda)(\delta^\lambda{}_\nu + n^\lambda n_\nu) = h_{\mu\nu}. \tag{7.8.19}$$

Now let us define the differential operator "D" by

$$D_\lambda T^{\alpha_1\cdots}{}_{\beta_1\ldots} = h_\lambda{}^\sigma h^{\alpha_1}{}_{\mu_1}\ldots h_{\beta_1}{}^{\nu_1}\ldots\nabla_\sigma T^{\mu_1\cdots}{}_{\nu_1\ldots} \tag{7.8.20}$$

i.e., by projecting the components of $\nabla\mathbb{T}$ using the projector, $\widehat{h}$. "D" defines the **intrinsic covariant derivative** operator on the hypersurface. It is easy to show that $D_\alpha h_{\beta\gamma} = 0$ and that "D" inherits all the conditions (eg., the Leibnitz rule) that would make it an acceptable derivative operator from the ordinary covariant derivative operator "∇". It is also torsion free *i.e.,*

$$[D_\mu, D_\lambda]\varphi(x) = 0, \tag{7.8.21}$$

therefore "D" defines a unique, torsion free connection in Σ_t. Now with the help of "D_μ" and n_μ we can define two properties of the spatial hypersurfaces, Σ_t.

The **extrinsic curvature** of Σ_t, or the **second fundamental form**, is defined as the spatial rate of change of n_μ,[16]

$$K_{\mu\nu} = D_\mu n_\nu = \nabla_\mu n_\nu + n_\mu a_\nu, \tag{7.8.22}$$

where $a_\mu = (n\cdot\nabla)n_\mu$, called the acceleration of the foliation, is perpendicular to n^μ and therefore tangent to Σ_t[17]. The extrinsic curvatures is a symmetric tensor, a property that is not obvious from its definition in (7.8.22). It follows because the connection is torsion free so that

$$K_{[\mu\nu]} = h_\mu{}^\alpha h_\nu{}^\beta\nabla_{[\alpha}n_{\beta]} = h_\mu{}^\alpha h_\nu{}^\beta(\partial_\alpha n_\beta - \partial_\beta n_\alpha). \tag{7.8.23}$$

Now, using the fact that $n_0 = -N,\ \ n_i = 0$, we have

$$K_{[\mu\nu]} = -h_\mu{}^\alpha h_\nu{}^0\partial_\alpha N + h_\mu{}^0 h_\nu{}^\beta\partial_\beta N, \tag{7.8.24}$$

but $h_\mu{}^0 = 0$, so it follows that $K_{[\mu\nu]} = 0$. Furthermore, all the indices of $\widehat{K}$ and of ${}^{(3)}\widehat{R}$ are perpendicular to $\widehat{n}$ by definition.

---

[16] Problem: Show that $K_{\mu\nu} = h_\mu{}^\alpha\nabla_\alpha n_\nu$.

[17] This follows from the fact that $\widehat{n}$ is a unit vector and therefore $n^\mu\nabla_\alpha n_\mu = 0$. Thus $n\cdot a = n^\alpha n^\mu(\nabla_\alpha n_\mu) \equiv 0$.

The **intrinsic curvature** of the hypersurface is defined in the usual way, but in terms of the derivative operator D, as

$$[D_\mu, D_\nu]A^\alpha = {}^{(3)}R^\alpha{}_{\beta\mu\nu}A^\beta \tag{7.8.25}$$

for all vectors A that "live" on Σ_t, *i.e.,* are perpendicular to n ($n \cdot A = 0$).

7.8.4 The Gauss equations

Let us now determine some relations between the four dimensional Riemann tensor, its three dimensional counterpart and the the extrinsic curvature of Σ_t. Starting from the definition of ${}^{(3)}\widehat{R}$, we have

$$\begin{aligned} D_{[\mu}D_{\nu]}A^\alpha &= h_\mu{}^\lambda h_\nu{}^\gamma h^\alpha{}_\kappa \nabla_\lambda(h^\kappa{}_\sigma h_\gamma{}^\rho \nabla_\rho A^\sigma) - (\mu \leftrightarrow \nu) \\ &= h_\mu{}^\lambda h_\nu{}^\gamma h^\alpha{}_\kappa[(\nabla_\lambda h^\kappa{}_\sigma h_\gamma{}^\rho)\nabla_\rho A^\sigma + h^\kappa{}_\sigma h_\gamma{}^\rho \nabla_\lambda \nabla_\rho A^\sigma] - (\mu \leftrightarrow \nu) \end{aligned} \tag{7.8.26}$$

The last term in square brackets on the right, when antisymmetrized in (μ, ν) gives $h_\mu{}^\lambda h_\nu{}^\gamma h^\alpha{}_\sigma \nabla_{[\lambda}\nabla_{\gamma]}A^\sigma = h^\alpha{}_\sigma h_\mu{}^\lambda h_\nu{}^\gamma R^\sigma{}_{\kappa\lambda\gamma}A^\kappa$. Again, because $n \cdot A = 0$ we can write $A^\kappa = A^\beta h_\beta{}^\kappa$ and express the contribution from this term as $h^\alpha{}_\sigma h_\beta{}^\kappa h_\mu{}^\lambda h_\nu{}^\gamma R^\sigma{}_{\kappa\lambda\gamma}A^\beta$.

To evaluate the first term, notice that

$$\begin{aligned} &h_\mu{}^\lambda h_\nu{}^\gamma h^\alpha{}_\kappa[(\nabla_\lambda h^\kappa{}_\sigma)h_\gamma{}^\rho + h^\kappa{}_\sigma(\nabla_\lambda h_\gamma{}^\rho)]\nabla_\rho A^\sigma \\ &\qquad = [h_\mu{}^\lambda h_\nu{}^\rho h^\alpha{}_\kappa n_\sigma \nabla_\lambda n^\kappa + h_\mu{}^\lambda h_\nu{}^\gamma h^\alpha{}_\sigma n^\rho \nabla_\lambda n_\gamma]\nabla_\rho A^\sigma \\ &\qquad = [h_\nu{}^\rho n_\sigma K_\mu{}^\alpha + h^\alpha{}_\sigma n^\rho K_{\mu\nu}]\nabla_\rho A^\sigma \end{aligned} \tag{7.8.27}$$

The second term in square brackets will vanish upon antisymmetrization because $\widehat{K}$ is symmetric. Moreover, the first term can be rewritten as

$$h_\nu{}^\rho K_\mu{}^\alpha n_\sigma \nabla_\rho A^\sigma = -h_\nu{}^\rho K_\mu{}^\alpha(\nabla_\rho n_\sigma)A^\sigma, \tag{7.8.28}$$

using the fact that $n \cdot A = 0$ and, because of the same identity, we can set $A^\sigma = A^\beta h_\beta{}^\sigma$ so the above expression turns into

$$-h_\nu{}^\rho h_\beta{}^\sigma(\nabla_\rho n_\sigma)K_\mu{}^\alpha A^\beta = -K_\mu{}^\alpha K_{\nu\beta}A^\beta.$$

Finally antisymmetrizing with respect to (μ, ν) we get the contribution $-K_{[\mu}{}^\alpha K_{\nu]\beta}A^\beta$ from this term. Therefore putting everything together, we recover the **Gauss** equations

$${}^{(3)}R^\alpha{}_{\beta\mu\nu} = h^\alpha{}_\sigma h_\beta{}^\kappa h_\mu{}^\lambda h_\nu{}^\gamma R^\sigma{}_{\kappa\lambda\gamma} - K_{[\mu}{}^\alpha K_{\nu]\beta} \tag{7.8.29}$$

relating the intrinsic and extrinsic curvatures of the three dimensional spatial hypersurfaces with the projection of the intrinsic curvature of the four dimensional manifold onto the spatial hypersurface. We can now contract appropriately to recover the Ricci tensor,

$$^{(3)}R_{\beta\nu} = R_{\beta\nu} + 2R_{\beta\gamma}n_\nu n^\gamma + R_{\kappa\gamma}n^\kappa n^\gamma n_\beta n_\nu + R^\sigma{}_{\beta\lambda\nu}n_\sigma n^\lambda - K_{[\alpha}{}^\alpha K_{\nu]\beta} \tag{7.8.30}$$

and the scalar curvature

$$^{(3)}R = R + 2R_{\beta\nu}n^\beta n^\nu - K_{[\alpha}{}^\alpha K_{\nu]}{}^\nu \tag{7.8.31}$$

7.8.5 The Codazzi-Mainardi equations

There is another set of useful identities, called the **Codazzi-Mainardi** equations, that are obtained by considering projections of the four dimensional Riemann tensor along the normal direction. To obtain this set, consider the derivative of the extrinsic curvature,

$$\begin{aligned} D_\mu K_{\nu\lambda} &= h_\mu{}^\alpha h_\nu{}^\beta h_\lambda{}^\sigma \nabla_\alpha K_{\beta\sigma} = h_\mu{}^\alpha h_\nu{}^\beta h_\lambda{}^\sigma \nabla_\alpha (h_\beta{}^\gamma h_\sigma{}^\rho \nabla_\gamma n_\rho) \\ &= h_\mu{}^\alpha h_\nu{}^\beta h_\lambda{}^\sigma h_\beta{}^\gamma h_\sigma{}^\rho \nabla_\alpha \nabla_\gamma n_\rho + h_\mu{}^\alpha h_\nu{}^\beta h_\lambda{}^\sigma \nabla_\alpha (h_\beta{}^\gamma h_\sigma{}^\rho)\nabla_\gamma n_\rho \\ &= h_\mu{}^\alpha h_\nu{}^\gamma h_\lambda{}^\rho \nabla_\alpha \nabla_\gamma n_\rho + K_{\mu\nu} h_\lambda{}^\rho (n\cdot\nabla) n_\rho \end{aligned} \tag{7.8.32}$$

Hence we find that

$$D_{[\mu}K_{\nu]\lambda} = -h_\lambda{}^\rho h_\mu{}^\alpha h_\nu{}^\gamma R_{\beta\rho\alpha\gamma} n^\beta \tag{7.8.33}$$

because of the symmetry of $K_{\mu\nu}$. Contracting,

$$D_{[\mu}K_{\nu]}{}^\nu = -h_\mu{}^\alpha h^{\gamma\rho} R_{\beta\rho\alpha\gamma} n^\beta = -h_\mu{}^\alpha R_{\alpha\beta} n^\beta \tag{7.8.34}$$

The Gauss-Codazzi-Mainardi equations are useful for decomposing the ten Einstein equations into components that are orthogonal to and components that are tangential to the spatial hypersurface. For instance, the orthogonal components of the Einstein tensor (time-time) are given as

$$2G_{\alpha\beta}n^\alpha n^\beta = 2R_{\alpha\beta}n^\alpha n^\beta + R = {}^{(3)}R + K_{[\alpha}{}^\alpha K_{\beta]}{}^\beta, \tag{7.8.35}$$

upon directly applying (7.8.31). Its mixed components (time-space),

$$h_\mu{}^\alpha G_{\alpha\beta} n^\beta = h_\mu{}^\alpha R_{\alpha\beta} n^\beta = -D_{[\mu}K_{\beta]}{}^\beta, \tag{7.8.36}$$

follow directly from (7.8.34) and so on.

Such a decomposition of the Einstein equations is a necessary prelude to setting up the initial value problem of general relativity. The field equations are second order, so any unique solution requires a specification of the metric *and* its first time derivative at some initial "time", *i.e.,* on some spatial hypersurface, Σ. However, on any Σ, the time-time and time-space components of the space-time metric cannot be given meaning in terms of the geometric properties of Σ alone, so only the six components of the first fundamental form can be specified. The remaining four components of the space-time metric are arbitrary and this reflects the freedom to choose how the four coordinates are laid down. Likewise, specifying the time derivative of the induced metric turns out to be a statement about the extrinsic curvature (the second fundamental form), as we will shortly see. The initial value problem of general relativity therefore consists of specifying the induced metric and the extrinsic curvature of some spatial hypersurface. They cannot be chosen freely but must satisfy the four constraints of general relativity, one given by (7.8.35),

$$^{(3)}R + K_{[\alpha}{}^{\alpha} K_{\beta]}{}^{\beta} = \frac{16\pi G}{c^4} T_{\alpha\beta} n^\alpha n^\beta \stackrel{\text{def}}{=} \frac{16\pi G\rho}{c^4} \tag{7.8.37}$$

and three by (7.8.36),

$$- D_{[\mu} K_{\beta]}{}^{\beta} = \frac{8\pi G}{c^4} h_\mu{}^\alpha T_{\alpha\beta} n^\beta \stackrel{\text{def}}{=} \frac{8\pi G j_\mu}{c^4}. \tag{7.8.38}$$

The remaining six (the space-space equations) provide the evolution equations for the first and second fundamental forms.

7.8.6 Action

With the help of the Gauss equation, we can now write the gravitational action (neglecting the boundary term). However, we must first find an expression for $R_{\mu\nu} n^\mu n^\nu$ in terms of $^{(3)}R$ and the extrinsic curvature, which we can do by using the fact that

$$[\nabla_\mu, \nabla_\nu] A^\alpha = R^\alpha{}_{\beta\mu\nu} A^\beta \tag{7.8.39}$$

holds for any vector and, in particular, for $\widehat{n}$. It follows that

$$\begin{aligned}
R_{\mu\nu} n^\mu n^\nu &= n^\nu [\nabla_\mu, \nabla_\nu] n^\mu \\
&= n^\nu (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) n^\mu \\
&= -(\nabla_\mu n^\nu)(\nabla_\nu n^\mu) + (\nabla_\nu n^\nu)(\nabla_\mu n^\mu) + \nabla_\mu (n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu) \\
&= K_{[\mu}{}^\mu K_{\nu]}{}^\nu + \frac{1}{2} \nabla_\mu Q^\mu.
\end{aligned} \tag{7.8.40}$$

where $Q^\mu = 2[a^\mu - K n^\mu]$ and $K = K^\alpha{}_\alpha$. Inserting this into (7.8.31)

$$^{(3)}R = R + K_{[\mu}{}^\mu K_{\nu]}{}^\nu - \nabla \cdot Q \tag{7.8.41}$$

and therefore the bulk term of the Hilbert action can be written as

$$S_G = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}R = \frac{c^4}{16\pi G}\int d^4x\ N\ \sqrt{\gamma}\ \left[{}^{(3)}R - K_{[\mu}{}^{\mu}K_{\nu]}{}^{\nu}\right] + S_{\partial\mathcal{M}} \tag{7.8.42}$$

where the surface term is

$$S_{\partial\mathcal{M}} = \frac{c^4}{8\pi G}\int d^4x\ N\ \sqrt{\gamma}\ \nabla_\mu(a^\mu - Kn^\mu). \tag{7.8.43}$$

Now ${}^{(3)}R$ contains no time derivatives of the metric, so it is a "potential" term. The bulk action therefore has the classic form of "kinetic energy", captured by the extrinsic curvature, minus "potential energy", captured by the intrinsic curvature.

7.8.7 Hamiltonian

The Lagrangian density (here, we include $\sqrt{\gamma}$) for the gravitational field is now

$$\mathfrak{L} = \frac{c^4 N}{16\pi G}\sqrt{\gamma}\ \left[{}^{(3)}R + K_{\mu\nu}K^{\mu\nu} - K^2\right] \tag{7.8.44}$$

Our configuration space consists of the lapse and shift functions together with the six components of the induced metric. We want to obtain the momenta conjugate to these variables.

The fact that all the indices of $K^{\mu\nu}$ are perpendicular to n_μ implies that $K^{0\mu} = 0 = K^{\mu 0}$, therefore the Lagrange density can be written as

$$\mathfrak{L} = \frac{c^4 N}{16\pi G}\sqrt{\gamma}\ \left[{}^{(3)}R + K_{ij}K^{ij} - K^2\right], \tag{7.8.45}$$

where $K = \gamma_{ij}K^{ij}$. We may work only with the spatial components of $\widehat{K}$; consider

$$\begin{aligned} K_{ij} &= \nabla_i n_j = N\Gamma^0_{ij} = \frac{N}{2}g^{0\alpha}\left[g_{\alpha i,j} + g_{\alpha j,i} - g_{ij,\alpha}\right] \\ &= -\frac{1}{2N}\left[N_{i,j} + N_{j,i} - \dot{\gamma}_{ij}\right] + \frac{N^l}{2N}\left[\gamma_{li,j} + \gamma_{lj,i} - \gamma_{ij,l}\right] \\ &= \frac{1}{2N}\left[\dot{\gamma}_{ij} - D_{(i}N_{j)}\right] \end{aligned} \tag{7.8.46}$$

where the braces as usual refer to the anticommutator.[18]

[18] Problem: Using the expressions for the unit hypersurface normal, show that

$$h_0{}^0 = 0 = h_i{}^0, \quad h_i{}^j = \delta_i^j, \quad h_0{}^i = N^i.$$

Use these expressions and the definition of the extrinsic curvature to demonstrate that

$$K_{0i} = K_{i0} = N^l K_{il}, \quad K_{00} = N^l N^m K_{lm},$$

and hence that $K^{0\alpha} = K^{\alpha 0} = 0$ and $K^{ij} = \gamma^{il}\gamma^{jm}K_{lm}$.

Notice that there are no time derivatives of the lapse and shift functions in the Lagrangian, so the momenta conjugate to these functions are identically zero. Let these be π_N, conjugate to N, and π^i_N conjugate to N_i. The vanishing of these momenta will be the primary constraints. There are time derivatives of the induced metric, however, so call π^{ij} the momentum conjugate to γ_{ij}, then

$$\pi^{ij} = \frac{\partial \mathfrak{L}}{\partial \dot{\gamma}_{ij}} = \frac{c^4}{16\pi G}\sqrt{\gamma}\ (K^{ij} - \gamma^{ij}K) \tag{7.8.47}$$

Taking the trace,

$$\pi = -\frac{c^4}{8\pi G}\sqrt{\gamma}\ K \tag{7.8.48}$$

and thus

$$\sqrt{\gamma}K^{ij} = \frac{16\pi G}{c^4}\left[\pi^{ij} - \frac{1}{2}\gamma^{ij}\pi\right]. \tag{7.8.49}$$

Now, from (7.8.46) we have

$$\dot{\gamma}_{ij} = 2NK_{ij} + D_{(i}N_{j)} \tag{7.8.50}$$

so putting it all together to get the Hamiltonian density, we find

$$\mathfrak{H}_p = \dot{\gamma}_{ij}\pi^{ij} + \pi_N\mu + \pi^i_N\nu_i - \mathfrak{L} = \pi_N\mu + \pi^i_N\nu_i + NH + N_iH^i \tag{7.8.51}$$

and thus the Hamiltonian

$$\mathcal{H}_p[\gamma_{ij}, \pi^{kl}] = \int d^3x \left[\pi_N\mu + \pi^i_N\nu_i + NH(\gamma_{ij}, \pi^{kl}) + N_iH^i(\gamma_{ij}, \pi^{kl})\right], \tag{7.8.52}$$

where we introduced Lagrange multipliers μ and ν_i to enforce the (four) primary constraints $\Phi = \pi_N \approx 0$, $\Phi^i = \pi^i_N \approx 0$, and where

$$\begin{aligned} H &= \left(\frac{c^4\sqrt{\gamma}}{16\pi G}\right)(K_{ij}K^{ij} - K^2 - {}^{(3)}R) = \left(\frac{16\pi G}{c^4}\right)G_{ijlm}\pi^{ij}\pi^{lm} - \left(\frac{c^4}{16\pi G}\right)\sqrt{\gamma}{}^{(3)}R, \\ H^i &= -\frac{c^4\sqrt{\gamma}}{8\pi G}(D^iK - D_jK^{ji}) = -2D_j\pi^{ji}, \end{aligned} \tag{7.8.53}$$

in which

$$G_{ijlm} = \frac{1}{2\sqrt{\gamma}}(\gamma_{il}\gamma_{jm} + \gamma_{im}\gamma_{jl} - \gamma_{ij}\gamma_{lm}) \tag{7.8.54}$$

and from which a total divergence,

$$\left(\frac{c^4}{8\pi G}\right)\partial_i[\sqrt{\gamma}(N^iK - K^{ij}N_j)] = 2\partial_i(N_j\pi^{ij}) \tag{7.8.55}$$

has been dropped. Notice that H is quadratic in the momenta and of the form $g_{ij}(q)p^i p^j + V(q)$, with G_{ijkl} playing the role of a "metric" on the configuration space of components of the induced metric. $\widehat{G}$ is called the **DeWitt metric**.

Given that π_N, π^i_N and π^{ij} are the conjugate momenta, one should define the fundamental Poisson brackets so that only

$$\{N(t,\vec{r}), \pi_N(t',\vec{r}')\}^{t=t'}_{P.B.} = \delta^3(\vec{r}-\vec{r}')$$

$$\{N_i(t,\vec{r}), \pi^j_N(t',\vec{r}')\}^{t=t'}_{P.B.} = \delta^j_i \delta^3(\vec{r}-\vec{r}')$$

$$\{\gamma_{ij}(t,\vec{r}), \pi^{kl}(t',\vec{r}')\}^{t=t'}_{P.B.} = (\delta^k_i\delta^l_j + \delta^l_i\delta^k_j)\delta^3(\vec{r}-\vec{r}') \tag{7.8.56}$$

are non-vanishing. Taking the Poisson brackets of π_N and π^i_N with $\mathcal{H}_p$ then shows that H and H_i are secondary constraints,

$$\dot{\pi}_N(t,\vec{r}) = \{\pi_N(t,\vec{r}), \mathcal{H}_p\}_{P.B.} = H \approx 0$$

$$\dot{\pi}^i_N(t,\vec{r}) = \{\pi^i_N(t,\vec{r}), \mathcal{H}_p\}_{P.B.} = H^i \approx 0 \tag{7.8.57}$$

and, after some tedium, it can be shown that the secondary constraints H and H_i form a closed algebra,

$$\{H(t,\vec{r}), H(t',\vec{r}')\}^{t=t'}_{P.B.} = g^{ij}\left(H_i(t,\vec{r}) + H_i(t,\vec{r}')\right)\partial_j\delta^{(3)}(\vec{r}-\vec{r}')$$

$$\{H(t,\vec{r}), H_i(t',\vec{r}')\}^{t=t'}_{P.B.} = \partial_i\left(H(t,\vec{r})\delta^{(3)}(\vec{r}-\vec{r}')\right)$$

$$\{H_i(t,\vec{r}), H_j(t',\vec{r}')\}^{t=t'}_{P.B.} = H_{(i}\partial_{j)}\delta^{(3)}(\vec{r}-\vec{r}') \tag{7.8.58}$$

There are four primary constraints and four secondary constraints, all of which are first class. Thus, the Lagrange multipliers μ and ν_i cannot be determined and we could simply set them to be zero. Furthermore, the vacuum gravitational field will have just $10-4-4=2$ local degrees of freedom. Just as the contraints of gauge theores generate local gauge transformations, the constraints of general relativity generate general coordinate transformations.

7.9 The Initial Value Problem

Chapter 8

The Weak Field

Finding solutions of the gravitational field equations is a difficult affair in general. Because of the non-linearity of the field equations, it is very useful to first consider approximations in which the gravitational field is weak, by linearizing the equations about a flat (Minkowski) background. Gravity being the weakest of the known forces, these approximate solutions can be of great practical importance in many situations of astrophysical interest such as, for example, in the study of the far field surrounding massive bodies and therefore of phenomena such as the precession of Mercury's perihelion, the bending of light about the sun, etc., the field within tenuous clouds of matter ,useful for studying the effects of Dark Matter, for example, gravitational waves and more. The weak field of slow moving bodies is the subject of the present chapter.

8.1 Linearization

In the absence of a cosmological constant, a weak gravitational field may be represented by a small perturbation of the Minkowski metric, *i.e.,* we take $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu} \ll |\eta_{\mu\nu}|$ (the inverse metric is then $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, where the indices on $h_{\mu\nu}$ ($h^{\mu\nu}$) will be raised/lowered by the Minkowski metric). This allows us to retain only terms that are first order in $h_{\mu\nu}$ in the gravitational equations of motion. We can then think of $h_{\mu\nu}$ as a symmetric second rank tensor field in Minkowski space and Einstein's equations as the equations of a free symmetric tensor field in Minkowski space, in this way enlarging our repertoire of free, Lorentz invariant field theories.

8.1.1 Field Equations

Up to first order, the Christoffel symbols are

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\left(h^{\lambda}{}_{\mu,\nu} + h^{\lambda}{}_{\nu,\mu} - h_{\mu\nu}{}^{,\lambda}\right) \tag{8.1.1}$$

and the Riemann curvature is

$$R^{\alpha}{}_{\mu\beta\nu} = \frac{1}{2}\left(-\partial^{\alpha}\partial_{\beta}h_{\mu\nu} - \partial_{\mu}\partial_{\nu}h^{\alpha}{}_{\beta} + \partial^{\alpha}\partial_{\nu}h_{\mu\beta} + \partial_{\mu}\partial_{\beta}h^{\alpha}{}_{\nu}\right). \tag{8.1.2}$$

Contracting, we find the Ricci curvature,

$$R_{\mu\nu} = \frac{1}{2}\left(\Box h_{\mu\nu} - h_{,\mu\nu} + h^{\alpha}{}_{(\mu,\nu)\alpha}\right) \tag{8.1.3}$$

and also the scalar curvature,

$$R = \Box h + h_{\alpha\beta}{}^{,\alpha\beta}. \tag{8.1.4}$$

Einstein's tensor becomes

$$E_{\mu\nu} = \frac{1}{2}\left(\Box h_{\mu\nu} - h_{,\mu\nu} + h^{\alpha}{}_{(\mu,\nu)\alpha} - \eta_{\mu\nu}\Box h - \eta_{\mu\nu}h_{\alpha\beta}{}^{,\alpha\beta}\right). \tag{8.1.5}$$

If we define $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, then $\overline{h} = -h$ and therefore $h_{\mu\nu} = \overline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overline{h}$, so $E_{\mu\nu}$ can be re-expressed in terms of $\overline{h}_{\mu\nu}$ as

$$E_{\mu\nu} = \frac{1}{2}\left(\Box\overline{h}_{\mu\nu} + \overline{h}^{\alpha}{}_{(\mu,\nu)\alpha} - \eta_{\mu\nu}\overline{h}_{\alpha\beta}{}^{,\alpha\beta}\right). \tag{8.1.6}$$

Now it is easy to see that $R^{\alpha}{}_{\mu\beta\nu}$ is invariant under the transformation

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \xi_{\{\mu,\nu\}}, \tag{8.1.7}$$

where ξ is any arbitrary function. This is the change in the metric induced by a coordinate transformation $x^{\mu} \to x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$ and corresponds to a "gauge freedom" within the linearized theory, under which

$$\overline{h}_{\mu\nu} \to \overline{h}'_{\mu\nu} = \overline{h}_{\mu\nu} + \xi_{\{\mu,\nu\}} - \eta_{\mu\nu}\xi^{\alpha}{}_{,\alpha}. \tag{8.1.8}$$

We may exploit this freedom to pick four functions, ξ^{μ}, so that $\overline{h}^{\mu\nu}{}_{,\nu} = 0$. To see that this is possible, suppose that we know a solution of the equations of motion, $\overline{h}_{\mu\nu}$, but it does *not* obey this condition. We will show that $\overline{h}'_{\mu\nu}$, given by (8.1.8), will satisfy this condition provided we choose ξ^{μ} judiciously. We have

$$\overline{h}'^{\mu\nu}{}_{,\nu} = \overline{h}^{\mu\nu}{}_{,\nu} + \xi^{\{\mu,\nu\}}{}_{,\nu} - \eta^{\mu\nu}\xi^{\alpha}{}_{,\alpha\nu} = \overline{h}^{\mu\nu}{}_{,\nu} - \Box\xi^{\mu} \tag{8.1.9}$$

so if we take ξ^{μ} to be a solution of the equation

$$\Box\xi^{\mu} = \overline{h}^{\mu\nu}{}_{,\nu} \tag{8.1.10}$$

then $\overline{h}'^{\mu\nu}{}_{,\nu} = 0$ and $\overline{h}'_{\mu\nu}$ is an equivalent solution of the equations of motion. This is called the "harmonic gauge" condition. Applying it from the start, we have the linearized Einstein equations in their simplest form,

$$\Box \overline{h}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu}. \tag{8.1.11}$$

This is a wave equation in a flat background, so one can directly write down the solution in terms of the retarded Green's function for the four dimensional Laplacian,

$$\overline{h}_{\mu\nu}(t, \vec{r}) = \frac{4G}{c^4} \int d^3\vec{r}' \; \frac{T_{\mu\nu}(\vec{r}', ct - |\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|} + \overline{h}^{(0)}_{\mu\nu}(t, \vec{r}) \tag{8.1.12}$$

where $\overline{h}^{(0)}(t, \vec{r})$ is an appropriate solution of Laplace's equation and the integration is over the past light cone of the observation point. Notice that $\overline{h}^{(0)}_{\mu\nu}$ must also satisfy the harmonic gauge condition, because the stress tensor is divergence free. Equation (8.1.12) is similar to the equation for electromagnetic waves except that the source is a second rank symmetric tensor, not a vector, and can be used to describe a variety of physical situations, including gravitational waves, in which the gravitational field is weak. The stress tensor may belong to one or more fields, as determined in the previous chapters, or a collection of particles.

8.1.2 Energy and Momentum

For small deviations about flat space, gravity is being described by a symmetric, second rank tensor field propagating on a flat background. This should lead us to ask for the energy and momentum carried by this field, just as we would for any other field Lorentz invariant field theory. There are several proposals out there. For example, Wald[1] suggests that we should consider the vacuum equations to second order. While the linearized vacuum equations require the first order Einstein tensor to vanish, this tensor will not generally vanish to second order in the metric perturbations. To continue to have a vanishing Einstein tensor at second order, imagine adding to $h_{\mu\nu}$ a higher order perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + h^{(2)}_{\mu\nu}. \tag{8.1.13}$$

To second order, the Einstein tensor will be quadratic in $h_{\mu\nu}$ and linear in $h^{(2)}_{\mu\nu}$. Any cross terms would be third order in the perturbations, so to second order,

$$G_{\mu\nu} \approx G^{(1)}_{\mu\nu}(h^{(2)}) + G^{(2)}_{\mu\nu}(h), \tag{8.1.14}$$

[1] Wald, R. M., *General Relativity*, U. Chicago Press (1984).

where the superscripts on G indicate the order of the expansion and we have used the fact that $G^{(1)}_{\mu\nu}(h) = 0$. We may then write the last equation as

$$G^{(1)}_{\mu\nu}(h^{(2)}) = -G^{(2)}_{\mu\nu}(h), \tag{8.1.15}$$

interpreting the right hand side as proportional to the stress energy tensor of the gravitational, according to

$$t_{\mu\nu} = -\frac{c^4}{8\pi G} G^{(2)}_{\mu\nu}(h). \tag{8.1.16}$$

It is certainly conserved, by the Bianchi identity, and symmetric by the properties of the Einstein tensor. It is not gauge invariant, as is to be expected (it is impossible to define a diffeomorphism invariant local energy and momentum density), and involves second derivatives of the linearized field, $h_{\mu\nu}$, a feature that is absent in other field theories.

It is possible, of course, to approach the question of the energy and momentum of the gravitational field in a more canonical way, following the techniques laid out in Chapter 2. Note that another approach to (8.1.5) is via the bulk action of (7.4.6). The bulk action is Lorentz invariant in the weak field limit *and* first order in field derivatives, which makes it straightforward to apply the machinery of Chapter 2. In the linear approximation the action[2]

$$S_{\text{bulk}} = \frac{c^4}{16\pi G} \int d^4x \left[h^{\alpha\beta}{}_{,\eta} h^{\eta}{}_{\alpha,\beta} - h_{,\beta} h^{\beta\eta}{}_{,\eta} - \frac{1}{2} h_{\alpha\beta,\eta} h^{\alpha\beta,\eta} + \frac{1}{2} h_{,\eta} h^{,\eta} \right], \tag{8.1.17}$$

is both Lorentz invariant and gauge invariant provided that the fields vanish rapidly enough at the boundary. The *Lagrangian*, however, is not gauge invariant, owing to the surface terms that have been dropped. [3],[4]

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[2]Problem: Derive (8.1.5) from (8.1.17).

[3]Problem: Show that S_{bulk} in (8.1.17) can be written as:

$$S_{\text{bulk}} = \frac{c^4}{16\pi G} \int d^4x \left[\overline{h}^{\alpha\beta}{}_{,\eta} \overline{h}^{\eta}{}_{\alpha,\beta} - \frac{1}{2} h_{\alpha\beta,\eta} \overline{h}^{\alpha\beta,\eta} \right].$$

where $\overline{h}_{\mu\nu}$ has been defined earlier.

[4]Problem: Show that so long as $h_{\mu\nu}$ vanishs rapidly enough at the boundary, the action is invariant under a *restricted* gauge transformation

$$\delta_g h_{\mu\nu} = -\xi_{\{\mu,\nu\}},$$

where ξ^μ preserves its asymptotic behavior. *Hint*: You will need to integrate by parts several times, eg., write the first term as

$$\begin{aligned} \delta_g S_{\text{bulk},1} &\sim \int d^4x \left[\delta_g h^{\alpha\beta}{}_{,\eta} h^{\eta}{}_{\alpha,\beta} + h^{\alpha\beta,\eta} \delta_g h^{\eta}{}_{\alpha,\beta} \right] \\ &\sim \int d^4x \left[\xi^{\{\alpha,\beta\}\eta} h_{\eta\alpha,\beta} + h^{\alpha\beta,\eta} \xi_{\{\eta,\alpha\}\beta} \right] \\ &\sim \int d^4x \left[2\xi^\alpha \Box h_{\eta\alpha}{}^{,\eta} + 2\xi^\beta h^{\eta\alpha}{}_{,\alpha\beta\eta} \right]. \end{aligned}$$

One finds

$$\frac{\partial \mathfrak{L}}{\partial h_{\mu\nu,\lambda}} = \frac{c^4}{16\pi G}\left[h^{\lambda\mu,\nu} + h^{\lambda\nu,\mu} - \eta^{\mu\nu}h^{\lambda}{}_{\sigma}{}^{,\sigma} - \frac{1}{2}\eta^{\lambda\mu}h^{,\nu} - \frac{1}{2}\eta^{\lambda\nu}h^{,\mu} - h^{\mu\nu,\lambda} + \eta^{\mu\nu}h^{,\lambda}\right]. \tag{8.1.18}$$

If we call this quantity $F^{\lambda\mu\nu}$ then it is clear that $\partial_\lambda F^{\lambda\mu\nu} = c^4 E^{\mu\nu}/(8\pi G)$ so Euler's equations are precisely (8.1.11). Therefore, while $F^{\lambda\mu\nu}$ itself is not gauge invariant, its derivative, $\partial_\lambda F^{\lambda\mu\mu}$ is gauge invariant and vanishes in the vacuum. It can be written as

$$F^{\lambda\mu\nu} = \frac{c^4}{16\pi G}\left[\overline{h}^{\lambda\mu,\nu} + \overline{h}^{\lambda\nu,\mu} - \overline{h}^{\mu\nu,\lambda} - \eta^{\mu\nu}\overline{h}^{\lambda\sigma}{}_{,\sigma}\right] \tag{8.1.19}$$

and is symmetric in (μ,ν), *i.e.*, $F^{\lambda[\mu\nu]} = 0$. The canonical stress energy tensor is obtained by applying (2.3.3),

$$\begin{aligned}\Theta^{\alpha\beta} &= \eta^{\alpha\beta}\mathfrak{L} - \frac{\partial \mathfrak{L}}{\partial h_{\mu\nu,\alpha}}h_{\mu\nu}{}^{,\beta} = \eta^{\alpha\beta}\mathfrak{L} - F^{\alpha\mu\nu}h_{\mu\nu}{}^{,\beta} \\ &= \eta^{\alpha\beta}\mathfrak{L} - \widetilde{F}^{\alpha\mu\nu}\overline{h}_{\mu\nu}{}^{,\beta} - \frac{c^4}{32\pi G}\overline{h}^{,\alpha}\overline{h}^{,\beta},\end{aligned} \tag{8.1.20}$$

where[5]

$$\widetilde{F}^{\alpha\mu\nu} = \frac{c^4}{16\pi G}\left[\overline{h}^{\alpha\mu,\nu} + \overline{h}^{\alpha\nu,\mu} - \overline{h}^{\mu\nu,\alpha}\right]. \tag{8.1.21}$$

It is not symmetric, because of the middle term, but we have already seen that the remedy for this was given by Belinfante and Rosenfeld. We must first construct the intrinsic angular momentum tensor density,

$$S^{\mu}{}_{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial h_{\lambda\rho,\mu}}G_{\lambda\rho\alpha\beta} = F^{\mu\lambda\rho}G_{\lambda\rho\alpha\beta} \tag{8.1.22}$$

where

$$\delta h_{\lambda\rho} = G_{\lambda\rho\alpha\beta}\delta\omega^{\alpha\beta}. \tag{8.1.23}$$

is the change in $h_{\lambda\rho}$ under a Lorentz transformation. As a symmetric, second rank tensor field, $h_{\lambda\rho}$ suffers the change

$$h^{\lambda\rho} \to h'^{\lambda\rho} = \frac{\partial x'^{\lambda}}{\partial x^{\sigma}}\frac{\partial x'^{\rho}}{\partial x^{\kappa}}h_{\sigma\kappa} = (\delta^{\lambda}_{\sigma} + \delta\omega^{\lambda}{}_{\sigma})(\delta^{\rho}_{\kappa} + \delta\omega^{\rho}{}_{\kappa})h^{\sigma\kappa}$$

Doing this for all the terms, show that they cancel. Notice that, while the action is invariant under the gauge transformation, the Lagrangian is not. This is because all the surface terms have been dropped in using (7.4.6) to write (8.1.17).

[5] Problem: Show that

$$\widetilde{F}^{\lambda\alpha\beta} \to \widetilde{F}'^{\lambda\alpha\beta} = \widetilde{F}^{\lambda\alpha\beta} - 2\xi^{\lambda,\alpha\beta} + \left(\eta^{\lambda\alpha}\delta^{\beta}_{\sigma} + \eta^{\lambda\beta}\delta^{\alpha}_{\sigma} - \eta^{\alpha\beta}\delta^{\lambda}_{\sigma}\right)\xi_{\kappa}{}^{,\kappa\sigma}.$$

under the transformation $\overline{h}_{\mu\nu} \to \overline{h}'_{\mu\nu} = \overline{h}_{\mu\nu} - \xi_{(\mu,\nu)} + \eta_{\mu\nu}\xi_{\alpha}{}^{,\alpha}$.

$$\delta h^{\lambda\rho} = \frac{1}{2}\left[\delta^{\lambda}_{\sigma}\delta^{\rho}_{[\alpha}\eta_{\beta]\kappa} + \delta^{\rho}_{\kappa}\delta^{\lambda}_{[\alpha}\eta_{\beta]\sigma}\right] h^{\sigma\kappa}\delta\omega^{\alpha\beta}, \tag{8.1.24}$$

where we have taken care to antisymmetrize appropriately. Thus we could take

$$G_{\lambda\rho\alpha\beta} = \frac{1}{2}\left[\eta_{\lambda[\alpha}\delta^{\sigma}_{\beta]}\delta^{\kappa}_{\rho} + \eta_{\rho[\alpha}\delta^{\sigma}_{\beta]}\delta^{\kappa}_{\lambda}\right] h_{\sigma\kappa} \tag{8.1.25}$$

and therefore

$$S^{\mu\alpha\beta} = F^{\mu\eta[\alpha}h^{\beta]}{}_{\eta} = \widetilde{F}^{\mu\eta[\alpha}\overline{h}^{\beta]}{}_{\eta}. \tag{8.1.26}$$

gives the intrinsic angular momentum tensor. The last equality above follows from the symmetries of F and $\overline{h}$. To verify that this expression makes sense, we must check that (2.3.8) holds. Now, because of the identity

$$\widetilde{F}^{\mu\eta\alpha} = -\widetilde{F}^{\alpha\eta\mu} + \frac{c^4}{8\pi G}\overline{h}^{\mu\alpha,\eta}, \tag{8.1.27}$$

it is easy to see that in the vacuum and on shell the left hand side is

$$\begin{aligned}
\partial_\mu S^{\mu\alpha\beta} &= \widetilde{F}^{\mu\eta[\alpha}\overline{h}^{\beta]}{}_{\eta,\mu} = \left[-\widetilde{F}^{\alpha\eta\mu} + \frac{c^4}{8\pi G}\overline{h}^{\mu\alpha,\eta}\right]\overline{h}^{\beta}{}_{\eta,\mu} - \alpha \leftrightarrow \beta \\
&= -\widetilde{F}^{\alpha\eta\mu}\overline{h}^{\beta}{}_{\eta,\mu} - \alpha \leftrightarrow \beta \\
&= -\frac{1}{2}\widetilde{F}^{\alpha\eta\mu}\left(\overline{h}^{\beta}{}_{\eta,\mu} + \overline{h}^{\beta}{}_{\mu,\eta} - \overline{h}_{\mu\eta}{}^{,\beta}\right) - \frac{1}{2}\widetilde{F}^{\alpha\eta\mu}\overline{h}_{\mu\eta}{}^{,\beta} - \alpha \leftrightarrow \beta \\
&= -\frac{1}{2}\widetilde{F}^{\alpha\eta\mu}\widetilde{F}^{\beta}{}_{\eta\mu} - \frac{1}{2}\widetilde{F}^{\alpha\eta\mu}\overline{h}_{\mu\eta}{}^{,\beta} - \alpha \leftrightarrow \beta \\
&\equiv -\frac{1}{2}\left(\widetilde{F}^{\alpha\eta\mu}\overline{h}_{\mu\eta}{}^{,\beta} - \widetilde{F}^{\beta\eta\mu}\overline{h}_{\mu\eta}{}^{,\alpha}\right)
\end{aligned}$$

where we have discarded terms that are symmetric in (α, β) in going from the first to the second line and again in going from the fourth to the fifth line. The last expression is precisely $(\Theta^{\alpha\beta} - \Theta^{\beta\alpha})/2$.

The Belinfante-Rosenfeld construction, (2.3.16), requires us to assemble the tensor

$$k^{\mu\alpha\beta} = -\left(S^{\mu\alpha\beta} + S^{\alpha\beta\mu} + S^{\beta\alpha\mu}\right) = -\left(\widetilde{F}^{\mu\eta[\alpha}\overline{h}^{\beta]}{}_{\eta} + \widetilde{F}^{\alpha\eta[\beta}\overline{h}^{\mu]}{}_{\eta} + F^{\beta\eta[\alpha}\overline{h}^{\mu]}{}_{\eta}\right) \tag{8.1.28}$$

and (2.3.14) gives the symmetric energy momentum tensor as follows: we have already seen that

$$\Theta^{\alpha\beta} - \partial_\mu S^{\mu\alpha\beta} = \eta^{\alpha\beta}\mathfrak{L} - \frac{c^4}{32\pi G}\overline{h}^{,\alpha}\overline{h}^{,\beta} - \frac{1}{2}\left(\widetilde{F}^{\alpha\eta\mu}\overline{h}_{\mu\eta}{}^{,\beta} + \widetilde{F}^{\beta\eta\mu}\overline{h}_{\mu\eta}{}^{,\alpha}\right),$$

Expanding the remaining terms,

$$
\begin{aligned}
-\partial_\mu\left(S^{\alpha\beta\mu}+S^{\beta\alpha\mu}\right) &= -\partial_\mu\left(\widetilde{F}^{\alpha\eta[\beta}\overline{h}^{\mu]}{}_\eta+\widetilde{F}^{\beta\eta[\alpha}\overline{h}^{\mu]}{}_\eta\right)\\
&= -\partial_\mu\left[\left(\widetilde{F}^{\alpha\eta\beta}+\widetilde{F}^{\beta\eta\alpha}\right)\overline{h}^{\mu}{}_\eta-\widetilde{F}^{\alpha\eta\mu}\overline{h}^{\beta}{}_\eta-\widetilde{F}^{\beta\eta\mu}\overline{h}^{\alpha}{}_\eta\right]\\
&= -\frac{c^4}{8\pi G}\left(\overline{h}^{\alpha\beta}{}_{,\eta\mu}\overline{h}^{\mu\eta}+\overline{h}^{\alpha\beta}{}_{,\eta}\overline{h}^{\mu\eta}{}_{,\mu}\right)+\\
&\quad+\left(\widetilde{F}^{\alpha\eta\mu}{}_{,\mu}\overline{h}^{\beta}{}_\eta+\widetilde{F}^{\alpha\eta\mu}\overline{h}^{\beta}{}_{\eta,\mu}+\widetilde{F}^{\beta\eta\mu}{}_{,\mu}\overline{h}^{\alpha}{}_\eta+\widetilde{F}^{\beta\eta\mu}\overline{h}^{\alpha}{}_{\eta,\mu}\right),
\end{aligned}
\tag{8.1.29}
$$

and applying (8.1.27) together with the vacuum equations, $\partial_\mu F^{\mu\eta\alpha}=0$, it follows that

$$
\widetilde{F}^{\alpha\eta\mu}{}_{,\mu}=\frac{c^4}{8\pi G}\overline{h}^{\mu\alpha,\eta}{}_{,\mu}
\tag{8.1.30}
$$

and the right hand side of (8.1.29) simplifies to

$$
\begin{aligned}
= \; & -\frac{c^4}{8\pi G}\left(\overline{h}^{\alpha\beta}{}_{,\eta\mu}\overline{h}^{\mu\eta}-\overline{h}^{\alpha\mu}{}_{,\eta\mu}\overline{h}^{\beta\eta}-\overline{h}^{\beta\mu}{}_{,\eta\mu}\overline{h}^{\alpha\eta}+\overline{h}^{\alpha\beta}{}_{,\eta}\overline{h}^{\mu\eta}{}_{,\mu}\right)\\
&+\widetilde{F}^{\alpha\eta\mu}\overline{h}^{\beta}{}_{\eta,\mu}+\widetilde{F}^{\beta\eta\mu}\overline{h}^{\alpha}{}_{\eta,\mu}
\end{aligned}
\tag{8.1.31}
$$

Putting all the terms together again,

$$
\begin{aligned}
t^{\alpha\beta} &= \mathfrak{L}\eta^{\alpha\beta}-\frac{1}{2}\widetilde{F}^{\alpha\eta\mu}\left(\overline{h}_{\mu\eta}{}^{,\beta}-2\overline{h}^{\beta}{}_{\eta,\mu}\right)-\frac{1}{2}\widetilde{F}^{\beta\eta\mu}\left(\overline{h}_{\mu\eta}{}^{,\alpha}-2\overline{h}^{\alpha}{}_{\eta,\mu}\right)\\
&\quad-\frac{c^4}{8\pi G}\left[\overline{h}^{\alpha\beta}{}_{,\eta\mu}\overline{h}^{\mu\eta}-\overline{h}^{\alpha\mu}{}_{,\eta\mu}\overline{h}^{\beta\eta}-\overline{h}^{\beta\mu}{}_{,\eta\mu}\overline{h}^{\alpha\eta}+\overline{h}^{\alpha\beta}{}_{,\eta}\overline{h}^{\mu\eta}{}_{,\mu}+\frac{1}{4}\overline{h}^{,\alpha}\overline{h}^{,\beta}\right]\\
&= \mathfrak{L}\eta^{\alpha\beta}+\frac{16\pi G}{c^4}\widetilde{F}^{\alpha\eta\mu}\widetilde{F}^{\beta}{}_{\eta\mu}\\
&\quad-\frac{c^4}{8\pi G}\left[\overline{h}^{\alpha\beta}{}_{,\eta\mu}\overline{h}^{\mu\eta}-\overline{h}^{\alpha\mu}{}_{,\eta\mu}\overline{h}^{\beta\eta}-\overline{h}^{\beta\mu}{}_{,\eta\mu}\overline{h}^{\alpha\eta}+\overline{h}^{\alpha\beta}{}_{,\eta}\overline{h}^{\mu\eta}{}_{,\mu}+\frac{1}{4}\overline{h}^{,\alpha}\overline{h}^{,\beta}\right].
\end{aligned}
\tag{8.1.32}
$$

The expression above involves second derivatives of field, in contrast with the energy momentum tensors of the other fields we have examined so far. It is Lorentz covariant but not gauge invariant. This is quite as it should be, considering that gauge transformations are diffeomorphisms and a diffeomorphism invariant definition of energy and momentum cannot exist.

If we employ vacuum solutions satisfying $\overline{h}^{\alpha\beta}{}_{,\beta} = 0$ the expression for the stress energy tensor is simplified,

$$t^{\alpha\beta} = \mathfrak{L}\eta^{\alpha\beta} + \frac{16\pi G}{c^4}\widetilde{F}^{\alpha\eta\mu}\widetilde{F}^{\beta}{}_{\eta\mu} - \frac{c^4}{8\pi G}\left[\overline{h}^{\alpha\beta}{}_{,\eta\mu}\overline{h}^{\mu\eta} + \frac{1}{4}\overline{h}^{,\alpha}\overline{h}^{,\beta}\right] \tag{8.1.33}$$

but a term involving second derivatives of the gravitational field persists. This term can, however, be expressed as a total derivative,

$$\overline{h}^{\alpha\beta}{}_{,\eta\mu}\overline{h}^{\mu\eta} = \left(\overline{h}^{\alpha\beta}{}_{,\eta}\overline{h}^{\mu\eta}\right)_{,\mu}$$

and therefore does not affect the *total* momentum, P^α, of (2.3.9), provided that the fields vanish at the boundary. Indeed, under these conditions, the total stress energy of the gravitational field on shell is also greatly simplified. Integrating by parts and dropping surface terms, we find

$$\langle t^{\alpha\beta}\rangle = \int d^3\vec{r}\; t^{\alpha\beta} = \frac{c^4}{16\pi G}\int d^3\vec{r}\left[\overline{h}^{\eta\mu,\alpha}\overline{h}_{\eta\mu}{}^{,\beta} - \frac{1}{2}\overline{h}^{,\alpha}\overline{h}^{,\beta}\right], \tag{8.1.34}$$

which has the vitrue of also being gauge invariant under restricted gauge transformations.[6]

8.2 Gravitoelectromagnetism

Of some astrophysical interest is the stress-energy tensor of ideal fluids,

$$T^{\mu\nu} = p\eta^{\mu\nu} + \left(\rho + \frac{p}{c^2}\right)U^\mu U^\nu, \tag{8.2.1}$$

where p is the pressure and ρ is the mass density. The non-relativistic limit of this tensor brings along some useful simplifications with interesting results. In this case the fluid flows slowly, $U^0 = 1$ and $U^i = v^i$, so the components of the ideal fluid stress tensor approach

$$T^{00} = \rho, \quad T^{0i} = \left(\rho + \frac{p}{c^2}\right)v^i, \quad T^{ij} = p\delta^{ij} + \left(\rho + \frac{p}{c^2}\right)v^i v^j. \tag{8.2.2}$$

Notice that $|T^{0i}/T^{00}| \sim \mathcal{O}(v)$ and $|T^{ij}/T^{00}| \sim \mathcal{O}(v^2)$, so to linear order in v^i we simply take

$$T^{00} = \rho, \quad T^{0i} \approx \rho v^i, \quad T^{ij} \approx 0. \tag{8.2.3}$$

With this stress energy, the equation of motion (8.1.12) lets us take $\overline{h}_{ij} = 0$, but note that this is a choice in boundary condition, because we have set $\overline{h}^{(0)}_{\mu\nu} = 0$, and will limit

[6]Problem: Derive (8.1.34) and show that it is gauge invariant under restricted gauge transformations.

its application to localized, stationary sources. To continue, let $\overline{h} = \overline{h}^0{}_0 = -h \stackrel{\text{def}}{=} 4\Phi/c^2$ and $\overline{h}_{0i} = h_{0i} = A_i/c$. These definitions mean that $h_{00} = \overline{h}_{00} - \frac{1}{2}\eta_{00}\overline{h} = -2\Phi$ and $h_{ij} = -(2\Phi/c^2)\delta_{ij}$. The metric becomes[7]

$$ds^2 = c^2\left(1+\frac{2\Phi}{c^2}\right)dt^2 - \frac{2A_i}{c}dx^i dt - \left(1-\frac{2\Phi}{c^2}\right)\delta_{ij}dx^i dx^j. \tag{8.2.4}$$

Furthermore,

$$\begin{aligned}\Phi(t,\vec{r}) &= -G\int d^3\vec{r}'\ \frac{\rho(\vec{r}', ct-|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|}\\ A_i(t,\vec{r}) &= -\frac{4G}{c}\int d^3\vec{r}'\ \frac{\rho(\vec{r}', ct-|\vec{r}-vr'|)\ v_i(\vec{r}', ct-|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|}\end{aligned} \tag{8.2.5}$$

and the gauge condition requires that $\dot{A}_i = 0$ and $\nabla\cdot\vec{A} = -4\dot{\Phi}/c$. The off-diagonal terms in the metric in (8.2.4) represent a genuine correction to Newton's law of gravitation but its effect is small for slowly moving objects, being suppressed by v/c. Conservation of the energy momentum tensor, $T^{\mu\nu}{}_{,\nu} = 0$, implies that

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) &= 0\\ \frac{\partial}{\partial t}(\rho\vec{v}) &= 0.\end{aligned} \tag{8.2.6}$$

The first is the continuity equation and guarantees the conservation of mass, the second says that momentum is conserved.

In this linear approximation, the Christoffel symbols are

$$\begin{aligned}&\Gamma^0_{00} = \frac{\dot{\Phi}}{c^2},\quad \Gamma^0_{0i} = \frac{\Phi_{,i}}{c^2},\quad \Gamma^0_{ij} = -\frac{1}{2c^3}\left(A_{i,j} + A_{j,i} + \frac{2\dot{\Phi}}{c}\delta_{ij}\right)\\ &\Gamma^i_{00} = \frac{\dot{A}^i}{c} + \Phi^{,i},\quad \Gamma^i_{0j} = \frac{1}{2c}\left(A^i{}_{,j} - A_j{}^{,i} - \frac{2\dot{\Phi}}{c}\delta^i_j\right)\\ &\Gamma^i_{jk} = -\frac{1}{c^2}\left(\Phi_{,k}\delta^i_j + \Phi_{,j}\delta^i_k - \Phi^{,i}\delta_{jk}\right).\end{aligned} \tag{8.2.7}$$

We will consider only the stationary case, taking $\vec{\nabla}\cdot\vec{A} = \dot{\Phi} = 0$. The geodesic equations for a test particle moving slowly relative to the source, $d\tau \approx dt$, $U^0 \approx 1$ and $U^i \approx u^i$ (we use u^i for the test particle velocity to distinguish it from the velocity distribution, $v^i(\vec{r}')$, of the source in (8.2.5)) will be

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{00} - 2\Gamma^i_{0j}u^j = -\Phi^{,i} - \frac{1}{c}\left[A^i{}_{,j} - A_j{}^{,i}\right]u^j, \tag{8.2.8}$$

[7] With our definitions, both gravitational potentials, Φ and A_i, have the mechanical dimension l^2/t^2.

or

$$\frac{d^2\vec{r}}{dt^2} = -\vec{\nabla}\Phi + \frac{\vec{u}}{c} \times (\vec{\nabla} \times \vec{A}). \tag{8.2.9}$$

It has the appearance of the Lorentz force of electrodynamics, with $\vec{A}$ playing the role of the vector potential. The analogy is completed by defining

$$\begin{aligned} \vec{E}_g &= -\vec{\nabla}\Phi \\ \vec{B}_g &= \vec{\nabla} \times \vec{A}, \end{aligned} \tag{8.2.10}$$

whence

$$\vec{a} = \vec{E}_g + \frac{\vec{u}}{c} \times \vec{B}_g. \tag{8.2.11}$$

In the same approximation, using (8.2.5), we also find that the field equations mirror Maxwell's equations,

$$\begin{aligned} &\vec{\nabla} \cdot \vec{E}_g = -4\pi G\rho, \quad \vec{\nabla} \times \vec{E}_g = 0 \\ &\vec{\nabla} \cdot \vec{B}_g = 0, \quad \vec{\nabla} \times \vec{B}_g = -\frac{16\pi G}{c}\vec{j}, \end{aligned} \tag{8.2.12}$$

thus completing the electromagnetic analogy, but *only* in the non-relativistic limit. The sign of the source terms is opposite to its electromagnetic counterpart; this because "like charges" attract in gravity. The extra factor of four in the magnetic source term reflects the fact that the gravitational field is spin-2 whereas the electromagnetic field is spin-1. Equations (8.2.10), (8.2.11), (8.2.12) and the gauge condition define "**Gravitoelectromagnetism**" or **GEM** for short.

8.2.1 Static and Stationary Sources

The linear approximation can be used to determine the gravitational field far from massive bodies, so long as they are not moving too fast. For an example, the gravitational potentials Φ and A_i, far from a spherical, static mass M, of radius R, located at the origin are found to be

$$\Phi(\vec{r}) = -\frac{GM}{r}, \quad A_i = 0, \tag{8.2.13}$$

to leading order, where $r = \sqrt{\sum_i {x^i}^2}$ and

$$M = \int d^3\vec{r}'\rho(\vec{r}') = 4\pi \int_0^R dr\rho(r). \tag{8.2.14}$$

It gives the linearized Schwarzschild line element,

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 - \left(1 + \frac{2GM}{c^2 r}\right)\delta_{ij}dx^i dx^j, \tag{8.2.15}$$

in "isotropic" coordinates.

For another example, consider the gravitational field far from a massive, rotating body. Applying (8.2.5) and the multipole expansion of the Green's function, because we're interested only in the far region, gives

$$\Phi(\vec{r}) = -\frac{G}{r}\int d^3\vec{r}'\rho(\vec{r}')\left[1 + \frac{\vec{r}\cdot\vec{r}'}{r^2} + \mathcal{O}(r^{-3}) + \ldots\right] \approx -\frac{GM}{r} \tag{8.2.16}$$

and

$$\begin{aligned} A_i(\vec{r}) &= -\frac{4G}{cr}\int d^3\vec{r}'\; T^0{}_i(\vec{r}')\left[1 + \frac{\vec{r}\cdot\vec{r}'}{r^2} + \mathcal{O}(r^{-3}) + \ldots\right] \\ &\approx -\frac{4G}{cr^3}\, x^l \int d^3\vec{r}'\; T^0{}_i(\vec{r}')\; x'_l \end{aligned} \tag{8.2.17}$$

to leading order. To evaluate the final integral, consider the integral

$$\int d^3\vec{r}'\partial_k(x'^i x'^l T^{0k}) = \int d^3\vec{r}'\left(x'^l T^{0i} + x'^i T^{0l} + x'^i x'^l T^{0k}{}_{,k}\right). \tag{8.2.18}$$

By the continuity equation, the last term on the right hand side above vanishes for a stationary source. The left hand side above also vanishes if the integral is taken over the entire source. Therefore the symmetric part of the integrand for $A_i(\vec{r})$ vanishes,

$$\int d^3\vec{r}' T^0{}_{(i}(\vec{r}')x'_{l)} = 0, \tag{8.2.19}$$

and $A_i(\vec{r})$ depends only on the antisymmetric part,

$$A_i(\vec{r}) = -\frac{2G}{cr^3}\, x^l \int d^3\vec{r}' T^0{}_{[i}(\vec{r}')\; x'_{l]} = -\frac{4G}{cr^3}\, x^l \int d^3\vec{r}' L^0{}_{il}(\vec{r}') \tag{8.2.20}$$

where $L^0{}_{il}(\vec{r}')$ are the spatial components of the orbital angular momentum tensor density as defined in (2.3.7). This term can be written in terms of the total angular momentum $\mathrm{L}^i = -\epsilon^{ijk}\mathrm{L}^0{}_{jk}$ (equivalently, $2\mathrm{L}^0{}_{il} = -\epsilon_{ilk}\mathrm{L}^k$),

$$A_i(\vec{r}) = \frac{2G}{cr^3}\;\epsilon_{ilk}x^l\mathrm{L}^k = \frac{2G}{cr^3}\;(\vec{r}\times\vec{\mathrm{L}})_i \tag{8.2.21}$$

The GEM metric is,

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 - \frac{4G}{c^2 r^3}\;(\vec{r}\times\vec{\mathrm{L}})_i dx^i dt - \left(1 + \frac{2GM}{c^2 r}\right)\delta_{ij}dx^i dx^j, \tag{8.2.22}$$

the gravitoelectric and gravitomagnetic fields are

$$\vec{E}_g = -\frac{GM}{r^2}\hat{r}$$
$$\vec{B}_g = \frac{2G}{cr^3}\left[\vec{\mathrm{L}} - 3(\hat{r}\cdot\vec{\mathrm{L}})\,\hat{r}\right] \tag{8.2.23}$$

and the gravitomagnetic acceleration of a test particle moving with velocity $\vec{u}$ will be

$$\vec{a} = \frac{\vec{u}}{c}\times(\vec{\nabla}\times\vec{A}) = \frac{2G}{c^2r^3}\left[\vec{u}\times\vec{\mathrm{L}} + 3(\hat{r}\times\vec{u})(\hat{r}\cdot\vec{\mathrm{L}})\right]. \tag{8.2.24}$$

A test particle moving in the equatorial plane, with an inward radial velocity, will experience a tangential acceleration in the direction of the rotation and, vice-versa, opposite the direction of the rotation if it is moving radially outward. This effect is known as rotational "**frame dragging**" (or the "Lense-Thirring" effect) and occurs whenever the off-diagonal components of the metric are non-vanishing.[8] The effect is stronger for closer objects, so imagine an extended test object falling into a massive, stationary body: the inner parts of the test object will be dragged more than the outer parts, resulting in a net torque that causes the the test body to rotate about itself, creating a locally rotating frame even as it is dragged as a whole around the rotating body.[9]

8.2.2 Hydrodynamics in GEM

An ideal fluid cannot avoid gravitational collapse without pressure. To find the non-relativistic Euler equations, we must consider the conservation equations on the space-time given by (8.2.22), taking $T^{\mu\nu}$ to order v^2

$$T^{00} \approx \rho\left(1+\frac{v^2}{c^2}\right),\quad T^{0i}\approx \rho v^i,\quad T^{ij}\approx p\delta^{ij}+\rho v^i v^j. \tag{8.2.25}$$

Using the connections in (8.2.7), we find, to lowest order,

$$\nabla_\mu T^{\mu 0} = 0 \;\Rightarrow\; \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0,$$

[8] You are already familiar with this effect as the "Coriolis Effect" in Newtonian mechanics.

[9] Problem: Suppose that the angular momentum vector points in the z direction with magnitude L. (i) Show that the metric in (8.2.22) can be given in isotropic coordinates as

$$ds^2 = c^2\left(1-\frac{2GM}{c^2r}\right)dt^2 + \frac{4GL}{c^2r}\sin^2\theta\; d\phi dt - \left(1+\frac{2GM}{c^2r}\right)\left(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right).$$

(ii) Now consider the conserved charges associated with diffeomorphism invariance of the metric in (7.5.19). Taking $\epsilon^\mu = (1,0,0,0)$ and S to be the two sphere of radius r (thus $n_\mu = (1,0,0,0)$ and $r_\mu = (0,1,0,0)$ and $\epsilon\epsilon' = -1$) in (6.7.67), show that, in the linear approximation, $Q_t = Mc^2$, *i.e.,* the mass of the body is conserved. (iii) Repeat the calculation, taking $\epsilon^\mu = (0,0,0,1)$ and show that $Q_\phi = 3L$, implying that the angular momentum is conserved.

$$\nabla_\mu T^{\mu i} = 0 \;\Rightarrow\; \rho\frac{dv_i}{dt} = -\partial_i p - \rho\partial_i\Phi - \rho\frac{v^j}{c}(A_{i,j} - A_{j,i}), \tag{8.2.26}$$

where we have included terms of linear order in v/c. These are the continuity and Euler equations respectively. In the static case, we recover the equation of hydrostatic equilibrium, $\vec{\nabla}p = -\rho\vec{\nabla}\Phi$. When rotation is present it gets modified by the gravitomagnetic potential,

$$\partial_i p = -\rho\partial_i\Phi - \rho\frac{v^j}{c}(A_{i,j} - A_{j,i}). \tag{8.2.27}$$

or

$$\vec{\nabla}p = \rho\vec{E}_g + \rho\frac{\vec{v}}{c}\times\vec{B}_g, \tag{8.2.28}$$

a result we should have expected from (8.2.11). These equations must be supplemented by an equation of state of the form $p = p(\rho)$.

What we have described so far are only the first steps in a systematic perturbative approach, called the Post Newtonian Expansion, in which both the field potentials ($h_{\mu\nu}$) and the speed of the matter (v^i and u^i) are assumed to be small (compared respectively to unity and the speed of light). This approach was used extensively in the early days of General Relativity to determine some of the observational consequences of Einstein's theory such as the precession of planetary orbits and the deflection of light in a gravitational field, among others.

8.2.3 Scalar Fields and GEM

Let us begin with the action for a scalar field in a gravitational field,

$$S = -\int d^4x\sqrt{-g}\left[g^{\alpha\beta}\partial_\alpha\phi^*\partial_\beta\phi + V(|\phi|)\right], \tag{8.2.29}$$

where $g_{\alpha\beta}$ is the space-time metric, $V(|\phi|)$ is the scalar potential, which we take to have the form

$$V(|\phi|) = \frac{m^2c^2}{\hbar^2}|\phi|^2 + \frac{2m}{\hbar^2}V_1(|\phi|). \tag{8.2.30}$$

In the weak field approximation, $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, the determinant of the metric is given to linear order in h by $\sqrt{-g} = 1 + \frac{1}{2}h$ and to the same order in h

$$\begin{aligned} S &= -\int d^4x\left(1 + \frac{1}{2}h\right)\left[\eta^{\alpha\beta}\partial_\alpha\phi^*\partial_\beta\phi - h^{\alpha\beta}\partial_\alpha\phi^*\partial_\beta\phi + V(|\phi|)\right] \\ &= S_0 - \frac{1}{2}\int d^4x h\left[\eta^{\alpha\beta}\partial_\alpha\phi^*\partial_\beta\phi + V(|\phi|)\right] + \int d^4x h^{\alpha\beta}\partial_\alpha\phi^*\partial_\beta\phi, \end{aligned} \tag{8.2.31}$$

where S_0 is the ordinary scalar field action in flat space. The remaining terms represent interactions between the scalar field and the gravitational field. Let us now go to the non-relativistic limit as we did in (2.5.12). Then the first term has the familiar form

$$S_0 = \int dt \int d^3\vec{r} \left[\frac{i\hbar}{2} \psi^* \overleftrightarrow{\partial_t} \psi - \frac{\hbar^2}{2m} |\vec{\nabla}\psi|^2 - V_1(|\psi|) \right]. \tag{8.2.32}$$

In the same approximation, the second term

$$S_1 = -\frac{1}{2} \int dt \int d^3\vec{r}\; h \left[\frac{i\hbar}{2} \psi^* \overleftrightarrow{\partial_t} \psi - \frac{\hbar^2}{2m} |\vec{\nabla}\psi|^2 - V_1(|\psi|) \right] \tag{8.2.33}$$

is ignorable, and the last term is

$$\begin{aligned} S_2 &= \int d^4x h^{\alpha\beta} \partial_\alpha \phi^* \partial_\beta \phi \\ &\approx \int dt \int d^3\vec{r} \left[\frac{1}{2} mc^4 h^{tt} |\psi|^2 + \frac{i\hbar c^2}{2} h^{ti} \left\{ \psi^* (\nabla_i \psi) - (\nabla_i \psi^*) \psi \right\} \right] \\ &= m \int dt \int d^3\vec{r} \left[-\Phi_G |\psi|^2 + \frac{1}{c} j \cdot A \right], \end{aligned} \tag{8.2.34}$$

where

$$j_i = -\frac{i\hbar}{2m} \left[\psi^* (\nabla_i \psi) - (\nabla_i \psi^*) \psi \right] \tag{8.2.35}$$

is the Schroedinger current (see (2.4.4)) in the non-relativistic limit.

To the desired order, therefore,

$$S = \int dt \int d^3\vec{r} \left[\frac{i\hbar}{2} \psi^* \overleftrightarrow{\partial_t} \psi - \frac{\hbar^2}{2m} |\vec{\nabla}\psi|^2 - V_1(|\psi|) - m\Phi_G |\psi|^2 + \frac{m}{c} j \cdot A \right]. \tag{8.2.36}$$

This may be put in an amusing form, if we use the "four-vector" $A_\mu = (\Phi, -A_i/c)$ and the "gauge covariant derivative" $D_\mu = \partial_\mu + \frac{im}{\hbar} A_\mu$, in terms of which our action reads

$$S = \int dt \int d^3\vec{r} \left[\frac{i\hbar}{2} \psi^* \overleftrightarrow{D_t} \psi - \frac{\hbar^2}{2m} |D_i \psi|^2 - V_1(|\psi|) \right], \tag{8.2.37}$$

dropping terms of $\mathcal{O}(A^2)$. The first two terms have the form of an "electromagnetic" coupling. Varying the action (8.2.36) yields the non-linear Schroedinger equation

$$i\hbar \mathcal{D}_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + \frac{m}{2} \Phi + V_1'(|\psi|)\psi \tag{8.2.38}$$

where we have used $\nabla \cdot A = 0$ and defined the transport derivative

$$\mathcal{D}_t \psi = \left(\partial_t - \frac{1}{2c} A \cdot \nabla \right) \psi, \tag{8.2.39}$$

which accounts for frame dragging due to the rotation. Finally, to this order, the stress energy tensor for the scalar field in (2.5.26) has the following components,

$$T^{\psi}_{00} = mc^4|\psi|^2, \quad T^{\psi}_{0i} = T^{\psi}_{i0} = -mc^2 j_i \tag{8.2.40}$$

therefore, the (8.2.38) must be supplemented by (8.1.12),

$$\begin{aligned} \nabla^2\Phi &= -4\pi G m|\psi(t,r)|^2 \\ \nabla^2 A_i &= \frac{8i\pi G\hbar}{c}\left[\psi^*(\nabla_i\psi) - (\nabla_i\psi^*)\psi\right] \end{aligned} \tag{8.2.41}$$

This completes the setup for non-relativistic scalar fields in GEM. In condensed matter physics, the non-relativistic scalar field action on a flat background describes a Bose-Einstein condensate in the mean field or Hartree-Fock approach, with ψ representing the single particle wave function. In the Hartree approximation, all bosons are in the state ψ. When the gravitational interaction is included, within the context of GEM, the Bose-Einstein condensate becomes weakly self-gravitating and so is of interest in astrophysical phenomena. This system has therefore been proposed in a description of dark matter as a Bose Einstein condensate, which successfully avoids several of the issues associated with dust ("cold" dark matter) models.

Unfortunately, equations (8.2.38) and (8.2.41) form a coupled system that is impossible to solve analytically. Nevertheless, a (remarkable) hydrodynamic analogy is obtained by performing a **Madelung** transformation,

$$\psi(t,\vec{r}) = \sqrt{\rho(t,\vec{r})}e^{iS(t,\vec{r})}, \tag{8.2.42}$$

on the GEM-scalar field system. The real and imaginary parts of the non-linear Schroedinger equation are then

$$\begin{aligned} \mathcal{D}_t\rho + \frac{\hbar}{m}\left(\nabla\rho\cdot\nabla S + \rho\nabla^2 S\right) &= 0 \\ \mathcal{D}_t S + \frac{\hbar}{2m}(\nabla S)^2 + \frac{m}{2\hbar}\Phi + \frac{1}{\hbar}V'(\rho) + Q &= 0 \end{aligned} \tag{8.2.43}$$

where

$$Q = -\frac{\hbar}{4m}\left[\frac{\nabla^2\rho}{\rho} - \frac{1}{2}\left(\frac{\nabla\rho}{\rho}\right)^2\right] \tag{8.2.44}$$

is Bohm's "Quantum Potential". If we call $v = \hbar\nabla S/m$, the first equation has the form of a continuity equation,

$$\mathcal{D}_t\rho + \nabla\cdot(\rho v) = 0, \tag{8.2.45}$$

so v has the interpretation of a velocity. It can be thought of as the velocity of a scalar field "fluid" for the second equation can also be put into a suggestive form, if we take its gradient,

$$\partial_t v - \frac{1}{2c}\nabla(A\cdot v) + \frac{1}{2}\nabla v^2 + \frac{1}{2}\nabla\Phi + \frac{1}{m}V''(\rho)\nabla\rho + \frac{\hbar}{m}\nabla Q = 0. \tag{8.2.46}$$

Now the velocity field is irrotational, therefore $\nabla_i v_j = \nabla_j v_i$ and it follows that $\nabla v^2 = 2(v\cdot\nabla)v$ and $\nabla_i(A_j v_j) = v_j\nabla_i A_j + (A\cdot\nabla)v_i$; this leads to the following generalization of Euler's equation,

$$\mathcal{D}_t v_i + (v\cdot\nabla)v_i - \frac{1}{2c}v_j\nabla_i A_j = -\frac{1}{\rho}\nabla_i P - \frac{1}{2}\nabla_i\Phi - \frac{\hbar}{m}\nabla_i Q \tag{8.2.47}$$

for a fluid of mass density ρ and pressure P, where

$$\nabla_i P = \frac{\rho}{m}V''(\rho)\nabla_i\rho. \tag{8.2.48}$$

serves as an equation of state. For example, if we take the interaction potential to be of the form $V(\rho) = \frac{\lambda}{4}\rho^2$ (*i.e.*, $V(\phi) = \frac{\lambda}{4}|\phi|^4$), we find

$$P = \frac{\lambda\rho^2}{4m}. \tag{8.2.49}$$

Thus repulsive self-interactions ($\lambda > 0$) lead to a positive pressure and, vice-versa, attractive self-interactions to a negative pressure.

8.3 Plane Gravitational Waves

In the absence of sources, or if the sources are very distant, we could expand $\overline{h}_{\mu\nu}$ in Fourier modes, with mode functions that look like plane waves,

$$\overline{h}_{\mu\nu}(k,x) = \overline{A}_{\mu\nu}(\vec{k})e^{ik\cdot x} \tag{8.3.1}$$

where $k_\mu = (-\omega_k, \vec{k})$ and $\omega_k = |\vec{k}|c$. The actual metric functions will be of a similar form,

$$h_{\mu\nu} = A_{\mu\nu}(\vec{k})e^{ik\cdot x}, \quad A_{\mu\nu} = \overline{A}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overline{A}^{\alpha}{}_{\alpha}. \tag{8.3.2}$$

From the gauge condition, we find that the amplitude components of $\overline{h}_{\mu\nu}$ must be perpendicular to k_μ, i.e., $\overline{A}_{\mu\nu}k^\nu = 0 = A_{\mu\nu}k^\nu - \frac{1}{2}k_\mu A^\alpha{}_\alpha$. This set of four conditions reduces the ten components of $A_{\mu\nu}$ to six independent ones. But there are further constraints on $A_{\mu\nu}$, due to the fact that the solutions of Poisson's equation in (8.1.10) are not unique. Indeed, one may add to ξ_μ any solution, χ_μ, of Laplace's equation with no change in (8.1.10) so that the gauge transformation still expresses the perturbations in the simplified, gauge fixed form. This is similar to the situation with the massless vector field, but this time we have *four* additional conditions on $h_{\mu\nu}$, so it is left with, once again, only two independent components, equivalently two independent polarization states. It was to be expected as we have already seen from counting the gravitational constraints that the pure gravitational field has only two degrees of freedom at each event.

8.3.1 The Transverse-Traceless gauge

Let us make this explicit, by taking (for a solution to Laplace's equation)

$$\chi_\mu = b_\mu e^{ik\cdot x} \tag{8.3.3}$$

with $k_\mu = (-\omega_k, \vec{k})$, as before. Let $\overline{h}_{\mu\nu}$ and $\overline{h}'_{\mu\nu}$ be two plane wave solutions satisfying the gauge condition (therefore both are perpendicular to k^μ) and let them be related by $\overline{h}'_{\mu\nu} = \overline{h}_{\mu\nu} - \chi_{\{\mu,\nu\}} + \eta_{\mu\nu}\chi^\alpha{}_{,\alpha}$. We will let $\overline{A}_{\mu\nu}$ be any set of coefficients (so long as they are perpendicular to k^μ) and ask if the additional freedom can be used to simplify $\overline{A}'_{\mu\nu}$. The two sets of coefficients are related by

$$\overline{A}'_{\mu\nu} = \overline{A}_{\mu\nu} - ik_{\{\mu}b_{\nu\}} + i\eta_{\mu\nu}k\cdot b \tag{8.3.4}$$

and therefore, taking the trace, ($\overline{A} = \overline{A}^\alpha{}_\alpha$)

$$\overline{A}' = \overline{A} + 2ik\cdot b \tag{8.3.5}$$

We could choose b_μ so that $\overline{A}' = 0$, in which case

$$k\cdot b = \frac{i}{2}\overline{A} = \frac{\omega_k}{c}^2 b_0 + \vec{k}\cdot\vec{b} \;\Rightarrow\; \vec{k}\cdot\vec{b} = \frac{i}{2}\overline{A} - \frac{\omega_k}{c}^2 b_0 \tag{8.3.6}$$

Three further conditions may be imposed on b_μ so we ask for $\overline{A}'_{0i} = 0$, which gives

$$\overline{A}_{0i} - ik_0 b_i - ik_i b_0 = 0 \tag{8.3.7}$$

or, "dotting" with k^i,

$$\overline{A}_{0i}k^i = -i\omega_k\vec{k}\cdot\vec{b} + \frac{i\omega_k^2}{c^2}b_0 = \frac{\omega_k}{2}\overline{A} + \frac{2i\omega_k^2}{c^2}b_0, \tag{8.3.8}$$

upon eliminating $\vec{k}\cdot\vec{b}$ by using the previous equation. This can be used to solve for b_0 in terms of the components of the matrix $\overline{A}$ and of the momentum k,

$$b_0 = -\frac{ic^2}{2\omega_k^2}\left[\overline{A}_{0i}k^i - \frac{\omega_k}{2}\overline{A}\right]. \tag{8.3.9}$$

Now because $\overline{A}_{\mu\nu}$ is perpendicular to k^μ, we have in particular $\overline{A}_{0\mu}k^\mu = 0$, or $\omega_k\overline{A}_{00}/c^2 = -\overline{A}_{0i}k^i$. Therefore b_0 can be re-expressed as

$$b_0 = \frac{i}{2\omega_k}\left[\overline{A}_{00} + \frac{1}{2}c^2\overline{A}\right], \tag{8.3.10}$$

which ensures that $\overline{A}'_{00} = 0$. The solution for b_0 can be reinserted into (8.3.7) to find b_i,

$$b_i = \frac{i}{\omega_k}\left[A_{0i} + \frac{k_i}{2\omega_k}\left(\overline{A}_{00} + \frac{1}{2}c^2\overline{A}\right)\right]. \tag{8.3.11}$$

Thus all the components of b_μ are fixed while $\overline{A}'_{\mu\nu}$ is constrained to be traceless and transverse, satisfying $\overline{A}'_{0\mu} = 0$. We could simply begin with a traceless and transverse (TT) form of the amplitude tensor, which we henceforth refer to as $A^{\mathrm{TT}}_{\mu\nu}$.

If, for example, the wave propagates in the z direction then $k^1 = k^2 = 0$ and

$$h^{\mathrm{TT}}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A(t,z) & B(t,z) & 0 \\ 0 & B(t,z) & -A(t,z) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A(t,z)\mathbf{e}_+ + B(t,z)\mathbf{e}_\times \tag{8.3.12}$$

where $\mathbf{e}_+$ and $\mathbf{e}_\times$ refer to the polarization states

$$\mathbf{e}_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{e}_\times = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8.3.13}$$

It is completely characterized by its energy, and the coefficients $A(t,z)$ and $B(t,z)$. Again, since $\overline{h}^{\mathrm{TT}}_{\mu\nu}$ is traceless, there is no difference between $\overline{h}^{\mathrm{TT}}_{\mu\nu}$ and $h^{\mathrm{TT}}_{\mu\nu}$.

8.3.2 Effect on Matter

We can understand the effect of this gravitational wave by considering the geodesic motion of particles as a gravitational wave passes by. Consider a wave traveling in the x^3 direction and passing a stationary particle ($U^0 = $ const.); the geodesic equations become simply

$$\frac{dU^i}{d\tau} = 0, \tag{8.3.14}$$

so a particle that was initially at rest, will experience no acceleration and remain at rest, *i.e.,* its coordinates do not change.

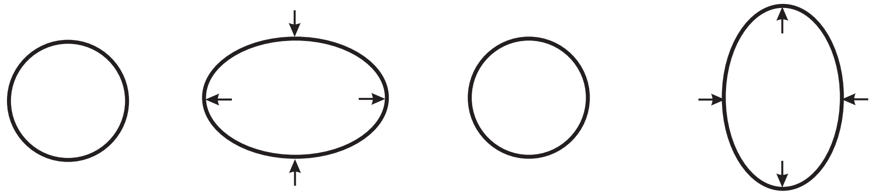

Figure 8.1: Effect of an $\mathbf{e}_+$ Gravitational Wave on a ring of particles in a plane $\perp$ to its propagation.

However, consider two particles at rest, say along the x axis, and separated by a coordinate distance Δ_x in the $z = 0$ plane. As a $\mathbf{e}_+$ wave passes them, the proper distance between the particles will be

$$\Delta l_x = \int |-g_{\mu\nu}dx^\mu dx^\nu|^{1/2} = \int_0^{\Delta_x} \sqrt{g_{xx}}dx = \Delta_x\sqrt{1 + A_0\cos(ckt)} \qquad (8.3.15)$$

Likewise, particles separated along the y axis will experience a changing proper distance according to

$$\Delta l_y = \int_0^{\Delta_y} \sqrt{g_{yy}}dy = \Delta_y\sqrt{1 - A_0\cos(ckt)} \qquad (8.3.16)$$

so particles along the x axis are "pushed out" while particles along the y axis are "pulled in", and vice versa (depending on the phase of the incoming wave). We can make this even more explicit by considering two particles in the $x-y$ plane, separated by a distance Δ along a line making an angle θ with the x axis, so that

$$\Delta l_+ = \int \sqrt{g_{xx}dx^2 + g_{yy}dy^2} = \Delta\sqrt{(1+\alpha(t))\cos^2\theta + (1-\alpha(t))\sin^2\theta} \qquad (8.3.17)$$

where $\alpha(t) = A_0\cos(ckt) \ll 1$. Instead of two particles, now consider a ring of particles for which diametrically opposite pairs make angles θ with the x axis, *i.e.,* $\theta \in [0, 2\pi)$, then we can view the above as the expression for the separation between diametrically opposite points on an ellipse of time-varying eccentricity

$$e = \sqrt{1 - \frac{1-|\alpha(t)|}{1+|\alpha(t)|}} = \sqrt{\frac{2|\alpha(t)|}{1+|\alpha(t)|}} \qquad (8.3.18)$$

This is shown in figure 8.1

Next we analyze the passage of an $\mathbf{e}_\times$ wave on particles lying in the plane perpendicular to its propagation. It is easy to see that particles separated along the x and y axes are not

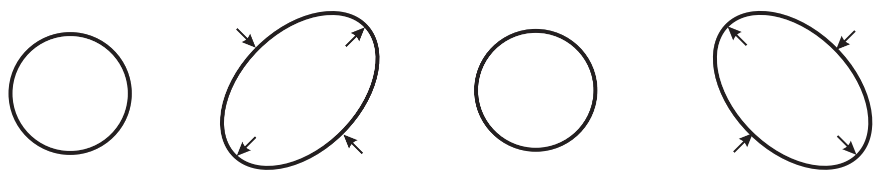

Figure 8.2: Effect of an $\mathbf{e}_\times$ Gravitational Wave on a ring of particles in a plane $\perp$ to its propagation.

affected by the wave, but particles separated along a line making an angle θ ($\neq 0, \pi/2$) with the x axis will again experience the "pushing" and "pulling". This time,

$$\Delta l_\times = \int \sqrt{dx^2 + dy^2 + 2g_{xy}dxdy} = \Delta\sqrt{1 + B_0 \cos(ckt)\sin 2\theta}. \tag{8.3.19}$$

This can be brought to the same form as (8.3.17) if we rotate the coordinate system by $\pi/4$, *i.e.*, define $\theta' = \theta + \pi/4$ and rewrite (8.3.19) as

$$\Delta l_\times = \Delta\sqrt{(1 + \beta(t))\cos^2\theta' + (1 - \beta(t))\sin^2\theta'}. \tag{8.3.20}$$

where $\beta(t) = B_0 \cos(kct)$. The conclusions are the same as before, the ellipsoidal perturbations are rotated by $\pi/4$ in the xy plane, as shown in figure 8.2.[10]

The changing proper distance between objects during the passage of a gravitational wave is measured by gravitational wave detectors. They are monitored by measuring the light travel time in the mutually perpendicular arms of a Michelson Morley interferometer. Differences in the light travel time produce measurable interference fringes in the output of the interferometer.

[10] Problem: Apply the equation for geodesic deviation (7.7.14), taking the non-relativistic limit, $U^0 \approx 1$, $U^i \approx 0$, to arrive at the same conclusions. First show that

$$\frac{\partial^2 V^\mu}{\partial t^2} = -c^2 R^\mu{}_{0\nu 0} V^\nu, \tag{8.3.21}$$

whence, for separations (V^x, V^y), in the xy plane,

$$\frac{\partial^2 V^x}{\partial t^2} \approx \frac{c^2}{2}\left[\ddot{A}V^x + \ddot{B}V^y\right], \quad \frac{\partial^2 V^y}{\partial t^2} \approx \frac{c^2}{2}\left[\ddot{A}V^x - \ddot{B}V^y\right].$$

Then argue for the ellipsoidal perturbations as follows: first let $V^i = V_0^i + \delta V^i$, where V_0^i is the separation before the passage of the wave and assume that $\delta V^i \ll V_0^i$; the equations become

$$\delta\ddot{V}^x \approx \frac{c^2}{2}\left[\ddot{A}V_0^x + \ddot{B}V_0^y\right], \quad \delta\ddot{V}^y \approx \frac{c^2}{2}\left[\ddot{A}V_0^x - \ddot{B}V_0^y\right].$$

Take $A = A_0 \cos(ckt)$ and $B = B_0 \cos(ctk)$.

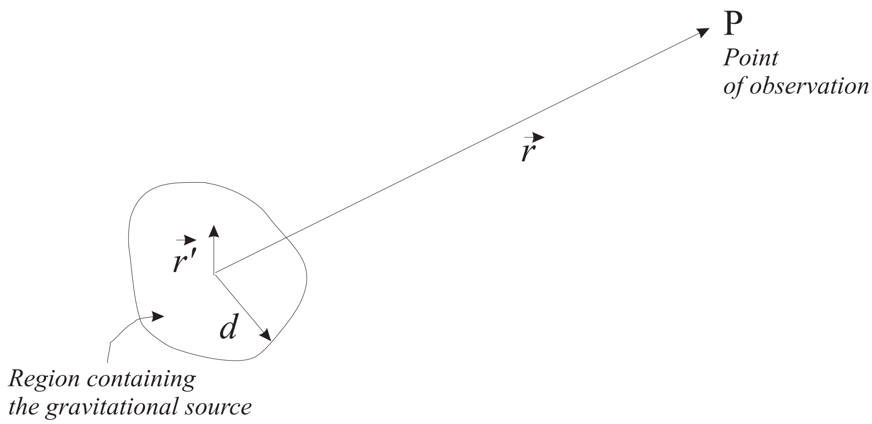

Figure 8.3: A distant, localized, gravitational source.

8.4 Sources of Gravitational Waves

We now give a general form of $h^{\rm TT}_{\mu\nu}$ for an arbitrary (null) wave vector, k^μ. To do this, we first construct a projection operator that projects perpendicular to the direction of propagation. However the wave vector is null, so we employ a second null vector m^μ that also satisfies $m \cdot k = -1$ and define

$$P_{\mu\nu} = \eta_{\mu\nu} + k_{(\mu}m_{\nu)}. \tag{8.4.1}$$

It is easy to see that $P_{\mu\nu}$ projects transverse to k^μ, is symmetric, of trace $P^\alpha{}_\alpha = 2$ and satisfies $P_{\mu\alpha}P^{\alpha\nu} = P_\mu{}^\nu$. With this projector we can construct the operator

$$\Lambda_{\mu\nu\alpha\beta} = P_{\mu\alpha}P_{\nu\beta} - \frac{1}{2}P_{\mu\nu}P_{\alpha\beta} \tag{8.4.2}$$

and define

$$h^{\rm TT}_{\mu\nu} = \Lambda_{\mu\nu}{}^{\alpha\beta}\overline{h}_{\alpha\beta} \tag{8.4.3}$$

where $\overline{h}_{\alpha\beta}$ is given in Lorentz gauge, but not necessarily in the TT gauge.[11]

We will be interested in the spectral distribution of the radiation from a distant, isolated source of finite size, $\mathfrak{d}$, much smaller than its distance, r, from the observer (see figure 8.3). In terms of the temporal (inverse) Fourier transforms

$$T_{\mu\nu}(t, \vec{r}) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} e^{-i\omega t} T_{\mu\nu}(\omega, \vec{r})$$

[11] Problem: Show that $h^{\rm TT}_{\mu\nu}$ defined by (8.4.3) is indeed traceless and transverse for a general $\overline{h}^{\alpha\beta}$. If we consider propagation in the z direction, then $k^\mu = \{c^{-1}, 0, 0, 1\}$ and we could take $m^\mu = \frac{1}{2}\{c^{-1}, 0, 0, -1\}$. Show that the projection gives precisely the form in (8.3.12).

$$\overline{h}_{\mu\nu}(t,\vec{r}) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} e^{-i\omega t}\overline{h}_{\mu\nu}(\omega,\vec{r}), \tag{8.4.4}$$

the wave equation will read

$$\overline{h}_{\mu\nu}(\omega,\vec{r}) = \frac{4G}{c^4}\int d^3\vec{r}'\ \frac{T_{\mu\nu}(\omega,\vec{r}')}{|\vec{r}-\vec{r}'|} e^{ik|\vec{r}-\vec{r}'|} \tag{8.4.5}$$

where $k = \omega/c$. Now, since the integration is over the source and $|\vec{r}| \gg |\vec{r}'|$, we can approximate

$$|\vec{r}-\vec{r}'| \approx r - \widehat{n}\cdot\vec{r}', \tag{8.4.6}$$

where $\widehat{n} = \widehat{r}$, so that

$$\overline{h}_{\mu\nu}(\omega,\vec{r}) \approx \frac{4G}{c^4}\frac{e^{ikr}}{r}\int d^3\vec{r}'\ T_{\mu\nu}(\omega,\vec{r}')e^{-ik\widehat{n}\cdot\vec{r}'}. \tag{8.4.7}$$

The Lorentz gauge ensures that we have to evaluate only the spacelike components of $\overline{h}_{\mu\nu}(\omega,\vec{r})$, for

$$\begin{aligned} \overline{h}_{\mu\nu}{}^{,\nu} = 0 \quad &\Rightarrow \quad \frac{i\omega}{c^2}\overline{h}_{\mu 0}(\omega,\vec{r}) + \overline{h}_{\mu k}{}^{,k}(\omega,\vec{r}) = 0 \\ &\Rightarrow \quad \overline{h}_{\mu 0}(\omega,\vec{r}) = \frac{ic^2}{\omega}\overline{h}_{\mu k}{}^{,k}(\omega,\vec{r}) \end{aligned} \tag{8.4.8}$$

and therefore

$$\overline{h}_{i0}(\omega,\vec{r}) = \frac{ic^2}{\omega}\overline{h}_{ik}{}^{,k}(\omega,\vec{r}), \quad \overline{h}_{00}(\omega,\vec{r}) = -\frac{c^4}{\omega^2}\overline{h}_{ik}{}^{,ik}(\omega,\vec{r}). \tag{8.4.9}$$

Now the spacelike components of $\widehat{\overline{h}}$ are obtained from the spacelike components of the stress energy tensor. If we define the wave vector $\vec{k} = k\hat{n}$, then

$$\overline{h}_{ij}(\omega,\vec{r}) \approx \frac{4G}{c^4}\frac{e^{ikr}}{r}\int d^3\vec{r}'\ T_{ij}(\omega,\vec{r}')e^{-i\vec{k}\cdot\vec{r}'}. \tag{8.4.10}$$

The exponent, above, has the Taylor expansion

$$e^{-i\vec{k}\cdot\vec{r}'} = \sum_{s=0}^{\infty}\frac{(-i\vec{k}\cdot\vec{r}')^s}{s!} \tag{8.4.11}$$

so in the long wavelength limit, *i.e.,* if the wavelength $\lambda = 2\pi/k \gg \mathfrak{d}$, we may replace the exponential by unity and

$$\overline{h}_{ij}(\omega,\vec{r}) \approx \frac{4G}{c^4}\frac{e^{ikr}}{r}\int d^3\vec{r}'\ T_{ij}(\omega,\vec{r}'). \tag{8.4.12}$$

Conservation of stress energy allows us to express the integral in a useful way. Using it we find

$$T^{\mu\nu}{}_{,\nu} = 0 \;\Rightarrow\; T^{\mu 0}{}_{,t} + T^{\mu k}{}_{,k} = 0 \;\Rightarrow\; i\omega T^{\mu 0}(\omega, \vec{r}) = T^{\mu k}{}_{,k}(\omega, \vec{r}). \tag{8.4.13}$$

With this in mind, consider the fact that

$$\begin{aligned} 0 &= \int d^3\vec{r}'\partial'_k \left[x'^i T^{jk}(\omega, \vec{r}') + x'^j T^{ik}(\omega, \vec{r}')\right] \\ &= \int d^3\vec{r}' \left[2T^{ij}(\omega, \vec{r}') + x'^i T^{jk}{}_{,k}(\omega, \vec{r}') + x'^j T^{ik}{}_{,k}(\omega, \vec{r}')\right] \\ &= \int d^3\vec{r}' \left[2T^{ij}(\omega, \vec{r}') + i\omega \left(x'^i T^{j0}(\omega, \vec{r}') + x'^j T^{i0}(\omega, \vec{r}')\right)\right] \end{aligned} \tag{8.4.14}$$

(being an integral over the entire source), which implies that

$$\begin{aligned} \int d^3\vec{r}'\; T^{ij}(\omega, \vec{r}') &= -\frac{i\omega}{2}\int d^3\vec{r}' \left(x'^i T^{j0}(\omega, \vec{r}') + x'^j T^{i0}(\omega, \vec{r}')\right) \\ &= -\frac{i\omega}{2}\int d^3\vec{r}' \left[\partial'_k \left(x'^i x'^j T^{0k}(\omega, \vec{r}')\right) - x'^i x'^j T^{0k}{}_{,k}(\omega, \vec{r}')\right] \\ &= -\frac{\omega^2}{2}\int d^3\vec{r}' x'^i x'^j T^{00}(\omega, \vec{r}') \end{aligned} \tag{8.4.15}$$

where, in the last step, we again dropped a total derivative over the source volume and used (8.4.13). The integral on the right is the **quadrupole moment tensor**, Q^{ij}, of the source, more precisely,

$$Q^{ij}(\omega) = 3\int d^3\vec{r}' x'^i x'^j T^{00}(\omega, \vec{r}') \tag{8.4.16}$$

so we find a simple expression for the generation of long wavelength gravitational waves from a distant source,

$$\overline{h}_{ij}(\omega, \vec{r}) \approx -\frac{2G\omega^2}{3c^4}\frac{e^{ikr}}{r}Q_{ij}(\omega) \;\Rightarrow\; \overline{h}_{ij}(t, \vec{r}) \approx \frac{2G}{3c^4 r}\ddot{Q}_{ij}(t - r/c). \tag{8.4.17}$$

and, from here, one could employ (8.4.9) to determine $\overline{h}_{0i}(t, \vec{r})$ and $\overline{h}_{00}(t, \vec{r})$.

The leading contribution to the long wavelength, gravitational wave produced by an isolated source and observed in the radiation zone depends only on the acceleration of the quadrupole moment of the source at the retarded time. On the other hand, the leading contribution to the *electromagnetic* radiation is from the dipole moment of the source. The reason for the absence of dipole radiation in gravity is that an accelerating gravitational (energy) dipole moment is forbidden by the conservation of momentum. There is no gravitational dipole radiation. The lowest contribution is from the quadrupole moment, which is weaker than the dipole moment, so gravitational radiation is generally weaker than electromagnetic radiation.

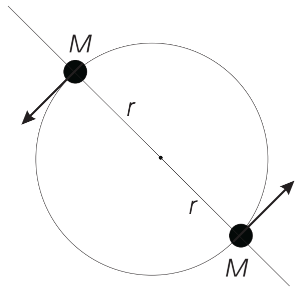

Figure 8.4: A simplified binary system.

8.4.1 Example

To illustrate (8.4.17), we consider the particularly simple but still very interesting system consisting of a binary star system. Assume that two equal masses are rotating slowly and uniformly in identical circular orbits in the $x-y$ plane about their center of mass, as shown in figure 8.4, and that it is sufficient to describe this system within the Newtonian approximation. Thus, if Ω is their angular velocity then their positions at time t may be given as

$$\mathfrak{r}_{A,B}(t) = R\langle\cos(\Omega t + \phi_{A,B}), \sin(\Omega t + \phi_{A,B})\rangle \tag{8.4.18}$$

(we are using the angular brackets to denote the vector displacement of each star from the center). Without loss of generality, take $\phi_A = 0$ and $\phi_B = \pi$. We may give the stress energy tensor as

$$T^{00}(t, \vec{r}') = M\left[\delta^2(\vec{r}' - \mathfrak{r}_A(t)) + \delta^2(\vec{r}' - \mathfrak{r}_B(t))\right]\delta(z'), \tag{8.4.19}$$

whence we compute the non-vanishing components of the quadrupole moment,

$$\begin{aligned} Q_{11}(t) &= 2MR^2\cos^2(\Omega t) \\ Q_{12}(t) &= 2MR^2\cos(\Omega t)\sin(\Omega t) \\ Q_{22}(t) &= 2MR^2\sin^2(\Omega t) \end{aligned} \tag{8.4.20}$$

The gravitational radiation follows from (8.4.17),

$$\overline{h}_{ij}(t, \vec{r}) = \frac{8GM}{3c^4 r}(\Omega R)^2 \begin{bmatrix} \cos(2\Omega t_r) & -\sin(2\Omega t_r) & 0 \\ -\sin(2\Omega t_r) & -\cos(2\Omega t_r) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{8.4.21}$$

where $t_r = t - r/c$ and the angular frequency of the orbits is given by

$$\Omega = \sqrt{\frac{GM}{2R^3}}. \tag{8.4.22}$$

The remaining components of $\overline{h}_{\mu\nu}$ are now easily recovered from (8.4.9).

Chapter 9

Static and Stationary Solutions

Within regions in which the gravitational field is too strong for the linearization to yield meaningful results, we must ask for exact solutions of the Einstein equations. These are ten coupled non-linear differential equations to be solved simultaneously, by no means an easy problem. To yield tractable results, we confine our search for solutions to situations involving a high degree of symmetry with idealized matter sources. Idealized as our solutions will be, they are still important as building blocks for our understanding of more complicated (and realistic) situations for which sophisticated numerical techniques must be applied. We categorize the solutions according to their symmetries and the kind of matter they describe, *i.e.,* according to the physical interpretation of the stress energy tensor. As such, we will consider (a) Vacuum solutions with and without a cosmological constant: $T_{\mu\nu} = 0$, (b) Electrovacuum solutions, for which the stress tensor describes the electromagnetic field, (c) Dust solutions, where the stress tensor describes pressureless dust, (d) Ideal Fluid solutions in a cosmological context, for which the stress tensor describes an ideal fluid with some equation of state and (e) solutions sourced by classical fields.

Before we proceed, it is well to say a few words about two essential concepts.

- *Singular Space-time:* The question of what constitutes a singular space-time is a very difficult one. Here we take the pragmatic view that, in a singular space-time, one or more of the curvature invariants becomes infinite somewhere, *i.e.,* at a "point" or at a set of points. Curvature invariants are scalars constructed from the Riemann tensor, its derivatives and its contractions. For example, the Ricci scalar, R, is a curvature invariant of the first order in the Riemann tensor and R^2, $R_{\alpha\beta}R^{\alpha\beta}$ and $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ are examples of curvature invariants to second order in the Riemann tensor. Higher order invariants can, of course, be likewise constructed. Because curvature serves as a measure of the energy density and pressure of matter via Einstein's equations, the physical properties of matter also become infinite at a space-time singularity. Space-time singularities therefore signal a breakdown of the laws of physics wherever they appear. What exactly is meant by "where" is, of course, ambiguous, because "where" the geometry breaks down the notion of time and place will have no meaning. One may detect these points by following a geodesic (lightlike or timelike), from any point where the curvature is well behaved and finding that it terminates at a finite value of the affine parameter, as it runs into the curvature singularity.

- *Geodesic Incompleteness:* If a geodesic of a space-time cannot be extended either into the past or into the future to arbitrary values of the affine parameter then it is said to be past or future incomplete. For a particular solution, geodesic incompleteness may simply signal a failure of the coordinate system being used (a *coordinate* singularity), in which case it can be extended by extending the coordinate system. It may also indicate the presence of a genuine curvature singularity (which has to be verified) at the point beyond which it cannot be extended. (For example, all infalling radial geodesics will be future incomplete in the space-time of a black hole and the geodesics of a Big Bang cosmology will be past incomplete.)

9.1 Spherical Symmetry

The earliest and arguably the most important solution of the Einstein equations, apart from Minkowski space, was obtained by Karl Schwarzschild in 1916, very shortly after Einstein published the gravitational field equations (in 1915). The solution describes a spherically symmetric vacuum and is, as we will soon prove, *unique* (subject, of course, to spherical symmetry). Whereas Schwarzschild solved Einstein's equations without a cosmological constant, we will here consider the more general case with a cosmological constant, Λ.

Spherical symmetry means that it is possible to foliate the three dimensional space by two spheres. The metric on each sphere of the foliation is

$$ds^2_{(2)} = d\theta^2 + \sin^2\theta d\varphi^2 \tag{9.1.1}$$

where (θ, ϕ) are coordinates on the two sphere and may be identified with the polar and azimuthal angle respectively, when the two sphere is viewed as embedded in a three dimensions. The two sphere is a maximally symmetric space.[1] We will now appeal to the following theorem (without proof): for a m dimensional space-time foliated by n dimensional maximally symmetric spaces with coordinates u^i and metric γ_{ij}, it is always

[1] Recall that a maximally symmetric n dimensional space is one that possesses the maximum number, $n(n+1)/2$, of linearly independent Killing vectors. You have already determined the three independent Killing vectors of the two sphere in a pervious exercise.

possible to find $m-n$ coordinates, v^a, such that the space-time metric can be expressed as

$$ds^2 = -g_{\mu\nu}dx^\mu dx^\nu = -g_{ab}dv^a dv^b - R^2(v)\gamma_{ij}du^i du^j. \tag{9.1.2}$$

where $R(v)$ is some function of the $m-n$ coordinates. A general, four dimensional, spherically symmetric metric will then take the form

$$ds^2 = -g_{11}(dv^1)^2 - 2g_{12}dv^1 dv^2 - g_{22}(dv^2)^2 - R^2(v^1, v^2)d\Omega^2 \tag{9.1.3}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ and the metric components g_{ab} are all functions of (v^1, v^2) (we have not written this dependence out explicitly). Notice that the area of a sphere at fixed (v^1, v^2) is $4\pi R^2$, so $R(v^1, v^2)$ is called the "area radius". Let us use the area radius as one of our coordinates, solving $R(v^1, v^2) = r$ for v^2, and call the $v^1 = T$. Replacing $v^2 = v^2(T, r)$ in the line element, it will take the form

$$ds^2 = -g_{TT}dT^2 - 2g_{Tr}dTdr - g_{rr}dr^2 - r^2 d\Omega^2, \tag{9.1.4}$$

where the metric components are all functions of (T, r). We now want to find a new function $t(T, r)$, if possible, which is such that the two dimensional (T, r) metric gets replaced by

$$ds^2_{(2)} = -g_{tt}dt^2 - g'_{rr}dr^2 \tag{9.1.5}$$

This would only be possible if we could satisfy the conditions:

$$\begin{aligned} g_{tt}\left(\frac{\partial t}{\partial T}\right)^2 &= g_{TT} \\ g_{tt}\left(\frac{\partial t}{\partial r}\right)^2 + g'_{rr} &= g_{rr} \\ g_{tt}\left(\frac{\partial t}{\partial T}\right)\left(\frac{\partial t}{\partial r}\right) &= g_{Tr} \end{aligned} \tag{9.1.6}$$

These are three equations for three unknown functions, $t(T, r)$, $g_{tt}(T, r)$ and $g'_{rr}(T, r)$. They could be solved, in principle, subject to initial conditions for $t(T, r)$, and the metric could be given as

$$ds^2 = -g_{tt}(t, r)dt^2 - g'_{rr}(t, r)dr^2 - r^2 d\Omega^2. \tag{9.1.7}$$

Now we assume that the manfold is locally Lorentzian and identify t with the time-like coordinate, so $g_{tt}(t, r)$ is negative definite. We have therefore concluded that the most general spherically symmetric metric has the form

$$ds^2 = A(t, r)dt^2 - B(t, r)dr^2 - r^2 d\Omega^2 \tag{9.1.8}$$

and will depend on two unknown functions, to be determined by Einstein's equations.

In order to be static as well, the metric must also admit a time-like Killing vector that is everywhere orthogonal to the spatial hypersurfaces. To satisfy this condition, the spherically symmetric metric of (9.1.8) must be independent of time, so

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\Omega^2 \tag{9.1.9}$$

is the most general spherically symmetric and static line element. For a static metric there are four Killing vectors, *viz.,* the three spacelike Killing vectors associated with the two spheres we used to foliate the three dimensional space and one timelike Killing vector. The three spacelike Killing vectors are associated with the conservation of the three components of the angular momentum and the timelike Killing vector with the conservation of energy. These can be obtained by applying (7.5.19). For an asymptotically flat (static) spacetime, integrating over the sphere at infinity, we find[2]

$$E = Q_\xi = \frac{c^4}{2G}\frac{r^2 A'(r)}{\sqrt{A(r)B(r)}}\bigg|_{r\to\infty} \tag{9.1.10}$$

We will now apply (9.1.9) with certain simple sources to obtain some remarkably useful exact solutions. But, first we remark on some quite general properties of geodesics in static, spherically symmetric space-times.

9.2 Geodesics and Redshift

9.2.1 Geodesics

For every Killing vector, ξ^μ, of any metric, the scalar $U\cdot\xi$ will be conserved along geodesics. This is easy to prove because, if λ is the affine parameter,

$$\frac{d}{d\lambda}(U\cdot\xi) = U^\mu \frac{d\xi_\mu}{d\lambda} + \frac{dU^\mu}{d\lambda}\xi_\mu = U^\mu U^\alpha \xi_{\mu,\alpha} - \Gamma^\mu_{\alpha\beta}\xi_\mu U^\alpha U^\beta \tag{9.2.1}$$

where we used the geodesic equation in the last equality. We can rewrite this result as follows:

$$\frac{d}{d\lambda}(U\cdot\xi) = \frac{1}{2}\left[\xi_{(\alpha,\beta)} - 2\Gamma^\mu_{\alpha\beta}\xi_\mu\right]U^\alpha U^\beta. \tag{9.2.2}$$

This is, of course, identically zero because ξ is a Killing vector, *i.e.,* $\xi_{(\alpha;\beta)} = 0$.

[2] Problem: Show that the charges due to the three Killing vectors on the two sphere:

$$\begin{aligned}\eta^\mu_{(1)} &= (0,0,0,1)\\ \eta^\mu_{(2)} &= (0,0,-\cos\phi,\cot\theta\sin\phi)\\ \eta^\mu_{(3)} &= (0,0,\sin\phi,\cot\theta\cos\phi)\end{aligned}$$

vanish.

As the test particle's angular momentum *vector* is conserved, the particle will move in the plane perpendicular to its angular momentum vector, which, without loss of generality, we can take to be the equatorial plane, $\theta = \pi/2$. With this choice, two of the components of the angular momentum will vanish and the two remaining non-vanishing conserved quantities will be the energy, related to the timelike Killing vector, $\xi^\mu_{(t)} = (1, 0, 0, 0)$, and the surviving component of the angular momentum, related to the azimuthal Killing vector, $\xi^\mu_{(\varphi)} = (0, 0, 0, 1)$. Thus, for geodesics in the general spherically symmetric and static metric (9.1.9), we have

$$\xi_{(t)} \cdot U = U_t = -A(r)\frac{dt}{d\lambda} = -E, \quad \xi_{(\varphi)} \cdot U = U_\varphi = r^2\frac{d\varphi}{d\lambda} = L, \tag{9.2.3}$$

where we set $\theta = \pi/2$. There is a preferred affine parameter for timelike geodesics, namely the proper time, τ. If we take λ to be the proper time along the geodesics, the constants are, respectively, the **Killing energy** and the angular momentum per unit mass of the test particle. The Killing energy has been defined with a negative sign because both $\xi_{(t)}$ and U are timelike, so their inner product is negative, and we want the particle energy (per unit mass) to be positive. The second conservation law is the relativistic version of Keppler's second law. Unfortunately, E and L do not have such a clear meaning for lightlike geodesics because of the freedom to rescale the affine parameter. However,

$$\frac{cL}{E} = \frac{cr^2}{A}\frac{d\varphi}{dt} \tag{9.2.4}$$

has dimension of length and is independent of any rescaling of λ. It can be related to the impact parameter for a lightlike geodesic that propagates to infinity, where we assume that the metric is almost flat.

The geodesic equations are,

$$\begin{aligned}
&\frac{dU^t}{d\lambda} + \frac{A'}{A}\,U^t U^r = 0 \\
&\frac{dU^r}{d\lambda} - \frac{r}{B}U^{\varphi 2} + \frac{A'}{2B}U^{t2} + \frac{B'}{2B}U^{r2} = 0 \\
&\frac{dU^\varphi}{d\lambda} + \frac{2}{r}U^\varphi U^r = 0
\end{aligned} \tag{9.2.5}$$

and equations (9.2.3) give the first integrals of the first and the third. The first integral of the second equation follows most easily from the constraint that the geodesics must satisfy; for any affine parameter, λ, we may write

$$g_{\mu\nu}U^\mu U^\nu = -AU^{t2} + BU^{r2} + r^2U^{\varphi 2} = -\left(\frac{ds}{d\lambda}\right)^2 \stackrel{\text{def}}{=} \epsilon c^2, \tag{9.2.6}$$

where ϵ is a constant. If, moreover, λ is the proper time then $\epsilon = \mp 1$ or zero depending on whether we are looking at timelike, spacelike or lightlike geodesics. For equatorial orbits therefore, taking into account our conserved quantities, this amounts to

$$U^r(r) = \pm\sqrt{\frac{\epsilon c^2}{B} + \frac{E^2}{AB} - \frac{L^2}{Br^2}}. \tag{9.2.7}$$

which completes the list of first integrals of the motion.[3] Knowing the velocities,

$$U_\mu = (-E, U_r(r), 0, L) \tag{9.2.8}$$

allows us to compute the shear, expansion and rotation of the geodesic congruences. Indeed, for timelike geodesics,

$$B_{\mu\nu} = U_{\mu;\nu} = \begin{pmatrix} -\frac{A'U_r}{2B} & \frac{EA'}{2A} & 0 & 0 \\ \frac{EA'}{2A} & U_r' - \frac{B'U_r}{2B} & 0 & -\frac{L}{r} \\ 0 & 0 & \frac{rU_r}{B} & 0 \\ 0 & -\frac{L}{r} & 0 & \frac{rU_r}{B} \end{pmatrix} \tag{9.2.9}$$

is symmetric, so we conclude that they are rotation free. In all static, spherically symmetric *vacuum* space-times, a certain combination of Einstein's equations will yield the relation $AB = c^2$. This simplifies the expression for the expansion,

$$\Theta = \frac{1}{r^2}\left(\frac{r^2 U_r}{B}\right)', \tag{9.2.10}$$

for either timelike or null geodesics.

9.2.2 Gravitational Redshift

Stationary observers, *i.e.,* at a fixed (r, θ, φ), in a static space-time are not in general inertial observers. This is easily seen by calculating their acceleration vector, a_μ. With $U^\mu = (c/\sqrt{A(r)}, 0, 0, 0)$ for such an observer, one finds

$$a_\mu = (U \cdot \nabla)U_\mu = \left(0, \frac{c^2 A'}{2A}, 0, 0\right). \tag{9.2.11}$$

Consider two radially separated, stationary observers, located at $r_1 < r_2$ (with the same angular coordinates). Suppose that observer O_1, at r_1, sends an electromagnetic wave,

[3] Problem: Derive the first integrals directly from the geodesic equations.

which is later received by observer O_2, at r_2. We would like to know how the frequencies of the emitted and received waves compare. To answer this question we must have a coordinate independent (*not* observer independent!) definition of the frequency of an electromagnetic wave. For this we appeal to the equivalence principle and the expression for the frequency measured by any observer in flat space: if $k_\mu = (-\omega, \vec{k})$ is the wave four vector in some inertial frame, S, then $\omega' = -k \cdot U$ gives the wave frequency measured by another inertial observer whose four velocity is U^μ relative to S. By the usual arguments we take this to hold in all coordinate systems. Now k^μ satisfies the null geodesic equation, which we have already integrated. In particular, we know from (9.2.3) that $k_t = -E$ is constant and the absence of a canonical affine parameter for null geodesics means that we can choose it so that E represents the proper frequency of the electromagnetic wave.

Now the four velocities of the two observers, O_1 and O_2, are $U^\mu_{(i)} = (c/\sqrt{A(r_i)}, 0, 0, 0)$, so we can give the frequencies measured by these observers as

$$\begin{aligned} \omega_1 &= -k \cdot U_{(1)} = \frac{Ec}{\sqrt{A(r_1)}} \\ \omega_2 &= -k \cdot U_{(2)} = \frac{Ec}{\sqrt{A(r_2)}} \end{aligned} \tag{9.2.12}$$

and it follows that

$$\omega_2 = \sqrt{\frac{A(r_1)}{A(r_2)}}\, \omega_1. \tag{9.2.13}$$

Suppose, for example, that the space-time is asymptotically flat, *i.e.,* it approaches flat space as $r \to \infty$. In this case, let observer O_2 be the asymptotic observer so that $A(r_2) \to c^2$ and O_1 be located at r. Then O_2 measures the frequency

$$\omega_\infty = \frac{1}{c}\sqrt{A(r)}\, \omega, \tag{9.2.14}$$

where ω is the frequency measured at r. Now in the cases of interest it will turn out that $0 < A(r) < c^2$, so the electromagnetic wave is red-shifted as it climbs out of a gravitational potential well. Conversely, it is blue-shifted as it falls into the well. We can also state this result in terms of the wavelength,

$$\lambda_2 = \sqrt{\frac{A(r_2)}{A(r_1)}}\, \lambda_1 \tag{9.2.15}$$

and give the redshift factor as

$$z = \frac{\lambda_2 - \lambda_1}{\lambda_1} = \sqrt{\frac{A(r_2)}{A(r_1)}} - 1. \tag{9.2.16}$$

Because $\omega = 2\pi/T$, where T is the proper period of the wave, gravitational redshifting is equivalent to the statement that proper time intervals measured at r are related to proper time intervals measured at infinity by

$$\Delta\tau_\infty = \frac{c\Delta\tau}{\sqrt{A(r)}}, \tag{9.2.17}$$

i.e., a proper time interval measured by a stationary clock at r corresponds to a larger proper time interval as measured on an identical stationary clock at infinity.[4] Therefore, any activity deep within the space-time appears to take place at a slower rate to the asymptotic observer. This phenomenon, known as gravitational time dilation, has even been observed in terrestrial experiments by comparing clocks at differing altitudes. The effect is small, being measured in nanoseconds.

9.3 Static Vacua

With a cosmological constant, the vacuum Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R - \Lambda) = 0, \tag{9.3.1}$$

and the metric (9.1.9) yield the following equations,

$$\begin{aligned}
&E_{tt} = 0 \;\Rightarrow\; -2B + (2 - r^2\Lambda)B^2 + 2rB' = 0\\
&E_{rr} = 0 \;\Rightarrow\; 2A - 2AB + \Lambda r^2 AB + 2rA' = 0\\
&E_{\theta\theta} = 0 \;\Rightarrow\; -rBA'^2 + 2A^2B' + 2\Lambda rA^2B^2 + A(-rA'B' + 2BA' + 2rBA'') = 0\\
&E_{\varphi\varphi} = \sin^2\theta E_{\theta\theta}
\end{aligned} \tag{9.3.2}$$

The last is a consequence of spherical symmetry. The first says that

$$B(r) = \left(1 + \frac{\alpha}{r} - \frac{\Lambda r^2}{6}\right)^{-1} \tag{9.3.3}$$

[4] A simpler approach to this particular case is the following: a proper time interval measured on a stationary clock in the static space-time will be given by

$$\Delta\tau = \frac{\sqrt{A(r)}}{c}\Delta t,$$

therefore, if $A(r) \to c$ as $r \to \infty$ then a proper time interval on a clock at spatial infinity will be $\Delta\tau_\infty = \Delta t$, hence

$$\Delta\tau_\infty = \frac{c\Delta\tau}{\sqrt{A(r)}}.$$

from which (9.2.14) and (9.2.16) follow. Employ similar reasoning to argue for length contraction.

for some integration constant, α, whose meaning we will shortly determine. We also find that

$$\frac{E_{tt}}{A} + \frac{E_{rr}}{B} = 0 = \frac{(AB)'}{rAB^2} \tag{9.3.4}$$

so that $A(r)B(r) = c^2\beta$, where β is some constant. With these solutions for $A(r)$ and $B(r)$ the last equation is an identity. Moreover, we can get rid of the constant β by simply rescaling the time coordinate, $t \to \beta t$. Our metric is therefore completely determined by a single integration constant,

$$ds^2 = c^2\left(1 + \frac{\alpha}{r} - \frac{\Lambda r^2}{6}\right)dt^2 - \left(1 + \frac{\alpha}{r} - \frac{\Lambda r^2}{6}\right)^{-1} dr^2 - r^2 d\Omega^2. \tag{9.3.5}$$

To discover the meaning of this constant, consider the diffeomorphism charges, taking ϵ^μ to be the time-like Killing vector, $\epsilon^\mu = (1,0,0,0)$, and $\Lambda = 0$. As we have seen before (eg. in our study of the weak field) the conserved charge associated with time translation invariance is the energy, specifically Mc^2. Applying (7.5.19) we find

$$Q_t = -\frac{\alpha c^4}{2G} = Mc^2 \;\Rightarrow\; \alpha = -\frac{2GM}{c^2} \tag{9.3.6}$$

where M now represents a point mass located at the origin.[5] We therefore give the solution in the form,

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{6}\right)dt^2 - \left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{6}\right)^{-1} dr^2 - r^2 d\Omega^2. \tag{9.3.7}$$

When $M = 0$ and $\Lambda > 0$ the metric describes **de-Sitter** space (dS) and when $M = 0$ and $\Lambda < 0$ it describes **anti-de-Sitter** space (AdS), both in static (or de-Sitter) coordinates.[6] When $\Lambda = 0$ and $M > 0$ this is the **Schwarzschild** metric. It represents the space-time of a point mass located at the origin of coordinates. Finally, when both M and Λ are not zero, the space-time is called the "Schwarzschild-de-Sitter" space (SdS) when $\Lambda > 0$ or the "Schwarzschild-Anti-de-Sitter" space (SAdS) when $\Lambda < 0$. Minkowski space, de-Sitter space and Anti-de-Sitter space are maximally symmetric spaces, *i.e.,* like the two sphere, they possess the maximum number of Killing vectors.[7]

9.3.1 Uniqueness

The solution in (9.3.7) represents the *unique* vacuum solution of Einstein's equations with a cosmological constant. Its uniqueness can be proved quite easily by considering the most

[5] Problem: Show that the conserved charges associated with the three Killing vectors of the two sphere vanish. These are associated with angular momementum.

[6] These were the coordinates that de-Sitter originally used.

[7] Problem: Obtain explicit expresions for the Killing vectors of dS and AdS space.

general spherically symmetric vacuum metric (not necessarily static),

$$ds^2 = A(t,r)dt^2 - B(t,r)dr^2 - r^2 d\Omega^2 \tag{9.3.8}$$

Of the four vacuum Einstein equations, $E_{\mu\nu} = 0$, the equation $E_{tr} = 0$ tells us that $B(t,r)$ is independent of time and

$$E_{tt} = 0 \;\Rightarrow\; -2B + (2 - r^2\Lambda)B^2 + 2rB' = 0 \;\;\Rightarrow\;\; B(r) = \left(1 + \frac{\alpha}{r} - \frac{\Lambda r^2}{6}\right)^{-1} \tag{9.3.9}$$

where α is some constant. Now we see that all of Einstein's equations are time independent. Also, inserting the solution for $E(r)$ into $E_{rr} = 0$ gives

$$E_{rr} = 0 \;\Rightarrow\; 2A - 2AB + \Lambda r^2 AB + 2rA' = 0 \;\;\Rightarrow\;\; A(t,r)B(r) = c^2\beta(t) \tag{9.3.10}$$

and we can get rid of $\beta(t)$ by simply defining a new time coordinate $t \to t' = \int^t \beta(t)dt$. The remaining Einstein equations are automatically satisfied, the metric becomes entirely time independent and agrees with (9.3.7). Thus, every vacuum solution is static and reduces to the SdS or SAdS solution. This is **Birkhoff's theorem.**

9.3.2 de-Sitter Space

The metric

$$ds^2 = c^2\left(1 - \frac{\Lambda r^2}{6}\right)dt^2 - \left(1 - \frac{\Lambda r^2}{6}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{9.3.11}$$

represents only a part of de-Sitter space, called the **static patch**, $r < \sqrt{6/\Lambda}$. De-Sitter space with d space dimensions is actually a hyperboloid in a higher dimensional flat, Lorentzian manifold with $d+1$ spatial dimensions, X_i, and one time dimension, T, defined by the constraint

$$c^2T^2 - \sum_i X_i^2 = -a^2. \tag{9.3.12}$$

where a is a real constant. The constraint preserves all of the symmetries of the $d+2$ dimensional Lorentzian manifold, *viz.,* $d(d+1)/2$ rotations and $d+1$ boosts, therefore $d+1$ dimensional de Sitter space will admit $d(d+1)/2 + (d+1) = (d+1)(d+2)/2$ Killing vectors and is maximally symmetric. The four dimensional metric in (9.3.11) is recovered by the following parameterization of the constraint:

$$\begin{aligned}
X_1 &= r\sin\theta\cos\varphi \\
X_2 &= r\sin\theta\sin\varphi \\
X_3 &= r\cos\theta \\
X_4 &= \sqrt{a^2 - r^2}\;\cosh(ct/a)
\end{aligned}$$

$$T = \frac{1}{c}\sqrt{a^2 - r^2}\ \sinh(ct/a), \tag{9.3.13}$$

which directly leads to (9.3.11) once we identify Λ with $6/a^2$. The space-time admits no curvature singularities, but the parameterization is valid only so long as $r < a$. On the hypersurface $r = a$, the norm of the timelike Killing vector vanishes. When this happens, the hypersurface is a called a **Killing Horizon**. In this particular case, $r = a$ is known as the **de-Sitter horizon**. Because de-Sitter space has no curvature singularity, the existence of the de Sitter horizon simply signals a breakdown of the coordinate system. Other parameterizations of the constraint are possible. [8]

Geodesics

The radial component of the velocity of equatorial geodesics for the de Sitter metric are

$$U^r = \frac{dr}{d\lambda} = \pm\sqrt{\frac{E^2}{c^2} - \left(\frac{L^2}{r^2} - \epsilon c^2\right)\left(1 - \frac{r^2}{a^2}\right)} \tag{9.3.16}$$

where we have adopted the notation $a^2 = 6/\Lambda$. When it is written as

$$\left(\frac{dr}{d\lambda}\right)^2 + \epsilon c^2 \frac{r^2}{a^2} + \frac{L^2}{r^2}\left(1 - \frac{r^2}{a^2}\right) = \frac{E^2}{c^2} + \epsilon c^2. \tag{9.3.17}$$

we notice the striking similarity with the energy equation describing orbits of test particles in a central force. If we take λ to be the proper time, this allows us to identify an effective potential measured by an observer attached to the test particle,

$$V_{\text{eff}} = \epsilon c^2 \frac{r^2}{a^2} + \frac{L^2}{r^2}\left(1 - \frac{r^2}{a^2}\right) \tag{9.3.18}$$

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[8] Problem: The parametrization

$$\begin{aligned} X_i &= e^{ct/a} x_i, \quad 1 \le i \le 3 \\ X_4 &= a\cosh(ct/a) - \frac{r^2}{2a} e^{ct/a} \\ T &= \frac{a}{c}\sinh(ct/a) + \frac{r^2}{2ac} e^{ct/a}, \end{aligned} \tag{9.3.14}$$

where $r^2 = \sum_i x_i^2$, is called a "flat slicing" of dS. Show that (i) the parametrization is faithful and (ii) it gives

$$ds^2 = c^2 dt^2 - e^{2ct/a} \sum_i dx_i^2. \tag{9.3.15}$$

It follows that the gravitational force is *repulsive* for timelike geodesics, attractive for spacelike geodesics and vanishes altogether for lightlike geodesics. In the usual way, provided that $L \neq 0$, we can give the solution of this equation as

$$\int_{r_0}^{r} \frac{dr/r^2}{\sqrt{\mathcal{E} - \frac{L^2}{r^2} - \frac{1}{2}\kappa r^2}} = \pm\frac{1}{L}(\varphi - \varphi_0), \tag{9.3.19}$$

where $\mathcal{E} = E^2/c^2 + L^2/a^2 + \epsilon c^2$ and $\kappa = 2\epsilon c^2/a^2$. This is precisely of the orbit of a body in a central force whose magnitude is proportional to its distance from the force center (like, eg., a harmonic oscillator). Therefore, borrowing the well known result, we give the solution as

$$\frac{\alpha^2}{r^2} = 1 + \varepsilon \cos 2(\varphi - \overline{\varphi}_0), \tag{9.3.20}$$

where

$$\alpha = \sqrt{\frac{2L^2}{\mathcal{E}}}, \quad \varepsilon = \sqrt{1 - \frac{2\kappa L^2}{\mathcal{E}}} \tag{9.3.21}$$

and $\overline{\varphi}_0$ is an integration constant. Timelike geodesics, for which $\varepsilon > 1$, are then hyperbolæ in the equatorial plane, spacelike geodesics are ellipses and lightlike geodesics ($\varepsilon = 1$) are straight lines.[9]

For radial, timelike geodesics ($L = 0$) it is more convenient to consider ($\dot{r} = dr/d\tau$)

$$\dot{r}^2 = \frac{E^2}{c^2} + \epsilon c^2 - \frac{1}{2}\kappa r^2 \;\Rightarrow\; \ddot{r} = \frac{1}{2}\frac{d\dot{r}^2}{dr} = -\frac{1}{2}\kappa r \tag{9.3.23}$$

so we have general solutions

$$r(\tau) = A\cosh(c\tau/a) + B\sinh(c\tau/a), \tag{9.3.24}$$

but they are subject to (9.3.23) and therefore the integration constants must satisfy

$$B^2 - A^2 = a^2\left(\frac{E^2}{c^4} - 1\right). \tag{9.3.25}$$

Lightlike geodesics are simply given by

$$r(\lambda) = \pm\frac{E}{c}\lambda + B, \tag{9.3.26}$$

[9] Problem: This can be made explicit by taking $x = r\cos\varphi$, $y = r\sin\varphi$ and $\overline{\varphi}_0 = 0$ (which amounts to choosing an orientation of the x and y axes). Show that the solution takes the form

$$\frac{x^2}{\alpha^2/(\varepsilon+1)} - \frac{y^2}{\alpha^2/(\varepsilon-1)} = 1, \tag{9.3.22}$$

which describes an ellipse when $\varepsilon < 1$, a straight line when $\varepsilon = 1$ and a hyperbola when $\varepsilon > 1$.

and the condition $0 < r < a$ implies that they are past and future incomplete, *i.e.,* geodesics will exit the static patch in the past and in the future at finite values of the affine parameter. Notice, however, that geodesics do not cross the horizon *in a finite coordinate time!* To see this, it is sufficient to consider radial null geodesics, for which

$$\frac{dr}{dt} = \frac{dr/d\lambda}{dt/d\lambda} = \pm c\left(1 - \frac{r^2}{a^2}\right). \tag{9.3.27}$$

Integrating the equation, setting $r(t_0) = 0$, we find

$$r(t) = a \tanh\left(\frac{c(t-t_0)}{a}\right) \tag{9.3.28}$$

showing that r only asymptotes to the horizon in coordinate time.

How can we understand the fact that gravity is repulsive when sourced by a positive cosmological constant? The reason is that both pressure and energy density contribute to the gravitational field in general relativity. Just as a negative energy density would lead to a repulsive gravitational field (the gravitational force due to a negative mass would be repulsive even in Newton's theory), so does negative pressure. If we think of the cosmological term as a source of the Einstein field equations, the effective stress tensor,

$$T_{\mu\nu} = -\frac{c^4\Lambda}{16\pi G}\, g_{\mu\nu}, \tag{9.3.29}$$

has the form of an ideal fluid with positive energy density and negative pressure. It satisfies (just barely) the weak, null and dominant energy conditions, while violating the strong energy condition. This violation of the strong energy condition is what leads to gravity acting repulsively on timelike geodesics.

Analytic Extension

Mindful of the range of r, we can define a new coordinate system for de Sitter space as follows. Letting

$$r_* = -\int \frac{dr}{1 - \frac{r^2}{a^2}} = -a \tanh^{-1}\frac{r}{a} = \frac{a}{2}\ln\frac{1 - r/a}{1 + r/a}, \tag{9.3.30}$$

the de Sitter metric takes the form

$$ds^2 = \left(1 - \frac{r^2(r_*)}{a^2}\right)\left(dt^2 - dr_*^2\right) - r^2(r_*)d\Omega^2, \tag{9.3.31}$$

where $r_* \in (-\infty, 0]$. This maneuver essentially pushes the horizon out to negative infinity, where the metric becomes degenerate, but it leaves a simple light cone structure everywhere. Exploiting this simplified light cone, introduce the null coordinates $u = ct - r_*$,

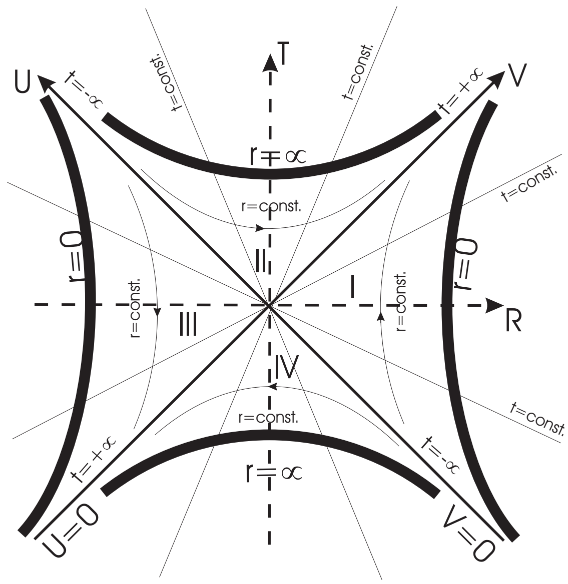

Figure 9.1: The maximally extended de Sitter space-time.

$v = ct + r_*$; then the deSitter metric can be written as

$$\left(1 - \frac{r^2(u,v)}{a^2}\right) du dv - r^2(u,v) d\Omega^2 \tag{9.3.32}$$

and we can get rid of the troublesome factor by defining a new set of null coordinates, $\overline{u} = -ae^{-u/a} \in (-\infty, 0)$ and $\overline{v} = ae^{v/a} \in (0, \infty)$. We find

$$\begin{aligned} ds^2 &= -\frac{a^2}{\overline{u}\overline{v}} \left(1 - \frac{r^2(\overline{u},\overline{v})}{a^2}\right) d\overline{u} d\overline{v} - r^2(\overline{u},\overline{v}) d\Omega^2 \\ &= e^{(u-v)/a} \left(1 - \frac{r^2(\overline{u},\overline{v})}{a^2}\right) d\overline{u} d\overline{v} - r^2(\overline{u},\overline{v}) d\Omega^2 \\ &= \left(\frac{1 + r(\overline{u},\overline{v})/a}{1 - r(\overline{u},\overline{v})/a}\right) \left(1 - \frac{r^2(\overline{u},\overline{v})}{a^2}\right) d\overline{u} d\overline{v} - r^2(\overline{u},\overline{v}) d\Omega^2 \\ &= \left(1 + \frac{r(\overline{u},\overline{v}))}{a}\right)^2 d\overline{u} d\overline{v} - r^2(\overline{u},\overline{v}) d\Omega^2. \end{aligned} \tag{9.3.33}$$

But $r(\overline{u},\overline{v})$ is found to be

$$r(\overline{u},\overline{v}) = a \left(\frac{1 + \overline{u}\overline{v}/a^2}{1 - \overline{u}\overline{v}/a^2}\right), \tag{9.3.34}$$

so we may write the de Sitter metric as

$$ds^2 = \frac{1}{(1 - \overline{uv}/a^2)^2}\left[4d\overline{u}d\overline{v} - a^2\left(1 + \overline{uv}/a^2\right)^2 d\Omega^2\right] \tag{9.3.35}$$

There is no pathology in the metric that prevents us from extending the ranges of the coordinates $\overline{u}$ and $\overline{v}$ to the entire real line, so long as $-a^2 < \overline{uv} < +a^2$. This is known as the **maximally extended** de Sitter space. As $r \to a^-$ both $\overline{u}$ and $\overline{v}$ approach zero. Thus the horizon is matched to two null surfaces, *viz.,* $\overline{u} = 0$ and $\overline{v} = 0$, whereas $r \to 0$ and $r \to \infty$ are mapped to the hypesurfaces $\overline{uv} \to -a^2$ and $\overline{uv} \to a^2$ respectively. Indeed, hypersurfaces of constant r are all characterized by $\overline{uv} = \alpha$, where $-a^2 \leq \alpha < +a^2$ is constant. Inside the horizon, $\alpha < 0$ and the surfaces are all timelike, but outside the horizon, $\alpha > 0$ and they are spacelike. Constant time (t) hypersurfaces are the straight lines $\overline{v}/\overline{u} = -\delta$, where δ is a constant. The null surfaces, $\overline{u} = 0$ and $\overline{v} = 0$, represent $t \to \pm\infty$ respectively and are called the future and past horizons, $\mathcal{H}^{\pm}$, respectively.

We may also define the new coordinates, T and R by

$$\overline{u} = \frac{1}{2}(cT - R), \; \overline{v} = \frac{1}{2}(cT + R), \tag{9.3.36}$$

and express the metric as

$$ds^2 = \frac{16a^4}{(4a^2 - c^2T^2 + R^2)^2}\left[c^2dT^2 - dR^2 - \left(a + \frac{c^2T^2 - R^2}{4a}\right)^2 d\Omega^2\right] \tag{9.3.37}$$

The space-time diagram for maximal extension of de Sitter space-time is shown in figure 9.1. There we see that the space-time naturally divides into four regions, labeled I – IV. Regions I and III have $\overline{uv} < 0$ and are therefore covered by the static coordinates, but regions II and IV have $\overline{uv} > 0$. They represent the space-time beyond the de Sitter horizon. Region I is what we would think of as the static de Sitter space: every future directed, timelike geodesic in region I will eventually pass into region II and no future directed path from II will get us back to region I. An observer in region I may move in any radial direction, but must move forward in coordinate time; she can avoid crossing the horizon by accelerating appropriately. In region II, the observer can move in any direction in coordinate time, but must always move toward increasing r. Both regions II and IV lie outside the de Sitter horizon. What one perhaps did not anticipate is the "doubling" of the interior and exterior regions, *i.e.,* the existence of regions III and IV. These are an artifact of the analytic continuation and are not realized in nature.

9.3.3 Anti-de-Sitter Space

Like de Sitter space, Anti-de-Sitter space with d space dimensions is non-singular and can be thought of as a quasi-sphere embedded in a flat, higher dimensional space. However, this time the higher dimensional space has d space dimensions and *two* time dimensions and is $SO(d,2)$ invariant, while the quasi-sphere is defined via the constraint,

$$c^2(T_1^2+T_2)^2 - \sum_i X_i^2 = a^2. \tag{9.3.38}$$

The constraint preserves all of the $SO(d,2)$ symmetries of the original manifold, *viz.,* $d(d-1)/2$ spatial rotations, $2d$ boosts and one time rotation, therefore AdS will admit the maximum number, $d(d-1)/2+1+2d=(d+1)(d+2)/2$, of Killing vectors. The four dimensional metric with a negative cosmological constant ($\Lambda=-|\Lambda|$),

$$ds^2 = c^2\left(1+\frac{|\Lambda|r^2}{6}\right)dt^2 - \left(1+\frac{|\Lambda|r^2}{6}\right)^{-1}dr^2 - r^2 d\Omega^2, \tag{9.3.39}$$

can be obtained by parameterizing the above constraint as follows:

$$\begin{aligned} X_1 &= r\sin\theta\cos\varphi \\ X_2 &= r\sin\theta\sin\varphi \\ X_3 &= r\cos\theta \\ cT_1 &= \sqrt{a^2+r^2}\ \cos(ct/a) \\ cT_2 &= \sqrt{a^2+r^2}\ \sin(ct/a), \end{aligned} \tag{9.3.40}$$

where, as before, $a=\sqrt{6/|\Lambda|}$. This time there is no restriction on r, the coordinates are global but, as before, other parameterizations are possible.[10] Note, however, that the

[10]Problem: A useful parameterization leads to the Poincaré coordinates of AdS: consider the transformations

$$\begin{aligned} X_1 &= aux_1, \quad X_2 = aux_2, \\ T_2 &= aut \\ X_3 &= \frac{1}{2u}\left(1-u^2a^2-u^2[x_1^2+x_2^2-c^2t^2]\right) \\ cT_1 &= \frac{1}{2u}\left(1+u^2a^2+u^2[x_1^2+x_2^2-c^2t^2]\right). \end{aligned}$$

where $u>0$. Show that parameterization is faithful and that the line element is given as

$$ds^2 = a^2\left(u^2(c^2dt^2-dx_1^2-dx_2^2)-\frac{du^2}{u^2}\right).$$

Because $u>0$, this metric covers only half of AdS and the other half must be covered with a similar system, for which $u<0$. It enjoys the symmetries of the full Poincaré group, $ISO(2,1)$, in two space and one time dimension, as well as an additional dilatation: $\{t,\vec{r},u\}\to\{\lambda t,\lambda\vec{r},\lambda^{-1}u\}$. A further transformation $u\to z=1/u$ brings the metric into the form

$$ds^2 = \frac{a^2}{z^2}\left(c^2dt^2-dx_1^2-dx_2^2-dz^2\right).$$

showing that AdS is conformally flat.

periodic nature of the parametrization requires us to identify points that are separated by $\Delta t = 2\pi a/c$. Another way of saying this is that curves of constant r, θ, φ are closed curves. This is very unpleasant in physics because closed time curves could lead to all sorts of paradoxes, such as someone from the future returning to the past and killing their own grandfather. Happily, we notice that because the metric is static periodicity in time is not apparent. Therefore we simply unwrap all the time circles and extend them in a line, allowing t to range over $(-\infty, \infty)$ and discarding the original higher dimensional quasi-sphere model of AdS. This space is referred to as the **universal covering** of AdS, or CAdS.

Geodesics

The effective potential for geodesics now takes the form

$$V_{\text{eff}} = -\epsilon c^2 \frac{r^2}{a^2} + \frac{L^2}{r^2}\left(1 + \frac{r^2}{a^2}\right) \tag{9.3.41}$$

Timelike geodesics with non-vanishing angular momentum will be elliptic and spacelike geodesics will be hyperbolic. In particular, stable, circular, timelike orbits of radius

$$r = \sqrt{\frac{aL}{c}} \tag{9.3.42}$$

will exist. As was the case for dS, null geodesics experience no gravitational force in AdS and travel in straight lines. The solutions are given in (9.3.19), with $a^2 \to -a^2$. Radial timelike geodesics, which are clearly given by

$$r(\tau) = A\cos(c\tau/a) + B\sin(c\tau/a) \tag{9.3.43}$$

together with the constraint,

$$B^2 + A^2 = a^2\left(\frac{E^2}{c^4} - 1\right), \tag{9.3.44}$$

just oscillate about $r = 0$ and never get to the boundary, $r \to \infty$. On the other hand, radial null geodesics are straight lines given by (9.3.26) and eventually do reach the boundary of AdS (as $\lambda \to \infty$). What about the coordinate time? Using

$$\frac{dr}{dt} = \pm c\left(1 + \frac{r^2}{a^2}\right) \tag{9.3.45}$$

we find

$$r(t) = a\tan\left(\frac{c(t - t_0)}{a}\right) \tag{9.3.46}$$

showing that the boundary of AdS is reached in a finite coordinate time, $\Delta t = \pi a/(2c)$.

9.3.4 Schwarzschild Black Hole

The Schwarzschild metric,

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{9.3.47}$$

should be thought of as covering only a portion (the "static patch") of the space-time describing the gravitational field of a point mass, located at the origin.[11] The coordinates break down at $r = r_s$, where

$$r_s = 2GM/c^2 \approx 3 \text{ km } \left(\frac{M}{M_\odot}\right), \tag{9.3.48}$$

given in terms of the solar mass, $M_\odot \approx 1.99 \times 10^{30}$ kg, and at $r = 0$. A genuine (curvature) singularity of this space-time occurs only at $r = 0$. This is verified by computing the **Kretschmann** scalar,

$$K = R^\alpha{}_{\beta\mu\nu} R_\alpha{}^{\beta\mu\nu} = \frac{12 r_s^2}{r^6}. \tag{9.3.49}$$

Thus, while static coordinates for the Schwarzschild metric are valid only so long as $r > r_s$, we should expect to be able to extend the coordinate system to cover the entire space-time, up to the singularity at $r = 0$. The surface $r = r_s$ is nevertheless interesting because the norm of the timelike Killing vector vanishes there. It is therefore a Killing horizon and is called the **Schwarzschild horizon**. Its area radius, r_s, is called the **Schwarzschild radius**. This is similar to the case of dS, where the static patch is valid so long as $r < a$, and the surface $r = a$ is a Killing horizon called the deSitter horizon.

Outside the horizon, $B(r) = (1 - r_s/r)^{-1}$ is positive and the normals to hypersurfaces of constant r are all spacelike. This is just as it should be because hypersurfaces of constant r are expected to be timelike. On the other hand, inside the horizon they are all timelike, so that surfaces of constant r are spacelike! One might argue, of course, that the coordinates cannot be used to draw any conclusions about the nature of hypersurfaces when $r < r_s$. However, the coordinates may be extended across the horizon and all the way up to the singularity and this statement, being coordinate invariant, will continue to hold true. Because surfaces of constant r are spacelike when $r < r_s$, the singularity at $r = 0$ is *spacelike.*

[11] Problem: Show that the exact solution in (9.3.47) is compatible with the GEM solution in (8.2.22) with $\vec{L} = 0$. Do this by finding a coordinate transformation from the radial coordinate, r, in (9.3.47) to a new radial coordinate, r', that transforms (9.3.47) to isotropic form:

$$ds^2 = \left(\frac{1 - GM/2c^2 r'}{1 + GM/2c^2 r'}\right)^2 dt^2 - \left(1 + \frac{GM}{2c^2 r'}\right)^4 (dr'^2 + r'^2 d\Omega^2).$$

Then verify that the linearized version of the above is precisely (8.2.22) when $\vec{L} = 0$.

The gravitational redshift formula of (9.2.16) shows that an electromagnetic wave emitted by a stationary source at a radius r in the Schwarzschild space-time will get redshifted by the factor

$$z = \sqrt{\frac{r}{r - r_s}} - 1. \tag{9.3.50}$$

Thus the wavelength of light emitted from a source closer to the horizon suffers a greater redshift by the time its gets to the asymptotic observer, until the redshift factor simply "blows up" as the source approaches the horizon. Therefore, the Schwarzschild horizon is also an **infinite redshift** surface and appears black, hence this solution is commonly known as the (eternal) **Black Hole**.[12] The solution is asymptotically flat and, by (7.5.19), its energy is $Q_\xi = Mc^2$.

Geodesics

The geodesics of a particle outside the horizon are again given by (9.2.8) with

$$U^r = \frac{dr}{d\lambda} = \pm\sqrt{\frac{E^2}{c^2} - \left(\frac{L^2}{r^2} - \epsilon c^2\right)\left(1 - \frac{r_s}{r}\right)}. \tag{9.3.51}$$

From here we find the effective potential,

$$V_{\text{eff}} = \epsilon c^2 \frac{r_s}{r} + \frac{L^2}{r^2}\left(1 - \frac{r_s}{r}\right), \tag{9.3.52}$$

the first two terms of which are familiar from the Newtonian case (apart from a factor of a half). The last term, which is absent from the Newtonian effective potential, represents the "spin-orbit" interaction between the spin of the gravitational field and the orbital angular momentum. It is responsible for, among other phenomena, the precession of Mercury's perihelion and the bending of light by massive bodies (apart from this spin-orbit interaction, the gravitational force is exactly zero for massless particles). Circular timelike orbits ($V'_{\text{eff}} = 0$) exist at

$$r_\pm = \frac{L^2}{c^2 r_s}\left[1 \pm \sqrt{1 - \frac{3c^2 r_s^2}{L^2}}\right]. \tag{9.3.53}$$

The orbit of radius r_+ is stable and the one of radius r_- is unstable. As $L \to \infty$, the radius of the stable circular orbit grows,

$$r_+ \to \frac{L^2}{c^2 r_s}, \tag{9.3.54}$$

[12] Problem: Show that it is impossible for an observer to remain stationary on the horizon.

but the radius of the unstable orbit approaches $r_- \to 3r_s/2$, which is also the radius of the only circular lightlike orbit.[13] A minimum angular momentum, $L^2_{\rm min} = 3c^2r_s^2$, is required for a timelike circular orbit. It gives a minimum radius for stable circular orbits of $r_{+,\rm min} = 3r_s$, which is also a *maximum* radius, $r_{-,\rm max}$, for unstable circular orbits. Thus we see that stable timelike circular orbits have $r \geq 3r_s$ and unstable (timelike) orbits lie between $3r_s/2 \leq r < 3r_s$.

The radial equation (9.3.51) indicates that there will be a turning point of the motion ($\dot{r}^2 = 0$), if there exists a positive, real root of

$$\frac{E^2}{c^2}r^3 + (\epsilon c^2 r^2 - L^2)(r - r_s) = 0. \tag{9.3.55}$$

This is a cubic equation unless one considers a massive particle dropped from rest at infinity. Such a particle is said to be **marginally bound** and has $E = c^2$. In this case, a turning point will exist if and only if

$$c^2 r_s r^2 - L^2(r - r_s) = 0, \tag{9.3.56}$$

which has real roots if and only if $|L| \geq 2cr_s = 4GM/c$. Provided that the angular momentum satsifies this criterion, the turning points will lie outside the horizon. Otherwise, we know that the polynomial on the left hand side of (9.3.55) must have at least one negative root because it approaches $-\infty$ as r approaches $-\infty$ and a positive constant as r approaches zero. Therefore, if it is to have a real, positive root then *all three* of its roots must be real (*i.e.,* the cubic discriminant should be non-negative). Turning points for massless particles (photons) occur when

$$\frac{E^2}{c^2}r^3 - L^2(r - r_s) = 0. \tag{9.3.57}$$

In this case, we can set $L = Ea/c$, where a is the impact parameter; then the smallest value of a for which the orbit does not terminate at the black hole singularity is $a_{\rm min} = 3\sqrt{3}r_s/2 \approx 2.60r_s$. This is the radius of the "**black hole shadow**", because only rays of light incident on one side of the hole, whose impact parameter is greater than a, can make it to an observer on the other side of the hole. Thus a black hole will appear to cast a shadow against a bright backdrop. If the impact parameter is smaller than $a_{\rm min}$, the particles are captured by the black hole. If it is larger than but close to $a_{\rm min}$ the particles may rotate around the black hole in the neighborhood of $r = 3r_s/2$ several times before escaping. The deflection of light around massive bodies leads to the phenomenon of **gravitational lensing**, in which one or more distorted images of an object in the

[13] Problem: Show that there is only one possible circular photon orbit at $r = 3r_s/2$. This is called the **photon sphere**.

background are formed, depending on the observer's location. By analyzing the images one can estimate the mass of the gravitational source.

As a function of the azimuthal angle, provided that $L \neq 0$,

$$\int_{r_0}^{r} \frac{dr/r^2}{\sqrt{\mathcal{E} - \frac{\epsilon c^2 r_s}{r} - \frac{L^2}{r^2} + \frac{L^2 r_s}{r^3}}} = \pm\frac{1}{L}(\varphi - \varphi_0) \tag{9.3.58}$$

where $\mathcal{E} = E^2/c^2 + \epsilon c^2$. Its solutions can be given in terms of the Weierstrass elliptic function, $\wp$, which is the general solution of the equation

$$\left(\frac{dy}{dx}\right)^2 = 4y^3 - g_2 y - g_3, \tag{9.3.59}$$

written as $y(x) = \wp(x + x_0 \mid g_2, g_3)$, where x_0 is an an arbitrary constant. Substituting $u = r_s/r$, the orbital equation can be turned into

$$\left(\frac{du}{d\varphi}\right)^2 = \left(\frac{r_s^2\mathcal{E}}{L^2} - \frac{\epsilon c^2 r_s^2}{L^2}u - u^2 + u^3\right), \tag{9.3.60}$$

but the cubic on the right hand side becomes a depressed cubic by the additional transformation[14]

$$w = u - \frac{1}{3}$$

and the orbital equation,

$$\left(\frac{dw}{d\overline{\varphi}}\right)^2 = 4w^3 - g_2 w - g_3, \tag{9.3.61}$$

now has the standard form defining the Weierstrass $\wp$-function in (9.3.59), where $\overline{\varphi} = \varphi/2$ and

$$\begin{aligned} g_2 &= 4\left(\frac{1}{3} + \frac{\epsilon c^2 r_s^2}{L^2}\right) \\ g_3 &= 4\left(\frac{\epsilon c^2 r_s^2}{3L^2} - \frac{r_s^2\mathcal{E}}{L^2} + \frac{2}{27}\right). \end{aligned} \tag{9.3.62}$$

Thus

$$\frac{3r_s}{r} = 1 + 3\wp\left(\frac{1}{2}\varphi + \varphi_0 \mid g_2, g_3\right). \tag{9.3.63}$$

[14] Any cubic $ax^3 + bx^2 + cx + d$ can be brought to "depressed" form by the transformation $y = x + b/3a$, which turns the expression into

$$ay^3 + \left(c - \frac{b^2}{3a}\right)y + \left(d - \frac{bc}{3a} + \frac{2b^3}{27a^2}\right).$$

provides an exact solution of the orbits. These are not exact conic sections, owing to the spin-orbit coupling.

Radial timelike geodesics are the same as they are for the Newtonian case because of the absence of the spin-orbit coupling. Lightlike geodesics,

$$r(\lambda) = \pm \frac{E}{c}\lambda + B, \tag{9.3.64}$$

are future incomplete if they are infalling and past incomplete if they are outgoing because they cannot be extended beyond r_s; there is no curvature singularity of space-time at $r = r_s$, however, so the incompleteness signals a breakdown of the spherical coordinates. One can find regular coordinates that extend the solution beyond the Schwarzschild horizon, where the solution is no longer static but time dependent and spatially homogeneous. Again, we see that the lightlike geodesics do not cross the horizon in a finite coordinate time,

$$\frac{dr}{dt} = \pm c\left(1 - \frac{r_s}{r}\right) \;\Rightarrow\; r_* = r + r_s \ln\left(\frac{r}{r_s} - 1\right) = \pm c(t - t_0) \tag{9.3.65}$$

showing that r only asymptotes to the horizon in coordinate time.

A particle at rest in the gravitational field of a black hole will have a four velocity

$$U^\mu = \frac{c}{\sqrt{A}}(1, 0, 0, 0), \tag{9.3.66}$$

and its associated Killing energy per unit mass will depend on its position, $E(r) = -\xi_{(t)} \cdot U = -U_t = c\sqrt{A}$. It is conserved along geodesics, but a particle at rest is not in free fall. Thus, a particle at rest very far from the hole has $E_\infty = c^2$ and a particle at rest near the horizon has energy $E(r_s) = 0$. Imagine lowering the particle quasi-statically into a black hole from infinity. The quasi-static process is taken to mean that the particle does not follow a geodesic, rather it is instantaneously at rest relative to the hole at all times. Thus, while lowering the partcile to the horizon from infinity quasi-statically, *all its rest mass energy could be extracted as useful work at infinity.* Conversely, raising a particle quasi-statically from the horizon to infinity would require an input of energy equal to the rest mass energy of the particle. Upon lowering the particle to the horizon, one could simply let it fall in; the black hole mass remains unchanged because the particle energy is zero: $\delta M = 0$. Since the area, $\mathcal{A}_s$, of the horizon depends only on the mass, this also implies that $\delta\mathcal{A}_s = 0$ in this adiabatic process.

Analytic Extension

The Schwarzschild coordinates can be extended across the horizon. In the coordinates $(t, r_*, \theta, \varphi)$, the Schwarzschild metric reads

$$ds^2 = \left(1 - \frac{r_s}{r}\right)\left(dt^2 - dr_*^2\right) - r^2(r_*)d\Omega^2. \tag{9.3.67}$$

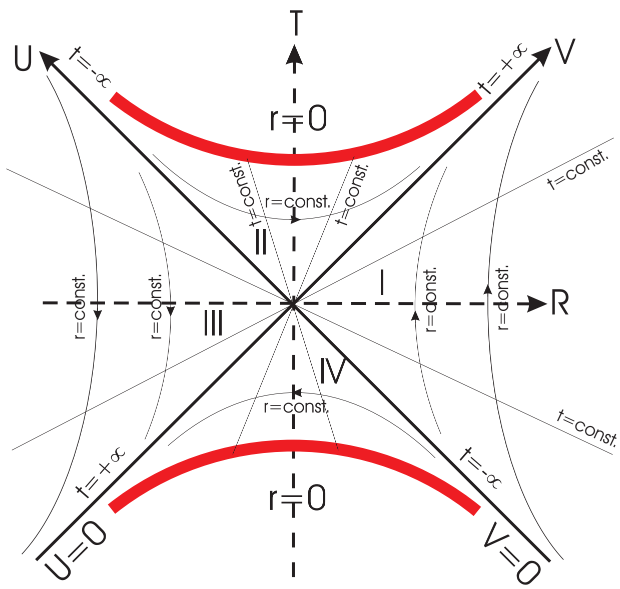

Figure 9.2: The maximally extended Schwarzschild space-time.

The coordinate r_* is called the **tortoise** coordinate. It ranges in from $-\infty$ to ∞ and is only related to r by (9.3.65) when $r > r_s$. Following our construction for de Sitter space, we introduce the lightcone coordinates, $u = t - r_*$ and $v = t + r_*$, and

$$\overline{u} = -r_s e^{-u/2r_s} \in (-\infty, 0), \quad \overline{v} = r_s e^{v/2r_s} \in (0, \infty). \tag{9.3.68}$$

The Schwarzschild metric

$$ds^2 = \frac{4r_s}{r(\overline{u},\overline{v})} e^{-r(\overline{u},\overline{v})/r_s} d\overline{u}d\overline{v} - r^2(\overline{u},\overline{v}) d\Omega^2, \tag{9.3.69}$$

is now free of any explicitly bad behavior at the horizon. Now we simply extend the ranges of the coordinates $\overline{u}$ and $\overline{v}$ to cover the entire real line, subject to the condition $\overline{uv} < r_s^2$. This is the maximal extension of the Schwarzschild space-time: the event horizon is mapped to the two null surfaces $\mathcal{H}^+ : \overline{u} = 0$ and $\mathcal{H}^- : \overline{v} = 0$, and the singularity at $r = 0$ turns into the (spacelike) hypersurface $\overline{uv} = r_s^2$. The intersection of $\mathcal{H}^+$ and $\mathcal{H}^-$ is called the **bifurcation two-sphere**. On the bifurcation two sphere, the Killing vector vanishes, as opposed to being simply null.

Hypersurfaces of constant r are all characterized by $\overline{uv} = \alpha$, where α is a constant. When $r > r_s$, α is negative and the hypersurfaces are timelike, but when $r < r_s$, α is positive and the hypersurfaces are spacelike. (This has already been noted through an argument concerning the sign of $B(r)$.) Constant time (t) hypersurfaces are the straight lines $\overline{v}/\overline{u} = -\delta$, where δ is a constant.

As in the case of de Sitter space, one can make yet another transformation to coordinates, one of which is timelike. These are the **Kruskal-Szerkeres** coordinates for the black hole: letting $\overline{u} = cT - R$ and $\overline{v} = cT + R$, we find

$$ds^2 = \frac{4r_s}{r(T,R)} e^{-r(T,R)/r_s} \left(c^2 dT^2 - dR^2\right) - r^2(T,R) d\Omega^2, \tag{9.3.70}$$

The space-time diagram for the "maximal" Schwarzschild black hole is shown in figure 9.2. It is useful to divide the space-time into four regions, as shown. In region I, $\overline{u} < 0$ and $\overline{v} > 0$, and in region III, $\overline{u} > 0$ and $\overline{v} < 0$. In both these regions, $\overline{uv} < 0$, so these regions represent the space-time outside the horizon, where the Schwarzschild coordinates are adequate. On the other hand, in region II both $\overline{u}$ and $\overline{v}$ are positive whereas they are both negative in region IV. In these two regions, $\overline{uv} > 0$, so they represent the space-time within the horizon. The portion of the event horizon that serves as the boundary between regions I and II is called the future event horizon ($\overline{u} = 0$) and the portion that serves as the boundary between regions IV and I is called the past event horizon ($\overline{v} = 0$).

- Region I is called the "normal" region. It is asymptotically flat and observers may move in either direction in r, but always move forward in time. Radial, timelike geodesic observers in this region cannot evade the future horizon and will cross over into region II. Non-geodesic observers may, however, hold themselves at constant r. Once the future horizon is crossed, all communication with observers in region I is cut off. The future horizon represents the last null ray that is able to escape to infinity.

- Region II is called the "black hole" region. Particles may move in either direction in t, but always move toward decreasing r. This region is cut of by the space-time singularity at $r = 0$, which lies in the future of all trajectories, geodesic or otherwise, *i.e.,* it is not possible to maintain a fixed radial coordinate in this region: crashing into the singularity in the future is inevitable.

- Region III is a parallel or mirror universe, identical to region I in all respects, except that observers must always move *backward* in time.

- Region IV is called the "white hole" region. Just as region III is the time reverse of region I, so is region IV the time reverse of region II. Observers are required to always move toward larger values of r, but are free to move in either direction in t. In doing so, they may choose to enter regions I or III.

Only regions I and II are actually realized in stellar collapse. Regions III and IV are artifacts of the anaytical continuation.

9.4 The Electrovacuum

The electrovacuum solution is a static solution of the Einstein-Maxwell system describing the gravitational field of a non-rotating mass M of charge Q. Thus the field equations are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R-\Lambda) &= \frac{8\pi gG}{c^3}\left[F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right] \\ \nabla_\mu F^\mu{}_\nu &= 0, \end{aligned} \tag{9.4.1}$$

where g is the electromagnetic coupling and $F_{\alpha\beta} = \nabla_{[\alpha}A_{\beta]}$ is the Maxwell field strength tensor and we have used the expression

$$t_{\mu\nu} = gc\left[F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right] \tag{9.4.2}$$

for the electromagnetic stress tensor. For spherical symmetry to hold, the only non-vanishing components of the Maxwell tensor can be $F_{tr} = f(r)$ and $F_{\theta\varphi} = g(r)\sin\theta$. The Bianchi identities, $\partial_\alpha {}^*F^\alpha{}_\beta = 0$, require that $g(r) = Q_m$ (a constant) and the other two Maxwell's equations imply that

$$f' + \left[\frac{2}{r} - \frac{B'}{2B} - \frac{A'}{2A}\right]f = 0 \;\Rightarrow\; f(r) = Q_e\frac{\sqrt{AB}}{r^2}, \tag{9.4.3}$$

where Q_e ($[Q_e] = ml$) is also an arbitrary constant. The two constants of the integration, Q_e and Q_m, represent, respectively, electric and magnetic charges. (We include the magnetic charge, although no magnetic monopoles have been detected, to show that the electric and magnetic charges play the same role as far as the space-time is concerned. Of course, it can can be set to zero.) Thus we find the general static, spherically symmetric electromagnetic potential,

$$A_\mu = \left(-\int^r dr' f(r'), 0, 0, -Q_m\cos\theta\right) \tag{9.4.4}$$

and the stress energy tensor

$$t^\mu{}_\nu = \frac{1}{2}gc\begin{pmatrix} -\frac{Q^2}{r^4} & 0 & 0 & 0 \\ 0 & -\frac{Q^2}{r^4} & 0 & 0 \\ 0 & 0 & \frac{Q^2}{r^4} & 0 \\ 0 & 0 & 0 & \frac{Q^2}{r^4} \end{pmatrix}. \tag{9.4.5}$$

where $Q^2 = Q_e^2 + Q_m^2$. Now according to the Einstein equations,

$$E_{tt} = \frac{8\pi gG}{c}\frac{Q^2 A}{2r^4} \;\Rightarrow\; -B(2 + B[\Lambda r^2 - 2]) + 2rB' = \frac{8\pi gG}{c}\frac{Q^2B^2}{r^2}, \tag{9.4.6}$$

which is easily solved to give

$$B(r) = \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} - \frac{\Lambda r^2}{6}\right)^{-1}, \tag{9.4.7}$$

where we set the arbitrary integration constant to $r_s = 2GM/c^2$, in view of the fact that the diffeomorphism charge associated with time translations is the mass energy, and $r_Q^2 = 4\pi g G Q^2/c$. Again, it turns out that

$$\frac{E_{tt}}{A} + \frac{E_{rr}}{B} = 0 = \frac{(AB)'}{rAB^2} \Rightarrow A(r) = c^2\beta B^{-1}(r) \tag{9.4.8}$$

as we had for the vacuum solutions. The constant β is absorbed into a redefinition of t and the last independent equation is automatically satisfied. In this way we have found the **Reissner-Nordström-(A)dS** solution,

$$ds^2 = c^2\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} - \frac{\Lambda r^2}{6}\right)dt^2 - \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} - \frac{\Lambda r^2}{6}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{9.4.9}$$

along with the electrostatic potential,

$$\Phi(r) = \frac{c^2 Q_e}{r}. \tag{9.4.10}$$

The solution has the same overall form as our previous spherically symmetric vacua, with an additional contribution from the electric and magnetic fields, and is valid so long as $g_{tt} > 0$,.

Consider the Reissner-Nordström solution (with $\Lambda = 0$). Its curvature invariants are singular only as $r \to 0$, if they are singular at all (eg., the scalar curvature vanishes everywhere). An example of an curvature invariant that is singular is the Kretschmann scalar,

$$K = R^{\alpha}{}_{\beta\mu\nu} R_{\alpha}{}^{\beta\mu\nu} = \frac{12r_s^2}{r^6} - \frac{48r_s^2 r_Q}{r^7} + \frac{56r_Q^4}{r^8}. \tag{9.4.11}$$

If $r_s > 2r_Q$, there are two Killing horizons at the real roots of g_{tt},

$$r = r_{\pm} = \frac{r_s \pm \sqrt{r_s^2 - 4r_Q^2}}{2}, \tag{9.4.12}$$

called the "outer horizon" and the "inner horizon" respectively. If $r_s = 2r_Q$ (this is known as the **extremal** Reissner-Nordström solution) there is one Killing horizon, and there is no horizon if $r_s < 2r_Q$. Surfaces of constant r are timelike outside r_+, spacelike between

r_- and r_+ and timelike within r_-. The singularity at $r = 0$ is therefore timelike. In the second case, the singularity is hidden behind just one horizon located at half the Schwarzschild radius. This horizon is a root (of g_{tt}) of multiplicity two, therefore the sign of g_{rr} does not change as it is crossed and surfaces of constant r never become spacelike, so that the singularity at $r = 0$ is timelike in this case as well. It is also timelike in the third case, because no horizon is present. Thus the Reissner-Nordström singularity is *always timelike.* As it is asymptotically flat, the energy of the Reisner-Nordstrom solution is given by (7.5.19) as $Q_\xi = Mc^2$.

As we did with the Schwarzschild black hole, we will now imagine lowering a charge (of unit mass) quasi-statically down from infinity to the outer horizon of the Reissner-Nordstrom black hole. Because the four momentum of the charged particle is $p_\mu = U_\mu + \delta q A_\mu$, its Killing energy will depend on its position through the four velocity *and* the electrostatic potential, $E(r) = -\xi_{(t)} \cdot p = c\sqrt{A} + \delta q \Phi$. At infinity, where the electrostatic potential is zero, $E = c^2$, but very near the outer horizon, $E = \delta q \Phi(r_+)$. This is negative if the black hole and the particle are oppositely charged, implying that we have could have extracted an amount of energy greater than the mass energy of the particle to do useful work, $W = c^2 - \delta q \Phi(r_+)$. Dropping the mass into the black hole will have changed the black hole mass by $\delta M c^2 = -|\delta q \Phi(r_+)|$ so the additional energy that was extracted has come at the cost of the mass of the black hole. Again, by dropping in an opposite charge, the black hole's charge has been reduced by $\delta Q_e = -(4\pi g c)^{-1} |\delta q|$ and we have the relation

$$\delta M c^2 - 4\pi g c \delta Q_e |\Phi(r_+)| = 0, \tag{9.4.13}$$

which implies that the area of the horizon, $\mathcal{A}(r_+) = 4\pi r_+^2$, does not change in this adiabatic process, *i.e.,* $\delta \mathcal{A}(r_+) = 0$.[15]

9.5 Static Interiors

Whether or not a given amount of matter will collapse into a black hole depends on its mass and its equation of state. The Schwarzschild radius of the earth, for example, is about 1 cm, so, to form a black hole, it would have to be compressed to a density far exceeding the nuclear density of roughly 3×10^{17} kg/m^3. All the mass of a ten solar mass star, however, would have to be compressed to under nuclear densities for it to cross the star's Schwarzschild radius of about 30 km. Let us now consider the metric inside

[15] Problem: Show that W is the maximum amount of energy that it is possible to extract from a marginally bound, charged particle. (For example, if the particle is not dropped quasi-statically but is allowed to fall freely, its Killing energy would be conserved and the amount of energy that could be extracted is exactly zero.) Verify that the area of the horizon does not change in the adiabatic process. Suppose that the energy extracted to do useful work were *less than* the maximum possible energy. How do the mass and charge of the black hole change? Is the change in horizon area still zero, less than zero or greater than zero?

a static ideal fluid. We will take its energy momentum tensor to be given by (7.4.36), with $U^\mu = (c/\sqrt{A(r)}, 0, 0, 0)$ to ensure a static condition. Then Einstein's equations are sourced by the stress tensor

$$T^\mu{}_\nu = \begin{pmatrix} -\rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{9.5.1}$$

and must be supplemented by some equation of state, $p = p(\rho)$. Spherical symmetry requires the energy density and pressure to be functions only of r.

The Einstein equations for this system,

$$\begin{aligned} E_{tt} &= \frac{2}{B} + 2r\left(\frac{1}{B}\right)' - (2 - \Lambda r^2) = -\frac{16\pi G}{c^2} r^2 \rho(r), \\ \frac{E_{tt}}{A} + \frac{E_{rr}}{B} &= \frac{(AB)'}{rAB^2} = \frac{8\pi G}{c^4}(\rho c^2 + p), \\ E_{\theta\theta} &= \frac{rA''}{2AB} - \frac{rA'^2}{4A^2B} + \frac{A'}{2AB}\left(1 - \frac{B'}{2B}\right) - \frac{B'}{2B^2} + \frac{\Lambda r}{2} = \frac{8\pi G}{c^4} p, \end{aligned} \tag{9.5.2}$$

are readily solved by quadratures. Formally integrating the time component of the gravitational field equations, we find

$$B(r) = \left(1 + \frac{\alpha}{r} - \frac{8\pi G}{c^2 r}\int^r dr' r'^2 \rho(r') - \frac{\Lambda r^2}{6}\right)^{-1}, \tag{9.5.3}$$

where the integration constant, α, represents a mass point at the center. To avoid a singularity there, we take $\alpha = 0$ and write

$$B(r) = \left(1 - \frac{2GM(r)}{c^2 r} - \frac{\Lambda r^2}{6}\right)^{-1}, \tag{9.5.4}$$

where we have interpreted

$$4\pi \int^r dr' r'^2 \rho(r') = M(r) \tag{9.5.5}$$

as the mass contained within a radius r. $M(r)$ is known as the **Misner-Sharp** mass. The **proper mass** within the body is given, over a constant time hypersurface and within a volume bounded by r, by

$$4\pi \int^r dr' r'^2 \sqrt{B(r')}\, \rho(r') = M_p(r). \tag{9.5.6}$$

It is always greater than the Misner-Sharp mass and the difference, $M_p(r) - M(r)$, called the **mass defect**. The mass defect represents the gravitational binding energy, which must be subtracted from the proper mass to get the gravitational mass.

Next we appeal to the special combination of the time and radial components of the Einstein equations, which can be written as

$$\frac{A'}{A} = \frac{8\pi GrB}{c^4}\left[p + \rho c^2\right] - \frac{B'}{B} \tag{9.5.7}$$

and formally integrated,

$$A(r) = \beta c^2 \left(1 - \frac{2GM(r)}{c^2 r} - \frac{\Lambda r^2}{6}\right) \exp\left[\frac{8\pi G}{c^4}\int^r dr' r' \left(\frac{p(r') + \frac{M'(r')c^2}{4\pi r'^2}}{1 - \frac{2GM(r')}{c^2 r'}}\right)\right], \tag{9.5.8}$$

where β is an arbitrary integration constant. If we now plug these results for $A(r)$ and $B(r)$ into the angular components of the Einstein tensor we get one additional constraint that must be satisfied by $p(r)$ and $\rho(r)$. This is the equation of hydrostatic equilibrium, which is entirely equivalent to and can be recovered more directly from the conservation of stress energy ($\nabla_\mu T^\mu{}_\nu = 0$),

$$p' + \frac{A'}{2A}(p + \rho c^2) = 0, \tag{9.5.9}$$

or, using (9.5.8),

$$p'(r) + \frac{GB(r)}{c^4 r^2}\left[M(r)c^2 + 4\pi r^3 p(r) - \frac{\Lambda c^4 r^3}{6G}\right]\left(p(r) + \rho(r)c^2\right) = 0. \tag{9.5.10}$$

Equation (9.5.10) is the **Tolman-Oppenheimer-Volkoff** (TOV) equation.

Suppose that the mass sphere has a sharp boundary at r_b. Then $\rho(r) = 0 = p(r)$ when $r \geq r_b$ and the mass function is constant outside the sphere. By Birkhoff's theorem, we know that the unique vacuum space-time is Schwarzschild, therefore the the two solutions must be compatible at r_b. In fact, whenever two regions of space-time, described by two different metrics, meet at a sharp boundary, it is necessary to match the two regions at that boundary in such a way that the entire space-time is at least $C^{(1)}$. This implies the continuity of the first and second fundamental forms across the boundary, called the Darmois-Israel junction conditions. In the case of the static interior and the Schwarzschild exterior, we use the same coordinates – the Schwarzschild coordinates – on both sides and match at $r = r_b$, so the induced metric (first fundamental form) can be written as

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu = \begin{pmatrix} -A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \tag{9.5.11}$$

where $n^\mu = (0, 1/\sqrt{B}, 0, 0)$ is the unit normal to hypersurfaces of constant r, and the extrinsic curvature (second fundamental form) is

$$K_{\mu\nu} = h_\mu{}^\alpha \nabla_\alpha n_\nu = \begin{pmatrix} -\frac{A'}{2\sqrt{B}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{r}{\sqrt{B}} & 0 \\ 0 & 0 & 0 & \frac{r\sin^2\theta}{\sqrt{B}} \end{pmatrix}, \tag{9.5.12}$$

To distinguish between the interior and exterior metrics, let "$-$" represent the interior and "$+$" the exterior. Earlier we had seen that both $A(r)$ and $B(r)$ are continuous across the boundary if the pressure vanishes there, therefore the first fundamental form is continuous. The second fundamental form is continuous if

$$\lim_{r\to r_b-} \frac{A^{-\prime}}{\sqrt{B^-}} = \lim_{r\to r_b+} \frac{A^{+\prime}}{\sqrt{B^+}}, \tag{9.5.13}$$

which is a condition on the arbitrary constant of integration we encountered in (9.5.8), and implies that[16]

$$A(r) = c^2\left(1 - \frac{2GM(r)}{c^2 r} - \frac{\Lambda r^2}{6}\right)\exp\left[\frac{8\pi G}{c^4}\int_{r_b}^{r} dr' r' \left(\frac{p(r') + \frac{M'(r')c^2}{4\pi r'^2}}{1 - \frac{2GM(r')}{c^2 r'}}\right)\right]. \tag{9.5.15}$$

This guarantees that any perfect fluid interior can be matched to a Schwarzschild exterior.

For particular models, we must either specify $\rho(r)$ or $p(r)$ or an equation of state, $p = p(\rho)$. If $\rho(r)$ is specified, $B(r)$ is obtained by integrating the density to find $M(r)$ while the TOV equation (9.5.9) can be rewritten as a Riccati equation for $z(r) = p(r) + \rho(r)c^2$, using (9.5.7) to eliminate all dependence on $A(r)$,

$$z' + \frac{4\pi G r B}{c^4} z^2 - \frac{B'}{2B} z - c^2\rho' = 0 \tag{9.5.16}$$

and gives, in principle, the pressure as a function of r. If this step is successful, knowing the pressure and the energy density allows us to use (9.5.9) to recover $A(r)$. On the other hand, if an equation of state $p(\rho)$ is specified, the TOV equation in the form (9.5.9) may be integrated to recover both $p(r)$ and $\rho(r)$ in terms of $A(r)$, then the time component in (9.5.2) and (9.5.7) solved as a coupled system to obtain $B(r)$ and $A(r)$. Notice that

[16] Problem: Show that the second fundamental form is continuous across the boundary if $p(r_b) = 0$ and

$$\beta = \exp\left[-\frac{8\pi G}{c^4}\int^{r_b} dr' r' \left(\frac{p(r') + \frac{M'(r')c^2}{4\pi r'^2}}{1 - \frac{2GM(r')}{c^2 r'}}\right)\right] \tag{9.5.14}$$

the TOV equation is inconsistent for pressureless dust, implying that a stable dust ball is impossible. This is expected because, bereft of pressure, dust is helpless against the inexhorable pull of gravity.

In all but the simplest cases, only a numerical solution of the TOV equation will be possible. However, consider the following example, with $\Lambda = 0$, for which the TOV is readily integrated. Taking the mass density to be constant within the ball,

$$\rho = \begin{cases} \rho_0 & r \leq r_b \\ 0 & r > r_b. \end{cases} \tag{9.5.17}$$

According to (9.5.4),

$$B(r) = \left(1 - \frac{8\pi G\rho_0 r^2}{3c^2}\right)^{-1}. \tag{9.5.18}$$

and the TOV equation (9.5.16) is a Bernoulli equation whose solution is,

$$z(r) = \rho c^2 \left[\frac{2z_0}{3\left(z_0 - \sqrt{\frac{3c^2}{B(r)}}\right)}\right], \tag{9.5.19}$$

where z_0 is a constant, which can be fixed by demanding that the pressure vanishes at the boundary, $p(r_b) = 0$. After some algebra, one finds

$$p(r) = \rho_0 c^2 \left[\frac{\sqrt{1 - 8\pi G\rho_0 r^2/3c^2} - \sqrt{1 - 8\pi G\rho_0 r_b^2/3c^2}}{3\sqrt{1 - 8\pi G\rho_0 r_b^2/3c^2} - \sqrt{1 - 8\pi G\rho_0 r^2/3c^2}}\right] \tag{9.5.20}$$

and

$$A(r) = \frac{9}{4}A(r_b)\left[1 - \frac{1}{3}\sqrt{\frac{1 - \frac{8\pi G\rho_0 r^2}{3c^2}}{1 - \frac{8\pi G\rho_0 r_b^2}{3c^2}}}\right]^2. \tag{9.5.21}$$

The central pressure, required to sustain the ball,

$$p_c = \rho_0 c^2 \left[\frac{1 - \sqrt{1 - 2GM/c^2 r_b}}{3\sqrt{1 - 2GM/c^2 r_b} - 1}\right]. \tag{9.5.22}$$

will be positive if

$$3\sqrt{1 - 2GM/c^2 r_b} > 1 \;\Rightarrow\; r_b > \frac{9}{4}\frac{GM}{c^2}. \tag{9.5.23}$$

Therefore, a star of uniform density, sustained by a positive central pressure, cannot be smaller that 1.125 times its Schwarzschild radius. This places an upper limit on the star's density, which behaves as $\rho_{\max} \sim M^{-2}$.

The TOV equation is also easily integrated for a polytropic fluid, with equation of state

$$p = \begin{cases} \alpha(\rho c^2)^{1+1/n} & r \leq r_b \\ 0 & r > r_b \end{cases} \tag{9.5.24}$$

where α and $n \neq -1$ are constants (n is called the polytropic index). Polytropic fluids are useful in describing a variety of astrophysical objects, ranging from rocky planets, to main sequence stars, white dwarfs and neutron stars. In this case, equation (9.5.9) is a Bernoulli equation

$$p' + \frac{A'}{2A}p = -\frac{A'}{2A}\left(\frac{p}{\alpha}\right)^{\frac{n}{n+1}}, \tag{9.5.25}$$

and may be integrated to obtain

$$\begin{aligned} p(r) &= \alpha^{-n}\left[\left(\frac{A(r_b)}{A(r)}\right)^{\frac{1}{2(n+1)}} - 1\right]^{n+1}, \\ \rho(r)c^2 &= \alpha^{-n}\left[\left(\frac{A(r_b)}{A(r)}\right)^{\frac{1}{2(n+1)}} - 1\right]^{n}. \end{aligned} \tag{9.5.26}$$

Now that the pressure and the energy density are known as functions of $A(r)$, one may attempt to solve the time component of (9.5.2) and (9.5.7),

$$\begin{aligned} \left(\frac{r}{B}\right)' - 1 &= -\frac{8\pi G}{c^2}r^2\rho(A(r)) \\ \frac{A'}{A} &= \frac{8\pi GrB}{c^4}z(A(r)) - \frac{B'}{B} \end{aligned} \tag{9.5.27}$$

as a coupled system of first order differential equations for $A(r)$ and $B(r)$. This is best done numerically.

9.6 Axial Symmetry

Intuitively, axial symmetry (or axisymmetry) refers to the symmetry of rotations about a fixed axis. More precisely, it is an isometry generated by a spacelike Killing vector, ξ, with compact orbits. The "axis of symmetry" is the set of all fixed points of the isometry.

Consider some general coordinates, (t, X^i) on the (axisymmetric) manifold and let $X^3 = \varphi$ be the coodinate adapted to the Killing vector that generates the isometry, so that $\xi^\mu_{(\varphi)} = (0, 0, 0, 1)$ in this chart. This means that all the metric functions are independent of φ, as can be shown by applying the Killing equation,

$$\nabla_{\{\mu}\xi_{\nu\}} = 0 = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - 2\Gamma^\lambda_{\mu\nu}\xi_\lambda \tag{9.6.1}$$

and expanding the right hand side, noting that $\xi_\mu = g_{\mu 3}$,

$$0 = g_{\mu 3,\nu} + g_{\nu 3,\mu} - g_{\lambda 3} g^{\lambda\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}) = g_{\mu\nu,3}. \tag{9.6.2}$$

According to the ADM decomposition, the line element can always be written in the form

$$ds^2 = N^2 dt^2 - \gamma_{ij}(dX^i + N^i dt)(dX^j + N^j dt) \tag{9.6.3}$$

where N and N^i are the "lapse" and "shift" functions, and γ_{ij} is the first fundamental form, *i.e.,* the metric on the spatial submanifold, Σ_t. All the metric functions will be depend on (t, X^1, X^2).

Let us focus on the three dimensional hypersurfaces, foliating them by surfaces of constant φ,

$$ds^2_{(2)} = \gamma_{11} dX^{1^2} + 2\gamma_{12} dX^1 dX^2 + \gamma_{22} dX^{2^2}, \tag{9.6.4}$$

and exploiting a most useful theorem, which says that *every two dimensional metric is conformally flat*. What is meant by this statement is that new coordinates, (r, z) can always be found so that (9.6.4) takes the form

$$ds^2_{(2)} = B(r,z)(dr^2 + dz^2), \tag{9.6.5}$$

where $B(r,z)$ is called the **conformal factor**. The proof of this statement goes as follows: from the transformation properties of the metric,

$$g'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} g^{\alpha\beta}, \tag{9.6.6}$$

we find the following relations:

$$\begin{aligned} B^{-1} &= \gamma^{ij}\partial_i r \partial_j r \\ B^{-1} &= \gamma^{ij}\partial_i z \partial_j z \\ 0 &= \gamma^{ij}(\partial_i r)(\partial_j z). \end{aligned} \tag{9.6.7}$$

Direct substitution shows that the last of these equations is satisfied by

$$\partial_i z = \epsilon_{ik}\gamma^{km}\partial_m r \tag{9.6.8}$$

(because of the antisymmetry of the two dimensional Levi-Civita tensor, ϵ_{ij}). The integrability condition, $\partial_1\partial_2 z = \partial_2\partial_1 z$ implies that

$$-\partial_1(\sqrt{\gamma}\gamma^{1m}\partial_m r) = \partial_2(\sqrt{\gamma}\gamma^{2m}\partial_m r) \tag{9.6.9}$$

or simply $\Box_{(2)} r(X^1, X^2) = 0$. With this solution the second equation becomes equivalent to the first, which in turn defines the conformal factor, $B(r, z)$. Therefore *any* solution of Laplace's equation on the two dimensional surface provides a coordinate transformation to the conformally flat metric in (9.6.5).[17]

The two dimensional surfaces are generally not maximally symmetric, so the proper distance on Σ_t will involve three additional functions and have the form

$$ds^2_{(3)} = B(r, z)(dr^2 + dz^2) + 2D_1(r, z)drd\varphi + 2D_2(r, z)dzd\varphi + C(r, z)d\varphi^2, \tag{9.6.10}$$

i.e.,

$$\gamma_{ij} = \begin{pmatrix} B(r,z) & 0 & D_1(r,z) \\ 0 & B(r,z) & D_2(r,z) \\ D_1(r,z) & D_2(r,z) & C(r,z) \end{pmatrix} \tag{9.6.11}$$

Recall that a space-time is stationary if it admits a time-like Killing vector. If t is adapted to this Killing vector, the metric functions will depend only on r and z. Moreover, if we assume that the proper distance is invariant under the simultaneous reflections $t \to -t$ and $\varphi \to -\varphi$, then four of the metric functions, *viz.*, $N^r(r, z)$, $N^z(r, z)$, $D_1(r, z)$ and $D_2(r, z)$ will vanish. We conclude that the most general stationary, axially symmetric metric in four dimensions obeying the reflection symmetry will be given in terms of four functions,

$$ds^2 = N^2(r, z)dt^2 - B(r, z)(dr^2 + dz^2) - C(r, z)\left(d\varphi - J(r, z)dt\right)^2. \tag{9.6.12}$$

Using (7.7.19) one can see that, so long as $J(r, z) \neq 0$, neither $\xi^\mu_{(t)} = (1, 0, 0, 0)$ nor $\xi^\mu_{(\varphi)} = (0, 0, 0, 1)$ are hypersurface orthogonal, although

$$\xi^\mu = \xi^\mu_{(t)} + J(r, z)\xi^\mu_{(\varphi)} \tag{9.6.13}$$

is hypersurface orthogonal but not a Killing vector field unless $J(r, z)$ is constant.

In order to be static as well the time-like Killing vector must be hypersurface orthogonal, which implies that $J(r, z) = 0$. Then the most general *static*, axially symmetric metric is given in terms of three functions,

$$ds^2 = N^2(r, z)dt^2 - B(r, z)(dr^2 + dz^2) - C(r, z)d\varphi^2. \tag{9.6.14}$$

If we are interested in vacuum solutions, it is convenient to redefine the three functions according to

$$ds^2 = c^2 e^{2\mu(r,z)}dt^2 - e^{-2\mu(r,z)}\left[e^{2\sigma(r,z)}(dr^2 + dz^2) + \alpha^2(r, z)d\varphi^2\right], \tag{9.6.15}$$

[17] Problem: Find a transformation that turns (9.1.1) into a conformally flat metric.

as this simplifies Einstein's vacuum equations. Indeed, a remarkable consequence is that

$$\alpha\left(R^t{}_t + R^\varphi{}_\varphi\right) = -e^{-2(\sigma-\mu)}\left(\partial_r^2 + \partial_z^2\right)\alpha = 0, \tag{9.6.16}$$

showing that α is harmonic in the flat, (r,z) plane. We could now take $\alpha(r,z) = r$ to get the canonical, **Weyl** form of the static, axisymmetric metric:

$$ds^2 = e^{2\mu(r,z)}dt^2 - e^{-2\mu(r,z)}\left[e^{2\sigma(r,z)}(dr^2+dz^2) + r^2 d\varphi^2\right], \tag{9.6.17}$$

which depends on just two unknown functions, $\mu(r,z)$ and $\sigma(r,z)$.

The vacuum equations, $R_{\mu\nu} = 0$, will now read

$$\begin{aligned}
R^t{}_t &= -e^{-2(\sigma-\mu)}\left(\mu_{,rr} + \mu_{,zz} + \frac{\mu_{,r}}{r}\right) = 0\\
R^r{}_r &= e^{-2(\sigma-\mu)}\left(\mu_{,rr} + \mu_{,zz} + \frac{\mu_{,r}}{r} - \sigma_{,rr} - \sigma_{,zz} + \frac{\sigma_{,r}}{r} - 2\mu_{,r}{}^2\right)\\
R^z{}_z &= e^{-2(\sigma-\mu)}\left(\mu_{,rr} + \mu_{,zz} + \frac{\mu_{,r}}{r} - \sigma_{,rr} - \sigma_{,zz} - \frac{\sigma_{,r}}{r} - 2\mu_{,z}{}^2\right)\\
R^r{}_z &= e^{-2(\sigma-\mu)}\left(\frac{\sigma_{,z}}{r} - 2\mu_{,r}\mu_{,z}\right)\\
R^\varphi{}_\varphi &= -R^t{}_t.
\end{aligned} \tag{9.6.18}$$

It follows from the first equation that

$$\nabla^2\mu(r,z) = 0, \tag{9.6.19}$$

where the Laplacian is on a *flat, three dimensional manifold* in cylindrical coordinates. The remaining three equations are not all independent and reduce to the pair

$$\frac{\sigma_{,r}}{r} = \mu_{,r}{}^2 - \mu_{,z}{}^2, \quad \frac{\sigma_{,z}}{r} = 2\mu_{,r}\mu_{,z}, \tag{9.6.20}$$

whose integrability condition is just (9.6.19). Thus, once a suitable solution for $\mu(r,z)$ has been found, the solution for $\sigma(r,z)$ follows by quadratures.

9.7 Weyl Vacua

The existence of two Killing vectors, guarantees that geodesics in the static, axisymmetric space-time will be possess two conserved quantities, *viz.,*

$$\xi_{(t)}\cdot U = -e^{2\mu(r,z)}\frac{dt}{d\lambda} = -E \tag{9.7.1}$$

and

$$\xi_{(\varphi)}\cdot U = e^{-2\mu(r,z)}r^2\frac{d\varphi}{d\lambda} = L. \tag{9.7.2}$$

The remaining geodesic equations read

$$\frac{dU^r}{d\lambda} + 2(\sigma_{,z} - \mu_{,z})U^r U^z + (U^{r2} - U^{z2})(\sigma_{,r} - \mu_{,r}) + P = 0$$
$$\frac{dU^z}{d\lambda} + 2(\sigma_{,r} - \mu_{,r})U^r U^z + (U^{r2} - U^{z2})(\mu_{,z} - \sigma_{,z}) + \mu_{,z} Q = 0 \tag{9.7.3}$$

where

$$P = e^{-2\sigma}\left[E^2\mu_{,r} - \frac{L^2}{r^2}e^{4\mu}\left(\frac{1}{r} - \mu_{,r}\right)\right]$$
$$Q = e^{-2\sigma}\left[E^2 + \frac{L^2}{r^2}e^{4\mu}\right]. \tag{9.7.4}$$

If we take $\lambda = \tau$ (the proper time), then the geodesic constraint reads

$$-E^2 e^{-2\mu} + \frac{L^2}{r^2}e^{2\mu} + e^{2(\sigma-\mu)}(U^{r2} + U^{z2}) = \epsilon c^2 \tag{9.7.5}$$

where $\epsilon = \mp 1$ or zero, depending on whether we are examining timelike, spacelike or null geodesics, respectively. These equations are evidently difficult to solve except in very special cases. We now examine some Weyl space-times.

9.7.1 The Chazy-Curzon Metrics

Consider a single mass point, M, located at $r = 0 = z$, with density

$$\rho(x') = \frac{M}{r'}\delta(r')\delta(z')\delta(\varphi').$$

acting as a source for (9.6.19). The solution of Laplace's equation is clearly

$$\mu(r, z) = -\frac{GM}{c^2\sqrt{r^2 + z^2}} \tag{9.7.6}$$

and we easily find from (9.6.20) that

$$\sigma(r, z) = -\frac{G^2M^2r^2}{2c^4(r^2 + z^2)^2} \tag{9.7.7}$$

The solution is regular on the axis ($r = 0$) but not at the origin and is not equivalent to the Schwarzschild metric, which is the unique *spherically symmetric* vacuum solution of General Relativity.

Consider also two masses, $M_\pm$, located on the $z-$axis at $z = \pm a$. The linearity of the equation for $\mu(r,z)$ allows for superposition, so the solution for $\mu(r,z)$ is now

$$\mu(r,z) = -\frac{GM_+}{c^2 R_+} - \frac{GM_-}{c^2 R_-} \tag{9.7.8}$$

where $R_\pm = \sqrt{r^2 + (z \mp a)^2}$, and (9.6.20) shows that

$$\sigma(r,z) = \frac{G^2}{2c^4}\left[-\frac{M_+^2 r^2}{R_+^2} - \frac{M_-^2 r^2}{R_-^2} + \frac{M_+ M_-(r^2+z^2-a^2)}{a^2 R_+ R_-}\right] \tag{9.7.9}$$

Curzon has also given solutions for N point masses located symmetrically on the z axis.[18]

Notice that for the single particle solution $\mu(r,z) \to -GM/c^2|z|$ and $\sigma(r,z) \to 0$ along the z axis. On the other hand, for the two particle solution in the same limit,

$$\begin{aligned} \mu(r,z) &\to -\frac{G}{c^2}\left[\frac{M_+}{|z-a|} + \frac{M_-}{|z+a|}\right] \\ \sigma(r,z) &\to \frac{G^2 M_+ M_-}{2c^4 a^2}\,\mathrm{sgn}(z^2 - a^2) = \sigma_0 \end{aligned} \tag{9.7.10}$$

Thus, σ approaches a negative constant between the mass points and a positive constant everywhere else outside $z = \pm a$. Considering the conformally scaled spatial metric very near the axis, we see that

$$d\tilde{s}^2_{(3)} = dr^2 + dz^2 + r^2 e^{-2\sigma_0} d\phi^2, \tag{9.7.11}$$

which reveals that the geometry admits a **deficit angle**. A deficit angle is the difference between 2π and the angle subtended by a closed loop about a point. For example, starting with a flat sheet of paper, cut off a wedge and identify the edges of the wedge. The result is a cone. The angle subtended by the wedge is the "deficit angle", δ, and the apex of the cone is a topological defect, called a conical singularity. Note, however, that the cone is locally identical to the sheet, *i.e.,* the curvature invariants of a cone vanish everywhere except at the conical singularity, where they are undefined. In the situation at hand,

$$\delta = 2\pi\left(1 - e^{-\sigma_0}\right) \tag{9.7.12}$$

occurs at all points on the z axis. This line defect is referred to as a cosmic string (or "strut") laid along the z axis. The strut holds the two masses apart. A deficit angle will occur whenever $\sigma_0 \neq 0$ and $\mu(r,z)$ is regular on the axis.

[18] See the original papers: J. Chazy, Bull. Soc. Math. France, **52** (1924) 17; H. Curzon, Proc. Math. Soc. London, **25** (1924) 477.

9.7.2 The Zipoy-Voorhees Metrics

The Zipoy-Voorhees metrics are defined by a uniform rod of finite length located along the z axis, from $z = -a$ to $z = +a$, which acts as a source for (9.6.19). The density function for this source is

$$\rho(x') = \frac{M}{2ar'}\delta(r')\Theta(a+z')\Theta(a-z')\delta(\varphi') \tag{9.7.13}$$

where Θ is the Heaviside function. Applying the spatial Green function,

$$G(\vec{r},\vec{r}') = -\frac{1}{4\pi|\vec{r}-\vec{r}'|},$$

we find the appropriate solution to Laplace's equation,

$$\begin{aligned}\mu(r,z) &= -\frac{GM}{2c^2a}\int_{-a}^{a}\frac{dz'}{\sqrt{r^2+(z-z')^2}} \\ &= \frac{GM}{4c^2a}\left[\ln\frac{R_+-(z+a)}{R_--(z-a)}+\ln\frac{R_-+(z-a)}{R_++(a+z)}\right],\end{aligned} \tag{9.7.14}$$

where $R_\pm = \sqrt{r^2+(z\pm a)^2}$. Now direct computation reveals that

$$\frac{R_+-(z+a)}{R_--(z-a)} = \frac{R_-+(z-a)}{R_++(a+z)} = \frac{R_++R_--2a}{R_++R_-+2a}, \tag{9.7.15}$$

therefore

$$\mu(r,z) = \frac{GM}{2c^2a}\ln\frac{R_++R_--2a}{R_++R_-+2a} \tag{9.7.16}$$

To find σ, we turn to (9.6.20) and find

$$\sigma(z,r) = \frac{1}{2}\left(\frac{GM}{c^2a}\right)^2\ln\left[\frac{(R_++R_-)^2-4a^2}{4R_+R_-}\right]. \tag{9.7.17}$$

These solutions were first given by Bach and Weyl[19] and later explored and interpreted by Zipoy[20] and Voorhees.[21] Notice that $\sigma(r,z)\to 0$ as $r\to 0$ when $|z|>a$, so, in this case, there is no strut on the z axis outside the rod.

[19] R. Bach and H. Weyl, Mathematische Zeitschrift **13** (1922) 134.
[20] D. M. Zipoy, Jour. Math. Physics **7** (1966) 1137.
[21] B. H. Voorhees, Phys. Rev D **2** (1970) 2119.

9.7.3 Weyl-Bach Ring Metrics

As a final example, we consider an uniform ring of mass M and radius a, laid out in the $z = 0$ plane and centered at the origin. The density function for this metric is

$$\rho(x') = \frac{M}{2\pi r'}\delta(r' - a)\delta(z') \tag{9.7.18}$$

and the solution to Laplace's equation for the metric function $\mu(r, z)$ is

$$\begin{aligned} \mu(r,z) &= -\frac{GM}{2\pi c^2}\int_0^{2\pi} \frac{d\varphi}{\sqrt{r^2 + a^2 + z^2 - 2ar\cos\varphi}} \\ &= -\frac{GM}{\pi c^2}\left[\frac{1}{R_+}K\left(\frac{4ar}{R_+^2}\right) + \frac{1}{R_-}K\left(-\frac{4ar}{R_-^2}\right)\right] \end{aligned} \tag{9.7.19}$$

where $R_\pm = \sqrt{(r \pm a)^2 + z^2}$ and $K(x)$ is the complete elliptic integral of the first kind. From here, the solution for $\sigma(r, z)$ is found by quadratures.

The solutions described are difficult to interpret because, as we have seen, the space-time geometry does not generally reflect the geometry of the source. For example, we do not recover the Schwarzschild black hole in the case of a single point mass. Motivated by the need to directly compare the symmetries of the space-time with those of the source, Zipoy and Voorhees suggested that the coordinates chosen to express any Weyl metric should be adapted to the symmetries of the source.

9.7.4 Prolate and Oblate Spheroidal Coordinates

Prolate spheroidal coordinates are obviously best suited to the line metrics of Zipoy and Voorhees. These may be defined by

$$r = a\sinh u \sin\theta, \quad z = a\cosh u \cos\theta, \tag{9.7.20}$$

where $u \geq 0$ and $\theta \in [-\pi/2, +\pi/2]$. The level curves $u =$ const. and $\theta =$ const. represent confocal *prolate* ellipses,

$$\frac{z^2}{a^2\cosh^2 u} + \frac{r^2}{a^2\sinh^2 u} = 1 \tag{9.7.21}$$

and (orthogonal) hyperbolæ

$$\frac{z^2}{a^2\cos^2\theta} - \frac{r^2}{a^2\sin^2\theta} = 1 \tag{9.7.22}$$

respectively, with foci at $z = \pm a$ as shown in figure 9.3. One finds that the gravitational equipotentials, $e^{\mu(r,z)} =$ const. are equivalent to the statement that $u =$ const., as

$$\mu(u,\theta) = \frac{GM}{2c^2 a}\ln\frac{R_+ + R_- - 2a}{R_+ + R_- + 2a} = \frac{GM}{c^2 a}\ln\tanh\frac{u}{2} \tag{9.7.23}$$

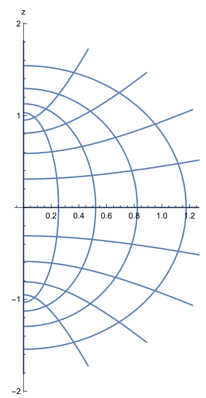

Figure 9.3: Prolate Coordinates in the $r - z$ plane.

and

$$\sigma(u, \theta) = \frac{1}{2}\left(\frac{GM}{c^2 a}\right)^2 \ln\left[\frac{\sinh^2 u}{\cosh^2 u - \cos^2\theta}\right]. \tag{9.7.24}$$

Now suppose we define the radial coordinate

$$\rho = a\cosh u, \tag{9.7.25}$$

then

$$\begin{aligned} \mu(\rho) &= \frac{GM}{2c^2 a}\ln\left[\frac{\rho - a}{\rho + a}\right] \\ \sigma(\rho, \theta) &= \frac{1}{2}\left(\frac{GM}{c^2 a}\right)^2 \ln\left[\frac{\rho^2 - a^2}{\rho^2 - a^2\cos^2\theta}\right] \end{aligned} \tag{9.7.26}$$

and the line element reads,

$$ds^2 = e^{2\mu}dt^2 - e^{-2\mu}\left[e^{2\sigma}\left(\rho^2 - a^2\cos^2\theta\right)\left(\frac{d\rho^2}{\rho^2 - a^2} + d\theta^2\right) + (\rho^2 - a^2)\sin^2\theta\ d\varphi^2\right]. \tag{9.7.27}$$

A special case clearly arises when $a = GM/c^2$, for then

$$\begin{aligned} e^{2\mu(\rho)} &= \frac{\rho - a}{\rho + a} \\ e^{2\sigma(\rho,\theta)} &= \frac{\rho^2 - a^2}{\rho^2 - a^2\cos^2\theta} \end{aligned} \tag{9.7.28}$$

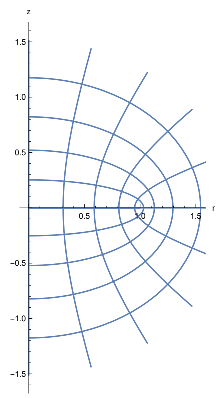

Figure 9.4: Oblate Coordinates in the $r - z$ plane.

so that (9.7.27) can be written as

$$ds^2 = \frac{\rho - a}{\rho + a}\, dt^2 - \frac{\rho + a}{\rho - a}\, d\rho^2 - (\rho + a)^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{9.7.29}$$

If we shift the radial coordinate by a, *i.e.,* let $R = \rho + a \geq 2a$, we may express (9.7.27) as

$$ds^2 = \left(1 - \frac{2a}{R}\right) dt^2 - \left(1 - \frac{2a}{R}\right)^{-1} dR^2 - R^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{9.7.30}$$

which is explicitly the exterior line element of a Schwarzschild black hole of mass M. Notice the counter-intuitive fact that the (spherically symmetric) Schwarzschild black hole arises as the Weyl vacuum solution sourced by a *rod* of a particular length (equal to the Schwarzschild radius of the hole). The mapping from Weyl coordinates to the spherically symmetric ones is one-to-one and covers the entire region outside the line mass.

Oblate spheroidal coordinates are likewise defined as

$$r = a \cosh u \sin\theta, \quad z = a \sinh u \cos\theta \tag{9.7.31}$$

with the same ranges as before, *viz.,* $u \geq 0$ and $\theta \in [-\pi/2, \pi/2]$. The level curves $u =$ const. are once again confocal ellipses whereas the level curves $\theta =$ const. are the orthogonal hyperbolæ, with focus at $r = a$ as shown in figure 9.4. Instead of guessing at an appropriate source for this metric symmetry, let us consider Laplace's equation in the coordinates (u, θ, φ) defined in (9.7.31). Assuming that the gravitational potential depends only on u, we find that $\mu(u)$ must satisfy

$$\mu''(u) + \tanh(u)\mu'(u) = 0 \tag{9.7.32}$$

outside of sources, with solution (C is an arbitrary constant)

$$\mu(u) = C\tan^{-1}\left(\tanh\frac{u}{2}\right) = \frac{C}{2}\mathrm{gd}(u), \tag{9.7.33}$$

representing the gravitational field outside an oblate homoeioid, *i.e.*, a thin shell bounded by two confocal ellipsoids. (Note that this same approach could be taken in the case of the prolate system of (9.7.20) to arrive at (9.7.23).[22])

If we define $R_\pm = \sqrt{(r\pm a)^2 + z^2}$, then

$$\tanh\frac{u}{2} = \sqrt{\frac{R_+ + R_- - 2a}{R_+ + R_- + 2a}}$$

and therefore

$$\mu(r,z) = C\tan^{-1}\sqrt{\frac{R_+ + R_- - 2a}{R_+ + R_- + 2a}}. \tag{9.7.34}$$

We can now integrate the second of (9.6.20) to find that

$$\sigma(r,z) = \frac{C^2}{8}\ln\left[\frac{2R_+R_-}{(R_+ + R_-)^2}\right] + D(r) \tag{9.7.35}$$

and the first requires that $D(r) = D$ (constant), which we take to be zero. Thus,

$$\sigma(r,z) = \frac{C^2}{8}\ln\left[\frac{2R_+R_-}{(R_+ + R_-)^2}\right], \quad \sigma(u,\theta) = \frac{C^2}{8}\ln\left[\frac{\cosh 2u + \cos 2\theta}{4\cosh^2 u}\right], \tag{9.7.36}$$

gives the metric function "σ" in the original cylindrical (r,z) coordinates and the oblate (u,θ) coordinates.

9.8 Kerr Metric

The discovery of the Schwarzschild solution only a few months after the publication of Einstein's field equations began a search for other exact solutions. Particularly sought after was a solution that could describe a rotating mass carrying angular momentum. This, however, took almost fifty years to discover, even though the weak gravitational field (8.2.22) due to a rotating massive body had been discovered by Lense and Thirring

[22] Problem: (a) Determine the flat, three dimensional metric in the prolate system. (b) Show that, if $\mu = \mu(u)$, Laplace's equation reads

$$\mu''(u) + \coth(u)\mu'(u) = 0$$

and (c) show that the solution of this equation is precisely (9.7.23).

within about two years. The result is the **Kerr** metric, named after its discoverer, R. Kerr.

Kerr's original derivation of the metric is complicated, so we will here follow a more intuitive path to it. A rotating mass tends to flatten at the poles, so it is reasonable to expect that the coordinates best suited to describe the gravitational field outside the mass belong to an oblate, spheroidal system, which we may define as

$$\begin{aligned} x &= a\cosh u \sin\theta\cos\phi \\ y &= a\cosh u\sin\theta\sin\phi \\ z &= a\sinh u\cos\theta, \end{aligned} \tag{9.8.1}$$

where $u \geq 0$, $\theta \in [0,\pi]$ and $\varphi \in [0,2\pi]$. In this way, $u =$ const. describes a family of oblate, confocal ellipsoids. Now let $r = a\sinh u \in [0,\infty)$. Then

$$\begin{aligned} x &= \sqrt{r^2+a^2}\sin\theta\cos\varphi \\ y &= \sqrt{r^2+a^2}\sin\theta\sin\varphi \\ z &= r\cos\theta, \end{aligned} \tag{9.8.2}$$

and the flat line element in this system is

$$ds^2 = c^2dt^2 - \frac{\rho^2}{r^2+a^2}\,dr^2 - \rho^2 d\theta^2 - (r^2+a^2)\sin^2\theta d\varphi^2 \tag{9.8.3}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$. By simply rearranging terms, this metric may be written as

$$ds^2 = \frac{r^2+a^2}{\rho^2}(cdt - a\sin^2\theta d\varphi)^2 - \frac{\rho^2}{r^2+a^2}dr^2 - \rho^2 d\theta^2 - \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)d\varphi - acdt]^2. \tag{9.8.4}$$

The cross term vanishes, *i.e.,* the coefficient of the $dtd\varphi$ term is zero because this is only the flat space metric.

We may rewrite the GEM line element (8.2.22), with the angular momentum, J, pointing in the $+z$ direction, as

$$\begin{aligned} ds^2 &= c^2\left(1-\frac{2GM}{c^2r}\right)dt^2 + \frac{4GJ}{c^2r}\sin^2\theta d\varphi dt - \left(1+\frac{2GM}{c^2r}\right)dr^2 + r^2d\Omega^2 \\ &= c^2dt^2 - \left(1+\frac{2GM}{c^2r}\right)dr^2 - r^2d\Omega^2 - \frac{2GM}{c^2r}\left(cdt - \frac{a'}{c}\sin^2\theta d\varphi\right)^2 \end{aligned} \tag{9.8.5}$$

where $a' = J/M$ ($[a'] = l^2/t$) is the angular momentum per unit mass of the body and the second expression is only valid up to terms *linear* in a'. Comparing (9.8.5) and (9.8.4), we are able to guess that the oblateness parameter a in (9.8.4) must be related to the angular momentum per unit mass by $a' = ac$ or $a = J/Mc$. Again, taking a cue from

Schwarzschild's spherically symmetric solution, we ask if (9.8.5) would be recovered as the linear approxmation of a line element of the form (9.8.4), with r^2+a^2 replaced by $F(r)$ in the first and second terms. The function, F, if it exists, would be obtained from Einstein's vacuum equations and the line element for the solution would read

$$ds^2 = \frac{F(r)}{\rho^2}(cdt - a\sin^2\theta d\varphi)^2 - \frac{\rho^2}{F(r)}dr^2 - \rho^2 d\theta^2 - \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)d\varphi - acdt]^2 \quad (9.8.6)$$

or, letting $F(r) = r^2 + a^2 + f(r)$,

$$\begin{aligned} ds^2 &= c^2dt^2 + \frac{f(r)}{\rho^2}(cdt - a\sin^2\theta d\varphi)^2 - \frac{\rho^2}{F(r)}dr^2 - \rho^2 d\theta^2 - (r^2+a^2)\sin^2\theta d\varphi^2 \\ &= c^2\left(1+\frac{f(r)}{\rho^2}\right)dt^2 - \frac{2f(r)}{\rho^2}ac\sin^2\theta dt d\varphi - \frac{\rho^2}{r^2+a^2+f(r)}dr^2 - \rho^2 d\theta^2 \\ &\qquad - \left(r^2+a^2-\frac{f(r)a^2}{\rho^2}\sin^2\theta\right)\sin^2\theta d\varphi^2 \end{aligned} \quad (9.8.7)$$

Note that we recover the flat metric (in oblate spheroidal coordinates) as $f(r)\to 0$ and the Schwarzschild metric if $a\to 0$ and $f(r) = -r_s r$. Let us now look at Einstein's equations for the geometry described by the line element given above. We find

$$E_{\theta\theta} = \frac{1}{\rho^2}\left(f - rf' + \frac{1}{2}\rho^2 f''\right) = 0 \;\Rightarrow\; f''=0, \quad rf' = f \quad (9.8.8)$$

which establishes that $f = \alpha r$ for some constant α and in fact solves *all* of Einstein's vacuum field equations. Comparing the solution to the GEM line element (8.2.22) then reveals that $\alpha = -2GM/c^2 = -r_s$, so we have arrived at the **Kerr** solution in "Boyer-Lindquist" coordinates,

$$\begin{aligned} ds^2 &= c^2\left(1-\frac{r_s r}{\rho^2}\right)dt^2 + \frac{2r_s rac\sin^2\theta}{\rho^2}dt d\varphi - \frac{\rho^2}{r^2+a^2-r_s r}dr^2 - \rho^2 d\theta^2 \\ &\qquad - \left(r^2+a^2+\frac{r_s ra^2}{\rho^2}\sin^2\theta\right)\sin^2\theta d\varphi^2. \end{aligned} \quad (9.8.9)$$

It can be put in the ADM form

$$ds^2 = c^2\frac{\rho^2\Delta}{\Sigma}dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \frac{\Sigma}{\rho^2}\sin^2\theta(d\varphi - \omega dt)^2 \quad (9.8.10)$$

where

$$\rho^2 = r^2 + a^2\cos\theta^2, \quad \Delta = r^2 + a^2 - r_s r$$

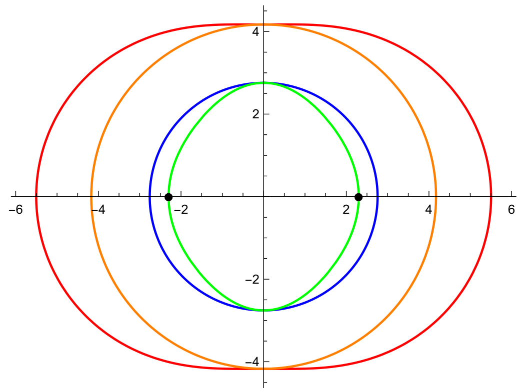

Figure 9.5: Surfaces of the Kerr metric

$$\Sigma = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \quad \omega = \frac{acr_s r}{\Sigma}. \tag{9.8.11}$$

One should be careful in interpreting the coordinates. For example, $r = 0$ is not a point. On a constant t hypersurface it represents the double cover of a disk whose boundary, at $\theta = \pi/2$, is a ring of physical radius a, which is a curvature singularity. The solution is asymptotically flat, but it is not unique. There is no equivalent of Birkhoff's theorem for the Kerr metric, therefore, to assume that it necessarily describes the space-time near a very massive rotating body would not be correct. What *is* true is that the vacuum geometry of a massive rotating body *asymptotically* approaches the Kerr geometry. To include charge (electric and magnetic), simply replace r_s by $r_s - Q^2/r$ in (9.8.9), where $Q^2 = Q_e^2 + Q_m^2$. The metric is then known as the **Kerr-Newman** metric.

Let us now take a closer look at the Kerr geometry. It admits two Killing vectors, one timelike, $\xi^\mu_{(t)} = (1, 0, 0, 0)$ and the other spacelike, $\xi^\mu_{(\varphi)} = (0, 0, 0, 1)$. The norm of the spacelike (azimuthal) Killing vector, being the sum of non-negative terms, never vanishes, but the norm of the timelike Killing vector vanishes when

$$1 - \frac{r_s r}{\rho^2} = 0, \quad \Rightarrow \quad r_\pm^{(E)} = \frac{1}{2}\left[r_s \pm \sqrt{r_s^2 - 4a^2 \cos^2 \theta}\right], \tag{9.8.12}$$

which defines two Killing horizons when $r_s > 2a$. These are the outer and inner **ergosurfaces** of the Kerr black hole, shown in red and green in the figure 9.5. There is no ergosurface when $r_s < 2a$ and, when the angular momentum vanishes, $r = r_+^{(E)}$ becomes the Schwarzschild horizon. There is also a coordinate singularity when

$$r^2 + a^2 - r_s r = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\left[r_s \pm \sqrt{r_s^2 - 4a^2}\right] \tag{9.8.13}$$

and $r_s \geq 2a$. The two solutions describe the outer and inner **event horizons**, shown in orange and blue in figure 9.5. We will soon see that the event horizons are also Killing horizons. There are no event horizons when $r_s < 2a$ and, when $r_s = 2a$ (the extremal case), only one event horozon exists at half the Schwarzschild radius. (This is also the case when $a = 0$, but now this is just the Schwarzschild black hole). Finally, the Kretschmann scalar reads

$$K = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = \frac{12r_s^2(r^2 - a^2\cos^2\theta)(\rho^4 - 16a^2r^2\cos^2\theta)}{\rho^{12}}, \tag{9.8.14}$$

so there is a curvature singularity when $\rho = 0$, which happens only when $r = 0$ *and* $\theta = \pi/2$. Referrng back to the coordinate definitions in (9.8.2), this turns out to be not a point but the *ring*

$$x^2 + y^2 = a^2, \quad z = 0. \tag{9.8.15}$$

Thus the angular momentum "stretches" the point-like Schwarzschild singularity into a **ring singularity**. It is easy to see that

$$r_+^{(E)} \geq r_+ \geq r_- \geq r_-^{(E)} \tag{9.8.16}$$

and that the ring singularity is surrounded by three of the four surfaces we have described, *viz.,* $r = r_+^{(E)}$ and $r = r_\pm$, as shown in figure 9.5. It coincides with the inner ergosurface in the equatorial plane. The region between the outer ergosurface and outer horizon is called the **erogoregion**. We will soon see that in the ergoregion it is impossible for an observer to remain stationary.

The Killing vectors show the space-time is characterized by its mass energy, $Q_t = Mc^2$ and angular momentum, $Q_\varphi = -2Mac = -2J$, as may be calculated by evaluating (7.5.19) on the two sphere at infinity.

9.8.1 Equatorial Geodesics

It is not difficult to verify that motion remains confined to the equatorial plane if the initial velocity is in the plane. To simplify matters, therefore, take $\theta = \pi/2$ and we have the following conservation laws,

$$\begin{aligned} \xi_{(t)} \cdot U &= -c^2\left(1 - \frac{r_s}{r}\right)U^t - \frac{acr_s}{r}U^\varphi = -E \stackrel{\text{def}}{=} AU^t + BU^\varphi \\ \xi_{(\varphi)} \cdot U &= -\frac{acr_s}{r}U^t + \left[r^2 + a^2\left(1 + \frac{r_s}{r}\right)\right]U^\varphi = L \stackrel{\text{def}}{=} BU^t + CU^\varphi \end{aligned} \tag{9.8.17}$$

This system is readily solved for U^t and U^φ by inversion,

$$U^t = \frac{CE + BL}{B^2 - AC}, \quad U^\varphi = -\frac{AL + BE}{B^2 - AC} \tag{9.8.18}$$

or

$$U^t = \frac{-acLr_s/r + E[r^2 + a^2(1 + r_s/r)]}{c^2(r^2 + a^2 - r_s r)}$$
$$U^\varphi = \frac{cL(1 - r_s/r) + aEr_s/r}{c(r^2 + a^2 - r_s r)} \tag{9.8.19}$$

where we used $B^2 - AC = c^2(r^2 + a^2 - r_s r) = c^2\Delta$.

It is immediately evident that a particle carrying zero angular momentum (a ZAMO, or inertial, Zero Angular Momentum Object) will possess a non-zero angular velocity,

$$\omega(r) = \frac{d\varphi}{dt} = \frac{U^\varphi}{U^t} = \frac{acr_s/r}{r^2 + a^2(1 + r_s/r)}, \tag{9.8.20}$$

in the direction of the hole and increasing as the hole is approached. This property of the space-time is called the "dragging of inertial frames". The effect vanishes at large distances from the hole as r^{-3}. Another consequence is that, no matter what its angular momentum or energy, the particle's angular velocity on the outer ergosurface will necessarily be,

$$\omega_+^E = \left.\frac{d\varphi}{dt}\right|_{r=r_s} = \frac{ac}{r_s^2 + 2a^2} = \lim_{r\to r_s} \omega(r). \tag{9.8.21}$$

Similarly, regardless of its angular momentum and energy, it will have the angular velocity

$$\omega_+ = \left.\frac{d\varphi}{dt}\right|_{r=r_+} = \frac{ac}{r_s r_+} \tag{9.8.22}$$

on the outer event horizon.

Taking λ to be the proper time for spacelike or timelike geodesics we find

$$g_{\mu\nu}U^\mu U^\nu = AU^{t2} + 2BU^tU^\varphi + CU^{\varphi 2} + \frac{r^2}{\Delta}U^{r2} = \epsilon c^2 \tag{9.8.23}$$

where $\epsilon = \mp 1$ or zero, depending on whether the geodesics are timelike, spacelike or null respectively. The expression may be simplified and written in terms of the conserved quantities with the help of (9.8.17); we find

$$-EU^t + LU^\varphi + \frac{r^2}{\Delta}U^{r2} = \epsilon c^2 \tag{9.8.24}$$

from which, using (9.8.18), it follows that

$$U^{r2} = \frac{CE^2 + AL^2 + 2BEL}{c^2r^2} + \frac{\epsilon c^2\Delta}{r^2} \tag{9.8.25}$$

One may frame the above equation as follows:

$$U^r = \frac{dr}{d\lambda} = \pm\sqrt{\mathcal{E} - V_{\text{eff}}} \tag{9.8.26}$$

where

$$\mathcal{E} = \frac{E^2}{c^2} + \epsilon c^2 \tag{9.8.27}$$

and

$$V_{\text{eff}} = \frac{\epsilon c^2 r_s}{r} + \frac{1}{r^2}\left[L^2 - a^2\left(\frac{E^2}{c^2} + \epsilon c^2\right)\right] - \frac{r_s}{r^3}\left(L - \frac{aE}{c}\right)^2 \tag{9.8.28}$$

The first term represents the ordinary Newtonian gravitational potential, the second represents the centrifugal force and the last the spin-orbit interaction. Notice that the structure of the effective potential is identical to (9.3.52) and in the Schwarzschild limit ($a \to 0$) we recover the geodesic equation in (9.3.51)[23]. Equatorial geodesics can be recovered by the same methods.

9.8.2 Equatorial Static and Stationary Particles

The outer ergosphere is a Killing horizon of the time like Killing vector $\xi_{(t)} = (1, 0, 0, 0)$, which implies that, within the outer ergosphere and the outer horizon, the particle cannot remain *static*. To be static its velocity four vector would have to point along the time direction, *i.e.,* $U^\mu = \gamma\xi^\mu$, where $\xi^\mu = (1, 0, 0, 0)$ is the Killing vector and γ is a normalization factor (ensuring $U^2 = -c^2$). Static particles are, naturally, not inertial and their motion is not geodesic. They require some external agent to keep them in place, but we see that they cannot exist within the ergoregion, where ξ^μ becomes spacelike! What happens is that frame dragging within the erogoregion compels the particle to rotate with the hole.

We may then ask if it is possible for a particle to remain *stationary* within the ergoregion. To remain stationary, the particle should possess a velocity that looks like

$$U^\mu = \gamma\chi^\mu = \gamma\left(\xi^\mu_{(t)} + \Omega\xi^\mu_{(\varphi)}\right) \tag{9.8.30}$$

where γ is again a (constant) normalization and Ω is a constant angular velocity. Like static particles, stationary particles are not inertial and their motion is not geodesic. Because Ω must be constant, χ^μ is a linear combination of Killing vectors and is therefore also a

[23] Problem: Consider circular, timelike orbits. Show that for any value of E and L there are two circular orbits at radii

$$V'_{\text{eff}} = 0 \;\Rightarrow\; r_\pm = \frac{L^2 - a^2\mathcal{E}}{c^2 r_s}\left[1 \pm \sqrt{1 - \frac{3c^2(L - aE/c)^2 r_s^2}{(L^2 - a^2\mathcal{E})^2}}\right]. \tag{9.8.29}$$

What is the radius of the only lightlike orbit?

Killing vector. Our only requirement is that it should be timelike, and this is possible ($r_s > 2a$) if and only if $r < r_-$ or $r > r_+$ and

$$\Omega_- < \Omega < \Omega_+ \tag{9.8.31}$$

where

$$\Omega_\pm = \omega(r) \pm \frac{c\sqrt{\Delta}}{r^2 + a^2(1 + r_s/r)} = \omega(r)\left[1 \pm \frac{r\sqrt{\Delta}}{ar_s}\right]. \tag{9.8.32}$$

From here we learn that:

- it is possible to remain stationary but not static within the outer ergosphere, outside the outer event horizon and inside the inner event horizon,
- the stationary particle on the outer ergosurface will have an angular velocity between zero and $2\omega_+^E$,
- Ω_- is negative outside the outer ergosphere and positive inside it,
- as r decreases, Ω_+ decreases and Ω_- increases, and
- $\Omega_+ = \Omega_-$ on the outer horizon, where the stationary particle will have an angular velocity precisely equal to ω_+.

Owing to the last, we associate

$$\Omega_{\mathcal{H}^+} = \omega_+ = \frac{ac}{r_s r_+} \tag{9.8.33}$$

with the angular velocity of the black hole. Stationary particles enter into a state of corotation with the black hole as they approach the outer horizon. The vector χ^μ is null on the outer and inner horizons, which are therefore also Killing horizons. Within the outer and inner even horizons stationary particles do not exist. Thus, the outer ergosphere serves as the inner boundary of the static region, which extends to infinity, and the outer horizon as the inner boundary of the stationary region of the Kerr geometry.

9.8.3 The Penrose Process

Penrose discovered a way in which energy could be extracted from a rotating black hole via a fully classical process. The idea hinges on the conservation of energy and momentum and the fact that that the outer ergosphere is the boundary of the static region of the Kerr geometry.

Consider an inertial particle originating at infinity and falling into the Kerr black hole. If p^μ is its momentum, its energy will be $E = -\xi_{(t)} \cdot p > 0$ and its angular momentum

$L = \xi_{(\varphi)} \cdot p$. Both of these quantities are conserved during its (geodesic) motion. Once it enters the horizon, imagine that it decays into two particles. Conservation of momentum implies that $p_\mu = p_\mu^{(1)} + p_\mu^{(2)}$, which means that the original energy and momentum get split into two parts according to

$$E = -\xi_{(t)} \cdot p = -\xi_{(t)} \cdot p^{(1)} - \xi_{(t)} \cdot p^{(2)} = E_1 + E_2$$

$$L = \xi_{(\varphi)} \cdot p = \xi_{(\varphi)} \cdot p^{(1)} + \xi_{(\varphi)} \cdot p^{(2)} = L_1 + L_2. \tag{9.8.34}$$

In the outer ergoregion $\xi_{(t)}$ is spacelike and therefore the signs of $E_{1,2}$ are individually indefinite although the sum is required to be positive and equal to E. Suppose that E_2 is negative then the other particle's energy would have to be greater than the original, $E_1 > E$. But, because E_2 is negative, the second particle cannot exit into the static region, for then its four velocity would be spacelike. The first particle, however, can return to the static region, where its energy can be extracted, with an energy that is greater than the energy that fell into the hole. Of course, we cannot create energy, so the excess energy must have come from the black hole. Some of the energy of the rotating black hole has been extracted. This is the Penrose process.

More concretely, for the second particle to fall into the outer event horizon, it must satisfy

$$\chi_{\mathcal{H}^+} \cdot p_2 = -(\xi_{(t)} + \omega_+ \xi_{(\varphi)}) \cdot p_2 = E_2 - \omega_+ L_2 > 0 \tag{9.8.35}$$

and, because E_2 is negative, this implies that

$$L_2 < -\frac{|E_2|}{\omega_+}, \tag{9.8.36}$$

so the particle must have a negative angular momentum, *i.e., opposite* the hole. Once this particle crosses the outer event horizon it is effectively absorbed by the hole. Then the mass and angular momentum of the hole will change according to $\delta M c^2 = -|E_2|$ and $\delta J = -|L_2| < -|E_2|/\omega_+$, where we let $J = acM$ be the angular momentum of the Kerr black hole. It follows that

$$\delta M c^2 - \omega_+ \delta J > 0, \tag{9.8.37}$$

which ensures that the area of the outer horizon increases during such a process.[24]. This result and its analogue for the Reissner-Nordstrom black hole in (9.4.13) can be viewed as consequences of an interesting set of theorems concerning the physics of horizons.

9.9 Classical Black Hole Thermodynamics

We have seen, in each of the special cases treated above, that processes involving black holes can (even on the classical level) allow for exchanges of energy in which useful work

[24] Problem: Show that the area of the outer horizon increases as a consequence of the Penrose process.

may be extracted from physical systems, with black holes acting as intermediaries. As far as the extraction of energy goes, adiabatic process, in which the area of the black hole horizon does not change, are the most efficient. This is strikingly similar to isentropic processes in thermodynamics, if the area of the horizon plays the role of entropy. This analogy is the subject of the present section.

9.9.1 Surface Gravity

The proper acceleration of a body at rest at r in a spherically symmetric vacuum is

$$a^\mu = \frac{dU^\mu}{d\tau} = (U\cdot\nabla)U^\mu = -\frac{1}{2}A'\delta^\mu_r \;\Rightarrow\; a = \sqrt{a^\mu a_\mu} = \frac{cA'}{2\sqrt{A}}. \tag{9.9.1}$$

When r represents the surface radius of an astronomical object, we call it the object's **surface gravity**. For example, the surface gravity of the Earth is calculated to be 9.81 m/s^2 and that of a 2 solar mass neutron star of radius 11 km is 2.2×10^{12} m/s^2. Within strong gravitational fields (such as on the surface of a neutron star) time dilation plays a significant role. The effect of time dilation will only grow as the horizon of a black hole ($A(r) = 0$) is approached and the definition will fail near the event horizon of a black hole.

Suppose we ask a different question: what force would have to be applied to a unit mass, by an agent at infinity, to suspend an object (by an infinitely long, light string) near the horizon of a black hole? One way to think about this would be to note that dt represents an infinitesimal interval of proper time at infinity, therefore this observer exerts a force per unit mass of $a^\mu_\infty = dU^\mu/dt$. But because

$$a^\mu = \frac{dU^\mu}{d\tau} = \frac{dU^\mu}{dt}\frac{dt}{d\tau}, \tag{9.9.2}$$

we have

$$a^\mu_\infty = -\frac{\sqrt{A}A'}{2c}\delta^\mu_r \;\Rightarrow\; a_\infty = \frac{1}{2}A', \tag{9.9.3}$$

which is well defined on the horizon of a black hole. We wish to construct a scalar that reproduces a_∞. If we are successful, then its value on the horizon is what we shall mean by the "horizon surface gravity".

More generally, a Killing horizon is defined as a surface on which a Killing vector, which is timelike in a region of space-time bounded by it, becomes null. The surface gravity of a Killing horizon can be expressed in terms of its defining Killing vector, $\xi_{(t)}$. Because $\xi_{(t)}$ is null on the horizon, it is both tangent and perpendicular to it. The normal to the hypersurface $\Phi = \xi_{(t)}\cdot\xi_{(t)} =$ const. is $n_\mu = \Phi_{,\mu}$, which *on the horizon (only!)* is proportional to $\xi_{(t)}$. Therefore we define the surface gravity, κ, of the Killing horizon as (dropping the subscript (t)),

$$(\xi^\alpha\xi_\alpha)_{,\mu}|_{\mathcal{H}} = -\frac{2\kappa}{c}\,\xi_\mu|_{\mathcal{H}}\,. \tag{9.9.4}$$

We will now check that κ is precisely a_∞, but first let us work with the above definition. An equivalent definition of κ is via

$$(\xi \cdot \nabla)\xi_\mu = -\frac{\kappa}{c}\xi_\mu \tag{9.9.5}$$

on the horizon. This can be shown by expanding the left hand side of (9.9.4) and using the fact that $\xi_{(\alpha;\mu)} = 0$ because ξ is a Killing vector. Now, the Killing vector is hypersurface orthogonal,[25] so it satisfies the identity

$$\xi_{\mu;\nu}\xi_\lambda + \xi_{\lambda;\mu}\xi_\nu + \xi_{\nu;\lambda}\xi_\mu = 0 \tag{9.9.6}$$

Contracting with $\xi^{\mu;\nu}$,

$$\begin{aligned} \xi^{\mu;\nu}\xi_{\mu;\nu}\xi_\lambda &= -\xi^{\mu;\nu}\left[\xi_{\lambda;\mu}\xi_\nu + \xi_{\nu;\lambda}\xi_\mu\right] \\ &= -\frac{\kappa}{c}\left[\xi^\mu\xi_{\lambda;\mu} - \xi^\nu\xi_{\nu;\lambda}\right] \\ &= -\frac{2\kappa^2}{c^2}\xi_\lambda \end{aligned} \tag{9.9.7}$$

Comparing the two sides, we get yet another equation for κ,

$$\kappa^2 = -\frac{c^2}{2}\xi^{\mu;\nu}\xi_{\mu;\nu} \tag{9.9.8}$$

on the horizon. All three equations (9.9.4), (9.9.5) and (9.9.8), are equivalent definitions of κ.

For example, using (9.9.8) in the static background and taking the positive root of (9.1.8) we find

$$\kappa = a_\infty = \left.\frac{A'(r)}{2}\right|_{\mathcal{H}}. \tag{9.9.9}$$

Thus, the surface gravity of the Schwarzschild horizon is

$$\kappa = \frac{c^2}{2r_s} = \frac{c^4}{4GM} \tag{9.9.10}$$

and of the outer horizon of the Reissner-Nordstrom black hole it is

$$\kappa = \frac{c^2(r_+ - r_-)}{2r_+^2} = \frac{c^4\sqrt{G^2M^2 - c^4r_Q^2}}{\left(GM + \sqrt{G^2M^2 - c^4r_Q^2}\right)^2}. \tag{9.9.11}$$

[25] Problem: Verify that the rotation of the timelike Killing vector vanishes everywhere.

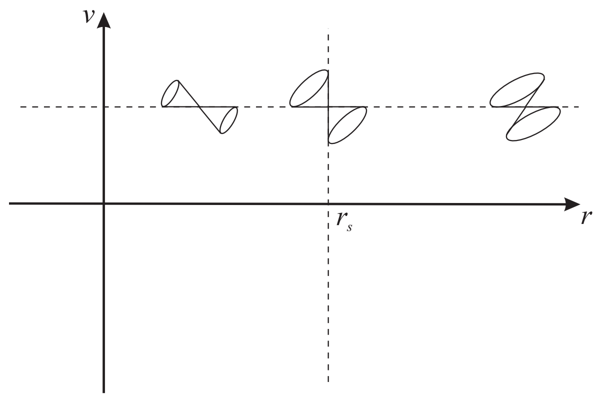

Figure 9.6: The light-cone in the $v - r$ system.

These are both finite and constant.

One can also use (9.9.4) and (9.9.5) to obtain the same result, but Schwarzschild coordinates do not serve us well because they are ill defined on the horizon; in particular, the light-cone "closes" at $A(r) = 0$,

$$\frac{dt}{dr} = \pm A(r)^{-1}. \tag{9.9.12}$$

To evaluate the left hand side of either of the equations on, say, the future horizon, we transform to the advanced time coordinate, $t \to v = t + r_*$, in which the static solution takes the form

$$ds^2 = \frac{1}{c^2} A(r) dv^2 - 2dvdr - r^2 d\Omega^2. \tag{9.9.13}$$

The light cones do not close on the future horizon in this system, for the radial null curves are given by

$$\frac{dv}{dr} = \begin{cases} 0 \\ 2c^2 A^{-1}(r) \end{cases} \tag{9.9.14}$$

as shown in figure 9.6. The timelike Killing vector transforms to $\xi^\mu = (c, 0, 0, 0)$, but now $\xi_\mu = (-A(r)/c, c, 0, 0)$ does not vanish when $A(r) = 0$. Then

$$\Phi = \xi \cdot \xi = -A(r) \ \to \ \Phi_{,\mu} = -A'(r)\delta^r_\mu \tag{9.9.15}$$

so, applying (9.9.4), we find (again) that $\kappa = A'(r)/2|_{\mathcal{H}}$.[26]

[26] Problem: Repeat the calculation of the surface gravity on the past horizon, by using the retarded time coordinate, $t \to u = t - r_*$. You should find that $\kappa = -A'(r)/2|_{\mathcal{H}}$. What is the significance of the change in sign? Sketch a few light-cones in this system.

We can also use (9.9.8) and (9.8.10) to directly calculate the surface gravity of the outer horizon of the Kerr space-time, taking

$$\xi^\mu = (1, 0, 0, \omega_+), \tag{9.9.16}$$

where ω_+ is the surface gravity of the outer horizon. The result is

$$\kappa = \left.\frac{c^2\Delta'}{2\sqrt{\Sigma}}\right|_{r=r_+} = \frac{c^2(2r_+ - r_s)}{2(r_+^2 + a^2)}, \tag{9.9.17}$$

which is also finite and constant.

9.9.2 Zeroeth Law

We have seen that the surface gravity is a constant on a (Killing) horizon in three examples. In fact, this can be proved rigorously, assuming only (i) stationarity, (ii) Einstein's equations and (iii) the dominant energy condition.[27]. The statement that:

- *the surface gravity of a Killing horizon is constant*

is known as the zeroeth law of black hole thermodynamics. The law itself is reminiscent of the implicit assumption, made in classical Thermodynamics, that thermal equilibrium is a reflexive property of equilibrium thermodynamic systems. It is therefore saying that the black hole horizon is in a kind of "thermal" equilibrium with itself but it is not a statement about the transitivity of this "thermal" equilibrium, as is the zeroeth law of thermodynamics. The analogy between the surface gravity and the "temperature" gets better with the first law.

9.9.3 First Law

Let us return to the Schawarzschild black hole, whose event horizon has an area $\mathcal{A} = 4\pi r_s^2$, and notice that

$$\frac{\kappa}{2\pi}\, d\left(\frac{c^2\mathcal{A}}{4G}\right) = d(Mc^2). \tag{9.9.18}$$

Likewise, for the Reissner-Nordstrom black hole, where the outer horizon has the area $\mathcal{A} = 4\pi r_+^2$, we find

$$\frac{\kappa}{2\pi}\, d\left(\frac{c^2\mathcal{A}}{4G}\right) = d(Mc^2) - 4\pi gc\Phi(r_+)dQ_e \tag{9.9.19}$$

[27] J.M. Bardeen, B. Carter and S.W. Hawking, "The Four Laws of Black Hole Mechanics", Commun. Math. Phys. **31** 161 (1973).

and, for the Kerr black hole, whose outer horizon has the area $\mathcal{A} = 4\pi(r_+^2 + a^2)$,

$$\frac{\kappa}{2\pi}\, d\left(\frac{c^2\mathcal{A}}{4G}\right) = d(Mc^2) - \omega_+ dJ \tag{9.9.20}$$

where ω_+ is the angular velocity of the black hole. In all three cases, we find equations analogous to the first law of thermodynamics,

$$TdS = dE + \sum_i p_i dX^i \tag{9.9.21}$$

where E is the energy, X^i are the extensive variables of the system and p_i their conjugate momenta, *provided* we think of the surface gravity of the Killing horizon as a temperature and its area as an entropy. Then we can think of $đW = 4\pi gc\Phi(r_+)dQ_e$ and $đW = \omega_+ dJ$ as thermodynamic work done by a black hole on charges and/or objects falling into it.

The Smarr Formula

All of this can be made more rigorous by considering the conserved charges in (7.5.19). Recall that we had

$$\int_{\Sigma_t} d\Sigma_\mu \nabla_\nu \mathcal{J}^{\mu\nu} = \frac{c^4}{8\pi G}\oint_S dS_{\mu\nu}\nabla^{[\mu}\varepsilon^{\nu]} \tag{9.9.22}$$

and that the left hand side can be written as[28]

$$\nabla_\nu \mathcal{J}^{\mu\nu} = \frac{c^4}{8\pi G}\nabla_\alpha \nabla^{[\mu}\varepsilon^{\alpha]} = \frac{c^4}{4\pi G} R^\mu{}_\alpha \varepsilon^\alpha, \tag{9.9.23}$$

therefore

$$2\int_{\Sigma_t} d\Sigma_\mu R^\mu{}_\alpha \varepsilon^\alpha = \oint_S dS_{\mu\nu}\nabla^{[\mu}\varepsilon^{\nu]} \tag{9.9.24}$$

For stationary black holes, we apply the equation above to a spatial hypersurface stretching from the (outer) event horizon to spatial infinity. The spatial hypersurface then has two boundaries: one is the two sphere at infinity, S_∞, and the other is the horizon two sphere, $S_{\mathcal{H}^+}$. With the help of Einstein's equations we may write

$$\int_{\Sigma_t} d\Sigma_\mu (2T^\mu{}_\alpha - \delta^\mu_\alpha T)\varepsilon^\alpha = \frac{c^4}{8\pi G}\left[\oint_{S_\infty} dS_{\mu\nu}\nabla^{[\mu}\varepsilon^{\nu]} - \oint_{S_{\mathcal{H}^+}} dS_{\mu\nu}\nabla^{[\mu}\varepsilon^{\nu]}\right] \tag{9.9.25}$$

or

$$\frac{c^4}{8\pi G}\oint_{S_\infty} dS_{\mu\nu}\nabla^{[\mu}\varepsilon^{\nu]} = \int_{\Sigma_t} d\Sigma_\mu (2T^\mu{}_\alpha - \delta^\mu_\alpha T)\varepsilon^\alpha + \frac{c^4}{8\pi G}\oint_{S_{\mathcal{H}^+}} dS_{\mu\nu}\nabla^{[\mu}\varepsilon^{\nu]} \tag{9.9.26}$$

[28] Problem: Show that *if* ε^μ *is a Killing vector* then $\nabla^2\varepsilon^\mu = -R^\mu{}_\alpha\varepsilon^\alpha$.

If ε^μ is the timelike Killing vector, the left hand side represents the total mass energy of the spacetime, the first term on the right is the matter contribution to the mass energy and the second term is the energy of the black hole. An analogous interpretation in terms of the angular momentum can be made if ε^μ is the azimuthal Killing vector. In the case of vacuum spacetimes there is no distinction between the mass and angular momentum of the spacetime and that of the black hole.

Let us therefore take the timelike Killing vector, $\xi^\mu_{(t)}$, in (9.9.26). The left hand side yields Mc^2. To evaluate the horizon integral on the right, we note that

$$\xi^\mu_{(t)} = \chi^\mu - \omega_+ \xi^\mu_{(\varphi)} \tag{9.9.27}$$

so that

$$\begin{aligned} \frac{c^4}{8\pi G}\oint_{S_{\mathcal{H}^+}} dS_{\mu\nu}\nabla^{[\mu}\varepsilon^{\nu]} &= \frac{c^4}{8\pi G}\left[\oint_{S_{\mathcal{H}^+}} dS_{\mu\nu}\nabla^{[\mu}\chi^{\nu]} - \omega_+ \oint_{S_{\mathcal{H}^+}} dS_{\mu\nu}\nabla^{[\mu}\xi^{\nu]}_{(\varphi)}\right] \\ &= \frac{c^4}{8\pi G}\oint_{S_{\mathcal{H}^+}} dS_{\mu\nu}\nabla^{[\mu}\chi^{\nu]} + \omega_+ J_{\mathcal{H}} \end{aligned} \tag{9.9.28}$$

where we have simply called $J_{\mathcal{H}}$ the angular momentum of the hole and

$$\frac{c^4}{8\pi G}\oint_{S_{\mathcal{H}^+}} dS_{\mu\nu}\nabla^{[\mu}\xi^{\nu]}_{(\varphi)} = -J_{\mathcal{H}}.$$

The integral over χ^μ is straightforward because because $\mathcal{H}^+$ is a Killing horizon of χ^μ so the area element on the horizon two sphere may be expressed $dS_{\mu\nu} = dS\chi_{[\mu}n_{\nu]}$, where n_ν is any vector that is null on $\mathcal{H}^+$ and satisfies $n\cdot\chi = -1$ there. Then

$$\frac{c^4}{8\pi G}\oint_{S_{\mathcal{H}^+}} dS_{\mu\nu}\nabla^{[\mu}\chi^{\nu]} = \frac{c^4}{8\pi G}\oint_{S_{\mathcal{H}^+}} dS\chi_{[\mu}n_{\nu]}\nabla^\mu\chi^\nu = \frac{c^2}{8\pi G}\oint_{S_{\mathcal{H}^+}} \kappa dS \tag{9.9.29}$$

where we have used (9.9.5). As the surface gravity of the horizon is constant, we have the remarkable result (for the Kerr black hole) that

$$Mc^2 = \frac{\kappa}{2\pi}\left(\frac{c^2 A_{\mathcal{H}}}{4G}\right) + \omega_+ J_{\mathcal{H}} \tag{9.9.30}$$

where $A_{\mathcal{H}}$ is the area of the horizon (we have discarded the source term because we are in a vacuum).

The source term cannot be discarded when $T^\mu{}_\nu$ does not vanish, which is what happens when the black hole is charged. In that case, we must include the stress tensor term in (9.9.26). The integral of (9.4.5) over the spatial hypersurface gives $Q_e\Phi(r_+)$, so we get the general result

$$Mc^2 = \frac{\kappa}{2\pi}\left(\frac{c^2 A_{\mathcal{H}}}{4G}\right) + Q_e\Phi_{\mathcal{H}} + \omega_+ J_{\mathcal{H}}. \tag{9.9.31}$$

9.9.4 Second and Third Laws

The analogy with thermodynamics is made complete if it can be said that:

- *the area of the Killing horizon never decreases in any process.*

In fact, this statement is true on the classical level and was proved by Hawking[29]. This proof is summarized in the Appendix.

The third law of of black hole thermodynamics is similar to the Nernst statement of the third law in ordinary thermodynamics:

- *the surface gravity of a black hole horizon cannot be reduced to zero in a finite advanced time.*

To see what this means, notice that κ would be zero for either the extremal charged or rotating black holes, *i.e.,* $r_s = 2r_Q$ in the charged case and $r_s = 2a$ in the rotating case. The extremal black holes are boundaries between "covered" singularities (those that are protected by an event horizon) and "naked" ones (with no event horizon for protection). Naked singuarities present a problem for describing time evolution in their futures because no initial data can be given on a singular surface. The third law can be viewed as prohibiting the existence of naked singularities, a statement about "cosmic censorship". However, there are many theoretical counterexamples to this and the issue remains open. There is no statement of the third law analogous to Planck's.

9.9.5 Information Loss

The association of the surface gravity with temperature and the horizon area with entropy may seem strange at first glance. After all, we have used the classical, time reversal invariant, equations of a field theory to obtain these space-times and there was never a hint of anything approaching (Fermi's) master equation along the way. Moreover, the mechanical dimension of the surface gravity is acceleration and that of the horizon area is, well, area. Finally, thermodynamic objects are supposed to radiate but classical black holes are black. In yet another brilliant work, Hawking showed that *quantum* black holes do radiate and that the temperature that can be associated with this radiation is indeed proportional to their surface gravity. In fact, using Planck's constant, dimensional analysis shows that temperature can be related to the acceleration by

$$k_B T = \frac{\hbar\kappa}{2\pi c} \tag{9.9.32}$$

and the thermodynamic entropy (called the black hole horizon entropy) to the horizon area by

$$S = k_B \frac{c^3 \mathcal{A}}{4\hbar G}. \tag{9.9.33}$$

[29] S. W. Hawking, Phys. Rev. Lett., 26 (1971) 1344.

These relations are entirely compatible with the statement of the first law:

$$\frac{\kappa}{2\pi} d\left(\frac{c^2\mathcal{A}}{4G}\right) = d(Mc^2) + \text{đ}W. \tag{9.9.34}$$

If a quantum black hole does radiate, then the area of a black hole horizon must actually decrease in some processes and such processes would violate the underlying assumptions of Hawking's proof of the area law, but they are possible in the quantum theory. However, the black hole cannot be thought of as an isolated system and one must account for the entropy of other fields that are also present in space-time. In that case, one may expect that a *generalized* second law of black hole thermodynamics, which states that

- *the total entropy, the sum of the radiation entropy and the black hole horizon entropy, does not decrease*

would remain inviolate. All of this brings about what is perhaps one of the greatest puzzles of modern theoretical physics: information loss.

This is most easily seen in the case of the Schwarzschild black hole. Suppose that the black hole evaporates (radiates away its energy) over time, shrinking as it does so. Its temperature keeps rising in this process, naïvely implying that the process accelerates. If it continues, the black hole should end up evaporationg completely into *thermal* radiation. If what is left is only a density matrix and not a wave function then the unitarity of the quantum theory is challenged. However, unitarity may be preserved if some unknown mechanism (arising, perhaps, from a more complete theory of quantum gravity) terminates the evaporation at some stage leaving behind a highly degenerate remnant. Alternatively, it is possible that Hawking's radiation from the black hole is not thermal in a more complete treatment. It should also be remembered that this problem arises because of the presence of a Killing horizon, which leads to a third possibility: quantum gravity may *prevent* the formation of a horizon in gravitational collapse.

Chapter 10

Time Dependent Solutions

10.1 Gravitational Collapse

We begin with the simplest form of matter obeying the energy conditions, *i.e.,* time-like, pressureless dust, whose energy momentum tensor is given simply by

$$T_{\mu\nu} = \rho U_\mu U_\nu \tag{10.1.1}$$

where ρ represents the mass density of the dust in its comoving frame and U^μ its four velocity. This stress tensor can be derived by varying the action

$$S_{\text{dust}} = -\frac{1}{2}\int d^4x\sqrt{-g}\ \rho(x)\left(g_{\alpha\beta}U^\alpha U^\beta + c^2\right) \tag{10.1.2}$$

with respect to $g_{\mu\nu}$ and ρ. The first returns the energy momentum tensor and the second yields the constraint $U^2 + c^2 = 0$, which enforces the time-like nature of the dust. Dust obeys all the energy conditions trivially if $\rho > 0$. We will look for spherically symmetric solutions of the Einstein equations sourced by (10.1.1). The equations we want to solve will therefore represent a spherically symmetric dust cloud (a dust ball) that may be collapsing under its own gravitational field, or expanding by virtue of its energy. Their solutions were first obtained by G. LeMaître in 1933 and later developed by Tolman (1934) and Bondi (1947).

Although the spherically symmetric metric can be given in terms of just two functions as in (9.1.8), it is more convenient to express it in terms of three functions,

$$ds^2 = -g_{\mu\nu}dx^\mu dx^\nu = c^2 e^{2\mu(t,r)}dt^2 - e^{2\lambda(t,r)}dr^2 - R^2(t,r)d\Omega^2, \tag{10.1.3}$$

and use the additional freedom to choose a suitable frame. The function $R(t,r)$ is the physical (area) radius and our objective is to determine the functions $\mu(t,r)$, $\lambda(t,r)$ and

$R(t,r)$ from Einstein's equations, once a suitable choice of frame has been made. We take the **comoving frame**, ($U^0 = e^{-\mu(t,r)}$, $U^i = 0$) and begin by exploiting the conservation of energy and momentum, $\nabla_\mu T^{\mu\nu} = 0$. With the metric in (10.1.3) conservation leads to two equations

$$\dot{\rho} + \rho\left(\dot{\lambda} + \frac{2\dot{R}}{R}\right) = 0$$
$$\rho\mu' = 0 \tag{10.1.4}$$

where the overdot refers to a time derivative and the prime to a space derivative. The first can be integrated in terms of one arbitrary function of r,

$$\rho(t,r) = \frac{Q(r)e^{-\lambda(t,r)}}{R^2(t,r)}. \tag{10.1.5}$$

The second says that $\mu = \mu(t)$, but this is equivalent to the statement that it is possible to make the gauge choice $\mu = 0$, for if μ is a function of time, we could define a new time coordinate, $t' = \int^t e^{\mu(t)}dt$. Thus we take

$$ds^2 = c^2dt^2 - e^{2\lambda(t,r)}dr^2 - R^2(t,r)d\Omega^2 \tag{10.1.6}$$

and

$$\rho(t,r) = \frac{Q(r)e^{-\lambda(t,r)}}{R^2(t,r)}. \tag{10.1.7}$$

Coordinates for which $g_{0i} = 0$ and $g_{00} = -c^2$ are called **synchronous**. In synchronous coordinates, the proper time, t, is the same for all comoving observers. "Synchronous", therefore refers to the fact that all comoving observers' clocks are synchronized.

10.1.1 Dust Einstein Equations

Of the four non-trivial Einstein equations, consider

$$G_{01} = 0 \Rightarrow \dot{\lambda}R' - \dot{R}' = 0 \tag{10.1.8}$$

which can readily be put in the form $e^{\lambda}\partial_t(e^{-\lambda}R') = 0$ and whose solution is clearly

$$e^{\lambda(t,r)} = \frac{R'(t,r)}{\sqrt{H(r)}}, \tag{10.1.9}$$

where $H(r)$ is a positive definite but otherwise arbitrary function of r. Then we can write the mass density as

$$\rho(t,r) = \frac{F'}{4\pi c^2 R^2 R'} \tag{10.1.10}$$

where we defined $Q(r) = F'(r)/4\pi c^2\sqrt{H(r)}$ for reasons that will soon become clear. The energy conditions require that $\rho \geq 0$, which means that $F'/R' \geq 0$. It seems reasonable to impose the condition that the area radius increases with the label coordinate, so that $R' > 0$. Then we must also require that $F' \geq 0$ to ensure that the positive energy condition holds.[1] Also with this identification, the equation

$$G_{00} = \frac{8\pi G}{c^4} T_{00} = 8\pi G\rho \tag{10.1.11}$$

gives

$$[R\{\dot{R}^2 + c^2 - c^2 H\}]' = \frac{2GF'}{c^2} \tag{10.1.12}$$

or

$$\dot{R}^2 + c^2(1 - H) = \frac{2GF}{c^2 R}. \tag{10.1.13}$$

In addition we have the equations,

$$\begin{aligned} &G_{11} = 0 \Rightarrow 2R\ddot{R} + \dot{R}^2 + c^2(1 - H) = 0 \\ &G_{22} = 0 \Rightarrow (2R\ddot{R} + \dot{R}^2 - c^2 H)' = 0 \\ &G_{33} = \sin^2\theta G_{22} \end{aligned} \tag{10.1.14}$$

Of the last three only one is independent and we take that to be the first. It is compatible with (10.1.13), for if we multiply (10.1.13) throughout by R and take a derivative with respect to t, we get $\dot{R}G_{11} = 0$. Thus the "equation of motion" for the area radius is

$$\dot{R} = \pm\sqrt{c^2(H - 1) + \frac{2GF}{c^2 R}} \tag{10.1.15}$$

and the sign that is chosen will depend on whether we wish to describe an expanding or a collapsing dust ball: the positive sign represents an expanding dust ball and the negative sign a collapsing one.

At this stage, it's a good idea to take stock of where we are: we have seen that the most general solutions of the gravity-dust system are given in terms of two functions, $F(r)$ and $f(r) = H(r) - 1$. The function $F(r)$ must be non-negative and monotonically increasing; the function $f(r)$ must be everywhere larger than -1. The metric is given by

$$ds^2 = c^2 dt^2 - \frac{R'^2(t, r)}{1 + f(r)} dr^2 - R^2(t, r) d\Omega^2, \tag{10.1.16}$$

[1] Situations in which $R'(t, r) = 0$ for any r must be excluded since they would correspond to a singular energy density for that r. These are called "shell crossing singularities" and are generally avoided by carefully choosing $F(r)$ and $f(r)$.

where the area radius satisfies

$$\frac{1}{2}\dot{R}^2(t,r) - \frac{GF(r)}{c^2 R(t,r)} = \frac{c^2}{2} f(r). \tag{10.1.17}$$

Notice that (10.1.17) has the traditional Newtonian form: "Kinetic Energy" + "Potential Energy" = "Total Energy", if we interpret $M(r) = F(r)/c^2$ as the gravitational mass contained within r, and $c^2 f(r)/2$ as the total energy contained within that radial label. The gravitational mass energy of the dust is determined by $F(r)$ according to (10.1.10). These solutions obviously become singular when $R(t,r) = 0$ and $F'(r) \neq 0$, or when $R'(t,r) = 0$ and $F'(r) \neq 0$. A "**central singularity**" forms when $R(t,r) = 0$ and $F'(r) \neq 0$. A "**shell crossing**" singularity occurs when $R'(t,r) = 0$ and $F'(r) \neq 0$. Shell crossing singularities can be avoided by appropriate choices of the initial data (the functions $F(r)$ and $f(r)$) but a central singularity cannot be avoided without the introduction of a cosmological constant.

Let us now turn to the interpretation of these solutions. In particular we want to give a physical meaning to $F(r)$ and $f(r)$ in terms of familiar concepts and then explore the properties of the space-times described by the solutions.

10.1.2 Meaning of the Integration Functions

The function $M(r)$ can be obtained by integrating the energy density (10.1.10) at a fixed time,

$$M(r) = \frac{F(r)}{c^2} = M_0 + \int_0^r dr\ R^2(t,r) R'(t,r) \rho(t,r). \tag{10.1.18}$$

where $M_0 = F(0)/c^2$ is a constant. If M_0 is non-vanishing, it leads to a singular initial configuration, so it is usually set to zero. Now, according to (10.1.17), $M(r)$ is naturally interpreted as that total gravitational mass energy contained within the shell labeled by r; it is called the **Misner-Sharp** mass function or simply the **mass function**. The gravitational mass contained within any shell does not change with time.

The gravitational mass is not the same thing as the sum of shell masses within the body. Over a constant time hypersurface, and within a volume bounded by r, the latter quantity is given by

$$M_{\text{shell}}(r) = \int_0^r dr\ \sqrt{\gamma(t,r)}\rho(t,r) = \int_0^r dr\ \frac{R^2(t,r)R'(t,r)}{\sqrt{1+f(r)}}\rho(t_0,r) \tag{10.1.19}$$

or, simply

$$M_{\text{shell}}(r) = \int_0^r dr\ \frac{F'(r)/c^2}{\sqrt{1+f(r)}}. \tag{10.1.20}$$

Evidently the sum of shell masses is also constant in time, but, depending on the sign of $f(r)$, the sum of shell masses may be greater than ($f(r) < 0$), equal to ($f(r) = 0$) or less than ($f(r) > 0$) the gravitational mass. If $f(r) < 0$ the system is gravitationally bound and the energy difference, $M_{\text{shell}}(r) - M(r)$, called the **mass defect**, represents the binding energy of the system which must be subtracted from the simple sum of shell masses to get the gravitational mass. If $f(r) = 0$, the **marginally bound** case, the two masses are equal. On the other hand, when $f(r) > 0$, the system is unbound and the gravitational mass is greater than the simple sum of shell masses indicating that the excess energy contributes to the gravitational mass.

The function $f(r)$, which represents the total energy in the Newtonian equation, is called the **energy function**.

10.1.3 Geodesics

Let us take a quick look at the radial geodesics of (10.1.16); these are described by the equations

$$\begin{aligned} &\frac{d^2t}{d\lambda^2} + \frac{R'\dot{R}'}{c^2(1+f)}\left(\frac{dr}{d\lambda}\right)^2 = 0 \\ &\frac{d^2r}{d\lambda^2} + \frac{2\dot{R}'}{R'}\left(\frac{dt}{d\lambda}\right)\left(\frac{dr}{d\lambda}\right) + \left[\frac{R''}{R'} - \frac{f'}{2(1+f)}\right]\left(\frac{dr}{d\lambda}\right)^2 = 0, \end{aligned} \tag{10.1.21}$$

where λ is any affine parameter and must be subjected to one of the two conditions,

$$\frac{dt}{ds} = +\sqrt{1 + \frac{R'^2}{c^2(1+f)}\left(\frac{dr}{ds}\right)^2}, \tag{10.1.22}$$

where s is the proper distance, for time-like geodesics, or

$$\frac{dt}{d\lambda} = \pm\frac{R'}{\sqrt{c^2(1+f)}}\frac{dr}{d\lambda}, \tag{10.1.23}$$

for null geodesics. The positive sign refers to outgoing geodesics and the negative sign to infalling ones. It is clear by inspection that the future directed time-like solution,

$$\frac{dt}{ds} = +1, \quad \frac{dr}{ds} = 0 \tag{10.1.24}$$

is valid for *any* time-like matter distribution, *i.e.,* any choice of $F(r)$ and $f(r)$, and that it represents the dust flow lines. Because of this, a shell at rest in a comoving frame will stay at rest and it is often useful to think of the dust ball as made up of "shells", each at rest in this frame and labeled by the coordinate r. Although each shell remains at a fixed r during its entire evolution, its physical distance from the center, given by the area radius, is a function of time.

If $dr/d\lambda \neq 0$, we can set $t = t(r)$ so

$$\frac{dt}{d\lambda} = t' \frac{dr}{d\lambda}, \quad \frac{d^2 t}{d\lambda^2} = t' \frac{d^2 r}{d\lambda^2} + t'' \left(\frac{dr}{d\lambda}\right)^2 \tag{10.1.25}$$

and using the second geodesic equation in the first,

$$\frac{d^2 t}{d\lambda^2} = t'' \left(\frac{dr}{d\lambda}\right)^2 - t' \left\{ \frac{2\dot{R}'}{R'} t' + \left[\frac{R''}{R'} - \frac{f'}{2(1+f)} \right] \right\} \left(\frac{dr}{d\lambda}\right)^2, \tag{10.1.26}$$

gives the equation for $t(r)$,

$$t'' - \frac{2\dot{R}'}{R'} t'^2 - \left[\frac{R''}{R'} - \frac{f'}{2(1+f)} \right] t' + \frac{R'\dot{R}'}{c^2(1+f)} = 0. \tag{10.1.27}$$

This is nonlinear and one can expect to be able to construct other families of geodesics, which will naturally depend on the properties of a particular solution of the field equations.

It may be verified by using (10.1.23) that (10.1.27) is automatic for all null geodesics, so (for null geodesics only) it is sufficient to find a solution of the first order equation (10.1.23). Outgoing null geodesics must satisfy

$$t' = \frac{R'}{\sqrt{c^2(1+f)}}. \tag{10.1.28}$$

Therefore, letting $t_n(r)$ represent a solution of this equation, the physical radius along an outgoing null geodesic will be $R_n(r) = R(t_n(r), r)$ and

$$\frac{dR_n}{dr} = \dot{R}\Big|_{t=t_n(r)} t_n' + R'\big|_{t=t_n(r)} = R' \left(1 + \frac{\dot{R}}{\sqrt{c^2(1+f)}}\right)_{t=t_n(r)}. \tag{10.1.29}$$

If the solution represents a collapsing dust ball, using the negative sign in (10.1.17) one finds

$$\frac{dR_n}{dr} = R' \left(1 - \sqrt{\frac{c^2 f + 2GF/c^2 R}{c^2(1+f)}}\right)_{t=t_n(r)}, \tag{10.1.30}$$

which clearly vanishes when $R_n = 2GF/c^4$ (assuming no shell crossing singularities). Thus outgoing light rays starting out, say, at the origin, may not cross the surface $R(t_{\text{ah}}, r) = 2GF/c^4$. This closed surface is called the **apparent horizon**. Since outgoing light itself may not cross this surface, it is a surface of no return. Anything that crosses the apparent horizon from the exterior will forever be confined within it. An apparent horizon forms during a collapse and not during an expansion.

10.1.4 Marginal Models

While the shell label does not change during collapse or expansion, the radius of the shell changes according to the evolution equation (10.1.13). This equation can actually be solved in general, but here, as an illustration, we will consider the **marginally bound** case, for which $f(r) = 0$. Eq. (10.1.13) integrates to

$$R^{3/2} = \psi(r)\left[1 \pm \frac{3}{2}\sqrt{\frac{2GF(r)}{\psi^2(r)}}(t - t_0)\right], \tag{10.1.31}$$

the function $\psi(r)$ being another integration function and t_0 being the initial time. We can exploit our freedom to choose r by asking for $R(t_0, r) = r$, which makes $\psi(r) = r^{3/2}$ and

$$R(t, r) = r\left[1 \pm \frac{3}{2}\sqrt{\frac{2GF(r)}{c^2 r^3}}(t - t_0)\right]^{2/3}. \tag{10.1.32}$$

If $f(r) \neq 0$ the solution is more complicated, but it still can be given implicitly.[2]

Notice that, in the case of collapse (the negative sign), $R(t, r)$ in (10.1.32) will approach zero in a finite amount of proper time,

$$R(t_f, r) = 0 \Rightarrow t_f - t_0 = \frac{2}{3}\sqrt{\frac{c^2 r^3}{2GF(r)}}. \tag{10.1.33}$$

This says that the shell labeled by r will approach zero physical radius in the time $t_s(r) = t_f - t_0$. $R(t, r) = 0$ is a singularity of the space-time and $t_s(r)$ is called the **singularity curve** of the collapse. Since all the shells will eventually approach zero physical radius, every dust ball with $f(r) = 0$ will eventually reach a singular configuration, there being no pressure to counter the collapse.[3] However, as we have seen in the previous subsection, an apparent horizon also forms. It turns out that the singularity *may* be covered by the apparent horizon, in which case no information (light ray) from the singularity will be accessible to observers outside it. Covered singularities are generally considered benign. If the singularity is not covered by the apparent horizon for any interval of time then it is called a **naked** singularity. Naked singularities are generally considered unacceptable because information from the singularity is capable, in principle, of reaching an external

[2] Problem: Determine a solution of (10.1.13) when $f(r) \neq 0$.

[3] Problem: Show that the condition for the avoidance of shell-crossing singularities in the marginal model is

$$t - t_0 \neq \sqrt{\frac{2c^2 r F(r)}{GF'^2(r)}}.$$

If this condition is violated for $t < t_f$, shell crossing singularities will form.

observer. But singularities are, by definition, unpredictable therefore a naked singularity would destroy causality in its future. The **cosmic censorship hypothesis** declares that one should simply consider all initial data leading to naked singularities as unphysical.

10.1.5 Constant Mass Function

The particular solution with $F(r) = F_0$ and $f(r) = 0$ is very interesting. A constant mass function should represent a point mass located at the center, which is just a Schwarzschild black hole. We will now show that this is indeed the case by finding the transformation from the LeMaître-Tolman-Bondi coordinates to Schwarzschild coordinates, but only in a restricted portion of space-time. Suppose we retain the same angular coordinates that we have been using and label the time and radial coordinates of the static metric $T(t,r)$ and $R(t,r)$ respectively. We express the transformation between the coordinates by

$$\begin{pmatrix} dT \\ dR \end{pmatrix} = \begin{pmatrix} \dot{T} & T' \\ \dot{R} & R' \end{pmatrix} \begin{pmatrix} dt \\ dr \end{pmatrix} \tag{10.1.34}$$

or

$$\begin{pmatrix} dt \\ dr \end{pmatrix} = \frac{1}{\|\|} \begin{pmatrix} R' & -T' \\ -\dot{R} & \dot{T} \end{pmatrix} \begin{pmatrix} dT \\ dR \end{pmatrix} \tag{10.1.35}$$

where $\|\|$ refers to the Jacobian of the transformation matrix. The transformation from the marginally bound solution to the new coordinates takes the form

$$ds^2 = \left(\frac{\partial t}{\partial T} dT + \frac{\partial t}{\partial R} dR\right)^2 - R'^2 \left(\frac{\partial r}{\partial T} dT + \frac{\partial r}{\partial R} dR\right)^2 - R^2 d\Omega^2 \tag{10.1.36}$$

which we rewrite, using (10.1.35), as

$$ds^2 = \frac{c^2}{\|\|^2}(R'dT - T'dR)^2 - \frac{R'^2}{\|\|^2}(-\dot{R}dT + \dot{T}dR)^2 - R^2 d\Omega^2. \tag{10.1.37}$$

We want to choose $T(t,r)$ so that the off diagonal terms in the metric vanish,

$$-c^2 T' + R'\dot{R}\dot{T} = 0. \tag{10.1.38}$$

But, for a constant mass function, the marginally bound solution is

$$R(t,r) = r\left(1 - \frac{3}{2}\sqrt{\frac{2GF_0}{c^2 r^3}}(t - t_0)\right)^{2/3} \tag{10.1.39}$$

so that

$$\dot{R} = -\sqrt{\frac{2GF_0}{c^2 R}}, \quad R' = \sqrt{\frac{r}{R}} \tag{10.1.40}$$

and therefore our condition for no off diagonal terms becomes

$$-c^2T' - \sqrt{\frac{2GF_0r}{c^2R^2}}\dot{T} = 0. \tag{10.1.41}$$

Now if we ask for a solution of the form $T(t,r) = t + f(R(t,r))$, where $f(R(t,r))$ is some function to be determined, then

$$-c^2\frac{df}{dR}R' - \sqrt{\frac{2GF_0r}{c^2R^2}}\left(1 + \frac{df}{dR}\dot{R}\right) = 0 \tag{10.1.42}$$

or

$$\frac{df}{dR} = \frac{\sqrt{\frac{2GF_0}{c^2R}}}{\left(\frac{2GF_0}{c^2R} - c^2\right)}. \tag{10.1.43}$$

The solution is readily determined and we define the Killing time, $T(t,r)$, as

$$T(t,r) = t - 2\sqrt{\frac{2GF_0R(t,r)}{c^6}} - \frac{2GF_0}{c^5}\ln\left(\frac{\sqrt{R(t,r)} - \sqrt{2GF_0/c^4}}{\sqrt{R(t,r)} + \sqrt{2GF_0/c^4}}\right) \tag{10.1.44}$$

so that

$$\dot{T} = \frac{R}{R - \frac{2GF_0}{c^4}}, \quad T' = \frac{\sqrt{\frac{2GF_0r}{c^6}}}{\frac{2GF_0}{c^4} - R}. \tag{10.1.45}$$

Then, using (10.1.34) and (10.1.35) it is straightforward to show that line element of (10.1.16) can be written in terms of the new coordinates (T,R) as

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2R}\right)dT^2 - \left(1 - \frac{2GM}{c^2R}\right)^{-1}dR^2 - R^2d\Omega^2, \tag{10.1.46}$$

where we have let $Mc^2 = F_0$.

10.1.6 Matching the Solutions to an Exterior

The solutions we have discussed cover only a portion of the space-time, which is the interior of the collapsing dust ball. We must also concern ourselves with the exterior of the dust ball (for example, the dust ball may represent a "star" with a definite boundary). The "interior" solution, must be matched to an "exterior" solution in such way that the solution over the entire space-time is at least $C^{(1)}$. This means that one should ensure the continuity of metric and its first derivatives, equivalently the first and second fundamental forms, across the boundary between the "interior" and the "exterior".

If the exterior is taken to be a vacuum, the solution is uniquely a Schwarzschild solution of some mass M, by Birkhoff's theorem. We will now show that every dust interior can be matched to a Schwarzschild exterior at the boundary of the dust ball. In the process we will also determine the Schwarzschild mass, M, and thus the mass of the entire dust ball as viewed by an external observer. As we have seen, the exterior Schwarzschild metric can be written as

$$ds^2 = c^2 V(\widetilde{R})dT^2 - V^{-1}(\widetilde{R})d\widetilde{R}^2 - \widetilde{R}^2 d\Omega^2, \quad V(\widetilde{R}) = \left(1 - \frac{2GM}{c^2\widetilde{R}}\right). \tag{10.1.47}$$

Let us choose the synchronous and comoving coordinates (t, θ, ϕ) to describe the boundary $\Sigma_b := r = r_b$ of the dust cloud. Then, as seen from the exterior, Σ_b will be described by the parametric equations $T = T(t)$, $\widetilde{R} = \widetilde{R}(t)$ and the metric of the boundary as viewed from the exterior will be

$$ds^2_{\Sigma_b +} = (c^2 V\dot{T}^2 - V^{-1}\dot{\widetilde{R}}^2)dt^2 - \widetilde{R}^2 d\Omega^2 \tag{10.1.48}$$

From the interior, its metric will be

$$ds^2_{\Sigma_b -} = c^2 dt^2 - R^2(r_b, t)d\Omega^2,$$

so identifying the angular coordinates and comparing the two tells us that the area radii must be equal on Σ_b, *i.e.*,

$$\widetilde{R}(t) = R(t, r_b) = R_b(t) \tag{10.1.49}$$

and

$$(c^2 V\dot{T}^2 - V^{-1}\dot{\widetilde{R}}^2) = c^2. \tag{10.1.50}$$

Thus

$$V\dot{T} = \sqrt{V(\widetilde{R}) + \dot{\widetilde{R}}^2/c^2} = \sqrt{1 + f(r_b) - \frac{2G}{c^2\widetilde{R}}\left(M - \frac{F(r_b)}{c^2}\right)} \stackrel{\text{def}}{=} A \tag{10.1.51}$$

which can be solved for $T(t)$ because we already know that $\widetilde{R}(t) = R_b(t)$. Thus we have matched the first fundamental form on Σ_b.

We now turn to the second fundamental form. Viewed from the interior the unit (outward) normal to Σ_b will be

$$n^-_\mu = \left(0, \frac{R'(t, r_b)}{\sqrt{1 + f(r_b)}}, 0, 0\right)_{r_b} \tag{10.1.52}$$

and the non-vanishing components of the extrinsic curvature of the boundary (using (7.8.22)) turn out to be

$$K^-_{\theta\theta} = \frac{K^-_{\varphi\varphi}}{\sin^2\theta} = \sqrt{1 + f(r_b)}R(t, r_b) = \sqrt{1 + f(r_b)}R_b(t). \tag{10.1.53}$$

Viewed from the exterior the condition defining the boundary can be found from the general transformation equations in (10.1.35) by setting $dr = 0$.[4] This gives the outward unit normal to the surface as

$$n_\mu^+ = (-\dot{\widetilde{R}}, \dot{T}, 0, 0) = (c\sqrt{A^2 - V(\widetilde{R})}, \frac{A}{V(\widetilde{R})}, 0, 0) \tag{10.1.54}$$

Given n_μ^+ it is straightforward to compute the extrinsic curvature, $K_{\mu\nu}^+$, viewed from the exterior. We find the non-vanishing components

$$\begin{aligned}
K_{TT}^+ &= \frac{c^2 G(F(r_b) - Mc^2)/(c^4 R_b)}{R_b}\sqrt{1 + f(r_b) + \frac{2G(F(r_b) - Mc^2)}{c^4 R_b}}, \\
K_{TR}^+ = K_{RT}^+ &= \frac{cG(F(r_b) - Mc^2)/(c^4 R_b)}{R_b\left(1 - \frac{2GM}{c^2 R_b}\right)}\sqrt{f(r_b) + \frac{2GF(r_b)}{c^4 R_b}} \\
K_{RR}^+ &= \frac{G(F(r_b) - Mc^2)/(c^4 R_b)\left(f(r_b) + \frac{2GF(r_b)}{c^4 R_b}\right)}{R_b\left(1 - \frac{2GM}{c^2 R_b}\right)^2 \sqrt{1 + f(r_b) + \frac{2G(F(r_b) - Mc^2)}{c^4 R_b}}} \\
K_{\theta\theta}^+ &= \frac{K_{\phi\phi}^+}{\sin^2\theta} = R_b\sqrt{1 + f(r_b) + \frac{2G(F(r_b) - Mc^2)}{c^4 R_b}}.
\end{aligned} \tag{10.1.55}$$

The angular part of $\widehat{K}^+$ agrees with its interior counterpart if $F(r_b) = Mc^2$. Furthermore, in this case, all the other components of $\widehat{K}^+$ will vanish so their counterparts in the (t, r) system will also vanish. Hence the components of the extrinsic curvatures on both sides of the boundary agree and we have confirmed that the mass of the Schwarzschild exterior geometry that smoothly joins with the collapsing geometry is the total *gravitational* mass of the dust ball.

10.1.7 Homogeneous Models

A very interesting class of models is obtained by requiring $R(t, r) = ra(t)$, where $a(t)$ is known as the scale factor.[5] According to (10.1.17) this would only be possible if

$$F(r) = \lambda r^3, \quad f(r) = -kr^2 \tag{10.1.56}$$

[4]Setting $r = r_b$ in (10.1.35) shows that $-\dot{\widetilde{R}}dT + \dot{T}d\widetilde{R} = 0$ on the hypersurface. Therefore, the unit normal to the surface will be proportional to $n_\mu^+ = (-\dot{\widetilde{R}}, \dot{T}, 0, 0)$. But one can check explicity that it n_μ^+ already has unit magnitude.

[5]A **homogeneous** space is translationally invariant. An **isotropic** space is rotationally invariant. A space that is *everywhere* isotropic is also homogeneous, but the converse is not true.

where λ and k are constants. We will take r to be dimensionless, so the scale factor has a length dimension. Then k is also dimensionless and $[\lambda] = ml^2/t^2$. The mass density of the dust ball depends only on time,

$$\rho(t) = \frac{3\lambda}{4\pi c^2 a^3(t)}, \tag{10.1.57}$$

the equation of motion can be written in the form,

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G\rho}{3}, \tag{10.1.58}$$

and the metric as

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\Omega^2\right]. \tag{10.1.59}$$

This is known as the Friedmann model.

Three distinct types of solutions exist, *viz.,* solutions with $k = 0$, $k > 0$ and $k < 0$. However, only the sign of k and not its magnitude is relevant since one can eliminate $|k|$ by simply rescaling the radial coordinate by the transformation $r \to \sqrt{|k|}r$ and letting $a(t) \to a(t)/\sqrt{|k|}$. It is sufficient to take $k = 0, \pm 1$. The curvature scalar of the spatial sections of the metric (10.1.59) is $R = 6k$. When $k = 0$ the spatial sections are all flat and they are positively and negatively curved when $k = +1$ and $k = -1$ respectively. The first case is obvious since when $k = 0$ the spatial section is just the metric of a flat three dimensional space in spherical coordinates. In order to gain some insight into the other two cases, consider the following coordinate transformation,

$$r \to \chi = \int^r \frac{dr}{\sqrt{1-kr^2}} = \begin{cases} \sin^{-1} r & k = +1 \\ r & k = 0 \\ \sinh^{-1} r & k = -1 \end{cases} \tag{10.1.60}$$

When $k = +1$, the coordinate r is not infinite in range but runs from zero to one. Otherwise, r extends over the entire non-negative real line. Consequently, χ runs from zero to $\pi/2$ when $k = +1$ and the spatial sections are compact, being equivalent to a three sphere but when $k = 0, -1$, they are non-compact. In terms of χ the metric can be expressed as

$$ds^2 = c^2dt^2 - a^2(t)\left[d\chi^2 + f^2(\chi)d\Omega^2\right] \tag{10.1.61}$$

where $f(\chi)$ is one of

$$f(\chi) = \begin{cases} \sin\chi & k = +1 & \chi \in [0, \frac{\pi}{2}) \\ \chi & k = 0 & \chi \in [0, \infty) \\ \sinh\chi & k = -1 & \chi \in [0, \infty) \end{cases} \tag{10.1.62}$$

The corresponding proper distance between two "radially" separated points is $a(t)\Delta\chi$, *i.e.,* it's time dependence arises only via the scale factor, $a(t)$, and the cross-sectional area subtended by a solid angle $d\Omega$ at time t is $dS(t) = a^2(t)f^2(\chi)d\Omega$.

Recalling the connection between the energy function and the constant k, we see that $k = 0$ describes a homogeneous, marginal expansion or collapse, $k = -1$ describes shells of positive total energy and $k = +1$ describes shells of negative total energy. If the dust ball is expanding, it will continue to expand (at an ever decreasing rate) forever if $k = 0$ or if $k = -1$, but it will eventually collapse if $k = +1$. If one transforms the time coordinate according to

$$t \to \eta = \int^t \frac{dt}{a(t)} \tag{10.1.63}$$

then $\eta(t)$ is called **conformal time** and the metric gets re-expressed as

$$ds^2 = a^2(\eta)\left[c^2 d\eta^2 - d\chi^2 - f^2(\chi)d\Omega^2\right], \tag{10.1.64}$$

which are conformally related to the metrics on (a) $\mathbb{R} \times \mathbb{S}^3$ ($k = +1$), (b) $\mathbb{R} \times \mathbb{R}^3$ ($k = 0$) and (c) $\mathbb{R} \times \mathbb{H}^3$ ($k = -1$) respectively.

Solutions to (10.1.58) can be chosen to be expanding ($\dot{a}(t) > 0$) or contracting ($\dot{a}(t) < 0$). The gravitational collapse of a homogeneous dust ball was first considered by J.R. Oppenheimer and H. Snyder (1939) and the collapsing solution with $k = +1$ is referred to as the **Oppenheimer-Snyder** model. Expanding (dust) solutions are studied in Cosmology, where they are found to be useful in describing the universe at late times. For the model with $k = 0$ the solution could simply be read off from (10.1.32), but it is often more useful to give it as a function of the conformal time. Noting that

$$\dot{a}(t) = \frac{da}{d\eta}\frac{d\eta}{dt} = \frac{a'}{a} \tag{10.1.65}$$

where the prime denotes a derivative with respect to η, we have the Friedman equation

$$a'^2 + c^2 k a^2 = \frac{2G\lambda a}{c^2}. \tag{10.1.66}$$

It follows that with $k = 0$, the solution is

$$a(\eta) = \left(\sqrt{a_0} \pm \sqrt{\frac{G\lambda}{2c^2}}(\eta - \eta_0)\right)^2 \tag{10.1.67}$$

where $a(\eta_0) = a_0$. The negative sign is for collapse and the positive sign is for expansion. On the other hand, with $k = 1$ (the Oppenheimer-Snyder model) we find the general solution

$$a(\eta) = \frac{G\lambda}{c^4}\left[1 \pm \sin\left\{c(\eta - \eta_0) \pm \sin^{-1}\left(\frac{c^4}{G\lambda}a_0 - 1\right)\right\}\right] \tag{10.1.68}$$

where $a(\eta_0) = a_0$ and, again, the negative sign is for collapse, the positive for expansion. With $k = -1$,

$$a(\eta) = \frac{G\lambda}{c^4}\left[\cosh\left\{c(\eta-\eta_0) + \cosh^{-1}\left(1 + \frac{c^4}{G\lambda}a_0\right)\right\} - 1\right], \tag{10.1.69}$$

with the same conventions as before. Collapse does not occur in this case.[6]

In the case of collapse, an apparent horizon will form at

$$a(\eta_{\rm ah}(\chi)) = 2G\lambda f^2(\chi)/c^4. \tag{10.1.70}$$

Each shell therefore arrives at the apparent horizon at a different time, but all the shells will collide into a central singularity the same time, given, in each case, by $a(\eta_s) = 0$.

An expanding dust ball is obtained by choosing the positive signs in the above expressions. In this case, because $a(\eta)$ is an increasing function, it will vanish at some past time and describes a **big bang** singularity. For example, in the marginally bound case ($k = 0$)

$$\eta_b - \eta_0 = -\sqrt{\frac{2c^2 a_0}{G\lambda}}. \tag{10.1.71}$$

If we agree to set the origin of time at the bang then $\eta_0 = \sqrt{\frac{2c^2a_0}{G\lambda}}$ represents the time at which the scale factor has the value $a(\eta_0)$, which could be taken to be the present moment. The solutions then become greatly simplified and we find

$$a(\eta) = \begin{cases} \frac{G\lambda}{c^4}[1 - \cos c\eta] & k = +1 \\ \frac{G\lambda}{2c^2}\eta^2 & k = 0 \\ \frac{G\lambda}{c^4}[\cosh c\eta - 1] & k = -1 \end{cases} \tag{10.1.72}$$

with commensurate simplifications in the expressions for $a(t)$.

10.2 Cosmological solutions

Cosmological solutions are geometries that describe an entire universe. They are founded on an extension of the Copernican principle. Recall that, in connection with the solar system, the Copernican principle simply states that Earth is in no privileged position in our solar system and that, in fact, the motion of the planets could be more simply understood and described if one treats the Sun and not the Earth as its center. The

[6] In each of these cases, the solutions may also be expressed in terms of the proper time by integrating (10.1.63). Find the general solutions in terms of t in each case.

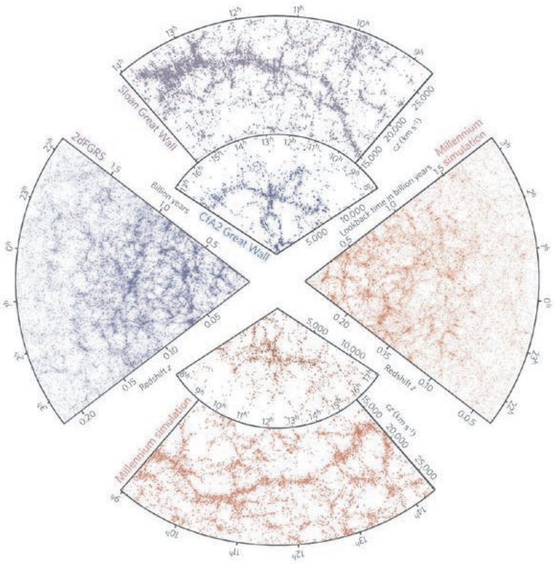

Figure 10.1: The large scale structure of the Universe.

Copernican principle represented, in its day, a major paradigm shift from the Ptolemaic system in which the Earth was taken as the center of the Universe. In cosmology this idea is extended to the universe as a whole: we assume that there is no privileged position within the universe and, by extension, no privileged observers.

Looking out from our position in the universe, we observe it to be roughly isotropic on distance scales larger than about 300 Mpc. By the Copernican principle, then, *every* observer should observe that the universe is roughly isotropic on the same distance scales and, if the Universe is everywhere isotropic, it follows that it must also be homogeneous. The two assumptions, that the universe is (a) homogeneous and (b) isotropic on large enough distance scales, form the **cosmological principle**. We will now begin a brief discussion of some exact solutions in General Relativity that are relevant to the study of cosmology, *i.e.,* that obey the cosmological principle.

10.2.1 Classical Perfect Fluid Cosmology

A perfect fluid is described by the energy momentum tensor

$$T_{\mu\nu} = p g_{\mu\nu} + \left(\rho + \frac{p}{c^2}\right) U_\mu U_\nu \tag{10.2.1}$$

where p is the pressure exerted by the fluid and ρ is its mass density. In a comoving frame this takes the simple form

$$T^\mu{}_\nu = \begin{pmatrix} -\rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{10.2.2}$$

and we seek solutions of Einstein's equations for a homogeneous expansion. If we take the metric to be of the form[7]

$$ds^2 = c^2 dt^2 - a^2(t)\left[d\chi^2 + f^2(\chi) d\Omega^2\right] \tag{10.2.3}$$

then Einstein's equations read

$$\begin{aligned} 3\left(\frac{\dot a}{a}\right)^2 + \frac{c^2(1 - f'^2 - 2ff'')}{f^2 a^2} &= 8\pi G\rho \\ -\frac{1}{c^2}(\dot a^2 + 2a\ddot a) + \frac{f'^2 - 1}{f^2} &= \frac{8\pi G}{c^4} p a^2 \\ -\frac{1}{c^2}(\dot a^2 + 2a\ddot a) + \frac{f''}{f} &= \frac{8\pi G}{c^4} p a^2 \end{aligned} \tag{10.2.4}$$

Since $f'^2 = 1 - kf^2$ and therefore $f'' = -kf$ we find two independent equations after rearranging terms,

$$\begin{aligned} \left(\frac{\dot a}{a}\right)^2 + \frac{kc^2}{a^2} &= \frac{8\pi G}{3}\rho \\ \frac{2\ddot a}{a} + \underbrace{\left(\frac{\dot a}{a}\right)^2 + \frac{kc^2}{a^2}} &= -\frac{8\pi G}{c^2} p \end{aligned} \tag{10.2.5}$$

and with the help of the first, which is the Friedmann equation we had for dust, the second can be put in the form

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right). \tag{10.2.6}$$

These two equations are not sufficient to determine the three functions, $\rho(t)$, $p(t)$ and $a(t)$ and must be supplemented by an equation of state.

The equation of state is generally taken to describe a **barotropic** flow, $p = p(\rho)$. A large class of fluids may be described by a *linear* barotropic flow,

$$p = c^2 w\rho, \tag{10.2.7}$$

[7] It can be shown that this is the most general homogeneous and isotropic metric.

where the weak energy condition requires that $w \geq -1$, the dominant energy condition requires, in addition, that $w \leq 1$ and the strong energy condition requires that $w > -1/3$. **Polytropic** flows, for which

$$p = c^2 w \rho^\alpha, \tag{10.2.8}$$

where $\alpha = 1 + 1/n$ have also been considered. The constant n is known as the polytropic index.

The conservation of energy and momentum can be obtained directly from the Bianchi identity, $\nabla_\nu T^\nu{}_\mu = 0$, but it is just as easily found by taking one derivative of the Friedmann equation and using the second equation in (10.2.5),

$$\frac{d\rho}{dt} + \frac{3\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0. \tag{10.2.9}$$

In relativistic flows, one must distinguish between the mass density, $\rho(t)$, and the *rest* mass density, $\varepsilon(t)$, which is defined in terms of the number density of particles, $n(t)$, and the rest mass of each particle, according to $\varepsilon = nm$. From the relativistic hydrodynamics of perfect fluids[8] we know that, given a fluid velocity U^μ, one can define the particle number density current, $N^\mu = nU^\mu$, which must be conserved if the total number of particles is constant. Combining the conservation of particle number with the conservation of energy and momentum then yields,

$$pd\left(\frac{1}{n}\right) + d\left(\frac{\rho c^2}{n}\right) = 0. \tag{10.2.10}$$

Because $1/n$ is the specific volume, v and $\rho c^2/n$ is the specific energy, u, this expression has the form of the first law of thermodynamics

$$Td\sigma = du + pdv \tag{10.2.11}$$

where T is the temperature and σ is the specific entropy. Of course, ideal fluids undergo isentropic flows, hence $d\sigma = 0$. The first law can also be written as

$$pd\left(\frac{1}{\varepsilon}\right) + d\left(\frac{\rho c^2}{\varepsilon}\right) = 0 \tag{10.2.12}$$

and therefore

$$d\rho = \left(\rho + \frac{p}{c^2}\right)\frac{d\varepsilon}{\varepsilon}. \tag{10.2.13}$$

This permits us to express (10.2.9) as

$$\frac{\dot{\varepsilon}}{\varepsilon} + 3\frac{\dot{a}}{a} = 0, \tag{10.2.14}$$

[8] See notes on Classical Mechanics

provided that $\rho + p/c^2 \neq 0$, which shows that

$$\varepsilon(t) = \varepsilon_0 \left(\frac{a_0}{a(t)}\right)^3 \tag{10.2.15}$$

and expresses the conservation of particle number in cosmology.

If the pressure is given in terms of the rest mass density, $p = p(\varepsilon)$, it is not difficult to see that (10.2.13) is a Bernouli equation giving the mass density, ρ in terms of the rest mass density,

$$\frac{d\rho}{d\varepsilon} - \frac{\rho}{\epsilon} = \frac{p(\varepsilon)}{c^2\varepsilon}, \tag{10.2.16}$$

whose solution is

$$\rho(\varepsilon) = A\varepsilon + \varepsilon \int^{\varepsilon} d\varepsilon' \frac{p(\varepsilon')}{c^2\varepsilon'^2}, \tag{10.2.17}$$

where A is an integration constant. On the other hand, if the pressure is given as $p = p(\rho)$, then the rest mass density is found from the mass density,

$$\varepsilon = \varepsilon_0 \exp\left[\int_{\rho_0}^{\rho} \frac{d\rho}{\rho + p(\rho)/c^2}\right], \tag{10.2.18}$$

where $\varepsilon(\rho_0) = \varepsilon_0$. In case $p(\rho) \ll \rho c^2$, we may approximate the relation above by

$$\varepsilon(\rho) = B\rho - B\rho \int^{\rho} d\rho' \ \frac{p(\rho')}{\rho'^2 c^2} \tag{10.2.19}$$

which can be compared with (10.2.17). Equations of state for barotropic flows may be specified in either form.

Returning to the linear barotropes in (10.2.7), we see that if $w = -1$ then $\rho = -p/c^2 = \Lambda$ (constant). This is, of course, the case of a cosmological constant, which one associates with space itself or a vacuum energy. On the other hand, with $w = 0$ we have pressureless dust with $\rho(t)$ given by (10.1.57). In general,

$$\rho(t) = \rho_0 \left(\frac{a(t)}{a_0}\right)^{-3(1+w)}, \tag{10.2.20}$$

where ρ_0 is a constant representing the dust energy density at time t_0, which can be taken to be the present, and $a_0 = a(t_0)$.[9]

[9] Problem: Show that the the first equation of (10.2.5) is the first integral of (10.2.6) by rewriting the latter as

$$\ddot{a} = \frac{1}{2}\frac{d\dot{a}^2}{da} = -\frac{4\pi G}{3}(1+3w)a\rho(a),$$

where $\rho(a)$ given in (10.2.20). Integrate the equation for $\dot{a}^2$ to recover the first. Interpret the integration constant.

The evolution of the scale factor will depend on whether space is flat, spherical or hyperbolic. We want to find general solutions of the equations of motion, but let us first transform them to conformal time, η. Then since $\dot{a} = a'/a$ and $\ddot{a} = a''/a - a'^2/a^2$, it follows that

$$\begin{aligned}\frac{a'^2}{a^2} + kc^2 &= \frac{8\pi G}{3}a^2\rho \\ \frac{a''}{a} - \frac{a'^2}{a^2} &= -\frac{4\pi G}{3}(1+3w)a^2\rho \end{aligned} \tag{10.2.21}$$

Defining the function $\mathfrak{h} = a'/a$, we have

$$\begin{aligned}\mathfrak{h}^2 + kc^2 &= \frac{8\pi G}{3}a^2\rho \\ \mathfrak{h}' &= -4\pi G(1+3w)a^2\rho. \end{aligned} \tag{10.2.22}$$

We can now use the first equation to eliminate $\rho(a)$ obtaining the following equation for $\mathfrak{h}$,

$$2\mathfrak{h}' + (1+3w)(\mathfrak{h}^2 + kc^2) = 0. \tag{10.2.23}$$

When $k = -1$, the weak energy condition requires that $\mathfrak{h} \geq c$. One possible non-trivial solution occurs when $k = -1$ and $\mathfrak{h} = c$. It describes an empty, hyperbolic space-time called the "Milne" universe (see the problem below). Again, $w = -1/3$ also implies that $\mathfrak{h}$ is constant for any value of k, but in this case the universe is not necessarily empty. If $\rho \neq 0$ and so long as $w \neq -1/3$,

$$\mathfrak{h}(\eta) = \begin{cases} c\tan\left[-\frac{c}{2}(1+3w)(\eta-\eta_0) + \tan^{-1}\frac{\mathfrak{h}_0}{c}\right] & k = +1 \\ \frac{2\mathfrak{h}_0}{(1+3w)\mathfrak{h}_0(\eta-\eta_0)+2} & k = 0 \\ c\coth\left[\frac{c}{2}(1+3w)(\eta-\eta_0) + \coth^{-1}\frac{\mathfrak{h}_0}{c}\right] & k = -1 \end{cases} \tag{10.2.24}$$

where $\mathfrak{h}_0 = \mathfrak{h}(\eta_0)$. We can integrate the above to find

$$a(\eta) = \begin{cases} a_0\left[\sqrt{\frac{\mathfrak{h}_0^2}{c^2}+1}\,\cos\left\{\frac{c}{2}(1+3w)(\eta-\eta_0) - \tan^{-1}\frac{\mathfrak{h}_0}{c}\right\}\right]^{\frac{2}{1+3w}} & k = +1 \\ a_0\left[\frac{1}{2}(1+3w)\mathfrak{h}_0(\eta-\eta_0)+1\right]^{\frac{2}{1+3w}} & k = 0 \\ a_0\left[\sqrt{\frac{\mathfrak{h}_0^2}{c^2}-1}\,\sinh\left\{\frac{c}{2}(1+3w)(\eta-\eta_0) + \coth^{-1}\frac{\mathfrak{h}_0}{c}\right\}\right]^{\frac{2}{1+3w}} & k = -1 \end{cases} \tag{10.2.25}$$

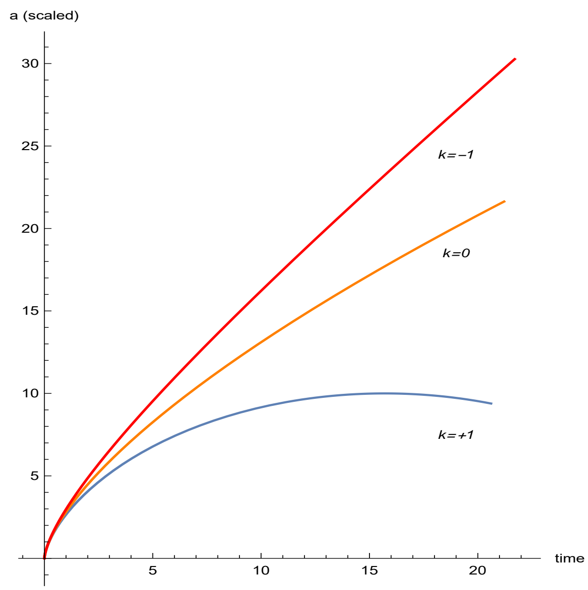

Figure 10.2: Behavior of the scale factor for dust with $k = 0, \pm 1$.

The special case of $w = -1/3$, for which the function $\mathfrak{h}$ is constant, $\mathfrak{h} = \mathfrak{h}_0$, has the solution $a(\eta) = a_0 e^{\mathfrak{h}_0(\eta - \eta_0)}$. Notice that, in this case, the scale factor only vanishes as $\eta \to -\infty$. Solutions satisfying $a(0) = 0$ can only be found for $w > -\frac{1}{3}$,

$$a(\eta) = \begin{cases} a_0 \left[\frac{1}{2}\left(\frac{\mathfrak{h}_0^2}{c^2} + 1\right)(1 - \cos(1+3w)c\eta)\right]^{\frac{1}{1+3w}} & k = +1 \\ a_0 \left[\frac{1}{2}(1+3w)\mathfrak{h}_0\eta\right]^{\frac{2}{1+3w}} & k = 0 \\ a_0 \left[\frac{1}{2}\left(\frac{\mathfrak{h}_0^2}{c^2} - 1\right)(\cosh(1+3w)c\eta - 1)\right]^{\frac{1}{1+3w}} & k = -1 \end{cases} \tag{10.2.26}$$

These generalize the solutions in (10.1.72) to linear barotropic ideal fluids and are shown in 10.2 for the case of dust, $w = 0$.[10]

In each of these cases, the proper time may be expressed in terms of the conformal time by using (10.1.63) but the expressions are complicated in all but the simplest case of

[10] Problem: Use the dust energy density in (10.1.57) to recover (10.1.72) for $w = 0$.

a spatially flat cosmology, for which

$$t = \int^{\eta} a(\eta) d\eta \sim \begin{cases} \eta^{\frac{3(1+w)}{1+3w}} & w \neq -1, -\frac{1}{3} \\ e^{\mathfrak{h}_0 \eta} & w = -\frac{1}{3} \\ \ln \mathfrak{h}_0 \eta & w = -1 \end{cases} . \tag{10.2.27}$$

Therefore, $a(t) \sim t^{2/3(1+w)}$, except when $w = -1$. In particular, for some standard forms of matter/energy we obtain

$$a(t) \sim \begin{cases} t^{2/3} & w = 0 \ \text{(dust)} \\ t^{1/2} & w = \frac{1}{3} \ \text{(radiation)} \\ t^{1/3} & w = 1 \ \text{(stiff matter)} \end{cases} \tag{10.2.28}$$

When $w = -1/3$ we find, using the solution for $a(\eta)$ given earlier, that

$$a(t) \sim \mathfrak{h}_0 t. \tag{10.2.29}$$

In this case there is neither acceleration nor deceleration of the scale factor and the universe expands at a constant rate.[11] Indeed this is the boundary between solutions with a decelerating scale factor ($\ddot{a} < 0$) and solutions with an accelerating scale factor ($\ddot{a} > 0$), which *always* occur when $w < -1/3$. In all cases, except when $w = -1$, the bang occurs at $t = 0$. When $w = -1$, which is the case of a cosmological constant, we find

$$a(t) \sim e^{(\mathfrak{h}_0/a_0)t}, \quad w = -1 \tag{10.2.30}$$

and the bang occurs *in the infinite past*.

10.2.2 Distance and Red-Shift

The proper spatial distance between two points is the distance between two events as measured *simultaneously* by an observer. Going back to (10.1.61) we see that the proper spatial distance between two points separated along the radial coordinate ($dt = d\Omega = 0$) depends on time via the scale factor according to $D(t) = a(t)\Delta\chi$ (the coordinate distance,

[11] Problem: Show that this same solution is obtained for the Milne universe. We end up with the metric solution

$$ds^2 = c^2 dt^2 - c^2 t^2 (d\chi^2 + \sinh^2 \chi d\Omega^2).$$

$\Delta\chi$, is often called the **comoving distance**). Consider an observer located at χ_0 and an object located at $\chi(t)$. The proper distance to the object, measured at time t, will be

$$D(t) = a(t)[\chi(t) - \chi_0] \tag{10.2.31}$$

and gives the proper velocity

$$v = \frac{dD(t)}{dt} = \dot{a}(t)[\chi(t) - \chi_0] + a(t)\dot{\chi}(t), \tag{10.2.32}$$

which is the sum of two parts. We call the first term on the right hand side the **Hubble flow** because it is a contribution that arises purely from the changing scale factor. The second term is called the **peculiar velocity** because it arises from changes in the position of the object in the comoving coordinate system, *i.e.,* changes in $\chi(t)$. We might write this as

$$v = \frac{\dot{a}(t)}{a(t)}D(t) + a(t)v_{\text{pec}}(t) \stackrel{\text{def}}{=} H(t)D(t) + a(t)v_{\text{pec}}(t) \tag{10.2.33}$$

where $H(t) = \dot{a}/a$ is called the **Hubble parameter**. The Hubble flow will be non-vanishing as long as the universe is not static. This contribution is referred to as **Hubble's Law**,

$$v = HD, \quad \text{when} \quad v_{\text{pec}} = 0. \tag{10.2.34}$$

Hubble's law is extremely powerful in determining distances to far-away stars and galaxies, assuming that their peculiar velocities are zero or negligible, since D is then determined directly from a measurement of v at the present time and the present value of the Hubble parameter according to $D = v_0/H_0$.

But how is v determined? Everything we know about the universe is determined from the electromagnetic waves reaching us from distant objects. In traveling through space, light emitted by any source will suffer a red-shift due to the universe's expansion. To see this, let λ be the wavelength emitted by the source at the time, t, that it was emitted. If there is no peculiar motion of the object then λ may be written in terms of of the scale factor as $\lambda = a(t)\Delta\chi$, where $\Delta\chi$ represents the coordinate separation between successive crests of the wave. Likewise, if λ_0 is the wavelength received at the present time, t_0, then $\lambda_0 = a(t_0)\Delta\chi$ and then

$$\frac{\lambda_0}{\lambda} = \frac{a(t_0)}{a(t)} \tag{10.2.35}$$

and one defines the red-shift factor by

$$z = \frac{\lambda_0 - \lambda}{\lambda} = \frac{a(t_0)}{a(t)} - 1. \tag{10.2.36}$$

It is positive for an expanding universe and negative for a contracting universe when we would expect blue-shifts, not red-shifts, for distant objects. For close enough objects we may expand $a(t)$ in a power series,

$$a(t) = a_0 \left[1 + H_0(t - t_0) + \ldots\right] \tag{10.2.37}$$

where $a_0 = a(t_0)$. The time $t_0 - t$ is the time it takes for light to travel from the object to the observer. From the equation for null geodesics,

$$c \int_t^{t_0} \frac{dt'}{a(t')} = \chi - \chi_0 \tag{10.2.38}$$

and, if we take the scale factor to be approximately constant in this time, $a(t) \approx a_0$, then $c(t_0 - t) \approx a_0(\chi - \chi_0) = D_0$ and the red-shift factor becomes

$$z \approx H_0 D_0/c = v_0/c. \tag{10.2.39}$$

The red-shift can be directly determined by comparing the spectrum of the received radiation with the known spectra of elements. Assuming there is an independent way to determine the proper distance D, the slope of red-shift distance diagrams for nearby stars will yield the present value of the Hubble parameter, which is today found to be

$$H_0 = 70.2 \pm 1 \text{ (km/s)/Mpc}. \tag{10.2.40}$$

From Hubble's law, we then conclude that the proper distance to the "edge" of the observable universe is $D_0 = c/H_0 \approx 4.27$ Gpc or 13.9 Gly. This is called the **Hubble radius** of the universe.

When we consider that only a miniscule fraction of the electromagnetic energy emitted by sources in the universe is actually collected in our telescopes, we realize that the another relevant measurable quantity is the flux of energy arriving at the present time on Earth. The inverse square law for the flux gets modified by the red-shift for three reasons, *viz.,*

- the proper area of a sphere centered at the source and passing through Earth at the present time is different from the proper area of a sphere centered on Earth and passing through the source in the past. The difference is captured by the **reciprocity theorem**.[12]

- the intrinsic luminosity of the source differs from its measured luminosity because of the red-shifting of photons and

[12] Etherington, I.M.H., "On the definition of distance in general relativity", Phil. Mag. Ser. 7 **15** (1933) 761.

- the expansion of the universe slows down the arrival rate of the photons leading to an additional drop in the measured luminosity.

For an observer at the present time, t_0, the area radius is defined to be the distance R_0 for which a *past* directed bundle of null rays, subtending an angle of $d\Omega_0$ at the observer has a cross-sectional area $dS_0 = R_0^2 d\Omega_0$ at R_0 (which is, say, the area radius to some observed galaxy G). On the other hand, the *galaxy* area radius R_G is defined as the radius at which a *future* directed bundle of null rays subtending a solid angle Ω_G would have a cross-sectional area $dS_G = R_G^2 d\Omega_G$ at the observer at time t_0. Because of the cosmic expansion, the two radii are not the same; one has

$$R_G = a(t_0)f(\chi), \quad R_0 = a(t)f(\chi) \tag{10.2.41}$$

so they are related to one another by

$$R_G = (1+z)R_0. \tag{10.2.42}$$

While the observer area radius is, in principle, measurable via direct astronomical observations, the galaxy area radius is not.

The **luminosity radius**, D_L, is defined as the distance satisfying the inverse square law

$$\mathcal{F}_0 = \frac{L}{4\pi D_L^2}, \tag{10.2.43}$$

where L is the *intrinsic* luminosity (also known as the bolometric luminosity) of the emitter at t and $\mathcal{F}_0$ is the *apparent* flux, *i.e.,* the luminosity per unit area received by the observer at t_0. The reason for this peculiar mixing of intrinsic and observed quantities is that the intrinsic luminosity is often known from theoretical considerations as, for example, in the case of the "standard candles" such as Type IA supernovæ. However, the received flux will depend on the received luminosity and the area radius as measured from the source according to

$$\mathcal{F}_0 = \frac{L_0}{4\pi R_G^2} = \frac{L_0}{4\pi R_0^2(1+z)^2} \tag{10.2.44}$$

but L_0 is not the same as the intrinsic luminosity, L, which also suffers a change due to the expansion as we can see from the following argument: photons emitted with a frequency ν in time intervals δt at time t are received with a frequency ν_0 in time intervals δt_0 at the present time. Wavelengths and time intervals are related by

$$\frac{\lambda_0}{\lambda} = \frac{\nu}{\nu_0} = \frac{\delta t_0}{\delta t} = 1+z, \tag{10.2.45}$$

therefore the emitted power (or Luminosity) is related to the received power by

$$\frac{L_0}{L} = \frac{\nu_0 \delta t}{\nu \delta t_0} = (1+z)^{-2} \quad \Rightarrow \quad L_0 = \frac{L}{(1+z)^2} \tag{10.2.46}$$

and it follows that the received flux, expressed in terms of the intrinsic luminosity of the emitter and the observer radius is

$$\mathcal{F}_0 = \frac{L_0}{4\pi R_G^2} = \frac{L}{4\pi R_0^2(1+z)^4}. \tag{10.2.47}$$

Hence we find $D_L = R_0(1+z)^2 = a_0 f(\chi)(1+z)^2$. Assuming thermal radiation, the wavelength at which the received flux is maximum gives the apparent temperature of the star according to Wein's law, $\lambda_{0,\max}T_0 = \text{const}$. As the constant on the right depends only on the fundamental constants, the intrinsic temperature of the source is then

$$T = (1+z)T_0. \tag{10.2.48}$$

Knowing the intrinsic temperature of the source, one may employ the Stefan-Boltzmann law, $L = \sigma A T^4$, where $A = \pi D_A^2$ is its surface area and D_A is its angular diameter, to determine the intrinsic luminosity of the source assuming that its angular diameter can be measured. This requires an accurate measurement of the angle, $\delta\theta$, subtended by the source at the observer, in terms of which $D_A = R_0\delta\theta = a(t)f(\chi)\delta\theta$.

10.2.3 Many Species

The universe consists not simply of one type of energy (by which we mean satisfying a particular equation of state) but of different types. In different epochs one type of energy may have dominated its evolution only to be superseded by another type at a later time. This follows by the fact that, as the universe expands, the energy density of the various species falls as $\rho(a) \sim a^{-3(1+w)}$ so, for example, the density of pressureless dust falls off as a^{-3} whereas that of radiation as a^{-4}. This implies that, at some stage, the universe will cease to be dominated by radiation energy and become dominated by dust. On the other hand, the energy density of the cosmological constant never changes, so the universe will *ultimately* become dominated by this vacuum energy (however small, but provided it is non-vanishing) no matter what its initial conditions were.

Consider the cosmological equations with many species, singling out the special case of the cosmological constant, which lies at the very boundary of what is permitted by the weak energy principle, and writing the Friedmann equations as

$$\begin{aligned} H^2 + \frac{kc^2}{a^2} &= \frac{8\pi G}{3}\left(\sum_i \rho_i + \Lambda\right) \\ \dot{H} + H^2 &= -\frac{4\pi G}{3}\left[\sum_i (1+3w_i)\rho_i - 2\Lambda\right] \end{aligned} \tag{10.2.49}$$

In a flat universe it must hold that

$$\sum_i \rho_i + \Lambda = \frac{3H^2}{8\pi G} \stackrel{\text{def}}{=} \rho_{\text{cr}} \tag{10.2.50}$$

which defines a "critical energy density" (including all forms of energy) of the universe for which it would be flat. We use this to define the fraction of the critical energy density contributed by each component species present as follows:

$$\Omega_i = \frac{\rho_i}{\rho_{\rm cr}}, \quad \Omega_\Lambda = \frac{\Lambda}{\rho_{\rm cr}} \tag{10.2.51}$$

All quantities above are time dependent, of course, so in the case of a non-flat universe

$$\frac{3kc^2}{8\pi G a^2 \rho_{\rm cr}} = \sum_i \Omega_i + \Omega_\Lambda - 1. \tag{10.2.52}$$

If we define

$$\Omega_k = -\frac{3kc^2}{8\pi G a^2 \rho_{\rm cr}},$$

the Friedmann equations become the statements that

$$\begin{aligned} &\sum_i \Omega_i + \Omega_k + \Omega_\Lambda = 1 \\ &\frac{\dot H}{H^2} + 1 = -\frac{1}{2}\left[\sum_i (1+3w_i)\Omega_i - 2\Omega_\Lambda\right] \end{aligned} \tag{10.2.53}$$

The fractions at the present time will be referred to, as usual, by the suffix "0". Current observations of the Cosmic Microwave Background (CMB) indicate that the universe is flat, $\Omega_k = 0$, and

$$\Omega_{m0} = 0.31, \quad \Omega_{r0} = 10^{-4}, \quad \Omega_{\Lambda 0} = 0.69 \tag{10.2.54}$$

where Ω_m refers to pressureless matter and Ω_r to radiation.[13] Pressureless matter is further categorized as baryonic (atoms) and non-baryonic (dark) matter,

$$\Omega_m = \Omega_b + \Omega_{\rm dm}, \quad \Omega_{b0} = 0.04, \quad \Omega_{\rm dm0} = 0.27 \tag{10.2.55}$$

so the vacuum energy, by all standards a bizarre form of energy that we have no direct (*i.e.,* non-gravitational) means to observe and study, clearly dominates the energy budget of our universe at the present time. This is the "Dark Energy" problem. Furthermore, only 4% of the matter energy content of the universe has been accounted for and not much is known of the properties of the remaining 27%. This is the "Dark Matter" problem. They are two of the most pressing problems in theoretical physics today.

[13] How data on the CMB is analyzed is beyond the scope of these notes. A clear description may be found in Weinberg, "Cosmology", Oxford University Press (2008).

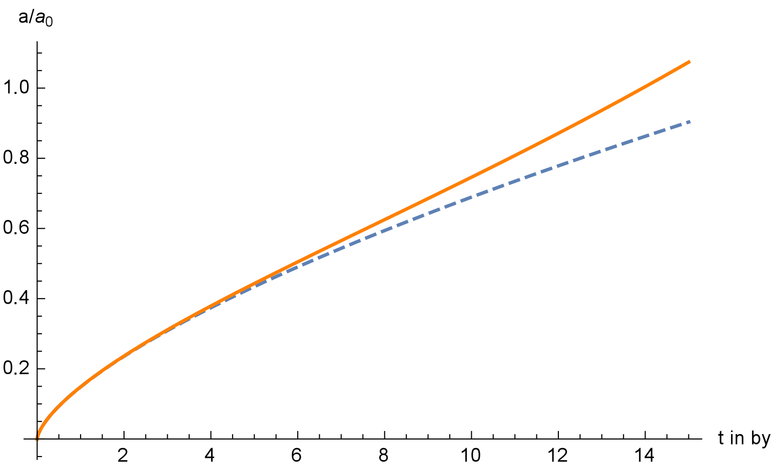

Figure 10.3: The scale factor in ΛCDM (solid) and CDM (dashed).

10.2.4 The ΛCDM Model

Let us now consider a two component, flat universe with a cosmological constant (Λ) and pressureless dust (cold dark matter, CDM). The equations we want to solve are then

$$\Omega_m + \Omega_\Lambda = 1$$

$$\dot{H} + H^2 = -\frac{H^2}{2}[\Omega_m - 2\Omega_\Lambda] \tag{10.2.56}$$

Using the first equation, the second may be put in the form

$$2\dot{H} + 3H^2 = 3\Omega_\Lambda H^2 = 8\pi G\Lambda \tag{10.2.57}$$

From (10.2.56), the absence of CDM implies that $\dot{H} = 0$ and we recover the $w = -1$ result of (10.2.30). For CDM obeying the weak energy condition, we must take $3H^2 > 8\pi G\Lambda$ we easily solve (10.2.57) to find

$$H(t) = \sqrt{\frac{8\pi G\Lambda}{3}} \coth\left[\sqrt{6\pi G\Lambda}\,(t - t_0) + \coth^{-1}\sqrt{\frac{3H_0^2}{8\pi G\Lambda}}\right] \tag{10.2.58}$$

Integrating once more gives the scale factor

$$a(t) = a_0\left[\sqrt{\frac{3H_0^2}{8\pi G\Lambda} - 1}\ \sinh\left\{\sqrt{6\pi G\Lambda}\,(t - t_0) + \coth^{-1}\sqrt{\frac{3H_0^2}{8\pi G\Lambda}}\right\}\right]^{2/3}, \tag{10.2.59}$$

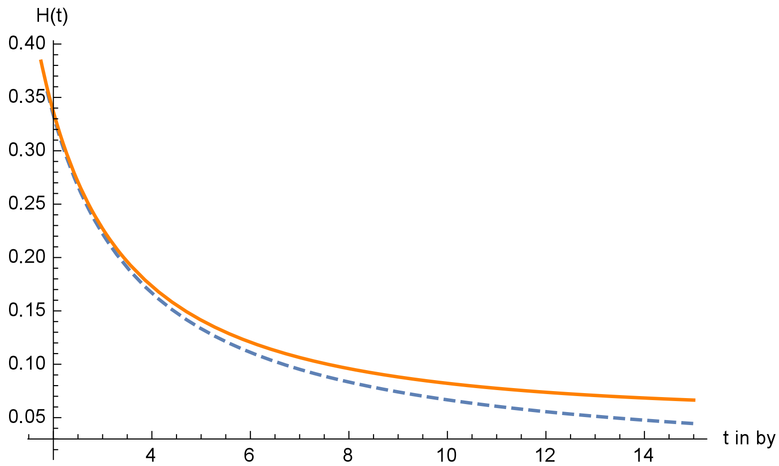

Figure 10.4: The Hubble parameter in ΛCDM (solid) and CDM (dashed).

so the solution with $a(0) = 0$ becomes

$$a(t) = a_0 \left[\sqrt{\frac{3H_0^2}{8\pi G\Lambda} - 1} \ \sinh \sqrt{6\pi G\Lambda}\ t \right]^{2/3}. \tag{10.2.60}$$

The solution reduces to ordinary CDM at "early" times,

$$a(t) \approx a_0 \left(1 - \frac{8\pi G\Lambda}{3H_0^2}\right)^{1/3} \left(\frac{3H_0 t}{2}\right)^{2/3}, \quad t \ll (6\pi G\Lambda)^{-1/2}, \tag{10.2.61}$$

with a negative acceleration, $\ddot{a} < 0$, but, at late times, it grows exponentially,

$$a(t) = \frac{a_0}{2^{2/3}} \left(\frac{3H_0^2}{8\pi G\Lambda} - 1\right)^{1/3} e^{\sqrt{\frac{8\pi G\Lambda}{3}}\ t}, \quad t \gg (6\pi G\Lambda)^{-1/2} \tag{10.2.62}$$

To estimate the time at which the cosmological constant begins to dominate it is sufficient to determine the inflection point in $a(t)$, which occurs at $\tau_0 = \cosh^{-1}(\sqrt{3/2})(6\pi G\Lambda)^{-1/2}$. Given the presently quoted values of H_0 and Ω_Λ, one finds $\Lambda \approx 0.67 \times 10^{-26}$ kg/m^3. This gives the estimate $\tau_0 \approx 7.2$ by for the transition from a matter dominated to a vacuum energy dominated universe, which is in fair agreement with the conclusions from various supernovae experiments and the CMB data. Figure 10.3 shows the evolution of the scale factor in the ΛCDM model as compared with its evolution in a dust (CDM) model and

Figure 10.4 shows the Hubble parameter as a function of time in each case.[14],[15]

The causal structures of the space-times discussed above are determined by the light cone structure, for which it is necessary to solve the null geodesic equation

$$dt^2 - a^2(t)d\chi^2 = 0 \quad \Rightarrow \quad \eta - \eta_i = \int_i^t \frac{dt'}{a(t')} = \pm\frac{1}{c}(\chi - \chi_i) \tag{10.2.63}$$

It is helpful to rewrite the time integral as

$$\int \frac{dt}{a(t)} = \int \frac{d\ln a}{\mathfrak{h}(a)} \tag{10.2.64}$$

and then to employ the Bianchi identity in (10.2.20) and the Friedmann equation in (10.2.22), which, in the simplest case of a flat universe, implies that

$$\mathfrak{h} = \mathfrak{h}_0 \left(\frac{a_0}{a}\right)^{(1+3w)/2} \tag{10.2.65}$$

and gives ($w \neq -1/3$)

$$\int \frac{dt}{a(t)} = \frac{2\mathfrak{h}_0^{-1}}{1+3w}\left(\frac{a}{a_0}\right)^{(1+3w)/2}. \tag{10.2.66}$$

Therefore

$$\eta - \eta_i = \frac{2\mathfrak{h}_0^{-1}}{1+3w}\left[\left(\frac{a(t)}{a_0}\right)^{(1+3w)/2} - \left(\frac{a_i}{a_0}\right)^{(1+3w)/2}\right] = \pm\frac{1}{c}(\chi - \chi_i). \tag{10.2.67}$$

Rewriting this in terms of $\mathfrak{h}$, using (10.2.65) gives

$$\chi_p = \chi - \chi_i = \pm\frac{2c}{|1+3w|}(\mathfrak{h}^{-1} - \mathfrak{h}_i^{-1}) \tag{10.2.68}$$

This is the light cone and therefore the greatest coordinate (or comoving) distance that light can travel between two times, t_i and t. If the initial time, t_i, is taken to be the time of the big bang, it is called the **particle horizon**. The physical distance to the

[14] Problem: Repeat the analysis for a two component evolution with a cosmological constant and matter with an arbitrary $w > -1/3$. Show that the solution with $a(0) = 0$ is

$$a(t) = a_0\left[\sqrt{\frac{3H_0^2}{8\pi G\Lambda} - 1}\ \sinh\sqrt{6\pi G\Lambda}\ (1+w)t\right]^{2/3(1+w)}$$

and determine the inflection point. How does the time τ_0 change with w?

[15] Problem: Consider a two component system consisting of radiation and CDM. Obtain the scale factor as a function of time. When does the transition from a radiation dominated universe to a matter dominated universe occur?

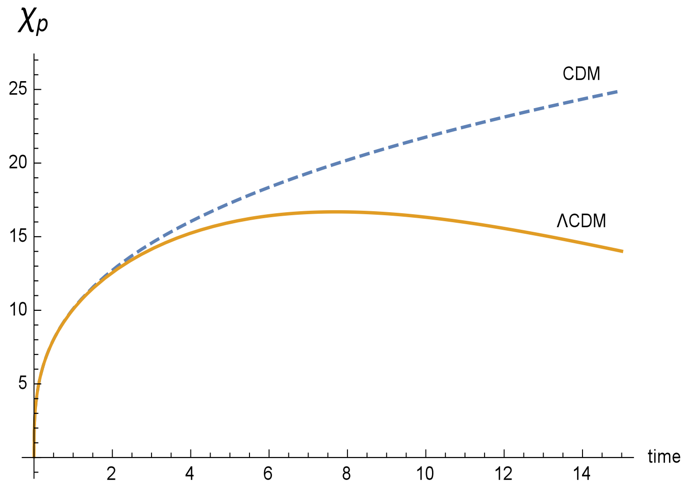

Figure 10.5: The particle horizon for the CDM and ΛCDM models.

particle horizon is, of course, $d_p = a(t)\chi_p$. So long as the strong energy condition is obeyed ($w > -1/3$), the scale factor is increasing and $\mathfrak{h}^{-1}$ also increases, causing the particle horizon to increase in time. However, if a component that violates the strong energy condition ($w < -1/3$) controls the evolution of the universe, then an increasing scale factor will cause the particle horizon to *decrease* in time. This is shown in figure 10.5 for the CDM and ΛCDM models worked out earlier.

In the ΛCDM model less and less of the universe is accessible to the observer in time. Therefore in the distant future an Earth bound astronomer will eventually be confined to observing just our Milky Way.

10.3 The Fine-Tuning Puzzles of Classical Cosmology

Classical Cosmology describes the evolution of the universe well and in close agreement with observations provided two assumptions are made concerning the initial conditions. These initial conditions cannot be specified at the big bang because there is a singularity there. Rather, we must give them on some later, smooth spatial hypersurface, Σ. It is customary to take this hypersurface to be at the time of last scattering, when the CMB was released, *i.e.,* when the universe became cool enough to become transparent to electromagnetic waves, about 300,000 years after the big bang. Physics is not so much about initial conditions as it is about evolution, so one could take the point of view that the initial conditions are given *à priori* and simply to be accepted, there being no context for arriving at them or "explaining" them within the confines of physics. However, one

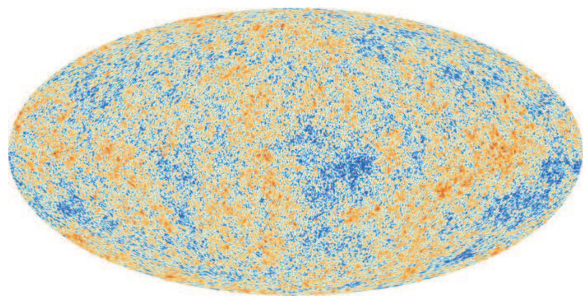

Figure 10.6: The Planck spacecraft's large telescope view of the CMB.

hopes that the initial conditions that lead to our present day universe would be, in some sense, generic. They do not appear to be. On the contrary it appears that we live in a very finely tuned universe.

10.3.1 The Horizon Problem

The first of puzzle is that the universe appears to have begun isotropic and homogeneous over regions that *should have been causally disconnected at the time of last scattering.* This is because, if the universe had a beginning, the light cone from any point on the last scattering surface cannot be traced "back" indefinitely.

Let us consider the behavior of the particle horizon, given by (10.2.68). We found that

$$\chi_p \sim a^{(1+3w)/2}. \tag{10.3.1}$$

The exponent is positive for matter obeying the strong energy condition (this is assumed in standard cosmology) and the particle horizon increases with time. As it does so, regions that were outside the horizon at the time of last scattering enter the horizon. But this means that portions of the CMB that we observe today were causally disconnected regions at the time it was released and so there is no dynamical mechanism by which they could achieve thermal equilibrium with each other. On the contrary, we observe an *almost* perfectly isotropic thermal background at approximately 2.7° K (see figure 10.6).

10.3.2 The Flatness Problem

The second puzzle is that the universe appears to be "flat" ($k = 0$), that is $\Omega_k \approx 0$ at the present time. But, according to (10.2.53),

$$\Omega_k = -\frac{3kc^2}{8\pi G a^2 \rho_{\rm cr}} = -\frac{kc^2}{(Ha)^2} = \sum_i \Omega_i + \Omega_\Lambda - 1. \tag{10.3.2}$$

In standard cosmology, $(Ha)^{-1}$ grows with time, therefore $|\Omega_k| = |\sum_i \Omega_i + \Omega_\Lambda - 1|$ should diverge in time if k deviated even slightly from zero in the early universe! In other words, the critical value $\sum_i \Omega_i + \Omega_\Lambda = 1$ is an *unstable* fixed point of the evolution. The fact that energy density of the universe at the present time is found to be very close to its critical value, means that the early universe must have had an energy density even closer to its critical value, numerically estimated to have departed from it by no more than one part in 10^{62}.

10.3.3 Inflation

Both of the problems above are connected to a common assumption: that the strong energy condition ($w > -1/3$) holds throughout the evolution of the universe. What if we relaxed this condition for some time interval? We have seen that when $-1 < w < -1/3$ the particle horizon shrinks in time,

$$\frac{d\chi_p}{dt} < 0, \tag{10.3.3}$$

the so-called "Hubble radius", $(Ha)^{-1}$, decreases,

$$\frac{d}{dt}(Ha)^{-1} < 0 \tag{10.3.4}$$

and the scale factor accelerates,

$$\frac{d^2a}{dt^2} > 0. \tag{10.3.5}$$

Generically, when these conditions are met, we say that the universe is in an **inflationary phase**.

Therefore imagine that, for some interval of time soon after after the big bang, the universe undergoes an inflationary phase, after which the component responsible for it becomes sub-dominant and allows the universe to follow a normal track. This early inflation would solve both the horizon and flatness problems.

- Provided that inflation occurred for a sufficiently long time, regions that enter the particle horizon during its later normal phase could have been inside the horizon before the early inflation began. Spatial homogeneity would then have been established *prior* to the start of the inflationary period. This solves the horizon problem.

- Again, through such an inflationary period, $(Ha)^{-1}$ *decreases*, driving Ω_k to zero and the energy density to its critical value. This solves the flatness problem.

The universe thus appears to have undergone roughly four principal eras in its expansion, up to the present time. Immediately after emerging from the quantum gravity (Planck) era, it is thought to have inflated, going from a length scale of about 10^{-35} m to about one micron in a small fraction of a second. Later, it passed through an era first dominated by radiation and followed, as its temperature cooled to below roughly 10^4 K, by one dominated by matter. During this time, the universe decelerated in its expansion. Finally, about 7 billion years ago, it appears to have entered the so-called Dark Energy (DE) dominated era and has been accelerating in its expansion ever since. Whereas the last three eras are observationally on solid ground, early inflation remains speculative. It is generally accepted, however, because it solves both the horizon problem and the flatness problems as well as the problem of the initial singularity.

10.4 Cosmology with Scalar Fields

Matter that satisfies the condition $w < -1/3$ is at best unfamiliar. The simplest models of inflation employ a single scalar field source, with action

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 2V(\phi)\right], \tag{10.4.1}$$

and stress tensor

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right]. \tag{10.4.2}$$

For a homogeneous solution, Einstein's equations are (10.2.5),

$$\begin{aligned} H^2 + \frac{kc^2}{a^2} &= \frac{8\pi G}{3}\rho_\phi \\ 2\dot{H} + 3H^2 + \frac{kc^2}{a^2} &= -\frac{8\pi G}{c^2}p_\phi \end{aligned} \tag{10.4.3}$$

where the field mass density, ρ_ϕ, and pressure, p_ϕ, are given by

$$\begin{aligned} \rho_\phi c^2 &= \frac{1}{2c^2}\dot{\phi}^2 + V(\phi) \\ p_\phi &= \frac{1}{2c^2}\dot{\phi}^2 - V(\phi). \end{aligned} \tag{10.4.4}$$

To these we may add the scalar field equation of motion,

$$\ddot{\phi} + 3H\dot{\phi} + c^2V'(\phi) = 0, \tag{10.4.5}$$

although it is not independent but follows from the conservation of energy and momentum. The middle term can be thought of in the same way as one thinks of the frictional "drag" on a particle moving in a medium. In this case, the drag coefficient is the Hubble parameter, therefore, the faster the universe expands the greater the drag on the field and the slower it evolves.

For simplicity, let us begin by considering the post inflationary era in which the universe is flat. The equation of state depends on the potential $V(\phi)$. Scalar potentials leading to prescribed equations of state may be found, but a given potential does not always lead to a unique equation of state. To see this, employ the first of (10.4.3) in (10.4.5) for a flat and expanding universe to express the latter as

$$2\frac{d}{d\phi}\sqrt{\left(\frac{1}{2}\dot{\phi}^2 + c^2V(\phi)\right)} + \sqrt{\frac{24\pi G}{c^4}}\dot{\phi} = 0. \tag{10.4.6}$$

Letting $y = \dot{\phi}$, $z^2 = \rho_\phi c^4$, the scalar field equation reduces to

$$\frac{dz}{d\phi} + \sigma y = 0 \tag{10.4.7}$$

where $\sigma = \sqrt{6\pi G/c^4}$. The dominant energy condition holds so long as $V(\phi) \geq 0$. In this case, we may set $z = c\sqrt{V(\phi)}\cosh u(\phi)$ and $y = c\sqrt{2V(\phi)}\sinh u(\phi)$. Rescaling ϕ, z and $V(\phi)$ according to[16]

$$\begin{aligned}\varphi &= \sqrt{\frac{12\pi G}{c^4}}\ \phi,\\ \tilde{z} &= \sqrt{\frac{12\pi G}{c^4}}\ z, \quad \widetilde{V}(\varphi) = \frac{12\pi G}{c^2}V(\phi)\end{aligned} \tag{10.4.8}$$

allows us to write the scalar field equation as

$$\frac{\tilde{z}'}{\tilde{z}} + \tanh u = 0 \tag{10.4.9}$$

or

$$u'(\varphi) + \frac{\widetilde{V}'(\varphi)}{2\widetilde{V}(\varphi)}\coth u(\varphi) + 1 = 0, \tag{10.4.10}$$

where the prime now refers to a derivative with respect to φ. We can take the equation of state to be a specification of $w_\phi(\phi)$, defined by

$$w_\phi(\varphi) = \frac{p_\varphi}{\rho_\varphi c^2} = \frac{\dot{\varphi}^2 - 2\widetilde{V}(\varphi)}{\dot{\varphi}^2 + 2\widetilde{V}(\varphi)} = 2\tanh^2 u(\varphi) - 1 \ \in\ [-1, 1). \tag{10.4.11}$$

The potential completely determines the equation of state (subject to boundary conditions) and, conversely, the equation of state (*i.e.,* a particular solution $u(\phi)$) determines $V(\varphi)$

[16]By this recaling, φ is dimensionless, but $[\tilde{z}] \sim 1/t$ and $[\widetilde{V}] \sim 1/t^2$.

up to a multiplicative constant. The former is a formidable problem because (10.4.10) is nonlinear. It can lead to multiple possible equations of state. However, if $u(\varphi)$ is known, all the quantities of interest may be expressed in terms of the energy density and its derivative, in particular,

$$\begin{aligned} \widetilde{V}(\varphi) &= \tilde{z}^2 - \tilde{z}'^2, \\ w_\varphi &= \frac{2\tilde{z}'^2}{\tilde{z}^2} - 1, \quad \text{and} \\ q_\varphi &= \frac{1}{2}(1 + 3w_\varphi) = \frac{3\tilde{z}'^2}{\tilde{z}^2} - 1, \end{aligned} \tag{10.4.12}$$

where Λ is some constant and q_φ is the "deceleration parameter". From Einstein's equations it follows that the universe accelerates in its expansion when $q_\varphi < 0$ and decelerates otherwise. The complete solution reduces to quadratures:

$$\begin{aligned} \dot{\varphi} &= -\sqrt{2}\,\epsilon\tilde{z}'(\varphi) \;\Rightarrow\; \int_{\varphi_0}^{\varphi} \frac{d\varphi}{\tilde{z}'(\varphi)} = \sqrt{2}\,\epsilon(t - t_0) \\ \frac{\dot{a}}{a} &= +\frac{\sqrt{2}\,\epsilon}{3}\tilde{z}(\varphi) \;\Rightarrow\; a(\varphi) = a(\varphi_0)\exp\left[-\frac{1}{3}\int_{\varphi_0}^{\varphi} d\varphi \frac{\tilde{z}(\varphi)}{\tilde{z}'(\varphi)}\right], \end{aligned} \tag{10.4.13}$$

where $\epsilon = \pm 1$ accounts for periods of expansion or contraction of the universe. We will now examine a few examples of the application of the above relations.

10.4.1 Linear Barotropes Revisited

Consider what scalar field potential would lead to linear barotropic behavior, *i.e.*, to w_ϕ (equivalently, u) is constant. If $u = u_0$ we find

$$\tilde{z} = \tilde{z}_0 e^{\gamma\varphi} \;\Rightarrow\; \widetilde{V}(\varphi) = \widetilde{V}_0 e^{2\gamma\varphi} \tag{10.4.14}$$

where $\gamma = -\tanh u_0 = \mp\sqrt{\frac{1+w_0}{2}}$ and we have used (10.4.12). For definiteness, let $\gamma < 0$ (an analogous solution is obtained if $\gamma > 0$). The corresponding scalar field equation in (10.4.13)

$$\dot{\varphi} = \epsilon\sqrt{\frac{2\widetilde{V}_0(1+w_0)}{1-w_0}}\,e^{\gamma\varphi} \tag{10.4.15}$$

has the solution

$$\varphi(t) - \varphi_0 = \ln\left[1 + \epsilon\sqrt{\frac{\widetilde{V}_0(1+w_0)^2}{1-w_0}}\,e^{\gamma\varphi_0}(t - t_0)\right]^{\sqrt{\frac{2}{1+w_0}}}. \tag{10.4.16}$$

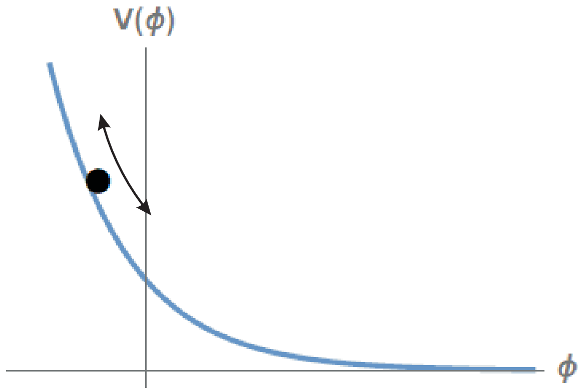

Figure 10.7: The exponetial potential, with $\gamma = -\sqrt{\frac{2}{3}}$.

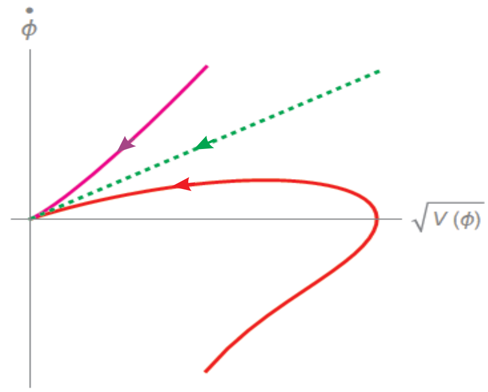

Figure 10.8: Scalar field vs. velocity plots for the three families of solutions for the exponetial potential.

When $\epsilon = +1$, the scalar field increases indefinitely, *i.e.*, descends, or "rolls down", the potential in time (because $\gamma < 0$) while, by the first of Einstein's equations, the scale factor grows as a power law,

$$a(t) = a_0 \left[1 + \epsilon\sqrt{\frac{\widetilde{V}_0(1+w_0)^2}{1-w_0}} e^{\gamma\varphi_0}(t - t_0)\right]^{\frac{2}{3(1+w_0)}}, \quad w_0 \neq -1, \tag{10.4.17}$$

arising out of a singularity at the time

$$t_s = t_0 - \sqrt{\frac{1-w_0}{\widetilde{V}_0(1+w_0)^2}} e^{-\gamma\varphi_0} \tag{10.4.18}$$

If $w_0 = -1$ both the potential and the the scalar field are constant and the universe undergoes an exponential expansion. We have obtained these results before, of course.

With an exponential potential, the nullcline ($u' = 0$) also solves equation (10.4.10) for u and serves as a separatrix between two families of solutions. This makes it very special. The general solution of (10.4.10) for the exponential potential is obtained in implicit form,

$$\varphi_0 - \varphi = \begin{cases} \frac{\gamma \ln|\gamma \cosh u + \sinh u| - u}{\gamma^2 - 1} & \gamma \neq \pm 1 \\ \\ \frac{1}{4}\left[2u + e^{-2u}\right] & \gamma = \pm 1 \end{cases} \tag{10.4.19}$$

but, because the solution naturally yields $\varphi(u)$, one can now seek to find $u(t)$, using

$$\dot{\varphi} = \dot{u}\frac{d\varphi(u)}{du} = c\sqrt{2\widetilde{V}_0 e^{\gamma\varphi(u)}} \sinh u. \tag{10.4.20}$$

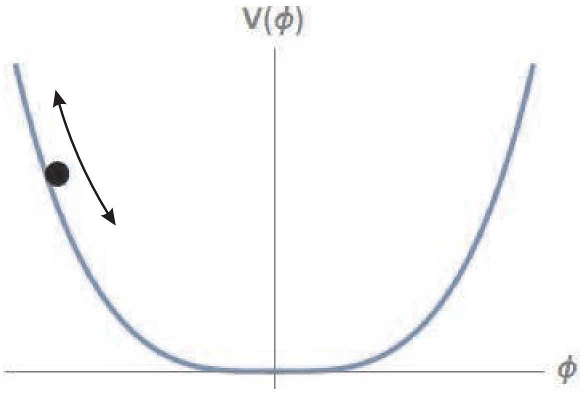

Figure 10.9: The scalar potential for the Chaplygin gas.

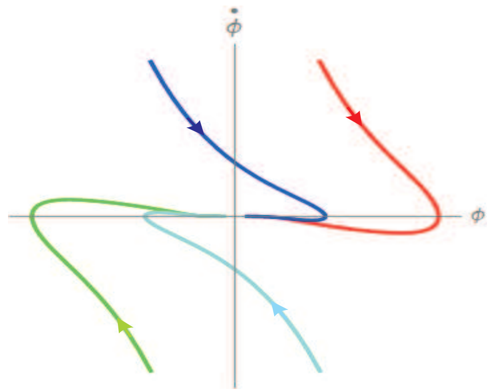

Figure 10.10: Scalar field vs. velocity plot for the evolution of the scalar field.

A simple, yet useful way to obtain a qualitative picture of the evolution is via a "phase" diagram for the evolution. Figure 10.8 is a plot of $\dot{\varphi}$ vs. $\sqrt{V(\varphi)}$ for the evolution of the scalar field with the exponential potential, where we have taken $\gamma = -\sqrt{\frac{2}{3}}$, showing the two families of solutions on either side of the separatrix. Bear in mind, however, that this is not a *true* phase diagram for the evolution of solutions (in a cosmological background, the scalar field momentumis $p_\phi \sim a^3\dot{\phi}$, and not simply $\dot{\phi}$, while the momentum conjugate to the scale factor is $p_a \sim -a\dot{a}$; because our system is Hamiltonian, Louville's theorem guarantees that the total phase space volume remains constant throughout the evolution). It only serves to illustrate the behavior of the field alone during the evolution. For all solutions, $\varphi \to \infty$ as $t \to \infty$, *i.e.,* the scalar field *eventually* "rolls down" the potential, with decreasing speed as φ grows indefinitely. Thus all solutions terminate at the origin. The separatrix, shown as the dashed line in green, is a straight line of slope $\sqrt{\frac{3\gamma^2}{1-\gamma^2}}$ and represents linear barotropic fluids. Solutions above the separatrix simply "roll down" the potential, whereas solutions below may begin by "rolling up" the potential but eventually stop and "roll down".

10.4.2 Polytropes

With the polytropic equation of state

$$p = K'c^2\rho^{1+1/n}, \quad w = K'\rho^{1/n} = K\tilde{z}^{2/n},$$

we find, by (10.4.11),

$$\tanh u = \pm\sqrt{\frac{1}{2}\left(1 + K\tilde{z}^{2/n}\right)}. \tag{10.4.21}$$

Thus the energy density is determined from

$$\int_{\tilde{z}_0}^{\tilde{z}} \frac{d\tilde{z}}{\tilde{z}\sqrt{1+K\tilde{z}^{2/n}}} = \pm\frac{\varphi-\varphi_0}{\sqrt{2}}. \quad (10.4.22)$$

Considering the special case for which $K=-|K|<0$ and $n=-|n|<0$, we find

$$\tilde{z} = |K|^{|n|/2}\cosh^{|n|}\left(\frac{\varphi\mp\varphi_0}{\sqrt{2}|n|}+\cosh^{-1}\frac{\tilde{z}_0^{1/|n|}}{\sqrt{|K|}}\right), \quad (10.4.23)$$

and the scalar field potential

$$\widetilde{V}(\varphi) = \widetilde{V}_0\cosh^{2(|n|-1)}X_\pm\left(1+\cosh^2 X_\pm\right), \quad (10.4.24)$$

where

$$X_\pm = \frac{\varphi\mp\varphi_0}{\sqrt{2}|n|}+\cosh^{-1}\frac{\tilde{z}_0^{1/|n|}}{\sqrt{|K|}}. \quad (10.4.25)$$

When $n=-1/2$, (10.4.21) describes the Chaplygin gas, for which the scalar field potential is

$$\widetilde{V}(\varphi) = \widetilde{V}_0\left(\operatorname{sech} X_\pm + \cosh X_\pm\right), \quad (10.4.26)$$

where $X=\sqrt{2}(\varphi\mp\varphi_0)+\cosh^{-1}\tilde{z}_0^2/\sqrt{|K|}$. Applying (10.4.13), one finds that

$$a(\varphi) = a(\varphi_0)\left(\frac{\sinh X}{\sinh X_0}\right)^{-1/6} \quad (10.4.27)$$

so that the energy density, as a function of the scale factor, behaves as

$$z^2 = \rho c^4 = \sqrt{|K|+A\left(\frac{a_0}{a}\right)^6}, \quad (10.4.28)$$

where $A=|K|\sinh X_0$. For small values of the scale factor $\rho\sim a^{-3}$, which behavior is characteristic of pressureless dust. On the other hand, for large values of a, the energy density $\rho\sim$ const., which is characteristic of a cosmological constant. This simple model therefore interpolates between an early dust phase and a late de Sitter phase, mimicking the ΛCDM model and has been suggested as a unification of Dark Matter and Dark Energy. Solutions for $\varphi(t)$ and $a(t)$ can be found in implicit form, and we leave that to the reader. From the shape of the potential (figure 10.9) we can deduce the shapes of the $\dot{\varphi}$ vs φ plots, shown in figure 10.10. For example, a field beginning at a positive value of φ and a positive velocity (the curve in red) slows to a stop, reverses course and rolls down the

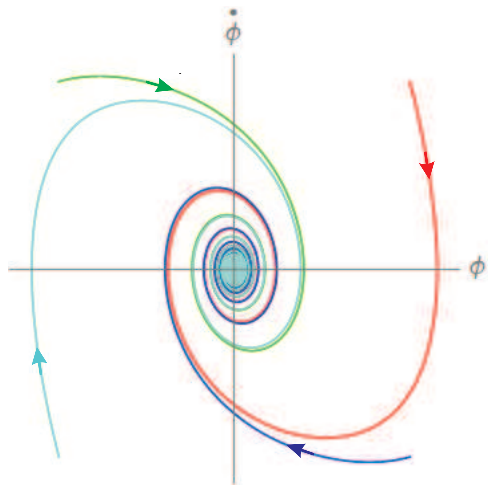

Figure 10.11: Evolution of a scalar field with a quadratic potential.

potential toward $\varphi = 0$, where all solutions eventually converge.[17] The late inflationary phase results from the fact that $\dot{\varphi} \to 0$ and $\widetilde{V}(\varphi) \to 2\widetilde{V}_0 \neq 0$ as $t \to \infty$, so that the scalar field behaves as a cosmological constant at late times.

10.4.3 The Potential $V(\varphi) = \mu^2\varphi^2$

The inverse problem, which is to determine the equation of state and evolution from a given problem is far more difficult. The reason is that it is most often extremely difficult to find a suitable integrating factor for (10.4.10). In such cases, a numerical approach must be taken. As an example, the evolution of the scalar field with a quadratic potential is shown in figure 10.11. Notice the existence of an "attractor" to which all solutions converge in time, as the scalar field oscillates about the $\varphi = 0$.

Although a closed-form solution is unavailable, we can distinguish some regions that are of interest. For example, consider what happens when the kinetic energy dominates over the potential energy, $\dot{\varphi} \gg \mu|\varphi|$. In this case, we may neglect the potential compared

[17]Problem: Analyze the equation of state

$$p = \rho\left(A - \frac{B}{\rho^{1/n}}\right)$$

where $-1 \leq A \leq 1$, and B and n are positive constants. Determine the scalar field potential and describe the evolution in the case $n = 1/2$; in particular, show that the evolution interpolates between a linear barotropic fluid, whose nature is determined by A, and a late de Sitter phase.

with $\dot{\varphi}$ and, from (10.4.5), find

$$\dot{\varphi} = \dot{\varphi}_0 e^{\mp(\varphi-\varphi_0)}, \tag{10.4.29}$$

where the negative (positive) sign refers to positive (negative) field velocities. Integrating,

$$\varphi(t) - \varphi_0 = \pm \ln\left(1 + \dot{\varphi}_0(t - t_0)\right) \tag{10.4.30}$$

and substituting this result into the Friedman equation reveals that $H(t) \sim 1/3t$, which describes decelerated expansion typical of stiff matter. The opposite limit, where the potential energy dominates over the kinetic energy, $\dot{\varphi} \ll \mu\varphi$, will describe a period of accelerated expansion.

10.5 Inflation Parameters

In practice, however, one is really interested in the conditions on $V(\phi)$ that would ensure that the universe undergoes an early inflationary phase *for a sufficiently long time.* Continuing with a flat universe, and returning to (10.4.3), we see that

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(1 + 3w_\phi)\rho_\phi = -\frac{1}{2}H^2(1 + 3w_\phi) \tag{10.5.1}$$

(we will no longer require w_ϕ to be constant, so $p_\phi = w_\phi(\rho_\phi)\rho_\phi c^2$ represents a general equation of state). If we define the *first* **slow roll parameter**, ε, by

$$\frac{\ddot{a}}{a} \stackrel{\text{def}}{=} H^2(1 - \varepsilon) \tag{10.5.2}$$

then $\varepsilon = \frac{3}{2}(1 + w_\phi) \geq 0$. Accelerated expansion occurs when $\varepsilon < 1$. It is easy to see that

$$\varepsilon = -\frac{\dot{H}}{H^2} \tag{10.5.3}$$

and so it is related to the evolution of the Hubble parameter. In the limit as $\varepsilon \to 0$, $\dot{\phi}^2 \ll V(\phi)$ and the potential energy dominates the evolution giving a constant Hubble parameter, H_0, which in turn implies that

$$a = a_0 e^{H_0 t}. \tag{10.5.4}$$

The solution is therefore a flat slicing of de-Sitter space, so it is known as the de-Sitter limit. Again, the accelerated expansion will be sustained only so long as the frictional term dominates over the acceleration in (10.4.5), *i.e.,*

$$|\ddot{\phi}| \ll |3H\dot{\phi}|. \tag{10.5.5}$$

We therefore define the *second* slow roll parameter, η, by

$$\eta = -\frac{\ddot{\phi}}{3H\dot{\phi}} \tag{10.5.6}$$

In a flat universe, the two parameters can be related. Using (10.5.3), we find that

$$\varepsilon - \frac{1}{2H}\frac{d}{dt}\ln\varepsilon = -\frac{\ddot{H}}{2H\dot{H}} \tag{10.5.7}$$

and $\ddot{H}$ can be associated with the acceleration of the scalar field in (10.5.6) by Einstein's equations. Combining the two equations of (10.4.3) we find

$$\dot{H} = -\frac{4\pi G}{c^4}\dot{\phi}^2, \tag{10.5.8}$$

and it follows that

$$\eta = -\frac{\ddot{\phi}}{3H\dot{\phi}} = -\frac{\ddot{H}}{6H\dot{H}} = \frac{1}{3}\left(\varepsilon - \frac{1}{2H}\frac{d}{dt}\ln\varepsilon\right). \tag{10.5.9}$$

If we call $dN = Hdt = d\ln a$, then dN represents the number of $e-$folds by which an exponentially growing scale factor would increase, and

$$\eta = \frac{1}{3}\left(\varepsilon - \frac{1}{2}\frac{d}{dN}\ln\varepsilon\right). \tag{10.5.10}$$

Thus $\eta \simeq \varepsilon/3$ ensures that the fractional change in ε per $e-$fold is small.

The conditions on the slow roll parameters, namely $3\eta \simeq \varepsilon < 1$, may be expressed as conditions on the shape of the potential as well. Within the slow roll regime ($\dot{\varphi} \approx 0 \approx \ddot{\phi}$) the first of (10.4.3) and the scalar field equation of motion respectively tell us that

$$H^2 \approx \frac{8\pi G}{3c^2}V(\phi) \quad \text{and} \quad \dot{\phi} \approx -\frac{c^2 V'(\phi)}{3H}. \tag{10.5.11}$$

In this approximation, the slow roll parameters of (10.5.3) and (10.5.6) become, respectively,

$$\varepsilon_V = \frac{c^4}{16\pi G}\left(\frac{V'}{V}\right)^2 \quad \text{and} \quad \eta_V = \frac{1}{3}\left[\varepsilon_V + \frac{c^4}{8\pi G}\frac{V''}{V}\right], \tag{10.5.12}$$

and the spacetime is almost de-Sitter (with an exponentially growing scale factor). Inflation ends when $\varepsilon_V \approx 1$. The number of $e-$folds before inflation ends is given by

$$N_{\text{tot}} = \ln\frac{a_{\text{end}}}{a_{\text{start}}} = \int_{t_{\text{start}}}^{t_{\text{end}}} H dt = \sqrt{\frac{4\pi G}{c^4}}\int_{\phi_{\text{end}}}^{\phi_{\text{start}}} \frac{d\phi}{\sqrt{\varepsilon_V}} \tag{10.5.13}$$

and is required to be between 40 and 60, based on the fluctuations observed in the CMB.[18] A homogeneous and isotropic universe has no structure. To describe the observed large scale structure of our universe one must turn to fluctuations of the field(s) responsible for inflation, Dark Matter and Dark Energy.

[18] Problem: Determine the slow roll conditions for the potentials considered in the previous section. In each case, (i) what is the relation between the number of $e-$folds and the field value before the end of inflation, and (ii) when are the fluctuations observed in the CMB created?

Appendix A

The Poincaré Group

If there is a physically measurable feature of a physical system that remains unchanged by a set of transformations, those transformations are symmetries of that system. Symmetries represent redundant descriptions of the system, but they may also point to particular useful descriptions. Here we will consider only continuous symmetries described by matrix groups.

A.1 Galilei Group

The Newtonian concept of "absolute time" and "absolute space" requires that time and space intervals and are independent of the (inertial) observer. As all distance measurements involve a *simultaneous* measurement of the end points, this is equivalent to saying that the spatial distance,

$$d\vec{s}^{\,2} = \delta_{ij} dx_i dx_j, \tag{A.1.1}$$

is invariant, and implies that there is a set of transformations of the coordinates, x_i, that leaves it unchanged.

A.1.1 Rotations

If those transformations act linearly on the coordinates, we let

$$x_i \;\to\; x'_i = R_{ij} x_j \tag{A.1.2}$$

where $\widehat{R} = R_{ij}$ represents a set of real, constant square matrices having the dimension of space (say, n). Then, because of the invariance of $d\vec{s}^{\,2}$

$$d\vec{s}^{\,2} = \delta_{ij} dx_i dx_j = \delta_{ij} dx'_i dx'_j = \delta_{ij} R_{il} R_{jm} dx_l dx_m \tag{A.1.3}$$

so that

$$R_{il}R_{im} = R^T_{li}R_{im} = \delta_{lm} \tag{A.1.4}$$

or, simply, $\widehat{R}^T \cdot \widehat{R} = \widehat{\delta}$, where $\widehat{\delta}$ is the unit matrix. We assume that $\widehat{R}$ depends continuously on a certain number of parameters (to be determined), α_I.The set of transformations satisfying (A.1.4) form a group under matrix multiplication, where the unit matrix is the identity element and the inverse of any transformation is its transpose. Associativity follows because matrix multiplication is associative.

This is the **orthogonal** group $O(3)$, of orthogonal matrices. It turns out that the group elements also form a manifold in which group multiplication and inversion are smooth maps. Such a set, combining both group and manifold structures, is called a **Lie group**. The orthogonality condition (A.1.4) implies that every orthogonal matrix will have determinant $\|\widehat{R}\| = \pm 1$. Orthogonal matrices of determinant $+1$ represent rotations and form a subgroup of $O(3)$ called the **special orthogonal** group, $SO(3)$. We will focus on the special orthogonal group.

We will build finite transformations from infinitesimal transformations, which means that all our group elements will be *connected to the identity.* An infinitesimal transformation can be written as

$$R_{ij} = \delta_{ij} + \delta U_{ij} \tag{A.1.5}$$

where ϵ is some small parameter. Then (A.1.4) requires that

$$\sum_i (\delta_{ij} + \delta U_{ij})((\delta_{ik} + \delta U_{ik}) = \delta_{jk} + (\delta U_{jk} + \delta U_{kj}) + \mathcal{O}(\epsilon^2) = \delta_{jk} \tag{A.1.6}$$

or $\delta\widehat{U}^T = -\delta\widehat{U}$. We conclude, therefore, that $\delta\widehat{U}$, being real, square and antisymmetric, can depend on at most $n(n-1)/2$ parameters, $\delta\alpha_I$. We therefore write a general $\delta\widehat{U}$ as the sum

$$\delta\widehat{U} = \sum_I \delta\alpha_I \widehat{U}^{(I)}, \tag{A.1.7}$$

where $\widehat{U}^{(I)}$ are constant matrices called the **generators** of the group, and construct a finite group element connected to the identity by simply iterating the group operation many times,

$$\widehat{R} = \lim_{N\to\infty}\left(\widehat{\delta} + \sum_I \frac{\alpha_I}{N}\widehat{U}^{(I)}\right)^N = \sum_{n=0}^{\infty}\frac{(\alpha_I \widehat{U}^{(I)})^n}{n!} \stackrel{\text{def}}{\equiv} e^{\alpha_I \widehat{U}^{(I)}} \tag{A.1.8}$$

As an example, consider an explicit representation of the rotation group in three dimensions. There are three parameters and and three generators, which we can choose to represent rotations about each of the three axes,

$$U^{(1)} = U_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$U^{(2)} = U_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$U^{(3)} = U_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.1.9}$$

(Notice that $[\widehat{U}_i]_{jk} = \epsilon_{ijk}$, where ϵ_{ijk} is the totally antisymmetric Levi-Civita tensor in three dimensions.) For example, if only $\alpha_z \neq 0$ ($\alpha_{x,y} = 0$) then the exponentiation in (A.1.8) gives a finite rotation about the z axis:

$$\begin{aligned} \widehat{R}_z &= e^{\alpha_z \widehat{U}_3} = \widehat{\delta} + \left(\frac{1}{2!}\alpha_z^2 - \frac{1}{4!}\alpha_z^4 + \ldots\right)\widehat{U}_3^2 + \left(\alpha_z - \frac{1}{3!}\alpha_z^3 + \frac{1}{4!}\alpha_z^4 - \ldots\right)\widehat{U}_3 \\ &= \begin{pmatrix} \cos\alpha_z & \sin\alpha_z & 0 \\ -\sin\alpha_z & \cos\alpha_z & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{A.1.10}$$

and we may similarly derive expressions for the rotation matrices about the other two axes. The generators form a closed commutator algebra,

$$[\widehat{U}_i, \widehat{U}_j] = -\epsilon_{ijk}\widehat{U}_k, \tag{A.1.11}$$

called the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$ and the components of the Levi-Civita symbol on the right hand side are called the structure constants of the algebra.

The matrices $\widehat{U}_i$ generate the action of $SO(3)$ on the finite dimensional vector space $\mathbb{R}^3$. They form a finite (in this case, three) dimensional representation of $\mathfrak{so}(3)$ called the **fundamental representation**. Let us now consider the action of $SO(3)$ on the infinite dimensional space of real valued, C^∞ functions, $\phi(x)$, on $\mathbb{R}^3$,

$$\delta\phi(x) = \delta_0\phi(x) + \partial_k\phi(x)\delta x_k = \delta_0\phi(x) - \delta\alpha_i\epsilon_{ijk}x_j\partial_k\phi(x), \tag{A.1.12}$$

where we made use of the fact that $[\widehat{U}_i]_{jk} = \epsilon_{ijk}$. The first term above represents a functional change in f. The second is a change in $\phi(x)$ induced by a change in its arguments and can be expressed as

$$\delta_1\phi(x) = -\delta\alpha_i(\widehat{L}_i\phi) \tag{A.1.13}$$

where $\widehat{L}_i = \epsilon_{ijk}x_j\partial_k$. It has the same form as δx_i and the set of operators $\widehat{L}_i$ satisfy the Lie algebra (A.1.11),

$$[\widehat{L}_i, \widehat{L}_j] = -\epsilon_{ijk}\widehat{L}_k. \tag{A.1.14}$$

An alternate form of these generators,

$$\widehat{L}_{ij} = \epsilon_{ijk}\widehat{L}_k = x_i\partial_j - x_j\partial_i \tag{A.1.15}$$

is also frequently used, in terms of which

$$\delta_1\phi(x) = -\frac{1}{2}\delta\omega_{ij}(\widehat{L}_{ij}\phi), \quad \delta\omega_{ij} = \epsilon_{ijk}\delta\alpha_k. \tag{A.1.16}$$

They satisfy the algebra[1]

$$[\widehat{L}_{ij}, \widehat{L}_{lm}] = \delta_{im}\widehat{L}_{jl} - \delta_{il}\widehat{L}_{jm} + \delta_{jl}\widehat{L}_{im} - \delta_{jm}\widehat{L}_{il} \tag{A.1.17}$$

Finally, note that $\widehat{L}^2 = \delta_{ij}\widehat{L}_i\widehat{L}_j$ is the only quadratic invariant of the algebra, *i.e.,*

$$[\widehat{L}^2, \widehat{L}_i] = 0. \tag{A.1.18}$$

$\widehat{L}^2$ is known as the quadratic **Casimir invariant** of $\mathfrak{so}(3)$.

A.1.2 Boosts

Galilean boosts,

$$t \rightarrow t' = t, \quad x_i \rightarrow x'_i = x_i - v_i t \tag{A.1.19}$$

act on a four dimensional space of vectors with components $x_\mu = \{t, x_i\}$, $\mu \in \{0,1,2,3\}$, with action

$$\begin{pmatrix} t \\ x_i \end{pmatrix} \rightarrow \begin{pmatrix} t' \\ x'_i \end{pmatrix} = \begin{pmatrix} 1 & 0_j \\ v_i & 1 \end{pmatrix} \begin{pmatrix} t \\ x_j \end{pmatrix}. \tag{A.1.20}$$

Therefore, two boosts with parameters $\vec{u}$ and $\vec{v}$ are simply equivalent to a single boost with parameter $\vec{u}+\vec{v}$, *i.e.,* the order in which they are applied is irrelevant. The following are the three generators of pure Galilean boosts:

$$\widehat{K}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{K}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{K}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{A.1.21}$$

They are obviously nilpotent of degree 2 and commute with one another,

$$[\widehat{K}_i, \widehat{K}_j] = 0 \tag{A.1.22}$$

but they do not commute with the generators of spatial rotations,

$$[\widehat{K}_i, \widehat{U}_j] = -\epsilon_{ijk}\widehat{K}_k \tag{A.1.23}$$

[1] Exercise: Find the structure constants of the algebra of the generators $\widehat{L}_{ij}$.

What about the action of boosts on the infinite dimensional vector space of C^∞ functions $\phi(x)$? We find

$$\delta\phi(x) = \delta_0\phi(x) + \delta x_k \partial_k \phi(x) = \delta_0\phi(x) - t\delta v_i \partial_i \phi(x) \tag{A.1.24}$$

shows that

$$\delta_1\phi(x) = -\delta v_i(\widehat{B}_i \phi) \tag{A.1.25}$$

with $\widehat{B}_i = t\partial_i$. It satisfies the same algebra as $\widehat{K}_i$, *i.e.*,

$$\begin{aligned} [\widehat{L}_i, \widehat{L}_j] &= -\epsilon_{ijk}\widehat{L}_k \\ [\widehat{B}_i, \widehat{L}_j] &= -\epsilon_{ijk}\widehat{B}_k \\ [\widehat{B}_i, \widehat{B}_j] &= 0, \end{aligned} \tag{A.1.26}$$

where we have included the algebra of pure rotations for completeness.

A.1.3 Translations

The Galilei group is a combination of rotations, boosts and translations, the last of which capture the homogeneity of space and time. Translations are given by

$$\begin{aligned} t &\to t' = t + s \\ x_i &\to x'_i = x_i + a_i. \end{aligned} \tag{A.1.27}$$

To obtain a finite dimensional representation of the Galilei group as a matrix group one must enlarge the dimension of the vector space on which it acts by taking a space-time event to be represented by the five dimensional vector $(t, x_i, 1)$. A suitable form of a general Galilei transformation would then be

$$\begin{pmatrix} t \\ x_i \\ 1 \end{pmatrix} \to \begin{pmatrix} t' \\ x'_i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0_j & s \\ v_i & \widehat{R} & a_i \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x_j \\ 1 \end{pmatrix} \tag{A.1.28}$$

We see that time translations are generated by

$$\widehat{E} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.1.29}$$

whereas the three generators of space translations are

$$\widehat{M}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{M}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{M}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.1.30}$$

These are all nilpotent of degree 2 and commute among themselves,

$$[\widehat{E}, \widehat{M}_i] = 0 = [\widehat{M}_i, \widehat{M}_j] \tag{A.1.31}$$

but not with the rotations and boosts. In fact, one can show that

$$[\widehat{E}, \widehat{U}_i] = 0, \quad [\widehat{E}, \widehat{K}_i] = \widehat{M}_i \tag{A.1.32}$$

and

$$[\widehat{M}_i, \widehat{K}_j] = 0, \quad [\widehat{M}_i, \widehat{U}_j] = \epsilon_{ijk}\widehat{M}_k \tag{A.1.33}$$

The Galilei group is ten dimensional, with generators $\widehat{E}$, $\widehat{M}_i$, $\widehat{K}_i$ and $\widehat{U}_i$ satisfying the algebra above. Its algebra is the extension of the group of rotations and boosts by translations. Acting on the functions of space and time, we find that $\widehat{E} \to -\widehat{H} = -\partial_t$ and $\widehat{M}_i \to \widehat{P}_i = \partial_i$. The infinite dimensional representations obey the same algebra, which we now state:

$$\begin{aligned}
&[\widehat{H}, \widehat{P}_i] = 0 \\
&[\widehat{H}, \widehat{B}_i] = \widehat{P}_i \\
&[\widehat{H}, \widehat{L}_i] = 0 \\
&[\widehat{P}_i, \widehat{P}_j] = 0 \\
&[\widehat{P}_i, \widehat{B}_j] = 0 \\
&[\widehat{P}_i, \widehat{L}_j] = \epsilon_{ijk}\widehat{P}_k \\
&[\widehat{B}_i, \widehat{B}_j] = 0 \\
&[\widehat{B}_i, \widehat{L}_j] = \epsilon_{ijk}\widehat{B}_k \\
&[\widehat{L}_i, \widehat{L}_j] = -\epsilon_{ijk}\widehat{L}_k
\end{aligned} \tag{A.1.34}$$

There are two Casimir invariants, *viz.,* $\widehat{P}^2$ and W^2, where $\vec{W} = \widehat{P} \times \widehat{B}$.[2]

One may consider a **central extension** of the algebra by a generator, $\widehat{M}$, which commutes with all the other generators (*i.e.,* which lies at the center of the algebra), by modifying the fifth commutation relation above as follows:

$$[\widehat{P}_i, \widehat{B}_j] = -\widehat{M}\delta_{ij}$$

This is called the Bargmann algebra. With the central extension, the two Casimir invariants get modified to

$$\widehat{M}\widehat{H} - \frac{\vec{P}^2}{2} \quad \text{and} \quad \vec{W} \cdot \vec{W}, \tag{A.1.35}$$

where $\vec{W} = \widehat{M}\vec{J} + \vec{P} \times \vec{B}$.

[2] Problem: Check this using the algebra of the Galilei group.

A.2 Poincaré Group

The Lorentz $SO(3,1)$, group keeps the (Minkowski) distance between events in space-time,

$$ds^2 = c^2 dt^2 - \delta_{ij} dx^i dx^j, \tag{A.2.1}$$

invariant. When we add space-time translations to the rotations and boosts, as we did for the Galilei group, we obtain the Poincaré group.

A.2.1 Lorentz Group

According to (1.3.51), boosts are captured by the transformations

$$\begin{pmatrix} ct' \\ x'^i \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma v_j}{c} \\ -\frac{v^i}{c} & \delta^i{}_j + (\gamma - 1)\frac{v^i v_j}{c^2} \end{pmatrix} \begin{pmatrix} ct \\ x^j \end{pmatrix}, \tag{A.2.2}$$

which lead to the following three generators of the boosts,

$$\widehat{K}_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{K}_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{K}_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{A.2.3}$$

These generators do not commute with one another (as they did in the Galilei algebra), rather the commutator of two boost generators yields a rotation generator, according to

$$[\widehat{K}_i, \widehat{K}_j] = \epsilon_{ijk} \widehat{U}_k \tag{A.2.4}$$

so the boosts do not form a subgroup of the Lorentz group, whose Lie algebra reads

$$\begin{aligned} [\widehat{U}_i, \widehat{U}_j] &= -\epsilon_{ijk} \widehat{U}_k \\ [\widehat{K}_i, \widehat{U}_j] &= -\epsilon_{ijk} \widehat{K}_k \\ [\widehat{K}_i, \widehat{K}_j] &= \epsilon_{ijk} \widehat{U}_k \end{aligned} \tag{A.2.5}$$

The generators of rotations can now be given in terms of the four dimensional Levi-Civita tensor as $[\widehat{U}_i]_{jk} = \varepsilon_{0ijk}$ and it follows that are

$$\delta_1 \phi = -\delta\alpha^i (\widehat{L}_i \phi) = -\frac{1}{2} \delta\omega_{ij} (\widehat{L}_{ij} \phi), \quad \omega_{ij} = \varepsilon_{0ijk} \delta\alpha^k \tag{A.2.6}$$

where

$$\widehat{L}_i = \epsilon_{0ijk} x_j \partial_k = \frac{1}{2} \varepsilon_{0ijk} \widehat{L}_{jk}, \quad \widehat{L}_{ij} = x_i \partial_j - x_j \partial_i. \tag{A.2.7}$$

The finite dimensional boost generators can also written in terms of the Levi-Civita tensor as

$$[\widehat{K}_i]^\mu{}_\nu = \frac{1}{2}\varepsilon_{0ijk}\varepsilon^{jk\mu}{}_\nu = \eta_{0\nu}\delta^\mu_i - \delta^\mu_0\eta_{i\nu} \tag{A.2.8}$$

where indices are raised and lowered by

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.2.9}$$

and its inverse. Therefore, acting on the infinite dimensional space of real valued, C^∞ functions,

$$\begin{aligned} \delta_1 f(x) &= \delta x^\mu \partial_\mu f(x) = \frac{\delta v^i}{c}[\widehat{K}_i]^\mu{}_\nu x^\nu \partial_\mu f(x) \\ &= -\frac{\delta v^i}{c}\left(ct\partial_i + \frac{1}{c}x_i\partial_t\right) f(x), \end{aligned} \tag{A.2.10}$$

we find that the boost generators are represented by

$$\widehat{B}_i = ct\partial_i + \frac{1}{c}x_i\partial_t \tag{A.2.11}$$

This has the same form as the rotations in (A.2.7) if we define $x^\mu = (ct, x^i)$ and $x_\mu = \eta_{\mu\nu}x^\nu = (-ct, x^i)$. In fact, by defining $\widehat{B}_i = \widehat{L}_{i0}$ both rotation and boost generators can be combined into

$$\widehat{L}_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu. \tag{A.2.12}$$

Moreover, (we check explicitly that) the Lorentz algebra is

$$[\widehat{L}_{\mu\nu}, \widehat{L}_{\alpha\beta}] = \eta_{\mu\beta}\widehat{L}_{\nu\alpha} - \eta_{\mu\alpha}\widehat{L}_{\nu\beta} + \eta_{\nu\alpha}\widehat{L}_{\mu\beta} - \eta_{\nu\beta}\widehat{L}_{\mu\alpha} \tag{A.2.13}$$

The Lorentz group admits two quadratic Casimirs, which can be conveniently expressed as

$$C_1 = \widehat{L}^2 = \widehat{L}^{\mu\nu}\widehat{L}_{\mu\nu}, \quad \text{and} \quad C_2 = {}^*\widehat{L}\cdot\widehat{L} = \epsilon_{\mu\nu\alpha\beta}\widehat{L}^{\alpha\beta}\widehat{L}^{\mu\nu}. \tag{A.2.14}$$

However, these are no longer invariant once translations are included, because translations do not commute with the rotations and boosts.

A.2.2 Translations

To complete the symmetries required by all classical field theories, we must include translations,

$$\begin{aligned} &t \to t' = t + s \\ &x_i \to x_i' = x_i + a_i. \end{aligned} \tag{A.2.15}$$

Finite dimensional representations can, once again, be given by enlarging the dimension of the vector space on which the group acts, so that the generators of time and space translations will be exactly given by (A.1.29) and (A.1.30) respectively. If we let $\widehat{E} = -\widehat{M}_0$, the algebra of all ten generators is now (A.2.13) together with

$$\begin{aligned} &[\widehat{M}_\mu, \widehat{M}_\nu] = 0 \\ &[\widehat{M}_\mu, \widehat{L}_{\alpha\beta}] = \eta_{\mu\alpha}\widehat{M}_\beta - \eta_{\mu\beta}\widehat{M}_\alpha \end{aligned} \tag{A.2.16}$$

Moreover, the infinite dimensional representation of $\widehat{M}_\mu$ is $\widehat{P}_\mu = -\partial_\mu$. We may cast the entire Poincaré algebra in the compact and covariant form

$$\begin{aligned} &[\widehat{L}_{\mu\nu}, \widehat{L}_{\alpha\beta}] = \eta_{\mu\beta}\widehat{L}_{\nu\alpha} - \eta_{\mu\alpha}\widehat{L}_{\nu\beta} + \eta_{\nu\alpha}\widehat{L}_{\mu\beta} - \eta_{\nu\beta}\widehat{L}_{\mu\alpha} \\ &[\widehat{P}_\mu, \widehat{P}_\nu] = 0 \\ &[\widehat{P}_\mu, \widehat{L}_{\alpha\beta}] = \eta_{\mu\alpha}\widehat{P}_\beta - \eta_{\mu\beta}\widehat{P}_\alpha \end{aligned} \tag{A.2.17}$$

There are again two quadratic Casimirs, but they are

$$C_1 = \widehat{P}^2, \quad \text{and} \quad C_2 = \widehat{W}^2 = \widehat{W}_\mu \widehat{W}^\nu \tag{A.2.18}$$

where $\widehat{W}_\mu$ is the Pauli-Lubanski vector,

$$\widehat{W}_\mu = \frac{1}{2}\varepsilon_{\mu\alpha\beta\gamma}\widehat{P}^\alpha \widehat{L}^{\beta\gamma}. \tag{A.2.19}$$

Hence, both $\widehat{P}^2$ and $\widehat{W}^2$ are not only Lorentz invariant but Poincaré invariant as well, whereas (A.2.14) are only Lorentz invariant.

A.3 Poincaré Invariance of Fields

Physical fields, ϕ^A, generically transform under the action of a Poincaré group element Λ according to some finite dimensional representation, $\widehat{D}(\Lambda)$,

$$\vec{\phi}(x) \to \vec{\phi}'(x') = \widehat{D}(\omega)\vec{\phi}(x) \tag{A.3.1}$$

so that an infinitesimal transformation will have the form (see (2.2.1))

$$\delta\vec{\phi}(x) = \vec{\phi}'(x') - \vec{\phi}(x) = \delta\omega^{\mu\nu}\widehat{G}_{\mu\nu}\vec{\phi} \tag{A.3.2}$$

where, explicitly, $[\widehat{G}_{\mu\nu}\vec{\phi}]^A = [\widehat{G}_{\mu\nu}]^A{}_B\phi^B$ and $[\widehat{G}_{\mu\nu}]^A{}_B$ is some finite dimensional representation of the Lorentz generators; their precise form will depend on the nature of the field. Furthermore, the change induced in the field may be decomposed into a functional change and a change brought about by the change in coordinates, $x^\mu \to x'^\mu = x^\mu + \delta x^\mu$, according to

$$\delta\vec{\phi}(x) \;=\; \overbrace{\vec{\phi}'(x) - \vec{\phi}(x)}^{\delta_0\vec{\phi}} + \overbrace{\delta x^\mu \partial_\mu\vec{\phi}(x)}^{\delta_1\vec{\phi}} = \delta_0\vec{\phi}(x) + \delta_1\vec{\phi}(x) \tag{A.3.3}$$

We are concerned principally with the functional change in the fields, on which the conserved quantities depend as shown in (2.2.10). Under a Lorentz transformation,

$$\delta_0\vec{\phi}(x) = \delta\vec{\phi}(x) - \delta_1\vec{\phi}(x) = \delta\omega^{\mu\nu}\left[\widehat{G}_{\mu\nu} + \frac{1}{2}\widehat{L}_{\mu\nu}\right]\vec{\phi}(x) \tag{A.3.4}$$

where, $\widehat{L}_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$ are the generators of the Lorentz group in the infinite dimensional representation. As we have seen, the action of $\widehat{G}_{\mu\nu}$ on $\vec{\phi}$ produces its spin whereas the action of $\widehat{L}_{\mu\nu}$ on $\vec{\phi}$ produces its orbital angular momentum. Physical fields are invariant under translations, therefore (under pure translations)

$$\delta_0\vec{\phi}(x) = \delta a^\mu \widehat{P}_\mu\vec{\phi}(x). \tag{A.3.5}$$

We may now state the total functional change in the field due to a general Poincaré transformation as

$$\delta_0\vec{\phi}(x) = \left[\delta\omega^{\mu\nu}\left(\widehat{G}_{\mu\nu} + \frac{1}{2}\widehat{L}_{\mu\nu}\right) + \delta a^\mu\widehat{P}_\mu\right]\vec{\phi}(x) \tag{A.3.6}$$

and, employing the second term in (2.2.10), find the conserved currents of (2.3.5).

A.4 Finite Dimensional Representations

Given the importance of the finite dimensional representations of the Lorentz algebra, let us classify them. These representations act on finite dimensional vector spaces and, as far as the standard model is concerned, only four of them participate. The simplest of these is the trivial representation, for which $\widehat{G}$ is zero. This is the scalar field, for which the transformation matrix $\widehat{D}(\omega) = 1$.

To construct the non-trivial representations, it is best to return to the non-covariant form of the generators, $\widehat{U}_i$ and $\widehat{K}_i$, which satisfy (A.2.5). If we define

$$\widehat{\mathcal{L}}_i = \frac{1}{2}\left(\widehat{U}_i + i\widehat{K}_i\right), \quad \widehat{\mathcal{R}}_i = \frac{1}{2}\left(\widehat{U}_i - i\widehat{K}_i\right) \tag{A.4.1}$$

then it follows that the generators $\{\widehat{\mathcal{U}}_i\}$ and $\{\widehat{\mathcal{K}}_i\}$ commute with one another, while each satisfies the same commutation relations,

$$\begin{aligned} [\widehat{\mathcal{L}}_i, \widehat{\mathcal{L}}_j] &= -\epsilon_{ijk}\widehat{\mathcal{L}}_k \\ [\widehat{\mathcal{R}}_i, \widehat{\mathcal{R}}_j] &= -\epsilon_{ijk}\widehat{\mathcal{R}}_k \\ [\widehat{\mathcal{L}}_i, \widehat{\mathcal{R}}_j] &= 0 \end{aligned} \tag{A.4.2}$$

In this way, the Lorentz algebra has turned into two copies of the algebra of the group $SU(2)$, *i.e.,* $\mathfrak{so}(3,1) \simeq \mathfrak{su}(2) \times \mathfrak{su}(2)$. The finite dimensional representations of $SU(2)$ are well known, *viz.,* the spin $j/2$ representations with j being any whole number. Let us consider how these come about.

Consider the algebra of either $\widehat{\mathcal{L}}$ or $\widehat{\mathcal{R}}$ (say $\widehat{\mathcal{L}}$), defining $\widehat{\mathcal{L}}_\pm = \frac{1}{\sqrt{2}}(\widehat{\mathcal{L}}_1 \pm i\widehat{\mathcal{L}}_2)$,

$$\begin{aligned} [\widehat{\mathcal{L}}_+, \widehat{\mathcal{L}}_-] &= i\widehat{\mathcal{L}}_3 \\ [\widehat{\mathcal{L}}_3, \widehat{\mathcal{L}}_\pm] &= \pm i\widehat{\mathcal{L}}_\pm \end{aligned} \tag{A.4.3}$$

and let $|l\rangle$ be an eigenstate of $\widehat{\mathcal{L}}_3$, with eigenvalue m, *i.e.,* $\widehat{\mathcal{L}}_3|l\rangle = im|m\rangle$ (the eigenvalues are imaginary because the generators, as defined, are antihermitean). It follows that $\widehat{\mathcal{L}}_+|m\rangle$ is an eigenstate of $\widehat{\mathcal{L}}_3$ of eigenvalue $m+1$ and $\widehat{\mathcal{L}}_-|m\rangle$ is an eigenstate of $\widehat{\mathcal{L}}_3$ of eigenvalue $m-1$, as follows:

$$\begin{aligned} \widehat{\mathcal{L}}_3\widehat{\mathcal{L}}_+|m\rangle &= \widehat{\mathcal{L}}_+\widehat{\mathcal{L}}_3|m\rangle + i\widehat{\mathcal{L}}_+|m\rangle = i(m+1)\widehat{\mathcal{L}}_+|m\rangle \\ \widehat{\mathcal{L}}_3\widehat{\mathcal{L}}_-|m\rangle &= \widehat{\mathcal{L}}_-\widehat{\mathcal{L}}_3|m\rangle - i\widehat{\mathcal{L}}_-|m\rangle = i(m-1)\widehat{\mathcal{L}}_+|m\rangle \end{aligned} \tag{A.4.4}$$

By succesively applying $\widehat{\mathcal{L}}_+$ to any state, we could generate an infinite set of eigenstates of $\widehat{\mathcal{L}}_3$, unless there is a state, $|l\rangle$, for which

$$\widehat{\mathcal{L}}_+|l\rangle = 0$$

This is called the highest weight state and will determine the dimension of the representation, so let us label the eigenstates of $\widehat{\mathcal{L}}_3$ with l as well as m, taking $\widehat{\mathcal{L}}_3|l,m\rangle = im|l,m\rangle$, $m \leq l$. Furthermore, we will have

$$\widehat{\mathcal{L}}_+|l,m\rangle = N^+_{l,m}|l,m+1\rangle, \quad \widehat{\mathcal{L}}_-|l\rangle = N^-_{l,m}|l,m-1\rangle \tag{A.4.5}$$

where $N^\pm$ are to be determined, and $N^+_{l,l} = 0$. Any state, $|l,m\rangle$ can be recovered from the highest weight state by a successive application of $\widehat{\mathcal{L}}_-$, since

$$\widehat{\mathcal{L}}_-^{\,l-m}|l,l\rangle = N^-_{l,m+1} \cdot N^-_{l,m+2} \cdot \ldots \cdot N^-_{l,l}|l,m\rangle \tag{A.4.6}$$

Now,

$$\begin{aligned}[\widehat{\mathcal{L}}_+, \widehat{\mathcal{L}}_-^{\,l-m}]|l,l\rangle &= \widehat{\mathcal{L}}_+ \cdot \widehat{\mathcal{L}}_-^{\,l-m}|l,l\rangle = N^-_{l,m+1} \cdot N^-_{l,m+2} \cdot \ldots \cdot N^-_{l,l}\widehat{\mathcal{L}}_+|l,m\rangle \\ &= N^-_{l,m+1} \cdot N^-_{l,m+2} \cdot \ldots \cdot N^-_{l,l} \cdot N^+_{l,m}|l,m+1\rangle \qquad \text{(A.4.7)}\end{aligned}$$

because $\widehat{\mathcal{L}}_+|l,l\rangle = 0$. However, applying the algebra, we find that

$$\begin{aligned}[\widehat{\mathcal{L}}_+, \widehat{\mathcal{L}}_-^{\,l-m}]|l,l\rangle &= \sum_{r=0}^{l-m-1} \widehat{\mathcal{L}}_-^{\,r} \cdot [\widehat{\mathcal{L}}_+, \widehat{\mathcal{L}}_-] \cdot \widehat{\mathcal{L}}_-^{\,l-m-r-1}|l,l\rangle \\ &= i\sum_{r=0}^{l-m-1} \widehat{\mathcal{L}}_-^{\,r} \cdot \widehat{\mathcal{L}}_3 \cdot \widehat{\mathcal{L}}_-^{\,l-m-r-1}|l,l\rangle \\ &= -\sum_{r=0}^{l-m-1} (m+r+1)\widehat{\mathcal{L}}_-^{\,l-m-1}|l,l\rangle \\ &= -\left[(m+1)(l-m) + \frac{(l-m)(l-m-1)}{2}\right]\widehat{\mathcal{L}}_-^{\,l-m-1}|l,l\rangle \\ &= -\frac{1}{2}(l-m)(l+m+1)\widehat{\mathcal{L}}_-^{\,l-m-1}|l,l\rangle \qquad \text{(A.4.8)}\end{aligned}$$

and putting this together with (A.4.7) we have

$$N^-_{l,m+1} \cdot N^+_{l,m} = -\frac{1}{2}(l-m)(l+m+1) \qquad \text{(A.4.9)}$$

But the generators are antihermitean, which implies that

$$(\langle l,m|\widehat{\mathcal{L}}_-|l,m+1\rangle)^* = \langle l,m+1|\widehat{\mathcal{L}}_-^{\dagger}|l,m\rangle = -\langle l,m+1|\widehat{\mathcal{L}}_+|l,m\rangle \qquad \text{(A.4.10)}$$

and therefore $(N^-_{l,m+1})^* = -N^+_{l,m}$. Again, since $N^+_{l,l} = 0$, it follows that

$$\begin{aligned}N^+_{l,m} &= \frac{i}{\sqrt{2}}\sqrt{(l-m)(l+m+1)} \\ N^-_{l,m} &= \frac{i}{\sqrt{2}}\sqrt{(l+m)(l-m+1)} \qquad \text{(A.4.11)}\end{aligned}$$

Notice $N^-_{l,m}$ vanishes when $m = -l$, so the representation is $2l+1$ dimensional. Furthermore, the state $|l,-l\rangle$ is arrived at after an integer number of applications of $\widehat{\mathcal{L}}_-$ to the highest weight state, so $2l$ is an integer and $2m$ must be an integer as well. There is exactly one irreducible representation for each $l \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ and, because the fields will transform in some representation of two copies of $\mathfrak{su}(2)$, we'll say that they transform in the (l,r) representation, where l will refer to the representation of $\widehat{\mathcal{L}}$ and r to the representation of $\widehat{\mathcal{R}}$ in which they transform.

A.5 Spinors and Vectors

As an example, let us construct the lowest non-trivial representation, $l = \frac{1}{2}$. The vector space is two dimensional, so we will take it to be spanned by the unit (basis) vectors

$$|\frac{1}{2}, \frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\frac{1}{2}, -\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for which

$$\widehat{\mathcal{L}}_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \widehat{\mathcal{L}}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad \widehat{\mathcal{L}}_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \tag{A.5.1}$$

and hence

$$\widehat{\mathcal{L}}_1 = \frac{i}{2}\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \widehat{\mathcal{L}}_2 = \frac{i}{2}\sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \widehat{\mathcal{L}}_3 = \frac{i}{2}\sigma_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{A.5.2}$$

Consider fields that transform under $\widehat{\mathcal{L}}$ but not under $\widehat{\mathcal{R}}$. In this case,

$$\widehat{U}_i = i\widehat{K}_i \;\Rightarrow\; \widehat{U}_i = \widehat{\mathcal{L}}_i = \frac{i}{2}\sigma_i \text{ and } \widehat{K}_i = \frac{1}{2}\sigma_i \tag{A.5.3}$$

They are called "left handed spinors" and are said to transform in the $\left(\frac{1}{2}, 0\right)$ representation of $SO(3,1)$,

$$\widehat{D}_L(\omega) = e^{\frac{i}{2}(a^i - ib^i)\widehat{\sigma}_i}, \tag{A.5.4}$$

where a^i are real rotation angles and b^i are parameters of the boosts. "Right handed spinors" would transform under $\widehat{\mathcal{R}}$, but are unchanged by $\widehat{\mathcal{L}}$, so that

$$\widehat{U}_i = -i\widehat{K}_i \;\Rightarrow\; \widehat{U}_i = \widehat{\mathcal{R}}_i = \frac{i}{2}\sigma_i \text{ and } \widehat{K}_i = -\frac{1}{2}\sigma_i. \tag{A.5.5}$$

They are in the $\left(0, \frac{1}{2}\right)$ representation of $SO(3,1)$,

$$\widehat{D}_R(\omega) = e^{\frac{i}{2}(a^i + ib^i)\widehat{\sigma}_i}. \tag{A.5.6}$$

Spinors are a pair of complex fields because the representations of $SU(2)$ are necessarily complex. Let ψ_L and ψ_R represent the left/right handed spinors, then

$$\begin{aligned} \psi_L(x) \to \psi'_L(x') &= \widehat{D}_L(\omega)\psi_L(x) = e^{\frac{i}{2}(a^i - ib^i)\widehat{\sigma}_i}\psi_L(x) \\ \psi_R(x) \to \psi'_R(x') &= \widehat{D}_R(\omega)\psi_R(x) = e^{\frac{i}{2}(a^i + ib^i)\widehat{\sigma}_i}\psi_R(x) \end{aligned} \tag{A.5.7}$$

Similarly,

$$\psi^*_L(x) \to \psi'^*_L(x') = (\widehat{D}_L(\omega)\psi_L(x))^* = e^{-\frac{i}{2}(a^i + ib^i)\widehat{\sigma}^*_i}\psi^*_L(x)$$

$$\psi_R^*(x) \to \psi_R'^*(x') = (\widehat{D}_R(\omega)\psi_R(x))^* = e^{-\frac{i}{2}(a^i - ib^i)\widehat{\sigma}_i^*}\psi_R^*(x) \tag{A.5.8}$$

but, if we define $\widehat{\epsilon} = \widehat{\sigma}_2$, we find that $\widehat{\epsilon}\ \widehat{\sigma}_i^* = -\widehat{\sigma}_i\ \widehat{\epsilon}$ and it follows that

$$\widehat{\epsilon}\psi_L^*(x) \to \widehat{\epsilon}\psi_L'^*(x') = \widehat{\epsilon}e^{-\frac{i}{2}(a^i+ib^i)\widehat{\sigma}_i^*}\psi_L^*(x) = e^{\frac{i}{2}(a^i+ib^i)\widehat{\sigma}_i}\widehat{\epsilon}\psi_L^*(x) \tag{A.5.9}$$

which is precisely the transformation of $\psi_R(x)$. Likewise, $\widehat{\epsilon}\psi_R^*(x)$ will transform as $\psi_L(x)$. Since a Lagrangian for the two component spinors would involve both the field and its conjugate, one can work with either left or right handed Weyl spinors.

Under a parity transformation, rotation generators remain unchanged but boost generators change sign, causing ψ_L to transform as ψ_R and vice-versa. A theory that is invariant under parity must therefore include both left and right handed spinors. These are the four component "bispinors" or Dirac spinors that transform in the direct sum, $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$, representation of $\mathfrak{su}(2) \times \mathfrak{su}(2)$,

$$\widehat{U}_i = \frac{i}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix},\quad \widehat{K}_i = \frac{1}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{pmatrix},\quad \psi_D = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \tag{A.5.10}$$

The boosts and rotations are readily verified by writing the matrix $\widehat{S}$ of (3.3.16) in the Weyl basis (3.2.4). It is now easy to see that the combination

$$\overline{\psi}\psi = \psi^\dagger\gamma^0\psi = \left(\psi_L^\dagger, \psi_R^\dagger\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = \psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L \tag{A.5.11}$$

is Lorentz invariant.

We now turn to the vector representations, $\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}, 0\right) \otimes \left(0, \frac{1}{2}\right)$ of $\mathfrak{su}(2) \times \mathfrak{su}(2)$. These are four dimensional representations, produced by taking the tensor product, $\widehat{D}_L \otimes \widehat{D}_R$, of the two dimensional representations and, by looking at the infinitesimal transformations, the generators are seen to be

$$\widehat{U}_i = \widehat{\mathcal{L}}_i \otimes \mathbf{1} + \mathbf{1} \otimes \widehat{\mathcal{R}}_i,\quad \widehat{K}_i = \widehat{\mathcal{L}}_i \otimes \mathbf{1} - \mathbf{1} \otimes \widehat{\mathcal{R}}_i. \tag{A.5.12}$$

Given $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{R}}$ in (A.5.2),

$$\widehat{U}_i = \frac{i}{2}\left(\sigma_i \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_i\right),\quad \widehat{K}_i = \frac{i}{2}\left(\sigma_i \otimes \mathbf{1} - \mathbf{1} \otimes \sigma_i\right), \tag{A.5.13}$$

which give

$$\widehat{U}_1 = \frac{i}{2}\begin{pmatrix}0 & 1 & 1 & 0\\ 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\end{pmatrix},\quad \widehat{U}_2 = \frac{1}{2}\begin{pmatrix}0 & 1 & 1 & 0\\ -1 & 0 & 0 & 1\\ -1 & 0 & 0 & 1\\ 0 & -1 & -1 & 0\end{pmatrix},\quad \widehat{U}_3 = i\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \tag{A.5.14}$$

and

$$\widehat{K}_1 = \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \quad \widehat{K}_2 = -\frac{i}{2}\begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}, \quad \widehat{K}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.5.15}$$

While these look nothing like the representations we had earlier for the action of rotations (in (A.1.9)) and boosts (in (A.2.3)) on four vectors, they are in fact related by a similarity transformation, $\widehat{G} \to \widehat{G}' = \widehat{W}\ \widehat{G}\ \widehat{W}^{-1}$, where

$$\widehat{W} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ i & 0 & 0 & i \\ 0 & -1 & -1 & 0 \end{pmatrix}. \tag{A.5.16}$$

$\widehat{W}$ is unitary and generates a change in basis,

$$\frac{1}{\sqrt{2}}\begin{pmatrix} x - iy \\ t - z \\ -t - z \\ -x - iy \end{pmatrix} \to \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \tag{A.5.17}$$

so the standard representation for $\widehat{D}\left(\frac{1}{2},\frac{1}{2}\right)$ is just given with respect to a different basis. In a coordinate basis, the finite dimensional vector representations of the Lorentz algebra can be given simply as $(\widehat{G}_{\mu\nu})^{\alpha\beta} = \delta^\alpha_\mu \delta^\beta_\nu - \delta^\beta_\mu \delta^\alpha_\nu$, which was used in (2.7.21) to define the spin of the vector field in (2.7.23) and (4.3.5).

Appendix B

Penrose Diagrams

Appendix C

The Area Theorem