Journal of Combinatorial Theory, Series B 151 (2021) 38–45

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Counterexamples to a conjecture of Erdős, Pach, Pollack and Tuza

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A R T I C L E I N F O

Article history:
Received 6 November 2020
Available online xxxx

Keywords:
Diameter
Minimum degree

A B S T R A C T

Erdős et al. (1989) [4] conjectured that the diameter of a K_{2r}-free connected graph of order n and minimum degree $\delta \geq 2$ is at most $\frac{2(r-1)(3r+2)}{(2r^2-1)} \cdot \frac{n}{\delta} + O(1)$ for every $r \geq 2$, if δ is a multiple of $(r-1)(3r+2)$. For every $r > 1$ and $\delta \geq 2(r-1)$, we create K_{2r}-free graphs with minimum degree δ and diameter $\frac{(6r-5)n}{(2r-1)\delta+2r-3} + O(1)$, which are counterexamples to the conjecture for every $r > 1$ and $\delta > 2(r-1)(3r+2)(2r-3)$.

1. Introduction

The following theorem was discovered several times [1,4,6,7]:

Theorem 1. *For a fixed minimum degree $\delta \geq 2$ and $n \to \infty$, for every n-vertex connected graph G, we have* $\operatorname{diam}(G) \leq \frac{3n}{\delta+1} + O(1)$.

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https://doi.org/10.1016/j.jctb.2021.06.001

Note that the upper bound is sharp (even for δ-regular graphs [2]), but the constructions have complete subgraphs whose order increases with δ. Erdős, Pach, Pollack, and Tuza [4] conjectured that the upper bound in Theorem 1 can be strengthened for graphs not containing complete subgraphs:

Conjecture 1 *([4]). Let $r, \delta \geq 2$ be fixed integers and let G be a connected graph of order n and minimum degree δ.*

(i) *If G is K_{2r}-free and δ is a multiple of $(r-1)(3r+2)$ then, as $n \to \infty$,*

$$\begin{aligned}\operatorname{diam}(G) &\leq \frac{2(r-1)(3r+2)}{(2r^2-1)} \cdot \frac{n}{\delta} + O(1)\\ &= \left(3 - \frac{2}{2r-1} - \frac{1}{(2r-1)(2r^2-1)}\right)\frac{n}{\delta} + O(1).\end{aligned}$$

(ii) *If G is K_{2r+1}-free and δ is a multiple of $3r-1$, then, as $n \to \infty$,*

$$\operatorname{diam}(G) \leq \frac{3r-1}{r} \cdot \frac{n}{\delta} + O(1) = \left(3 - \frac{2}{2r}\right)\frac{n}{\delta} + O(1).$$

Set $k = 2r$ or $k = 2r+1$ according the cases. As connected δ-regular graphs are $K_{\delta+1}$-free (apart from $K_{\delta+1}$ itself), we need $\delta \geq k$ (at least) to make improvement on Theorem 1. Furthermore, as the conjectured constants in the bounds are at most $3 - \frac{2}{k}$, Theorem 1 implies that the conjectured inequalities hold trivially, unless $\delta \geq \frac{3k}{2} - 1$.

Erdős *et al.* [4] constructed graph sequences for every $r, \delta \geq 2$, where δ satisfies the divisibility condition, which meet the upper bounds in Conjecture 1. We show these construction them in Section 2.

Part (ii) of Conjecture 1 for $r = 1$ was proved in Erdős *et al.* [4]. Conjecture 1 is included in the book of Fan Chung and Ron Graham [5], which collected Erdős's significant problems in graph theory.

No more progress has been reported on this conjecture, except that for $r = 2$ in (ii), under a stronger hypothesis (4-colorable instead of K_5-free), Czabarka, Dankelman and Székely [3] arrived at the conclusion of Conjecture 1:

Theorem 2. *For every connected 4-colorable graph G of order n and minimum degree $\delta \geq 1$, $\operatorname{diam}(G) \leq \frac{5n}{2\delta} - 1$.*

In Section 3, we give an unexpected counterexample for Conjecture 1 (i) for every $r \geq 2$ and $\delta > 2(r-1)(3r+2)(2r-3)$. The question whether Conjecture 1 (i) holds in the range $(r-1)(3r+2) \leq \delta \leq 2(r-1)(3r+2)(2r-3)$ is still open. The counterexample leads to a modification of Conjecture 1, which no longer requires cases:

Conjecture 2. *For every $k \geq 3$ and $\delta \geq \lceil \frac{3k}{2} \rceil - 1$, if G is a K_{k+1}-free (weaker version: k-colorable) connected graph of order n and minimum degree at least δ,* $\mathrm{diam}(G) \leq \left(3 - \frac{2}{k}\right)\frac{n}{\delta} + O(1)$.

For $k = 2r$, Conjecture 2 is identical to Conjecture 1(ii). For $k = 2r-1$, $3 - \frac{2}{k} = \frac{6r-5}{2r-1}$, and, although the conjectured bound is likely not tight for *any* δ, the fraction $\frac{6r-5}{2r-1}$ cannot be reduced for *all* δ according to the construction in Section 3.

2. Clump graphs and the constructions for Conjecture 1

We define a *(weighted) clump graph* H as follows: x is a vertex of maximum eccentricity, L_i is the set of vertices at distance i from x, $D = \mathrm{diam}(H)$, $L_0 = \{x\}$, $L_D = \{y\}$ the weight of x and y is 1 and all other vertices are weighted with positive integers. The vertices of H are referred to as clumps.

A weighted clump graph H gives rise to a simple (unweighted) graph G by blowing up vertices of H into as many copies as their weight is, i.e. every vertex of H corresponds to an independent set of G with size the same as the weight was. Two vertices in G are connected if they correspond to connected vertices in H. The degrees in G correspond to the sum of the weights of neighbors of the vertices in H, $\mathrm{diam}(G) = \mathrm{diam}(H)$, and the number of vertices in G is the sum of the weights of all vertices in H, and the maximum clique size and chromatic number of G and H are the same.

It is convenient to describe the constructions of Erdős *et al.* [4] in terms of clump graphs. Any two consecutive layers of the clump graphs will form a complete graph, and, as the order of these complete graphs will be at most $2r-1$ (resp. $2r$), the graphs will be $(2r-1)$-colorable and K_{2r}-free (resp. $2r$-colorable and K_{2r+1}-free).

For the construction for K_{2r}-free graphs, when δ is a multiple of $(r-1)(3r+2)$: For $2 \leq i \leq D-2$, for every odd i, layer L_i has r clumps, and for every even i, layer L_i has $r-1$ clumps The single clumps in L_0 and L_D get weight 1, for $3 \leq i \leq D-2$, for every odd i, the clumps in layer L_i get weight $\frac{r\delta}{(r-1)(3r+2)}$ and for $2 \leq i \leq D-1$, for every even i, the clumps in layer L_i get of weight $\frac{(r+1)\delta}{(r-1)(3r+2)}$. Use weight δ for clumps in L_1 and L_{D-1}.

For the construction for K_{2r+1}-free graphs, when δ is a multiple of $3r-1$: For $1 \leq i \leq D-1$, layer L_i has r clumps. The single clumps in L_0 and L_D get weight 1, and clumps in layers L_i for $2 \leq i \leq D-2$ get weight $\frac{\delta}{3r-1}$. Use weight δ for clumps in L_1 and L_{D-1}.

The diameters of these constructions obviously meet the upper bounds of Conjecture 1 within a constant term that depends on r, but not on n or δ.

3. Counterexamples

We will make use of a clump graph to create a $(2r-1)$-colorable (and hence K_{2r}-free) graphs with minimum degree δ for every $r \geq 2$ that refute Conjecture 1 (i).

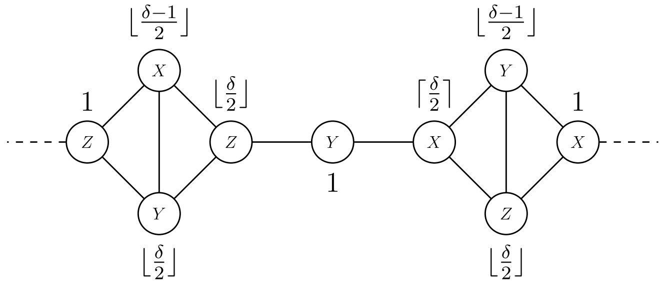

Fig. 1. The repetitive block $C_{1,\delta}$ for the weighted clump graph of the counterexample for 3-colorable/K_4-free graphs. The letters X, Y, Z give a 3-coloration and the label above the vertex gives the weight of the vertex.

To make our quantities slightly more palatable in the description, we make the shift $s = r - 1$, and work with $(2s + 1)$-colorable graphs for $s \geq 1$.

For positive integers p, s and $\delta \geq 2s$, we will create a weighted clump graph $H_{s,\delta,p}$ with $p(6s + 1)$ layers, such that the number of vertices in two consecutive layers is at most $2s+1$, each vertex is adjacent to all other vertices in its own layer and in the layers immediately before and after it. The layer structure of $H_{s,\delta,p}$ is basically *periodic*, up to a tiny *modification* in the weights. We are going to define a symmetric *block* $C_{s,\delta}$ of $6s + 1$ layers, and $H_{s,\delta,p}$ is the juxtaposition of p copies of $C_{s,\delta}$, with the modification of increasing by 1 the weight of one vertex in the second layer L_1 and one vertex in the next-to-last layer $L_{p(6s+1)-1}$.

Let $0 \leq d \leq 2s-1$ be the *remainder*, when we divide δ with $2s$. We define $C_{s,\delta}$ by the number of points and their weights in the layers L_m for $0 \leq m \leq 3s$ as detailed below; for $3s+1 \leq m \leq 6s$, L_m and the weights will be the same as in L_{6s-m}. In layers $L_{3i\pm1}$, every weight will be $\lfloor \frac{\delta}{2s} \rfloor$ or $\lceil \frac{\delta}{2s} \rceil$ before adjustment, and in layers L_{3i} the weights will be 1. More precisely:

(A) For each $i : 0 \leq i \leq s$, let the layer L_{3i} contain a single vertex with weight 1.
(B) For each $i : 0 \leq i \leq s-1$, let the layer L_{3i+1} contain $2s - i$ vertices, and assign them the following weights:
  (a) If $d = 0$, let the weight of each of these vertices be $\frac{\delta}{2s}$. The adjustment is that for a single vertex in L_1, whose weight is reduced to $\frac{\delta}{2s} - 1$. (By symmetry, the same adjustment happens in L_{6s-1}.)
  (b) If $d \geq 1$, then let $\min(2s-i, d-1)$ vertices have weight $\lceil \frac{\delta}{2s} \rceil$, and the rest have weight $\lfloor \frac{\delta}{2s} \rfloor$.
(C) For each $i : 0 \leq i < s-1$, let the layer L_{3i+2} contain $i+1$ vertices, and assign them the following weights:
  (a) If $d = 0$, let the weight of each of them be $\frac{\delta}{2s}$.
  (b) If $1 \leq d$, then let $d - \min(2s-i-1, d-1)$ vertices have weight $\lceil \frac{\delta}{2s} \rceil$, and the rest gets weight $\lfloor \frac{\delta}{2s} \rfloor$. (This weight assignment is feasible. Note that L_{3i+2} contains $i+1$ vertices, and, as $d \leq 2s-1$, $1 \leq d - \min(2s-i-1, d-1) \leq i$).

(D) Let layer L_{3s-1} (and symmetrically layer L_{3s+1}) have s vertices each. In these layers, let $\lfloor \frac{d}{2} \rfloor$ vertices (resp. $\lceil \frac{d}{2} \rceil$ vertices) have weight $\lceil \frac{\delta}{2s} \rceil$, and the remaining vertices get weight $\lfloor \frac{\delta}{2s} \rfloor$. (This weight assignment is feasible. Since $d \leq 2s-1$, $\lceil \frac{d}{2} \rceil \leq s$.)

Note that $\min(d-1, 2s-i) = d-1$ for $i \in \{0,1,2\}$. We use this minimization for $i \leq s-2$. When $s \leq 4$, we have $s-2 \leq 2$, consequently there is no need to use the minimization formula for $s \leq 4$. Therefore we show $C_{5,\delta}$ in Fig. 2, which is the first instance to show all complexities of the counterexamples. The case $s=1$, when $d \in \{0,1\}$, is even simpler: it is possible to describe the weights without reference to d, see Fig. 1 for $C_{1,\delta}$.

Lemma 1. *Let $p \geq 1$ and $s \geq 2$. The weighted clump graph $H_{s,\delta,p}$ has the following properties:*

(a) *$H_{s,\delta,p}$ is $(2s+1)$-colorable with diameter $p(6s+1)-1$.*
(b) *The sum of the weights of all vertices is $p\big((2s+1)\delta + 2s - 1\big)+2$.*
(c) *For any vertex $y \in V(H_{s,\delta,p})$, the sum of the weights of its neighbors is at least δ.*

Proof. (a) The statement on the diameter is trivial. As the number of vertices in any two consecutive layer of $H_{s,\delta,p}$ is at most $2s+1$, we can $(2s+1)$-color $H_{s,\delta,p}$ with $(2s+1)$ colors from left to right greedily.

(b) If W is the sum the weights of vertices in the block $C_{s,\delta}$, then the total sum of weights in $H_{s,\delta,p}$ is $pW+2$ (the 2 is due to the modification), so we need to show that $W = (2s+1)\delta + 2s - 1$.

Consider an i with $0 < i < s-1$. $L_{3i-1} \cup L_{3i+1}$ has $(i-1)+1+2s-i = 2s$ vertices. If $d=0$, each of them has weight $\frac{\delta}{2s}$, otherwise $d - \min(d-1, 2s-i) + \min(d-1, 2s-i) = d$ of them have weight $\lceil \frac{\delta}{2s} \rceil$, and the rest have weight $\lfloor \frac{\delta}{2s} \rfloor$. So the sum of the weight of the vertices in $L_{3i-1} \cup L_{3i} \cup L_{3i+1}$ is $\delta+1$, and so is in $L_{6s-3i+1} \cup L_{6s-3i} \cup L_{6s-3i-1}$.

$L_{3s-1} \cup L_{3s+1}$ contains $2s$ vertices, $\lfloor \frac{d}{2} \rfloor + \lceil \frac{d}{2} \rceil = d$ of them has weight $\lceil \frac{\delta}{2s} \rceil$, the rest $\lfloor \frac{\delta}{2s} \rfloor$, so the sum of the weights of the vertices in $L_{3s-1} \cup L_{3s} \cup L_{3s+1}$ is also $\delta+1$.

L_1 has $2s$ vertices. If $d=0$, one of these have weight $\frac{\delta}{2s}-1$ and the rest have weight $\frac{\delta}{2s}$. If $d>0$, $d-1$ of the vertices have weight $\lceil \frac{\delta}{2s} \rceil$, the rest has weight $\lfloor \frac{\delta}{2s} \rfloor$. The sum of the weights in $L_0 \cup L_1$ is $1+\delta-1=\delta$.

So $W = 2\delta + (2s-1)(\delta+1) = (2s+1)\delta + 2s - 1$, which finishes the proof of (b).

For (c): Let y be a vertex of $H_{s,\delta,p}$. Then for some j $(0 \leq j < p)$, y is in the j-th block $C_{s,\delta}$, and for some m $(0 \leq m \leq 6s)$, y is in the layer L_m of $C_{s,\delta}$. Because of symmetry, we may assume that $0 \leq m \leq 3s$. The weights in the layer L_{3s-1} are less or equal than the weights in the layer L_{3s+1}, but may not be equal, breaking the symmetry, but still handling cases with $0 \leq m \leq 3s$ gives a δ lower bound to the degrees of all vertices of $H_{s,\delta,p}$. In addition, layer $(j,m) = (p-1, 6s-1)$, where a modification happened, is symmetric to the layer $(j,m) = (0,1)$, where identical modification happened. Therefore checking the degrees of the vertices in the first half of the first (and modified) copy of

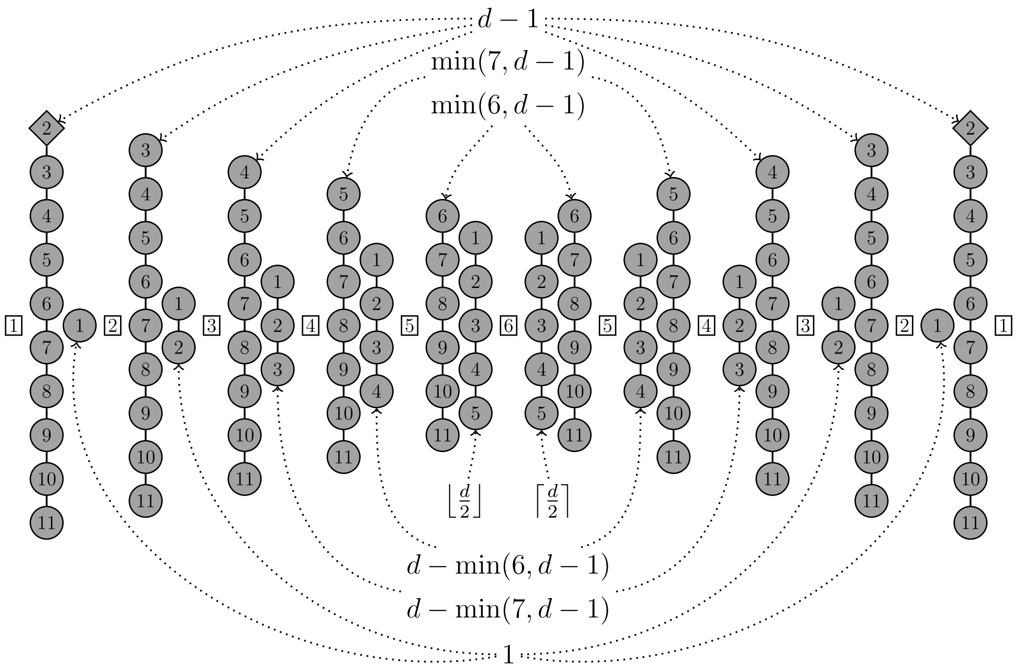

Fig. 2. The repetitive block $C_{5,\delta}$ of the weighted clump graph of the for 11-colorable/K_{12}-free counterexample graphs. The vertices within a layer are connected with a vertical line. Two vertices are connected, if they are in the same layer or in consecutive layers. The numbers in the vertices give a good 11-coloration. Before adjustment, white rectangular vertices have weight 1 and gray vertices have either weight $\lceil \frac{\delta}{10} \rceil$ or weight $\lfloor \frac{\delta}{10} \rfloor$; and the numbers, from which dotted arrows point to columns, give the number of vertices in the column that have weight $\lceil \frac{\delta}{10} \rceil$. Recall $d = \delta - 10\lfloor \frac{\delta}{10} \rfloor$. The adjustment: if $d = 0$, the weight of the two diamond shaped vertices are decreased by 1.

$C_{s,\delta}$ in $H_{s,\delta,p}$ covers checking the degrees in the second half of the last (and modified) copy of $C_{s,\delta}$ in $H_{s,\delta,p}$.

If $y \in L_{3i}$ for some $0 < i \le 2s$, then y has weight 1 and is adjacent to all vertices but itself in $L_{3i-1} \cup L_{3i} \cup L_{3i+1}$. As we have already shown in the proof of part (b), $L_{3i-1} \cup L_{3i} \cup L_{3i+1}$ has total weight $\delta + 1$, the neighbors of y have total weight δ.

If $y \in L_0$, then as we showed in the proof of part (b), the total weight of the vertices in L_0 in an unmodified block, which is not the first or the last block, is $\delta - 1$. Either y is adjacent to a vertex of weight 1 outside of its own block, or y is in a modified block where the total weight of L_1 got increased by 1: in both cases the sum of the weights of the neighbors of y is δ.

Assume now that y is a vertex of $L_{3i+1} \cup L_{3i+2}$ for some $0 \le i \le s-1$. Note $L_{3i+1} \cup L_{3i+2}$ contains $2s+1$ vertices, $2s$ of which is the neighbor of y, plus y has a neighbor of weight 1 outside of $L_{3i+1} \cup L_{3i+2}$. We consider two cases for d:

If $d = 0$ and $0 < i \le s-1$, then each neighbor of y in $L_{3i+1} \cup L_{3i+2}$ has weight $\frac{\delta}{2s}$, so the sum of the weights of the neighbors of y is $\delta + 1$. If $d = 0$ and $i = 0$, because of the adjustment, the sum of the weights of the neighbors may decrease by 1, and is still $\ge \delta$.

If $d > 0$, then all vertices in $L_{3i+1} \cup L_{3i+2}$ have weight at least $\lfloor \frac{\delta}{2s} \rfloor$. If $i < s-1$, then at least $\min(d-1, 2s-i) + d - \min(d-1, 2s-i-1) \ge d$ many vertices of $L_{3i+1} \cup L_{3i+2}$ have weight $\lceil \frac{\delta}{2s} \rceil$. If $i = s-1$, then, as $s+1 > \lceil \frac{d}{2} \rceil$, we have that $\lfloor \frac{d}{2} \rfloor \ge \max(1, d-(s+1))$, so $L_{3s-2} \cup L_{3s-1}$ has at least $\min(d-1, s+1) + \lfloor \frac{d}{2} \rfloor = d - \max(1, d-(s+1)) + \lfloor \frac{d}{2} \rfloor \ge d$ vertices with weight $\lceil \frac{\delta}{2s} \rceil$. Therefore, for any $0 \le i \le s-1$, any $y \in L_{3i+1} \cup L_{3i+2}$ has at least $d-1$ neighbors of weight $\lfloor \frac{\delta}{2s} \rfloor + 1$, the total weight of y's neighbors is at least $2s \lfloor \frac{\delta}{2s} \rfloor + d - 1 + 1 = \delta$. This finishes the proof of (c). □

Theorem 3. *Let $r \ge 2$, $\delta \ge 2r-2$, and for each positive integer p, let $G_{r,\delta,p}$ be the graph whose weighted clump graph is $H_{r-1,\delta,p}$. Then $G_{r,\delta,p}$ is $2r-1$ colorable (and hence K_{2r}-free), connected, with minimum degree δ, of order $n = p\big((2r-1)\delta + 2r - 3\big)+2$, and of diameter $\frac{(6r-5)n}{(2r-1)\delta+2r-3} + O(1)$. Consequently,* Conjecture 1 *fails for every $\delta > 12r^3 - 22r^2 - 2r + 12 = 2(r-1)(3r+2)(2r-3)$. Furthermore, the difference between the coefficient of $\frac{n}{\delta}$ in our construction and in* Conjecture 1(i) *is $\frac{1}{(2r^2-1)(2r-1)} + o(1)$, as $\delta \to \infty$.*

Proof. By Lemma 1, $G_{r,\delta,p}$ is $(2r-1)$-colorable with minimum degree δ, diameter $p(6r-5) - 1$, and it has $n = p\big((2r-1)\delta + 2r - 3\big)+2$ vertices. Therefore its diameter is $\frac{(6r-5)(n-2)}{(2r-1)\delta+2r-3} - 1 = \frac{(6r-5)n}{(2r-1)\delta+2r-3} + O(1)$. Consider the identity

$$\frac{(6r-5)\delta}{(2r-1)\delta + 2r - 3} - \frac{2(r-1)(3r+2)}{(2r^2-1)} = \frac{1}{(2r^2-1)(2r-1)} \cdot \frac{1 - \frac{12r^3-22r^2-2r+12}{\delta}}{(1 + \frac{2r-3}{(2r-1)\delta})}.$$

This shows both the fact that $\frac{(6r-5)n}{(2r-1)\delta+2r-3} \le \frac{2(r-1)(3r+2)}{(2r^2-1)} \cdot \frac{n}{\delta}$ iff $\delta \le 12r^3 - 22r^2 - 2r + 12$, and the statement about the difference. □

Acknowledgments

The last two authors were supported in part by the National Science Foundation contract DMS 1600811.

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