A universality property of Sobolev spaces in metric measure spaces

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Abstract

Current research on analysis in metric measure spaces has used alternative notions of Sobolev functions on metric measure spaces. We will show that under some mild geometric assumptions on the metric measure space, all these notions give the same class of functions.

1 Introduction

Motivated by the goal of understanding geometry of singular (non-Riemannian) spaces, the study of analysis on metric measure spaces in recent years has seen significant development. The metric spaces obtained as Gromov-Hausdorff limits of Riemannian manifolds may be non-Riemannian in nature, and the study of smooth analysis may be insufficient in understanding the geometry of such singular spaces. Furthermore, analysis on fractal-type sets would be incomplete if one is restricted to studying only smooth functions. Thus the current research on metric spaces utilizes notions of first-order Sobolev space and an appropriate formulation of Poincaré inequalities. In metric spaces where this machinery is available, some of the current research considers variational problems, partial differential equations, potential theory and other aspects of analysis.

The theory of Sobolev spaces of functions in Euclidean domains is now well-understood, see for example a standard reference [Maz]. Currently there exist

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various approaches to defining Sobolev-type spaces of functions in metric measure spaces. The geometric construction using the notion of upper gradients from [HeK] was first studied independently in [C] and [Sh1]; this construction yields the classical first-order Sobolev space $W^{1,p}(\Omega)$ for $1 \le p < \infty$ when Ω is an Euclidean domain or a Riemannian manifold, and under additional assumptions such as Poincaré inequality, allows for a comprehensive first-order theory of variational methods in potential theory and partial differential equations ([KiSh]). These Sobolev type spaces are called Newtonian spaces. A second notion of Sobolev type spaces, due to Hajłasz [Ha], gives an almost everywhere pointwise control of Sobolev type functions in terms of a function that plays the role of a maximal function of the gradient. A third avenue of development of first-order calculus in metric measure spaces is due to Korevaar and Schoen [KS]. The approaches of [Ha], [HaK], and [KS] yield notions of gradients that are not local; that is, from the fact that a Sobolev function is constant on a Borel set one cannot conclude that the corresponding notion of gradient is zero almost everywhere on that set. However, the Newtonian space, where the notion of gradient is the concept of upper gradient, has this locality property. Another concept of Sobolev-type space is obtained via a probabilistic approach using a symmetric bilinear form called a Dirichlet form. This concept yields a version of the Sobolev space $W^{1,2}$ which is naturally endowed with a Hilbert-space structure. The standard reference for this approach is Fukushima–Oshima–Takeda [FOT], and has been developed in the setting of fractals in [BB], [Ki1], [Ki2]. The literature on Dirichlet forms on self-similar sets is rapidly expanding and we cannot hope to provide a bibliography which does justice to the field here. Concurrently the abstract theory of Dirichlet forms on general metric spaces has been studied for example in Mosco [M], Biroli–Mosco [BM1], [BM2], Sturm [St1], [St2], [St3], Jost [J], and the references therein.

By examining the analytic properties satisfied by the above various notions of Sobolev spaces in metric measure spaces, in [GT] Goldshtein and Troyanov developed the foundational theory of axiomatic Sobolev spaces. Of the different approaches to the notion of Sobolev functions in metric measure spaces, in this article we will focus on the Newtonian spaces of [Sh1], the axiomatic Sobolev spaces of Goldshtein and Troyanov [GT], and the Drichlet forms developed in [FOT] and [J]. Under suitable conditions such as strong locality of the Sobolev-type spaces, doubling property of the measure, and the satisfaction of a Poincaré inequality, we will show that these three spaces are isomorphic. For an expository description of connections between the Hajłasz-Sobolev spaces, the Sobolev spaces of Korevaar-Schoen, and the Newtonian spaces we refer the interested reader to [ShMat].

This article is structured as follows. Section 2 provides the background material and basic concepts used in the main theorems. There are two main results in this article. The first, demonstrating that under certain conditions that are standard in analysis of Dirichlet forms, the domain of such a Dirichlet form, called the Dirichlet domain, coincides with the Newtonian space $N^{1,2}(X)$. Section 3 begins with a description of Dirichlet forms, and then provides a proof of Theorem 3.13. The final section gives a definition of the axiomatic Sobolev spaces of Goldshtein-Troyanov, and proves that under strong locality assumptions these spaces coincide with the Newtonian spaces (Theorem 4.3).

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2 Background

In this article (X, d, μ) will denote a set X equipped with a metric d and a Borel measure μ such that non-empty open sets have positive measure and bounded sets have finite measure. We will also assume that X is complete and that μ is doubling, that is, there is a constant $C > 0$ such that whenever $B(x,r) = \{y \in X \,:\, d(x,y) < r\}$ is a ball in X,

$$\mu(B(x,2r)) \le C\,\mu(B(x,r)).$$

It is easy to see that such a metric space X is proper, that is, closed and bounded subsets of X are compact.

In [HeK], Heinonen and Koskela proposed the following substitute for gradients in metric spaces.

Definition 2.1. Given a function $u : X \to [-\infty, \infty]$, we say that a non-negative Borel measurable function ρ on X is an *upper gradient* of u if whenever γ is a compact rectifiable path in X,

$$|u(x) - u(y)| \le \int_\gamma \rho\, ds,$$

where x, y denote the end points of γ. If $u(x)$ or $u(y)$ is not finite, the above inequality is interpreted to mean that $\int_\gamma \rho\, ds = \infty$. We say that ρ is a *p-weak upper gradient* of u for some $1 \le p < \infty$ if there is a non-negative Borel measurable function $\rho_0 \in L^p(X)$ such that whenever γ does not satisfy the above inequality, $\int_\gamma \rho_0\, ds = \infty$.

The uniform convexity of $L^p(X)$ when $1 < p < \infty$ implies that if u has a p-weak upper gradient $\rho \in L^p(X)$, then u has a *minimal p-weak upper gradient* $\rho_u \in L^p(X)$, see [Sh1]. If u is constant on a Borel set A and $\rho \in L^p(X)$ is a p-weak upper gradient of u, then $\rho\,\chi_{X\setminus A}$ is also a p-weak upper gradient of u; for a proof of this fact we refer the interested reader to [Sh2]. Such a property is called the *strong locality* property.

For functions $u \in L^p(X)$ with upper gradients $\rho \in L^p(X)$, we define

$$\|u\|_{N^{1,p}(X)} = \|u\|_{L^p(X)} + \inf_\rho \|\rho\|_{L^p(X)},$$

where the infimum is taken over all upper gradients of u. The function space $\widehat{N}^{1,p}(X)$ is the collection of all functions u on X such that $u \in L^p(X)$ and u has an upper gradient in $L^p(X)$. It is easy to see that $\widehat{N}^{1,p}(X)$ is a vector space with a lattice structure. The Newtonian space $N^{1,p}(X) = \widehat{N}^{1,p}(X)/\sim$, where the equivalence relation $\sim$ is given by the rule $u \sim v$ if $\|u - v\|_{N^{1,p}(X)} = 0$.

Observe that if X is a domain in an Euclidean space, equipped with the Euclidean metric and the standard Lebesgue measure, then $N^{1,p}(X) = W^{1,p}(X)$; see for example [Sh1].

Definition 2.2. We say that X supports a $(1,p)$-Poincaré inequality for Newtonian functions if there are constants $C > 0, \tau \ge 1$ such that for all $u \in N^{1,p}(X)$, all upper gradients ρ of u, and all balls $B(x,r) \subset X$,

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \le Cr \left(\fint_{B(x,\tau r)} \rho^p\, d\mu \right)^{1/p}. \tag{2.3}$$

Here $\fint_{B(x,r)} u\, d\mu = u_{B(x,r)} = \mu(B(x,r))^{-1} \int_{B(x,r)} u\, d\mu$.

The following deep result, due to Cheeger [C], was proven for a Sobolev-type space $H^{1,p}(X)$ that appears on the surface to be different from the Newtonian space. However, when $1 < p < \infty$, it can be seen that $H^{1,p}(X) = N^{1,p}(X)$; we direct interested readers to [Sh1].

Proposition 2.4 ([C, Section 6]). *If the measure on X is doubling and supports a $(1,p)$-Poincaré inequality for Newtonian functions with $p > 1$, and u is a Lipschitz function on X, then*

$$Lipu(x) := \limsup_{y \to x} \frac{|u(y) - u(x)|}{d(y,x)}$$

is the unique (up to sets of μ-measure zero) minimal p-weak upper gradient of u.

The following result, due to Keith, is a key tool in proving that the three candidate Sobolev spaces are isomorphic under certain conditions.

Proposition 2.5 ([Kei, Theorem 2]). *Let X be a complete metric measure space equipped with a doubling measure, and $p \geq 1$. The following are equivalent.*

1. *X supports a $(1,p)$-Poincaré inequality for all measurable functions.*
2. *X supports a $(1,p)$-Poincaré inequality for all compactly supported Lipschitz functions with compactly supported continuous upper gradients.*
3. *X supports a $(1,p)$-Poincaré inequality for all Newtonian functions.*

In proving one of the two focal theorems of this note, Theorem 3.13, we need the auxiliary candidate space called the Korevaar-Schoen space, first given in [KS] and also studied in [KMc]. The Korevaar-Schoen space $KS^{1,2}(X)$ is defined to be the collection of all functions $f \in L^2(X)$ for which the approximating energies

$$e_\epsilon^2(x; f) = \fint_{B(x,\epsilon)} \frac{d_Y(f(x), f(y))^2}{\epsilon^2} \, d\mu_X(y)$$

converge to a finite energy

$$E_{KS}(f) := \sup_{\text{balls } B} \limsup_{\epsilon \to 0} \int_B e_\epsilon^2(x; f) \, d\mu_X(x) < \infty.$$

3 Dirichlet forms and $N^{1,2}(X)$

We follow the spirit of [J] in defining Dirichlet forms. Recall that $L^2(X) = L^2(X,\mu)$ is a Hilbert space endowed with the inner product $(u,v) := \int_X uv\, d\mu$.

Definition 3.1. A symmetric bilinear form $\mathcal{E} : L^2(X) \times L^2(X) \to \mathbb{R}^+ \cup \{\infty\}$ is a *Dirichlet form* if the following conditions are satisfied:

1. *Quadratic contraction property* - If $\mathcal{E}(u,u) < \infty$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is L-Lipschitz with $\varphi(0) = 0$, then
$$\mathcal{E}(\varphi \circ u, \varphi \circ u) \leq L^2 \mathcal{E}(u,u). \tag{3.2}$$

2. *Closedness* - If $\{u_n\}$ is a Cauchy sequence in $L^2(X)$ so that $\mathcal{E}(u_n) < \infty$ for all n and $\mathcal{E}(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to \infty$, then $\mathcal{E}(u_n - u, u_n - u) \to 0$ as $n \to \infty$. Here u denotes the L^2-limit of the Cauchy sequence $\{u_n\}$.

3. *Density condition* - The domain $D(\mathcal{E})$ of the bilinear form, consisting of all functions u in $L^2(X)$ for which $\mathcal{E}(u,u) < \infty$, is dense in $L^2(X)$ with respect to its norm and is dense in the space $C_0(X)$ of continuous functions with compact support with respect to the supremum norm.

4. For every $u, v \in L^2(X)$, we have the *Minkowski inequality*
$$\sqrt{\mathcal{E}(u+v, u+v)} \leq \sqrt{\mathcal{E}(u,u)} + \sqrt{\mathcal{E}(v,v)}. \tag{3.3}$$

The approach of [FOT] begins with a Markovian condition which is weaker than the quadratic contraction property. However, as much of the current research on Dirichlet forms also require them to satisfy the quadratic contraction property, we include it as part of the definition.

It can be seen by the quadratic contraction property that $D(\mathcal{E})$ is a vector space, and indeed is a normed space when equipped with the norm
$$\|u\|_{\mathcal{E}} := \|u\|_{L^2(X)} + \sqrt{\mathcal{E}(u,u)}.$$
The closability condition implies that if $\{u_n\}$ is a sequence which is Cauchy in this norm with L^2-limit u, then $\mathcal{E}(u,u) = \lim_{n\to\infty} \mathcal{E}(u_n, u_n)$ is finite. Moreover, as a consequence of the closedness condition and the Minkowski inequality, the normed space $(D(\mathcal{E}), \|\cdot\|_{\mathcal{E}})$ is a Hilbert space equipped with the norm $\|\cdot\|_{\mathcal{E}}$.

Using the fact that $L^2(X)$ is a Hilbert space, Beurling and Deny [BD] construct a unique signed Radon measure-valued symmetric bilinear form associated with $\mathcal{E}$:

$$\eta : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathcal{M}(X)$$

(where $\mathcal{M}(X)$ is the collection of all finite signed Radon measures on X) which satisfies $\mathcal{E}(u,v) = \int_X d\eta(u,v)$. The measure $\eta(u,u)$ plays the role of the (square of the norm of the) point-wise derivative of u in the general setting, satisfying the quadratic contraction property

$$d\eta(\varphi \circ u, \varphi \circ u) \leq L^2 d\eta(u,u)$$

whenever φ is an L-Lipschitz function on $\mathbb{R}$, and for all Oriels sets $E \subset X$,

$$|\eta(u,v)(E)| \leq \sqrt{\eta(u,u)(E)\, \eta(v,v)(E)}. \tag{3.4}$$

If $\mathcal{E}$ is strongly local, then for all $u, v \in D(\mathcal{E})$ the singular (jump) part of $\eta(u,v)$ with respect to μ is zero.

For more discussion about Dirichlet forms, we refer the reader to [J], [BD], [St1], [St2], [M] and the text [FOT].

Definition 3.5. We say that $\mathcal{E}$ is *local* if whenever $u, v \in D(\mathcal{E})$ such that the support of u and the support of v are disjoint compact sets, then $\mathcal{E}(u,v) = 0$. We say that $\mathcal{E}$ is *strongly local* if whenever $U \subset X$ is an open set and $u \in D(\mathcal{E})$ is constant on U, then for all $v \in d(\mathcal{E})$, $\eta(u,v)$ is supported in $X \setminus U$.

If X is a Euclidean domain, the form $\mathcal{E}(u,v) = \int_X \langle \nabla u, \nabla v \rangle\, dx$ is a strongly local Dirichlet form in the above sense. Bilinear Dirichlet forms on certain fractals and more general self-similar sets have been constructed by Barlow–Bass [BB] and Kigami [Ki1], [Ki2]; these are not strongly local in general. If X is a metric measure space equipped with a doubling measure supporting a $(1,2)$-Poincaré inequality for Newtonian functions, then by a deep Rademacher type theorem of [C], it follows that $N^{1,2}(X) = D(\mathcal{E})$ for some strongly local Dirichlet form on X. Sturm [St3] proved that the Korevaar-Schoen space $KS^{1,2}(X)$ (a definition of this space is provided at the end of Section 2) is a Dirichlet domain corresponding to a local Dirichlet form.

In considering Dirichlet forms, traditionally the underlying space is only a Hausdorff topological space equipped with a Borel measure, and the Poincaré inequality for a such a Dirichlet form $\mathcal{E}$ is typically expressed using the so-called *intrinsic metric* associated with $\mathcal{E}$.

A subspace Γ of $D(\mathcal{E}) \cap C_0(X)$ is a *μ-separating core* if it is dense in $C_0(X)$ with respect to the supremum norm, is dense in $D(\mathcal{E})$ with respect to the norm $\|\cdot\|_{\mathcal{E}}$, and has the following separation property: for every pair of distinct points x, y in X there is a function $\varphi \in \Gamma$ so that $\varphi(x) \neq \varphi(y)$ and $d\eta(\varphi, \varphi) \leq d\mu$. The intrinsic metric $d_{\mathcal{E}}$ is defined as follows:

$$d_{\mathcal{E}}(x, y) = \sup\{\varphi(x) - \varphi(y) \ : \ \varphi \in \Gamma, d\eta(\varphi, \varphi) \leq d\mu\}, \qquad x, y \in X. \tag{3.6}$$

Following [J] and [BM1], we make the standing assumption that the topology induced by $d_{\mathcal{E}}$ coincides with the underlying topology on X and that the measure μ is doubling for balls in the new metric, i.e., there exists a constant $C \geq 1$ so that

$$\mu(2B_{\mathcal{E}}) \leq C\mu(B_{\mathcal{E}})$$

for every ball B_E in X. A change in the underlying metric results in a change in the corresponding Newtonian spaces. Hence we will from now on assume also that $d_{\mathcal{E}} = d$ and that every 1-Lipschitz function φ on X is in the μ-separating core Γ with $d\eta(\varphi, \varphi) \leq d\mu$. Given a Lipschitz function φ on X, let

$$Lip(u, x) := \lim_{r \to 0^+} \sup_{y, z \in B(x, r), y \neq z} \frac{|\varphi(y) - \varphi(z)|}{d(y, z)}. \tag{3.7}$$

Standard measure theoretic arguments show that $Lip(u, \cdot)$ is a bounded μ-measurable function on X.

Lemma 3.8. *Let $\mathcal{E}$ be a strongly local Dirichlet form such that every compactly supported 1-Lipschitz function φ on X is in $D(\mathcal{E})$ with $d\eta(\varphi, \varphi) \leq d\mu$. Then whenever u is a compactly supported Lipschitz function on X, $u \in D(\mathcal{E})$ and for μ-almost every $x \in X$,*

$$d\eta(u, u)(x) \leq Lip(u, x)^2 \, d\mu(x).$$

Proof. Fix $x \in X$, and $r > 0$. Let

$$L_u(x; r) := \sup_{y, z \in B(x, r), y \neq z} \frac{|\varphi(y) - \varphi(z)|}{d(y, z)}.$$

Then $Lip(u, x) = \lim_{r \to 0^+} L_u(x; r)$. Restricting u to the ball $B(x, r)$, it is easy to see that this restriction is $L_u(x; r)$-Lipschitz continuous on this ball. We can find a Lipschitz extension φ of the restriction of u to X (for example, via a

McShane construction [He]) such that φ is $L_u(x;r)$-Lipschitz on X. By modifying φ outside $B(x,r)$ (by multiplying φ by a compactly supported Lipschitz function that is 1 on $B(x,r)$ if necessary), we can ensure that φ is compactly supported. Hence $L_u(x;r)^{-1}\varphi \in D(\mathcal{E})$ with $d\eta(L_u(x;r)^{-1}\varphi, L_u(x;r)^{-1}\varphi) \leq d\mu$ by hypothesis. Now by the quadratic contraction property, $d\eta(\varphi,\varphi) \leq L_u(x;r)^2\, d\mu$.

Since $\mathcal{E}$ is strongly local, by (3.4) we have $d\eta(\varphi,\varphi) = d\eta(u,u)$ on $B(x,r)$. Hence $d\eta(u,u) \leq L_u(x;r)^2\, d\mu$ on $B(x,r)$. Now letting $r \to 0$ completes the proof for $x \in X$ that are Lebesgue points for $Lip(u,\cdot)$. $\square$

Definition 3.9. We say that X supports a *weak* $(1,2)$*-Poincaré inequality* for the form $\mathcal{E}$ if there are constants $\tau \geq 1$ and $C > 0$ so that for all $u \in D(\mathcal{E})$ and balls $B(x,r) \subset X$,

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \leq Cr\sqrt{\frac{\eta(u,u)(B(x,\tau r))}{\mu(B(x,\tau r))}}. \tag{3.10}$$

Remark 3.11. *The paper [KST] studies metric space-valued Sobolev-type spaces and Dirichlet domains, and demonstrates that if a strongly local Dirichlet form satisfies a Poincaré inequality and for all Lipschitz functions φ on X, $d\eta(\varphi,\varphi)(x) \approx Lip(\varphi,x)\, d\mu(x)$, then the corresponding Dirichlet domain embeds isomorphically into the Korevaar-Schoen space $K^{1,2}(X)$ ([KST, Proposition 4.2]). The condition that $d\eta(\varphi,\varphi)(x) \approx Lip(\varphi,x)\, d\mu(x)$ for all Lipschitz functions φ on X was used there only to ensure that the underlying metric used in constructing the Korevaar-Schoen space is biLipschitz equivalent to the intrinsic metric of the Dirichlet form. We will need this result of [KST], but as we consider only the intrinsic metric on the space ($d_{\mathcal{E}} = d$), this condition on the measures η will not be needed by us. The main theorem of this section, Theorem 3.13, is an improvement of Proposition 4.2 of [KST].*

Recall that X is said to be quasiconvex if there is a constant $C > 0$ such that whenever $x, y \in X$ there is a curve γ connecting x to y with length no more than $Cd(x,y)$.

Lemma 3.12. *Let (X,d,μ) be a complete metric measure space such that μ is doubling and supports the following Poincaré inequality: for every Lipschitz* *function φ on X, whenever $B(x,r)$ is a ball in X,*

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \le Cr \left(\fint_{B(x,\tau r)} Lip(u,y)^2 \, d\mu(y) \right)^{1/2} .$$

Then X is quasiconvex.

Proof. The proof of this lemma is along the lines of a proof of Semmes on quasiconvexity; see [C, Appendix]. For $\epsilon > 0$ and $x, y \in X$, we say that $x \sim_\epsilon y$ if there is a finite collection of points $x_1, \dots, x_k$ in X such that $d(x_1, x) < \epsilon, d(x_k, y) < \epsilon$, and for $j = 1, \dots, k-1$, $d(x_j, x_{j+1}) < \epsilon$. An easy topological argument shows that the equivalence relation $\sim_\epsilon$ partitions X into pairwise disjoint closed sets. If U_1 and U_2 are such sets, then if both are non-empty, the distance between them is at least ϵ, and hence these sets are open as well. Hence the characteristic function $u = \chi_{U_1}$ of U_1 is $1/\epsilon$-Lipschitz, with $Lip(u, \cdot) = 0$. Since U_2 is non-empty and open, for sufficiently large balls B in X the integral $\fint_B |u - u_B| \, d\mu$ is positive, thus violating the above version of Poincaré inequality. Hence every $x, y \in X$ have $x \sim_\epsilon y$.

For $\epsilon > 0$ and $x \in X$, let $\rho_\epsilon : X \to \mathbb{R}$ be the function given by

$$\rho_\epsilon(y) = \inf_{(x_1, \dots, x_k)} d(x, x_1) + d(x_k, y) + \sum_{j=1}^{k-1} d(x_j, x_{j+1}).$$

If $y_1, y_2 \in X$ such that $d(y_1, y_2) < \epsilon$, then $|\rho_\epsilon(y_1) - \rho_\epsilon(y_2)| \le d(y_1, y_2)$; that is, $Lip(\rho_\epsilon, y) \le 1$ on X. Hence an employment of the above version of Poincaré inequality together with the telescoping argument given in [C, Section 4] shows that $(\rho_\epsilon)_\epsilon$ is a bounded sequence of continuous functions in $L^1(B(x,R))$ for all $R > 0$. Since ρ_ϵ increases as ϵ decreases, the limit $\lim_{\epsilon \to \infty} \rho_\epsilon$ exists, and by the above boundedness converges to a finite-valued function ρ_∞ on X. The final part of the argument of [C, Appendix] now completes the proof. $\square$

The main theorem of this section is the following result.

Theorem 3.13. *Let $\mathcal{E}$ be a strongly local Dirichlet form on X such that the intrinsic metric d induced by $\mathcal{E}$ is compatible with the topology of X. Suppose that X supports a Borel measure μ that is doubling (with respect to the metric d). If every compactly supported 1-Lipschitz function on X is in $D(\mathcal{E})$ with $d\eta(\varphi, \varphi) \le d\mu$, and $\mathcal{E}$ satisfies a weak $(1,2)$-Poincaré inequality, then*

X supports a weak $(1,2)$-Poincaré inequality for Newtonian functions and $D(\mathcal{E}) = N^{1,2}(X)$ as a Banach space isomorphism.

Proof. By Proposition 2.5, it suffices to prove the Poincaré inequality for compactly supported Lipschitz functions u with compactly supported Lipschitz upper gradients ρ.

Since $\mathcal{E}$ satisfies a Poincaré inequality, by Lemma 3.8, whenever $B(x,r)$ is a ball in X,

$$\fint_{B(x,r)} |u - u_B|\, d\mu \le Cr \left(\fint_{B(x,\tau r)} Lip(u,y)^2\, d\mu(y) \right)^{1/2}.$$

Let ρ be a compactly supported Lipschitz upper gradient of u. By quasiconvexity of X (see Lemma 3.12), whenever $z, y \in B(x,r)$, there is a curve γ connecting z and y in $B(x, Cr)$ with length $\ell(\gamma) \le C_Q d(z,y)$. Hence

$$|u(z) - u(y)| \le \int_\gamma \rho\, ds \le \ell(\gamma)\rho(w_\gamma) \le C_Q d(z,y)\rho(w_\gamma)$$

for some w_γ in the trajectory of γ. Hence $|u(z) - u(y)|/d(z,y) \le C_Q \rho(w_\gamma)$. Therefore

$$L_u(x;r) \le C_Q \sup_{w \in B(x,Cr)} \rho(w).$$

Letting $r \to 0$ and noting that ρ is continuous, we obtain $Lip(u,x) \le C_Q \rho(x)$. It now follows that

$$\fint_{B(x,r)} |u - u_B|\, d\mu \le CC_Q r \left(\fint_{B(x,\tau r)} \rho^2\, d\mu \right)^{1/2}.$$

Since $\mathcal{E}$ supports a weak $(1,2)$-Poincaré inequality, by [KST, Proposition 4.2] (see Remark 3.11 above), $D(\mathcal{E})$ embeds via a Banach space isomorphism into the Korevaar-Schoen space $KS^{1,2}(X)$. Since X supports a $(1,2)$-Poincaré inequality for Newtonian functions, $KS^{1,2}(X) = N^{1,2}(X)$ as a Banach space isomorphism. Therefore, $D(\mathcal{E})$ is embedded isomorphically in $N^{1,2}(X)$.

It suffices now to prove that $N^{1,2}(X)$ embeds into $D(\mathcal{E})$. For $\epsilon > 0$ we can cover X by countably many balls $B_i = B(x_i, \epsilon)$ such that $\sup_{x \in X} \sum_{i \in \mathbb{N}} \chi_{5B_i}(x) \le$

C, and obtain a partition of unity φ_i subordinate to this cover; $0 \le \varphi_i \le 1$, φ_i is supported on B_i, $\sum_{i\in\mathbb{N}} \varphi_i = 1$, and φ_i is C/ϵ-Lipschitz (see [He] or [KMc]). As in the proof of [KMc, Lemma 4.6], we obtain a discrete convolution of $u \in N^{1,2}(X) = KS^{1,2}(X)$ as follows:

$$u_\epsilon(x) = \sum_{i\in\mathbb{N}} u_{B_i}\varphi_i(x).$$

By Lemma 4.6 of [KMc] and Hölder's inequality, whenever $y, z \in X$ with $d(y,z) < \epsilon$,

$$\frac{|u_\epsilon(y) - u_\epsilon(z)|}{d(y,z)} \le C \left(\fint_{B(y,2\epsilon)} e_{5\epsilon}^2(w;u)\,d\mu(w) \right)^{1/2},$$

and hence if $x \in X$ and $r < \epsilon/2$, we have

$$Lip_{u_\epsilon}(x;r) \le C \left(\fint_{B(x,4\epsilon)} e_{5\epsilon}^2(w;u)\,d\mu(w) \right)^{1/2}.$$

It follows that

$$Lip(u_\epsilon, x) \le C \left(\fint_{B(x,4\epsilon)} e_{5\epsilon}^2(w;u)\,d\mu(w) \right)^{1/2},$$

and hence by Lemma 3.8,

$$\eta(u_\epsilon, u_\epsilon)(B_i) \le C \int_{B_i} \fint_{B(x,4\epsilon)} e_{5\epsilon}^2(w;u)\,d\mu(w)\,d\mu(x) \le C \int_{B_i} e_{5\epsilon}^2(x;u)\,d\mu(x).$$

Hence by the bounded overlap property of the cover B_i,

$$\eta(u_\epsilon, u_\epsilon)(X) \le C \int_X e_{5\epsilon}^2(x;u)\,d\mu(x),$$

that is, for sufficiently small $\epsilon > 0$, $\mathcal{E}(u_\epsilon, u_\epsilon) \le CE_{KS}(u)$. By the fact that X supports $(1,2)$-Poincaré inequality for Newtonian functions, as in [KMc] we see that $u_\epsilon \to u$ in $KS^{1,2}(X) = N^{1,2}(X)$. Thus by the closability property of the Dirichlet form, we see that $\mathcal{E}(u,u) \le C\|u\|^2_{N^{1,2}(X)}$, that is, $N^{1,2}(X) \subset D(\mathcal{E})$. $\square$

4 Axiomatic Sobolev spaces and $N^{1,p}(X)$

The focus of this section is the axiomatic theory of Goldshtein-Troyanov [GT]. The starting point for this theory is the assumption of a class of "derivatives" in the metric space setting, called a *D structure*, which associates with each $u \in L^p_{loc}(X)$ a class $D[u]$ of non-negative measurable functions on X such that the following five axioms are satisfied:

1. (*non-triviality*) If u is a non-negative L-Lipschitz function supported on a set $A \subset X$, then $L\chi_A \in D[u]$.

2. (*upper linearity*) If $\rho_1 \in D[u_1]$, $\rho_2 \in D[u_2]$, and $\alpha, \beta \in \mathbb{R}$, and $g \geq |\alpha|\rho_1 + |\beta|\rho_2$, then $g \in D[\alpha u_1 + \beta u_2]$.

3. (*Leibnitz rule*) If $g \in D[u]$ and φ is a bounded L-Lipschitz function on X, then $\|\varphi\|_{L^\infty(X)} g + L|u| \in D[\varphi u]$.

4. (*lattice property*) If $\rho_1 \in D[u_1]$, $\rho_2 \in D[u_2]$, then $\max\{\rho_1, \rho_2\} \in D[\max\{u_1, u_2\}] \cap D[\min\{u_1, u_2\}]$.

5. (*completeness*) If $g_i \in D[u_i]$ for $i \in \mathbb{N}$, $u_i \to u$ in $L^p(X)$, and $g_i \to g$ in $L^p(X)$, then $g \in D[u]$.

The Axiomatic Sobolev space $\mathcal{L}^{1,p}(X)$ consists of all $u \in L^p(X)$ for which there exists $g \in D[u] \cap L^p(X)$. This space, equipped with the norm

$$\|u\|_{\mathcal{L}^{1,p}(X)} = \|u\|_{L^p(X)} + \inf_{g \in D[u]} \|g\|_{L^p(X)}$$

is a Banach space by the axioms above.

It was shown in [GT] that if $1 < p < \infty$, then for every $u \in \mathcal{L}^{1,p}(X)$ there is a unique function $g_u \in D[u]$, called the *minimal pseudo-gradient* of u, such that $\inf_{g \in D[u]} \|g\|_{L^p(X)} = \|g_u\|_{L^p(X)}$.

Definition 4.1. We say that the D structure supports a $(1,p)$-Poincaré inequality if there are constants $C > 0$ and $\tau \geq 1$ such that whenever $u \in L^p_{loc}(X)$ and $g \in D[u]$, for all balls $B(x,r)$ in X,

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq Cr \left(\fint_{B(x,\tau r)} g^p \, d\mu \right)^{1/p}.$$

We say that the D structure is *strongly local* if whenever $u_1, u_2 \in \mathcal{L}^{1,p}(X)$ and $u_1 = u_2$ on a Borel set $A \subset X$, then for all $g_1 \in D[u_1]$ and $g_2 \in D[u_2]$, $g_1\chi_A + g_2\chi_{X\setminus A} \in D[u_2]$.

The axiomatic regularity theory developed in [Tim] would not be applicable to solutions to variational problems based on the axiomatic Sobolev space theory without the assumption of the support of Poincaré inequality and strict locality. Hence the hypotheses of the following main theorem of this section are reasonable.

Lemma 4.2. *If the D structure is strongly local and $1 < p < \infty$, then whenever $u_\in\mathcal{L}^{1,p}(X)$ and $g \in D[u]$, $g \geq g_u$ μ-a.e. in X (g_u is the minimal pseudo-gradient of u).*

The proof is a direct application of the strong locality of the D structure, and is left to the reader.

Theorem 4.3. *Suppose that $\mathcal{L}^{1,p}(X)$ is an axiomatic Sobolev space on a complete metric measure space with doubling measure. If the associated D structure supports a $(1,p)$-Poincaré inequality and is strongly local, then X supports a $(1,p)$-Poincaré inequality for Newtonian functions, and $\mathcal{L}^{1,p}(X) \subset N^{1,p}(X)$ as a Banach space isomorphism.*

Proof. First we show that under the strict locality assumption, whenever $\varphi : X \to \mathbb{R}$ is a Lipschitz function on X, $Lip(\varphi, \cdot) \in D[\varphi]$. Without loss of generality we assume that φ is compactly supported, and hence by the non-triviality axiom, $\varphi \in \mathcal{L}^{1,p}(X)$. Fix $r > 0$, and for $x \in X$ let $\psi_{x,r}$ be a Lipschitz extension (given by McShane for example, see [He]) of $\varphi|_{B(x,r)}$ to X; then the global Lipschitz constant of $\psi_{x,r}$ is given as follows:

$$\begin{aligned} L_{\psi_{x,r}} &= \sup_{y,z\in X:y\neq z} \frac{|\psi_{x,r}(y) - \psi_{x,r}(z)|}{d(y,z)} \\ &= \sup_{y,z\in B(x,r):y\neq z} \frac{|\varphi(y) - \varphi(z)|}{d(y,z)} \\ &= Lip_\varphi(x;r). \end{aligned}$$

By the non-triviality axiom, $L_{\psi_{x,r}} \in D[\psi_{x,r}]$ and . Note that $Lip(\varphi, x) = \limsup_{r\to 0^+} Lip_\varphi(x;r)$. Since $\varphi = \psi_{x,r}$ on the open ball $B(x,r)$, by the strong locality property, if g_φ, is the minimal pseudo-gradient of φ, then $g_\varphi \leq \psi_{x,r}$ on $B(x,r)$ (see the lemma above).

Let Z be the collection of all non-Lebesgue points of g_φ. Then $\mu(Z) = 0$, and for all $x \in X \setminus Z$,

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} g_\varphi \, d\mu \leq Lip_\varphi(x;r),$$

and hence

$$g_\varphi(x) = \lim_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} g_\varphi \, d\mu \leq \limsup_{r \to 0^+} Lip_\varphi(x;r) = Lip(\varphi, x).$$

That is, almost everywhere on X, $g_\varphi \leq Lip(\varphi, \cdot)$; hence $Lip(\varphi, \cdot) \in D[\varphi]$ by the upper linearity axiom.

Now, if the axiomatic space satisfies a $(1,p)$-Poincaré inequality, then whenever φ is a Lipschitz function on X, for all balls $B \subset X$,

$$\frac{1}{\mu(B)} \int_B |\varphi - \varphi_B| \, d\mu \leq C \, \mathrm{rad}(B) \left(\fint_{\tau B} Lip(\varphi, y)^p \, d\mu(y) \right)^{1/p},$$

and then as in the proof of Theorem 3.13, we obtain that X supports a $(1,p)$-Poincaré inequality for Newtonian functions and that $N^{1,p}(X) = \mathcal{L}^{1,p}(X)$, thus concluding the proof. $\square$

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