UNIFORMITY FROM GROMOV HYPERBOLICITY

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ABSTRACT. We show that, in a metric space X with annular convexity, uniform domains are precisely those Gromov hyperbolic domains whose quasiconformal structure on the Gromov boundary agrees with that on the boundary in X. As an application we show that quasimöbius maps between geodesic spaces with annular convexity preserve uniform domains. These results are quantitative.

1. INTRODUCTION

Bonk, Heinonen and Koskela introduced uniform metric spaces in [BHK], and demonstrated a fundamental two-way correspondence between these and proper geodesic Gromov hyperbolic spaces. In addition to this connection with geometric group theory, uniform metric spaces have come to play a significant role in the program of doing analysis in the metric space setting. The importance of Euclidean uniform domains is well known and documented in [Geh87] and [V4]. Recently, uniform subspaces of the Heisenberg groups, as well as more general Carnot groups, have been a focus of study; see [CT95], [CGN00], [Gre01]. Many important concepts in potential theory are known to hold in uniform spaces; for example, see [Aik04] and [Aik06]. There are close ties between uniformity and extension of Sobolev functions; see [Jon81] for Euclidean space and [BSh] for the metric space setting.

An important problem is to determine which Gromov hyperbolic Euclidean domains (Euclidean domains which are Gromov hyperbolic when endowed with their quasihyperbolic metric) are quasiconformally equivalent to a uniform domain. Initial results were established in [BHK] (Theorem 7.11). A Banach space analog was corroborated by Väisälä [V1] (Theorem 3.27). The goal of this paper is to provide a similar characterization for uniform domains in metric spaces that satisfy some mild geometric conditions.

Let (X, d) be a metric space and $C_{\mathrm{a}} \geq 2$ a constant. We say that (X, d) is C_{a}-*annular convex* if for all $x \in X$, all $r > 0$, and every pair of points $y, z \in B(x, r) \backslash B(x, r/2)$, there is a path γ joining y, z and satisfying:

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(1) the length of γ is at most $C_{\mathrm{a}}\, d(y,z)$,
(2) the path γ lies in the annulus $B(x, C_{\mathrm{a}} r)\backslash B(x, r/C_{\mathrm{a}})$.

Examples of metric spaces possessing annular convexity include Banach spaces and Carnot groups, as well as metric spaces equipped with doubling measures that support Poincaré inequalities [K].

Let (X,d) be a proper metric space (that is, closed and bounded subsets are compact), and $\Omega \subset X$ a rectifiably connected open subset (every pair of points in Ω can be joined by a rectifiable path in Ω) with boundary $\partial\Omega \neq \emptyset$. We say Ω is a *Gromov hyperbolic domain* if Ω is Gromov hyperbolic with respect to the quasihyperbolic metric k on Ω. Given a bounded Gromov hyperbolic domain Ω, we obtain the Gromov closure $\Omega^* = \Omega \cup \partial^*\Omega$ of (Ω, k), where $\partial^*\Omega$ is the Gromov boundary of (Ω, k). The closure of Ω in (X,d) is denoted $\overline{\Omega}$; since X is proper and Ω is bounded, $\overline{\Omega}$ is compact. In general, the identity map $f : (\Omega, k) \to (\Omega, d)$ may not extend to a continuous map from Ω^* to $\overline{\Omega}$, and even if f does extend, the extension may not be injective. However, if Ω is a uniform domain, then f extends to a homeomorphism from Ω^* to $\overline{\Omega}$, and the restriction of the extension to the Gromov boundary is a quasimöbius map with respect to the visual metric on $\partial^*\Omega$ [BHK]. The main result (Theorem 9.1) of this paper is that in the setting of annular convex proper metric spaces, uniform domains are the only Gromov hyperbolic domains with the above property.

Theorem 9.1 provides a characterization of uniform domains in terms of Gromov hyperbolic spaces and the quasiconformal structure on the Gromov boundary. It makes it possible to study uniform domains using the theory of Gromov hyperbolic spaces. As an illustration we show that quasimöbius maps preserve uniform domains (Theorem 10.1): if Ω is a domain in an annular convex proper metric space and Ω is quasimöbius equivalent to a uniform domain in some metric space, then Ω is also uniform.

For domains in Euclidean spaces and spheres Theorem 9.1 was proved by Bonk, Heinonen, and Koskela [BHK] (Theorem 7.11), and for domains in Banach spaces by Väisälä [V1] (Theorem 3.27). The proof in [BHK] makes use of the notion of moduli of path families, and therefore does not extend to metric spaces that have no "nice" measure. The proof in [V1] uses only metric properties. Our proof follows the general outline of Väisälä's arguments, but our proof contains several new ingredients.

In [V1] the theorem was first proved for unbounded domains in Banach spaces, and then inversions in Banach spaces were used to reduce the study of bounded domains to the study of unbounded domains. To follow this strategy, we use a notion of "inversion" in general metric spaces, see Section 4 or [BHX] for more details.

We interpret the cross ratio in the Gromov boundary (with respect to a visual metric) in terms of distances between certain geodesics (see Section 5).

Let (Y,h) be a proper geodesic Gromov hyperbolic space, $Q=(\xi_1,\xi_2,\xi_3,\xi_4)$ a quadruple of distinct points in the Gromov boundary ∂^*Y and $h_{y,\varepsilon}$ ($y\in Y$, $\varepsilon>0$) a visual metric on ∂^*Y. Fix any geodesic $[\xi_i,\xi_j]$ ($1\le i,j\le 4$) from ξ_i to ξ_j. The cross ratio of Q with respect to $h_{y,\varepsilon}$, denoted $\operatorname{cr}(Q,h_{y,\varepsilon})$, satisfies

$$\operatorname{cr}(Q,h_{y,\varepsilon})\approx\begin{cases} e^{\varepsilon h([\xi_1,\xi_4],[\xi_2,\xi_3])} & \text{if } h([\xi_1,\xi_4],[\xi_2,\xi_3])\ge h([\xi_1,\xi_3],[\xi_2,\xi_4]),\\ e^{-\varepsilon h([\xi_1,\xi_3],[\xi_2,\xi_4])} & \text{otherwise.}\end{cases}$$

This interpretation of cross ratio is quite convenient in studying the quasiconformal structure of the Gromov boundary, and allows us to simplify some of the arguments found in [V1].

A crucial property used in Väisälä's proof is that spheres in Banach spaces are 2-quasiconvex. A consequence of this property is that each arc point lies on an anchor. (A point x in a domain Ω is an arc point if every point on the boundary of Ω that is closest to x is essentially not isolated. The technical definition of arc points and anchors are given in Section 7). This was first shown in [BHK] in the Euclidean setting. Annular convexity is our replacement for the quasiconvexity of spheres. Under the assumption of annular convexity we establish (see Section 7) a slightly weaker version of this fact sufficient for the proof of Theorem 9.1.

There is another difference between the Banach space setting of [V1] and our setting of proper quasiconvex space. Since infinite dimensional Banach spaces are not proper, one cannot assume the existence of quasihyperbolic geodesics in domains there; hence the tools of roads and biroads were developed in [V1]. In our setting, such tools are not needed, as we have the availability of quasihyperbolic geodesic rays and quasihyperbolic geodesic lines.

It is not clear whether Theorem 9.1 holds if the metric space is not annular convex. The examples in Section 11 show that even if the theorem holds without annular convexity, there can be no quantitative result.

Notation: Henceforth (X,d) denotes a metric space, $B(x,r)=\{y\in X: d(y,x)<r\}$ is the open ball and $S(x,r)=\{y\in X: d(y,x)=r\}$ is the sphere, with center $x\in X$ and radius $r>0$. The image of a path $\alpha:[a,b]\to X$ is denoted $|\alpha|$ and $\ell_d(\alpha)$ is the d-length of α. We simply use $\ell(\alpha)$ if the metric d in question is clear. We use $\alpha: x\frown y$ to indicate a path $\alpha:[a,b]\to X$ with $\alpha(a)=x$ and $\alpha(b)=y$. Given a path $\alpha:[a,b]\to X$ and $x,y\in|\alpha|$, then $\alpha[x,y]$ denotes an arbitrary but fixed subpath of α from x to y. If $A\subset X$ and $r>0$, then $N_d(A,r)=\{y\in X: d(y,x)\le r \text{ for some } x\in A\}$ denotes the closed r-neighborhood of A. For two bounded subsets $A,B\subset X$,

$$HD_d(A,B):=\inf\{r>0\,:\,B\subset N_d(A,r)\text{ and }A\subset N_d(B,r)\}$$

is the Hausdorff distance between A and B; if the metric d in question is clear, we write $HD(A,B)$. Given two real numbers a,b, we denote the smaller of these by $a \wedge b$. By $c = c(\delta, \eta, C_{\mathrm{a}})$ we mean a constant c that depends only on the parameters δ, η and C_{a}.

Throughout the entire paper: (X, d) **is a proper and quasiconvex metric space, and** $\Omega \subset X$ **is a nonempty rectifiably connected open subset with nonempty boundary.** See Section 2 for the relevant definitions.

The structure of this paper is as follows. In section 2 of this paper we discuss the quasihyperbolic metric on domains in a metric space, and recall some needed facts about the quasihyperbolic metric. The focus of Section 3 is Gromov hyperbolic spaces and some useful results about them. The tools of inversion and sphericalization in metric spaces are given in Section 4, while a discussion of cross ratios and boundary maps induced by quasiisometries can be found in Section 5. The focus of Section 6 is to prove that if a domain is uniform, then the natural map between the Gromov boundary of the domain, equipped with the quasihyperbolic metric, and metric boundary of the domain is quasi-Möbius. This fact was proven in [BHK] for bounded domains, and so the principal concern of Section 6 is unbounded domains. The geometric concepts of annular points, arc points, anchors, and star-likeness are discussed in Section 7, and in Section 8 a geometric "carrot" condition associated with quasihyperbolic geodesics is studied in the setting of the main theorem of this paper, Theorem 9.1. The proof of this main theorem is then given in Section 9. Finally, in Section 10 an application of the main theorem to quasi-Möbius maps is demonstrated, and some examples are studied in Section 11.

2. Quasihyperbolic metric

In this section we recall some basic facts about quasihyperbolic metric. While we do not give proofs for most of these facts, we do provide citations the reader can refer to for them.

A metric space is *c-quasiconvex* for some $c \geq 1$ if each pair of points x, y in the space can be joined by a path of length no more than $c\, d(x, y)$. A *geodesic space* is simply a 1-quasiconvex metric space.

Let U be an open subset of a metric space. We say U is *rectifiably connected* if each pair of points $x, y \in U$ can be joined by a rectifiable path in U. The boundary ∂U of U is the set $\overline{U} \backslash U$, where $\overline{U}$ is the closure of U.

Recall that (X, d) is a proper and quasiconvex metric space, and $\Omega \subset X$ is a nonempty rectifiably connected open subset with nonempty boundary. For $x \in \Omega$, we denote $\delta_\Omega(x) = d(x, \partial\Omega)$. The *quasihyperbolic metric* k on Ω is defined as follows: for $x, y \in \Omega$,

$$k(x, y) := \inf \int_\gamma \frac{ds(z)}{\delta_\Omega(z)},$$

where the infimum is taken over all rectifiable paths γ in Ω joining x and y, and ds denotes arc length along γ. It is well known that k is a metric [BHK, page 9].

The length metric l_Ω on Ω is given by $l_\Omega(x,y) = \inf_\gamma \ell_d(\gamma)$ for $x, y \in \Omega$, where the infimum is over paths in Ω joining x and y. Notice that for every $x \in \Omega$ there exists an $r_x > 0$ such that all $y, z \in B(x, r_x)$ can be joined by a path in Ω with length at most $c\, d(y,z)$. Hence the identity map $\mathrm{id} : (\Omega, d) \to (\Omega, l_\Omega)$ is a homeomorphism. By [BHK, Proposition 2.8], $\mathrm{id} : (\Omega, d) \to (\Omega, k)$ is also a homeomorphism and (Ω, k) is a proper geodesic space.

Lemma 2.1. ([GP]) *If $x, y \in \Omega$ and $\alpha : x \frown y$ is a rectifiable arc in Ω, then*

$$\ell_d(\alpha) \le \left(e^{\ell_k(\alpha)} - 1\right) \delta_\Omega(x).$$

Lemma 2.1 implies the following inequalities (see also [BHK, page 9]): for all $x, y \in \Omega$:

$$k(x,y) \ge \log\left(1 + \frac{d(x,y)}{\delta_\Omega(x) \wedge \delta_\Omega(y)}\right) \ge \left|\log \frac{\delta_\Omega(y)}{\delta_\Omega(x)}\right|. \tag{2.1}$$

Lemma 2.2. (Lemma 2.13 of [BHK]) *If $\gamma : [0,1] \to \Omega$ is a path that satisfies*

$$\min\{\ell_d(\gamma|_{[0,t]}),\ \ell_d(\gamma|_{[t,1]})\} \le A\, \delta_\Omega(\gamma(t))$$

for all $t \in [0,1]$, then with $x = \gamma(0)$ and $y = \gamma(1)$,

$$\ell_k(\gamma) \le 4A \log\left(1 + \frac{\ell_d(\gamma)}{\delta_\Omega(x) \wedge \delta_\Omega(y)}\right).$$

The following is a modification of Lemma 3.5 of [V1] to our setting. Since we replace the 2-quasiconvexity of spheres (in a Banach space) with the annular convexity property, our estimates are necessarily weaker than those in [V1].

Lemma 2.3. *Suppose (X, d) is C_a-annular convex for some C_a. Let $\alpha : x \frown y$ be a quasihyperbolic geodesic in Ω, $b \in \partial\Omega$, and $t > 0$.*

(i) *If $B(b, 16C_a^2\, t) \setminus B(b, e^{-4C_a^3}\, t/2) \subset \Omega$ and $x, y \in \Omega \setminus B(b, 8C_a\, t)$, then $|\alpha| \subset \Omega \setminus B(b, e^{-4C_a^3}\, t)$.*

(ii) *If $B(b, 8C_a\, t) \setminus B(b, t/C_a) \subset \Omega$ and $x, y \in \Omega \cap B(b, 4t)$, then $|\alpha| \subset \Omega \cap B(b, 8e^{4C_a^3}\, t)$.*

Proof. We first prove (i). Suppose that $|\alpha| \cap B(b, e^{-4C_\mathrm{a}^3} t) \ne \emptyset$. Then α must intersect $B(b, 8C_\mathrm{a} t)$ as well, and so we can choose $z_1, z_2 \in |\alpha| \cap S(b, 8C_\mathrm{a} t)$ such that the subpath $\alpha[z_1, z_2]$ of α satisfies both $|\alpha[z_1, z_2]| \subset \overline{B}(b, 8C_\mathrm{a} t)$ and $|\alpha[z_1, z_2]| \cap B(b, e^{-4C_\mathrm{a}^3} t) \ne \emptyset$.

As X is annular convex, there is a path $\gamma : z_1 \frown z_2$ with

$$\ell_d(\gamma) \le C_\mathrm{a} d(z_1, z_2) \le C_\mathrm{a} 2(8C_\mathrm{a} t) = 16C_\mathrm{a}^2 t$$

and $|\gamma| \subset B(b, 8C_{\rm a}^2 t) \setminus B(b, 8t) \subset \Omega$. Hence by the hypothesis of (i), for every $w \in |\gamma|$ we have

$$\delta_\Omega(w) \geq \min\{8C_{\rm a}^2 t, 8t - e^{-4C_{\rm a}^3} t/2\} = 8t - e^{-4C_{\rm a}^3} t/2 \geq 4t.$$

Therefore,

$$\ell_k(\gamma) = \int_\gamma \frac{1}{\delta_\Omega(w)}\, ds(w) \leq \frac{1}{4t} \ell_d(\gamma) \leq \frac{1}{4t} 16 C_{\rm a}^2 t = 4C_{\rm a}^2,$$

hence we see that $k(z_1, z_2) \leq 4C_{\rm a}^2$. Since $\alpha[z_1, z_2]$ is a quasihyperbolic geodesic, we have

$$\ell_k(\alpha[z_1, z_2]) = k(z_1, z_2) \leq 4C_{\rm a}^2. \tag{2.2}$$

By assumption, there is a point $z \in |\alpha[z_1, z_2]| \cap B(b, e^{-4C_{\rm a}^3} t)$. By Lemma 2.1,

$$\begin{aligned}
\ell_k(\alpha[z_1, z_2]) &= \ell_k(\alpha[z_1, z]) + \ell_k(\alpha[z, z_2]) \\
&\geq \log\left[\left(1 + \frac{\ell_d(\alpha[z_1, z])}{\delta_\Omega(z)}\right)\left(1 + \frac{\ell_d(\alpha[z, z_2])}{\delta_\Omega(z)}\right)\right] \\
&\geq \log\left(1 + \frac{\ell_d(\alpha[z_1, z_2])}{\delta_\Omega(z)}\right).
\end{aligned}$$

However, as $\ell_d(\alpha[z_1, z_2]) \geq 8C_{\rm a} t - e^{-4C_{\rm a}^3} t$ and $\delta_\Omega(z) \leq e^{-4C_{\rm a}^3} t$, we see that

$$\ell_k(\alpha[z_1, z_2]) \geq \log\left(1 + \frac{8C_{\rm a} t - e^{-4C_{\rm a}^3} t}{e^{-4C_{\rm a}^3} t}\right) = \log\left(8C_{\rm a} e^{4C_{\rm a}^3}\right) \geq 4C_{\rm a}^3. \tag{2.3}$$

Combining inequalities (2.2) and (2.3) we obtain $4C_{\rm a}^2 \geq 4C_{\rm a}^3$, a contradiction because $C_{\rm a} \geq 2$. Thus the path α cannot intersect the ball $B(b, e^{-4C_{\rm a}^3} t)$.

Now we prove (ii). To do so, suppose that $|\alpha| \cap S(b, 8e^{4C_{\rm a}^3} t) \neq \emptyset$. Then clearly $|\alpha|$ intersects the sphere $S(b, 4t)$, and so there are points $w_1, w_2 \in |\alpha| \cap S(b, 4t)$ satisfying $|\alpha[w_1, w_2]| \cap S(b, 8e^{4C_{\rm a}^3} t) \neq \emptyset$ and $|\alpha[w_1, w_2]| \cap B(b, 4t) = \emptyset$.

By the annular convexity of X, there is a path γ joining w_1 and w_2 in the annulus $B(b, 4C_{\rm a} t) \setminus B(b, 4t/C_{\rm a}) \subset \Omega$ with $\ell_d(\gamma) \leq C_{\rm a} d(w_1, w_2) \leq 8C_{\rm a} t$. For every $z \in |\gamma|$,

$$\delta_\Omega(z) \geq \min\left\{8C_{\rm a} t - 4C_{\rm a} t, \frac{4t}{C_{\rm a}} - \frac{t}{C_{\rm a}}\right\} = \frac{3t}{C_{\rm a}}.$$

Therefore,

$$k(w_1, w_2) \leq \ell_k(\gamma) \leq \frac{C_{\rm a}}{3t} \ell_d(\gamma) \leq \frac{C_{\rm a}}{3t}\, 8C_{\rm a} t = \frac{8}{3} C_{\rm a}^2.$$

Since α is a quasihyperbolic geodesic, we see that

$$\ell_k(\alpha[w_1, w_2]) = k(w_1, w_2) \leq \frac{8}{3} C_{\rm a}^2. \tag{2.4}$$

Meanwhile, Lemma 2.1 in conjunction with $\ell_d(\alpha[w_1,w_2]) \geq 8e^{4C_{\mathrm{a}}^3}t - 4t \geq 4e^{4C_{\mathrm{a}}^3}t$ and $\delta_\Omega(w_1) \leq 4t$ yields

$$\begin{aligned}\ell_k(\alpha[w_1,w_2]) \geq \log\left(1+\frac{\ell_d(\alpha[w_1,w_2])}{\delta_\Omega(w_1)}\right) &\geq \log\left(1+\frac{4e^{4C_{\mathrm{a}}^3}t}{4t}\right)\\ &\geq \log e^{4C_{\mathrm{a}}^3} = 4C_{\mathrm{a}}^3.\end{aligned}$$

By inequality (2.4), we now get $4C_{\mathrm{a}}^3 \leq \frac{8}{3}C_{\mathrm{a}}^2$, a contradiction as $C_{\mathrm{a}} \geq 2$. $\square$

Given $c \geq 1$, a path $\gamma : [0,1] \to \Omega$ is called a *c-uniform path* if $\ell_d(\gamma) \leq c\, d(\gamma(0),\gamma(1))$ and $c\,\delta_\Omega(\gamma(t)) \geq \min\{\ell_d(\gamma|_{[0,t]}),\ \ell_d(\gamma|_{[t,1]})\}$ for all $t \in [0,1]$. We say that Ω is a *c-uniform domain* for some $c \geq 1$ if every two points $x, y \in \Omega$ can be joined by a c-uniform path. If Ω is equipped with more than one metric, then to specify the metric d with respect to which Ω is uniform we say that (Ω, d) is a c-uniform domain.

Lemma 2.4. *Let $x_1 \in \Omega$ and $x_2 \in \overline{\Omega}$ be such that $\delta_\Omega(x_1) \geq d(x_1,x_2)$. Suppose γ is a d-geodesic in X connecting x_1 and x_2. Then $|\gamma|\backslash\{x_2\} \subset \Omega$, γ is a 1-uniform path in Ω, and furthermore, $\delta_\Omega(x) \geq \ell_d(\gamma[x_2,x]) = d(x,x_2)$ for all $x \in |\gamma|\backslash\{x_2\}$.*

Proof. By assumption, $x_2 \in \overline{B}(x_1,\delta_\Omega(x_1)) \cap \overline{\Omega}$. Let $\gamma : [0, d(x_1,x_2)] \to X$ be the arc-length parametrization of γ with $\gamma(0) = x_2$ and $\gamma(d(x_1,x_2)) = x_1$. Then, for every $z \in |\gamma| \setminus \{x_2\}$ we have $d(x_1,z) < d(x_1,x_2)$, and therefore $z \in B(x_1,\delta_\Omega(x_1)) \subset \Omega$.

Now, for $t \in (0, d(x_1,x_2)]$, we have $d(\gamma(t),x_1) = d(x_1,x_2) - t$, and so

$$\begin{aligned}\delta_\Omega(\gamma(t)) \geq \delta_\Omega(x_1) - d(x_1,\gamma(t)) &= \delta_\Omega(x_1) - d(x_1,x_2) + t\\ &= t + [\delta_\Omega(x_1) - d(x_1,x_2)] \geq t.\end{aligned}$$

$\square$

Proposition 2.5. *Let $x_0 \in \Omega$, $b \in \partial\Omega$ with $\delta_\Omega(x_0) = d(x_0,b)$, and γ be a d-geodesic in X joining x_0 and b. Then $|\gamma|\backslash\{b\}$ is a quasihyperbolic geodesic ray in Ω.*

Proof. Let $\gamma : [0,\delta_\Omega(x_0)] \to X$ be the arclength parametrization of γ with respect to d, with $\gamma(0) = b$ and $\gamma(\delta_\Omega(x_0)) = x_0$. Then $\delta_\Omega(\gamma(t)) = t$ for all $t \in (0,\delta_\Omega(x_0)]$. Let $0 < t_1 < t_2 \leq \delta_\Omega(x_0)$. Inequality (2.1) implies that $k(\gamma(t_1),\gamma(t_2)) \geq \log(t_2/t_1)$. On the other hand,

$$k(\gamma(t_1),\gamma(t_2)) \leq \ell_k(\gamma|_{[t_1,t_2]}) = \int_{t_1}^{t_2} \frac{1}{\delta_\Omega(\gamma(t))}dt = \int_{t_1}^{t_2} \frac{1}{t}dt = \log\left(\frac{t_2}{t_1}\right).$$

Hence γ is a quasihyperbolic geodesic ray in Ω. $\square$

Given two rectifiable paths α, β in a metric space (Y, d), we call a map $f : |\alpha| \to |\beta|$ a *length map* with respect to the metric d if for all $x, y \in |\alpha|$ we have $\ell_d(\beta[f(x), f(y)]) = \ell_d(\alpha[x, y])$.

Lemma 2.6 (Lemma 3.3 of [V1])**.** *If α and β are paths in (Ω, k) with $\ell_k(\alpha) \leq \ell_k(\beta)$, and $f : |\alpha| \to |\beta|$ is a length map (with respect to k) with $k(f(x), x) \leq c$ for all $x \in |\alpha|$, then*

$$e^{-c}\,\ell_d(\alpha) \leq \ell_d(f \circ \alpha) \leq e^{c}\,\ell_d(\alpha).$$

3. Gromov hyperbolic spaces

In this section we review some basic facts about Gromov hyperbolic spaces. See [CDP], [GdlH], [V3], and references therein for more details.

Let (Y, h) be a proper geodesic space and $\delta \geq 0$ a constant. We say that (Y, h) is *δ-hyperbolic* if geodesic triangles in Y are δ-thin. This means that for any $x, y, z \in Y$ and any geodesics $\gamma_1 : x \frown y$, $\gamma_2 : y \frown z$, $\gamma_3 : z \frown x$, we have $|\gamma_3| \subset N_h(|\gamma_1| \cup |\gamma_2|, \delta)$. A space (Y, h) is *Gromov hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$. Let $w \in Y$ be a (fixed) base point. The *Gromov product* of $x, y \in Y$ based at w is:

$$(x|y)_w = \frac{1}{2}\,[h(x, w) + h(y, w) - h(x, y)].$$

For the remainder of this section (Y, h) is always δ-hyperbolic. A sequence of points $\{y_i\}$ *tends to infinity* if $\lim_{i,j\to\infty}(y_i|y_j)_w = \infty$ for some (or any) base point $w \in Y$. Two sequences $\{x_i\}$ and $\{y_i\}$ both tending to infinity are *equivalent* if $\lim_{i,j\to\infty}(x_i|y_j)_w = \infty$. The *Gromov boundary* $\partial_h^* Y$ of Y is the set of equivalence classes of sequences tending to infinity, and the Gromov closure of Y is defined by $Y_h^* = Y \cup \partial_h^* Y$. If the metric h is clear from context, we simply write $\partial^* Y$ and Y^*. If $\xi \in \partial^* Y$ and a sequence of points $\{x_i\}$ represents ξ we write $\{x_i\} \to \xi$.

If $\gamma : [0, \infty) \to Y$ is a geodesic (ray), then one easily sees from the definition that $\{\gamma(t)\}$ tends to infinity as $t \to \infty$, and hence represents some $\xi \in \partial^* Y$. In this case we say $\gamma(0)$ and ξ are the endpoints of γ. Similarly, for any complete geodesic $\gamma : \mathbb{R} \to Y$ there are $\xi_+, \xi_- \in \partial^* Y$ such that $\{\gamma(t)\} \to \xi_+$ as $t \to \infty$ and $\{\gamma(t)\} \to \xi_-$ as $t \to -\infty$. We say ξ_+ and ξ_- are the endpoints of γ. A proper geodesic δ-hyperbolic space has the visibility property: given any two distinct points $a, b \in Y^*$, there is a geodesic γ with a and b as endpoints [CDP, Chapter 2, Proposition 2.1]. For three distinct points $a_1, a_2, a_3 \in Y^*$, and geodesics $\gamma_i : a_i \frown a_{i+1}$ $(i = 1, 2, 3)$ in Y with a_i and a_{i+1} as endpoints $(a_4 := a_1)$, the subset $|\gamma_1| \cup |\gamma_2| \cup |\gamma_3|$ of Y is called a geodesic triangle in Y^*. Geodesic triangles in Y^* are 24δ-thin [CDP, Chapter 2, Proposition 2.2].

Lemma 3.1. *Let $a, b, c \in Y^*$ be three distinct points, and $\alpha : a \frown b$, $\beta : b \frown c$, $\gamma : c \frown a$ be geodesics. Then there is a point $x \in |\gamma|$ satisfying $h(x, |\alpha|) \leq 24\delta$ and $h(x, |\beta|) \leq 24\delta$.*

Proof. Let $A = \{x \in |\gamma| : h(x, |\alpha|) \leq 24\delta\}$ and $B = \{x \in |\gamma| : h(x, |\beta|) \leq 24\delta\}$. Then both A and B are closed subsets of $|\gamma|$. Since geodesic triangles in Y^* are 24δ-thin, we have $A \cup B = |\gamma|$. Suppose $A \cap B = \emptyset$. Then the connectedness of $|\gamma|$ implies that either $A = |\gamma|$ or $B = |\gamma|$. We may assume $A = |\gamma|$; the case $B = |\gamma|$ can be handled similarly. Then $|\gamma|$ is contained in the 24δ-neighborhood of $|\alpha|$. If $c \in \partial^* Y$, then we must have $c = a$ or $c = b$, contradicting the assumption that a, b, c are distinct. To see this, note that if $c \in \partial^* Y$, then we can find $x_n \in |\gamma|$ such that $\{x_n\} \to c$ in the sense discussed above. But then for each n, we can find $z_n \in |\alpha|$ such that $h(z_n, x_n) \leq 24\delta$. It is clear that $\{z_n\}$ tends to infinity, and $\{x_n\}$ and $\{z_n\}$ are equivalent. Hence $\{z_n\} \to c$. Since $z_n \in |\alpha|$, we have either $c = a$ or $c = b$. Hence we must have $c \in Y$. But then $c \in A \cap B$, contradicting the assumption $A \cap B = \emptyset$.

□

Let $w \in Y$ be a base point. The Gromov product of two points $\xi, \eta \in \partial^* Y$ is defined as follows:

$$(\xi|\eta)_w = \sup \liminf_{i,j\to\infty} (x_i|y_j)_w,$$

where the supremum is taken over all sequences $\{x_i\} \to \xi$, $\{y_i\} \to \eta$. One can show that $(\xi|\eta)_w - 2\delta \leq \liminf_{i,j\to\infty} (x_i|y_j)_w \leq (\xi|\eta)_w$ for all $w \in Y$, all $\xi, \eta \in \partial^* Y$ and all sequences $\{x_i\} \to \xi$, $\{y_i\} \to \eta$; see Chapter 7 of [GdlH]. Similarly, the Gromov product of $x \in Y$ and $\eta \in \partial^* Y$ is defined to be

$$(x|\eta)_w = \sup \liminf_{i\to\infty} (x|y_i)_w,$$

where the supremum is taken over all sequences $\{y_i\} \to \eta$.

We define a topology on Y^* by specifying when a sequence of points $x_i \in Y^*$ converges to a point $\xi \in Y^*$: if $\xi \in Y$, then $x_i \to \xi$ means $h(\xi, x_i) \to 0$ as $i \to \infty$; and if $\xi \in \partial^* Y$, then $x_i \to \xi$ means $(\xi|x_i)_w \to \infty$ for some (equivalently, every) $w \in Y$ as $i \to \infty$. In this topology, Y^* is compact and Y is a dense open subset. The induced topology on Y agrees with the metric topology on Y.

Given $\varepsilon > 0$, $w \in Y$ and $\xi, \eta \in \partial^* Y$, let $\rho_{w,\varepsilon}(\xi, \eta) = e^{-\varepsilon(\xi|\eta)_w}$.

Proposition 3.2. [GdlH, Chapter 7, Proposition 10] *Let $\varepsilon_0(\delta) = \min\{1, \frac{1}{5\delta}\}$. Then for any δ-hyperbolic metric space Y, any base point $w \in Y$, and any $0 < \varepsilon \leq \varepsilon_0$, there is a metric $h_{w,\varepsilon}$ on $\partial^* Y$ such that for all $\xi, \eta \in \partial^* Y$,*

$$\frac{1}{2}\rho_{w,\varepsilon}(\xi, \eta) \leq h_{w,\varepsilon}(\xi, \eta) \leq \rho_{w,\varepsilon}(\xi, \eta).$$

A metric $h_{w,\varepsilon}$ satisfying the conclusion of Proposition 3.2 is called a *visual metric.*

Definition 3.3. *Let $L \geq 1$ and $A \geq 0$. A (not necessarily continuous) map $\gamma : I \to Y$ on an interval I is an (L, A)*-quasigeodesic *if for all $t_1, t_2 \in I$ we* *have*

$$L^{-1}|t_2 - t_1| - A \leq h(\gamma(t_1), \gamma(t_2)) \leq L|t_2 - t_1| + A.$$

Note that an $(1, 0)$-quasigeodesic is a geodesic. An important property of Gromov hyperbolic spaces is the stability of quasigeodesics. It says that quasigeodesics are close to geodesics (see also [V3]):

Lemma 3.4 (Theorem 1.2 and Theorem 3.1 of [CDP], Chapter 3)**.** *Given any $\delta \geq 0$, $L \geq 1$, and $A \geq 0$, there is a constant $M = M(\delta, L, A)$ such that whenever Y is a proper geodesic δ-hyperbolic space, the following conditions hold:*

(i) *If $\alpha : [a, b] \to Y$ and $\alpha' : [a', b'] \to Y$ are two (L, A)-quasigeodesics with $\alpha(a) = \alpha'(a')$ and $\alpha(b) = \alpha'(b')$, then $HD(|\alpha|, |\alpha'|) \leq M$;*
(ii) *If $\alpha : \mathbb{R} \to Y$ is an (L, A)-quasigeodesic, then there exists a geodesic $\alpha' : \mathbb{R} \to Y$ such that $HD(|\alpha|, |\alpha'|) \leq M$.*

Lemma 3.4 (ii) implies that every quasigeodesic $\alpha : \mathbb{R} \to Y$ has two endpoints ξ_+, ξ_- in $\partial^* Y$. Since two complete geodesics with the same endpoints in a δ-hyperbolic space have Hausdorff distance at most 2δ from each other, by replacing $2\delta + 2M$ with M, we have that $HD(|\alpha|, |\alpha'|) \leq M$ for any two (L, A)-quasigeodesics with the same endpoints.

We also recall the following two results.

Theorem 3.5 (Chapter 8 of [CDP])**.** *Let (Y, h) be a δ-hyperbolic space, $y_0 \in Y$, and $Y_0 = \{y_0, y_1, \cdots, y_n\}$ be a set of $n+1$ points in Y^*. For each $1 \leq i \leq n$, let $[y_0, y_i]$ be a fixed geodesic connecting y_0 and y_i. Let X denote the union of the geodesics $[y_0, y_i]$, and choose a positive integer k such that $2n \leq 2^k + 1$. Then there exists a simplicial tree, denoted $T(X)$, and a continuous map $u : X \to T(X)$ which satisfies the following properties:*

(i) *For each i, the restriction of u to the geodesic $[y_0, y_i]$ is an isometry;*
(ii) *For every x and y in X we have $h(x, y) - 2k\delta \leq d(u(x), u(y)) \leq h(x, y)$, where d is the metric on $T(X)$.*

Lemma 3.6 (Lemma 2.17 of [V3])**.** *Suppose (Y, h) is δ-hyperbolic and $\alpha_1 : a_1 \frown b_1$, $\alpha_2 : a_2 \frown b_2$ are geodesics with $\ell(\alpha_1) \leq \ell(\alpha_2)$. If $h(a_1, a_2) \leq \mu$ and $h(b_1, |\alpha_2|) \leq \mu$ for some $\mu \geq 0$, and $f : |\alpha_1| \to |\alpha_2|$ is the length map with $f(a_1) = a_2$, then for all $x \in |\alpha_1|$*

$$h(f(x), x) \leq 8\delta + 5\mu.$$

4. Inversions in metric spaces

In this section we recall the notion of inversions in metric spaces and collect related facts useful in this paper. See [BHX] for more details.

Let $Q = (x_1, x_2, x_3, x_4)$ be a quadruple of distinct points in (X, d). The *cross ratio* of Q with respect to d is the number

$$\mathrm{cr}(Q, d) = \frac{d(x_1, x_3)\, d(x_2, x_4)}{d(x_1, x_4)\, d(x_2, x_3)}.$$

Let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism. A homeomorphism $f : (X, d_1) \to (Y, d_2)$ between two metric spaces is called *η-quasimöbius* if for each quadruple of distinct points $Q = (x_1, x_2, x_3, x_4)$ in X,

$$\mathrm{cr}(f(Q), d_2) \le \eta(\mathrm{cr}(Q, d_1)),$$

where $f(Q) = (f(x_1), f(x_2), f(x_3), f(x_4))$. We say that a homeomorphism $f : (X, d_1) \to (Y, d_2)$ is quasimöbius if it is η-quasimöbius for some η. A homeomorphism $f : (X, d_1) \to (Y, d_2)$ between two metric spaces is called *η-quasisymmetric* if, for all triples of distinct points (x_1, x_2, x_3) in X,

$$\frac{d_2(f(x_1), f(x_2))}{d_2(f(x_1), f(x_3))} \le \eta\left(\frac{d_1(x_1, x_2)}{d_1(x_1, x_3)}\right).$$

We say that a homeomorphism $f : (X, d_1) \to (Y, d_2)$ is quasisymmetric if it is η-quasisymmetric for some η. A quasisymmetric homeomorphism is quasimöbius, but a quasimöbius homeomorphism may not be quasisymmetric. However, a quasimöbius homeomorphism between bounded metric spaces is quasisymmetric. See [V2] for more details.

Let $p \in X$. Set $I_p(X) = X \backslash \{p\}$ if X is bounded, and $I_p(X) = (X \backslash \{p\}) \cup \{\infty\}$ if X is unbounded (where ∞ is a point not in X). Define a function $f_p : I_p(X) \times I_p(X) \to [0, \infty)$ as follows:

$$f_p(x, y) = f_p(y, x) = \begin{cases} \frac{d(x,y)}{d(x,p)\, d(y,p)} & \text{if } x, y \in X \backslash \{p\}, \\ \frac{1}{d(x,p)} & \text{if } y = \infty \text{ and } x \in X \backslash \{p\}, \\ 0 & \text{if } x = \infty = y. \end{cases}$$

Theorem 4.1 ([BHX])**.** *There exists a metric d_p on $I_p(X)$ satisfying*

$$\frac{1}{4} f_p(x, y) \le d_p(x, y) \le f_p(x, y)$$

for all $x, y \in I_p(X)$. Furthermore, the identity map $(X \backslash \{p\}, d) \to (X \backslash \{p\}, d_p)$ is η-quasimöbius with $\eta(t) = 16t$.

Let $\Omega \subset (X, d)$ be an open subset and $p \in \partial\Omega$, and denote $d_0 = diam(\Omega, d)$ and $d_0' = diam(\partial\Omega, d)$.

- (i) *If (X, d) is c-quasiconvex and c-annular convex, then $(I_p(X), d_p)$ is c'-quasiconvex and c'-annular convex with $c' = c'(c)$;*
- (ii) *If (Ω, d) is c-quasiconvex, $d_0' > 0$ and $d_0 < \infty$, then the identity map $(\Omega, k) \to (\Omega, k_p)$ is M-bilipschitz with $M = \max\{40c,\, 4c\, d_0/d_0'\}$, where k_p denotes the quasihyperbolic metric on Ω induced by the metric d_p;*
- (iii) *If (Ω, d_p) is c_1-uniform and (X, d) is both c_2-quasiconvex and c_2-annular convex, then (Ω, d) is c-uniform with $c = c(c_1, c_2)$;*

(iv) *If* $d'_0 > 0$ *and* (Ω, d) *is* c*-uniform, then* (Ω, d_p) *is* c'*-uniform with* $c' = c'(c)$.

Under the assumptions of Theorem 4.1 (ii), (Ω, k_p) is δ'-hyperbolic for some δ' whenever (Ω, k) is δ-hyperbolic. In general, one cannot control δ' in terms of δ and c alone. However, we have the following result.

Proposition 4.2. *Let* (X, d) *be* c*-quasiconvex and* c*-annular convex. Let* d_0 *and* d'_0 *be as in Theorem 4.1. Suppose* $d_0 < \infty$ *and* $d'_0 > 0$. *If* (Ω, k) *is* δ*-hyperbolic and* $p \in \partial\Omega$*, then* (Ω, k_p) *is* δ'*-hyperbolic with* $\delta' = \delta'(\delta, c)$.

The proof of Proposition 4.2 is given after Lemma 4.5. We first establish some preliminary results.

Let $L \geq 1$ and $A \geq 0$. A (not necessarily continuous) map $f : (X, d_1) \to (Y, d_2)$ between two metric spaces is an (L, A)*-quasiisometry* if the following two conditions are satisfied:

(1) $d_1(x_1, x_2)/L - A \leq d_2(f(x_1), f(x_2)) \leq L\, d_1(x_1, x_2) + A$ holds for all $x_1, x_2 \in X$;
(2) For each $y \in Y$, there is some $x \in X$ with $d_2(f(x), y) \leq A$.

By definition an L-bilipschitz map is an $(L, 0)$ quasiisometry.

It is well-known (see for example Theorem 3.18 in [V3]) that if $f : Y_1 \to Y_2$ is an (L, A)-quasiisometry between geodesic spaces and Y_1 is δ-hyperbolic, then Y_2 is δ'-hyperbolic with $\delta' = \delta'(\delta, L, A)$. Hence Proposition 4.2 follows from Theorem 4.1 (ii) when $d_0 \leq 20c^2\, d'_0$. Therefore we assume $d_0 > 20c^2\, d'_0$ from now on. Since (Ω, d) is bounded ($d_0 < \infty$) and $p \in \partial\Omega$, the topological boundary of Ω in (X, d_p) is $\partial_p\Omega := \partial\Omega \setminus \{p\}$. We denote the Gromov boundary of (Ω, k_p) by $\partial^*_p\Omega$. For $x \in \Omega$, let $\delta_p(x) = d_p(x, \partial_p\Omega)$. However, $B = B(p, 10c^2 d'_0)$ will denote the ball with respect to the original metric d. Let $K = \Omega \backslash B$ and $S = \{x \in \Omega : d(x, p) = 10c^2 d'_0\}$. By the definition of d'_0 and the assumption that d_0 is finite, we see that K and S are compact subsets of Ω. Let $D_1 = \operatorname{diam}(S, k)$, $D_2 = \operatorname{diam}(S, k_p)$ and $d_2 = \operatorname{diam}(K, k_p)$. Then $D_1, D_2, d_2 < \infty$. Since $S \subset K$, we always have $D_2 \leq d_2$.

We remark that Proposition 4.2 does not follow from the above mentioned result (Theorem 3.18 in [V3]) since the bilipschitz constant for $\mathrm{id} : (\Omega, k) \to (\Omega, k_p)$ depends on the ratio d_0/d'_0. In fact, from Lemma 4.4 below we see that $\operatorname{diam}(K, k_p) \leq 2c$ while it can be seen that $\operatorname{diam}(K, k) \to \infty$ as $d_0/d'_0 \to \infty$.

To understand Propositions 4.2 and 5.6, it is useful to keep in mind the following geometric pictures of (Ω, k) and (Ω, k_p). On $\Omega \backslash K$, the two metrics k and k_p are roughly the same (meaning they are quasiisometric quantitatively, see Lemma 4.3). The set (K, k_p) is quantitatively bounded (Lemma 4.4), while (K, k) is a "long hair" with "root" (S, k) (the "hair" grows longer as $d_0/d'_0 \to \infty$). Hence (Ω, k) has a "long hair" sticking out and the transformation from (Ω, k) to (Ω, k_p) is "shrink a long hair to its root". Hence (Ω, k_p) is

Gromov hyperbolic quantitatively (Proposition 4.2), and the boundary map of $(\Omega, k) \to (\Omega, k_p)$ is quasimöbius, also quantitatively (Proposition 5.6).

The following result follows from the proof of Corollary 4.11 in [BHX].

Lemma 4.3. *Let K, D_1 and D_2 be as above. Under the assumptions of Proposition 4.2, there is a constant L depending only on c such that for all $x, y \in \Omega\backslash K$ we have $k_p(x,y) \le L\, k(x,y) + D_2$ and $k(x,y) \le L\, k_p(x,y) + D_1$.*

Recall that X is both c-quasiconvex and c-annular convex. The d_p-length of a path γ in $I_p(X)$ is denoted $\ell_p(\gamma)$.

Lemma 4.4. *The inequalities $D_1 \le 4c^2$ and $d_2 \le 8c/5$ hold.*

Proof. Let $x, y \in S$. Since X is c-annular convex, there is a path γ in X joining x and y such that $|\gamma| \subset B(p, 10c^3 d_0') \setminus B(p, 10cd_0')$ and $\ell(\gamma) \le c\, d(x,y)$. Since $\partial\Omega$ is a subset of $B(p, 10cd_0')$, the path γ does not intersect $\partial\Omega$. However, γ intersects $S \subset \Omega$ as it has both its end points in S. So $|\gamma| \subset \Omega$ and $\delta_\Omega(z) \ge 5cd_0'$ for all $z \in |\gamma|$. Now

$$\begin{aligned} k(x,y) \le \int_\gamma \frac{1}{\delta_\Omega(z)} ds(z) \le \int_\gamma \frac{1}{5cd_0'} ds(z) = \frac{1}{5cd_0'}\ell(\gamma) \le \frac{1}{5cd_0'} cd(x,y) \\ \le \frac{1}{5d_0'} 20c^2 d_0' = 4c^2. \end{aligned}$$

Now we prove the second inequality. We first prove that whenever $r \ge 10c^2 d_0'$, for every $x, y \in (B(p,2r) \setminus B(p,r)) \cap \Omega$,

$$k_p(x,y) \le \frac{8c^3 d_0'}{r}. \tag{4.1}$$

Assume $r \ge 10c^2 d_0'$ and let $x, y \in (B(p,2r) \setminus B(p,r)) \cap \Omega$. Since X is c-annular convex, there is a path γ connecting x and y with $|\gamma| \subset B(p, 2cr) \setminus B(p, 2r/c)$ and $\ell_d(\gamma) \le c\, d(x,y)$. Note again that $|\gamma| \subset \Omega$. For any $z_1, z_2 \in |\gamma|$ we have by Theorem 4.1,

$$d_p(z_1, z_2) \le \frac{d(z_1,z_2)}{d(z_1,p)\, d(z_2,p)} \le \frac{d(z_1,z_2)}{(2r/c)^2} = \frac{c^2 d(z_1,z_2)}{4r^2}.$$

It follows that

$$\ell_p(\gamma) \le \frac{c^2 \ell_d(\gamma)}{4r^2} \le \frac{c^2 \cdot c\, d(x,y)}{4r^2} \le \frac{c^3\, 4r}{4r^2} = c^3/r.$$

On the other hand, as $r \ge 10c^2 d_0'$ and $|\gamma| \subset B(p, 2cr) \setminus B(p, 2r/c)$, we have $d(z,w) \ge d(z,p)/2$ for all $z \in |\gamma|$ and $w \in \partial_p\Omega$. Hence by Theorem 4.1 again, for any $w \in \partial_p\Omega$ and $z \in |\gamma|$,

$$d_p(z,w) \ge \frac{d(z,w)}{4d(z,p)d(w,p)} \ge \frac{1}{8d(w,p)} \ge \frac{1}{8d_0'}.$$

It follows that $\delta_p(z) \geq \frac{1}{8d_0'}$ for all $z \in |\gamma|$. Consequently

$$k_p(x,y) \leq \int_\gamma \frac{1}{\delta_p(z)} ds_p(z) \leq 8d_0' \ell_p(\gamma) \leq 8d_0' \cdot c^3/r = \frac{8c^3 d_0'}{r},$$

where ds_p denotes the d_p-arc length along γ.

Set $r_0 = 10c^2 d_0'$ and let $n \geq 2$ be the integer such that $2^{n-1}r_0 < d_0 \leq 2^n r_0$. Then $K = \bigcup_{i=1}^{n+1} \Omega \cap (B(p, 2^i r_0) \setminus B(p, 2^{i-1}r_0))$. The inequality (4.1) now implies

$$d_2 \leq \sum_{i=0}^{n} \frac{8c^3 d_0'}{2^i r_0} \leq 2\frac{8c^3 d_0'}{r_0} = 8c/5.$$

□

Let $\alpha : I \to \Omega$ be a k_p-geodesic with I a closed (not necessarily compact) interval such that the endpoints of α do not lie in $K = \Omega \setminus B(p, 10c^2 d_0')$. We define a map $\alpha' : I \to (\Omega, k)$ as follows. If $|\alpha| \cap K = \emptyset$, then we let $\alpha' = \alpha$. If $|\alpha| \cap K \neq \emptyset$, then let $t_1 = \inf \alpha^{-1}(K)$ and $t_2 = \sup \alpha^{-1}(K)$; observe that $\alpha(t_1), \alpha(t_2) \in S$. Since $\operatorname{diam}(S, k_p) = D_2$, we have $t_2 - t_1 \leq D_2$. Let $\alpha'(t) = \alpha(t)$ if $t < t_1$ or $t > t_2$, and $\alpha'(t) = \alpha(t_1)$ if $t \in [t_1, t_2]$. Similarly, given any geodesic $\beta : I \to (\Omega, k)$ whose endpoints do not lie in K, we can define a map $\tilde{\beta} : I \to (\Omega, k_p)$.

Lemma 4.5. *The map α' is an (L, A)-quasigeodesic with respect to k, where L, A depend only on c. Similarly, $\tilde{\beta}$ is an (L, A)-quasigeodesic with respect to k_p.*

Proof. We only prove the claim for α', as the proof for $\tilde{\beta}$ is similar. We use Lemma 4.3. Let $s, t \in I$. First assume $s, t \in I \setminus [t_1, t_2]$. Then $\alpha'(s) = \alpha(s)$, $\alpha'(t) = \alpha(t)$ and hence

$$\begin{aligned} L^{-1}|s-t| - L^{-1}D_2 &= L^{-1}[k_p(\alpha(s), \alpha(t)) - D_2] \\ &\leq k(\alpha'(s), \alpha'(t)) \\ &\leq L\, k_p(\alpha(s), \alpha(t)) + D_1 = L|s-t| + D_1. \end{aligned}$$

Next assume $s, t \in [t_1, t_2]$. Then $|s-t| \leq t_2 - t_1 \leq D_2$ and $\alpha'(s) = \alpha'(t)$. We therefore see that the above chain of inequalities is again satisfied. Finally assume $s \in [t_1, t_2]$ and $t \notin [t_1, t_2]$. Then

$$\begin{aligned} k(\alpha'(s), \alpha'(t)) = k(\alpha'(t_1), \alpha'(t)) \leq L|t_1 - t| + D_1 &\leq L(|t_1 - s| + |s-t|) + D_1 \\ &\leq L|s-t| + LD_2 + D_1, \end{aligned}$$

and

$$\begin{aligned} k(\alpha'(s), \alpha'(t)) = k(\alpha'(t_1), \alpha'(t)) \geq \frac{|t_1 - t|}{L} - \frac{D_2}{L} &\geq \frac{|s-t|}{L} - \frac{|s-t_1|}{L} - \frac{D_2}{L} \\ &\geq \frac{|s-t|}{L} - \frac{D_2}{L} - \frac{D_2}{L}. \end{aligned}$$

Now the lemma follows from Lemma 4.4. □

Proof of Proposition 4.2. As we pointed out in the discussion following the statement of Proposition 4.2, we can assume that $d_0 > 20c^2 d_0'$. The goal is to prove that all geodesic triangles in (Ω, k_p) are δ'-thin for some $\delta' = \delta'(\delta, c)$.

Let $x_1, x_2, x_3 \in \Omega$ and set $x_4 := x_1$, let α_i $(i = 1, 2, 3)$ be a geodesic in (Ω, k_p) joining x_i and x_{i+1}. We want to find some $\delta' = \delta'(\delta, c)$ such that $|\alpha_1| \subset N_{k_p}(|\alpha_2| \cup |\alpha_3|, \delta')$. We consider several cases.

Case 1: $x_1, x_2, x_3 \notin K$. By Lemma 4.5 we obtain an (L, A)-quasigeodesic α_i' in (Ω, k) connecting x_i and x_{i+1}. Fixing a geodesic β_i in (Ω, k) joining x_i and x_{i+1}, note by Lemma 3.4 that $HD_k(|\beta_i|, |\alpha_i'|) \le c_1$ with $c_1 = c_1(\delta, L, A) = c_1(\delta, c)$. Let $x \in |\alpha_1|$, and fix $y \in |\alpha_1'|$ such that $k_p(x, y) \le d_2 + 1$. Since $HD_k(|\beta_1|, |\alpha_1'|) \le c_1$, there is some $y_1 \in |\beta_1|$ with $k(y, y_1) \le c_1$. By the δ-hyperbolicity of (Ω, k) there is some $y_2 \in |\beta_2| \cup |\beta_3|$ with $k(y_1, y_2) \le \delta$. We may assume $y_2 \in |\beta_2|$. The fact $HD_k(|\beta_2|, |\alpha_2'|) \le c_1$ implies that there is some $y_3 \in |\alpha_2'|$ with $k(y_2, y_3) \le c_1$. The triangle inequality implies that $k(y, y_3) \le 2c_1 + \delta$. By Lemma 4.3 we have

$$\begin{aligned} k_p(x, y_3) \le k_p(x, y) + k_p(y, y_3) &\le d_2 + 1 + L\, k(y, y_3) + D_2 \\ &\le d_2 + D_2 + 1 + L(2c_1 + \delta). \end{aligned}$$

As $y_3 \in |\alpha_2'| \subset |\alpha_2|$, we have shown $x \in N_{k_p}(|\alpha_2| \cup |\alpha_3|, \delta_1)$ with $\delta_1 = d_2 + D_2 + 1 + L(2c_1 + \delta)$.

Case 2: $x_1, x_2 \in K$. Since $\operatorname{diam}(K, k_p) = d_2$, we have

$$|\alpha_1| \subset N_{k_p}(\{x_2\}, d_2) \subset N_{k_p}(|\alpha_2| \cup |\alpha_3|, d_2).$$

Case 3: $x_3 \in K$ and exactly one of x_1, x_2 lies in K, say $x_1 \in K$ and $x_2 \notin K$. Let x_1' be the first point on α_1 (oriented from x_2 to x_1) that lies in K and x_3' the first point on α_2 (oriented from x_2 to x_3) that lies in K. Let γ' be a geodesic in (Ω, k_p) connecting x_1' and x_3'. Now by Case 1 (strictly speaking we need $x_1', x_3' \notin K$ in order to apply Case 1 here, but by the choice of x_1', x_3' we may employ a limiting argument together with Case 1 to get the desired inclusion),

$$\begin{aligned} |\alpha_1| \subset N_{k_p}(|\alpha_1[x_2, x_1']|, d_2) &\subset N_{k_p}(|\alpha_2[x_2, x_3']| \cup |\gamma'|, d_2 + \delta_1) \\ &\subset N_{k_p}(|\alpha_2[x_2, x_3']|, 2d_2 + \delta_1) \\ &\subset N_{k_p}(|\alpha_2|, 2d_2 + \delta_1). \end{aligned}$$

Case 4: $x_3 \notin K$ and exactly one of x_1, x_2 lies in K, say $x_1 \in K$ and $x_2 \notin K$. Let x_3' be the first point on α_3 (oriented from x_3 to x_1) that lies in K. Let γ' be a geodesic in (Ω, k_p) joining x_2 and x_3'. Again Case 1 implies that $|\gamma'| \subset N_{k_p}(|\alpha_3[x_3, x_3']| \cup |\alpha_2|, \delta_1)$ (strictly speaking we need $x_3' \notin K$ in order to apply Case 1 here, but by the choice of x_3' we may employ a limiting argument together with Case 1 to get the desired inclusion), and an application of Case 3 yields $|\alpha_1| \subset N_{k_p}(|\alpha_3[x_1, x_3']| \cup |\gamma'|, 2d_2 + \delta_1)$. It follows that

$$|\alpha_1| \subset N_{k_p}(|\alpha_2| \cup |\alpha_3|, 2d_2 + 2\delta_1).$$

Case 5: $x_1, x_2 \notin K$ and $x_3 \in K$. Let x_1' be the first point on α_3 (oriented from x_1 to x_3) that lies in K. Let γ' be a geodesic in (Ω, k_p) connecting x_2 and x_1'. Case 1 implies that $|\alpha_1| \subset N_{k_p}(|\alpha_3[x_1, x_1']| \cup |\gamma'|, \delta_1)$, and Case 3 implies that $|\gamma'| \subset N_{k_p}(|\alpha_2|, 2d_2+\delta_1)$. It follows that $|\alpha_1| \subset N_{k_p}(|\alpha_2| \cup |\alpha_3|, 2d_2+2\delta_1)$. □

We shall also need the following construction of Bonk–Kleiner [BK].

Let (X, d) be an unbounded metric space and $p \in X$. Let $S_p(X) = X \cup \{\infty\}$, where ∞ is a point not in X. We define a function $s_p : S_p(X) \times S_p(X) \to [0, \infty)$ as follows:

$$s_p(x, y) = s_p(y, x) = \begin{cases} \frac{d(x,y)}{[1+d(x,p)][1+d(y,p)]} & \text{if } x, y \in X, \\ \frac{1}{1+d(x,p)} & \text{if } x \in X \text{ and } y = \infty, \\ 0 & \text{if } x = \infty = y. \end{cases}$$

It was shown in [BK] that there exists a metric $\widehat{d}_p$ on $S_p(X)$ satisfying

$$\frac{1}{4} s_p(x, y) \le \widehat{d}_p(x, y) \le s_p(x, y) \quad \text{for all} \quad x, y \in S_p(X).$$

Furthermore, the identity map $(X, d) \to (X, \widehat{d}_p)$ is η-quasimöbius with $\eta(t) = 16t$.

Theorem 4.6 ([BHX])**.** *Let (X, d) be an unbounded proper metric space, $\Omega \subset X$ a rectifiably connected open subset of X, and $p \in \partial\Omega$. Denote by $\widehat{k}_p$ the quasihyperbolic metric on Ω induced by the metric $\widehat{d}_p$. Suppose Ω is unbounded.*

(i) *If (Ω, d) is c-quasiconvex, then the identity map $(\Omega, k) \to (\Omega, \widehat{k}_p)$ is $80c$-bilipschitz;*
(ii) *If (Ω, d) is c-uniform, then $(\Omega, \widehat{d}_p)$ is c'-uniform with $c' = c'(c)$;*
(iii) *If (X, d) is c-quasiconvex and c-annular convex, then $(S_p(X), \widehat{d}_p)$ is c'-quasiconvex and c'-annular convex with $c' = c'(c)$;*
(iv) *If $(\Omega, \widehat{d}_p)$ is c-uniform, then (Ω, d) is c'-uniform with $c' = c'(c)$.*

5. Boundary maps of quasiisometries

In this section we study the boundary maps of quasiisometries between Gromov hyperbolic spaces. Please see [P], [BS] and [V3] for related results. The result we desire (Proposition 5.6) does not follow from the previous results.

If (Ω, k) is Gromov hyperbolic, and the hypotheses of Theorem 4.1 (ii) hold, then (Ω, k_p) is also Gromov hyperbolic and the boundary map of the identity map $(\Omega, k) \to (\Omega, k_p)$ is η-quasimöbius for some η. In general, there is no control on η. In this section we prove Proposition 5.6 which provides quantitative estimates for η in the case (X, d) is annular convex.

If $f : Y_1 \to Y_2$ is an (L, A)-quasiisometry between geodesic spaces and Y_1 is δ-hyperbolic, then the boundary map $(\partial^* Y_1, h_{y_1,\varepsilon}) \to (\partial^* Y_2, h_{y_2,\varepsilon})$ of f is η-quasimöbius with $\eta = \eta(L, A, \delta)$; see Proposition 5.10. Proposition 5.6 does not follow from this general result since the bilipschitz constant of the identity map $(\Omega, k) \to (\Omega, k_p)$ depends on the ratio d_0/d_0', where $d_0 = \operatorname{diam}(\Omega, d)$ and $d_0' = \operatorname{diam}(\partial\Omega, d)$. See also the remark after Proposition 4.2.

We first study the cross ratio on the Gromov boundary of a Gromov hyperbolic space (Corollary 5.2).

Let (Y, h) be a proper geodesic δ-hyperbolic space and $Q = (\xi_1, \xi_2, \xi_3, \xi_4)$ a quadruple of distinct points in $\partial^* Y$. The *signed distance* $\operatorname{sd}(Q)$ of Q is the number

$$\operatorname{sd}(Q) = \inf\{h([\xi_1, \xi_4], [\xi_2, \xi_3]) - h([\xi_1, \xi_3], [\xi_2, \xi_4])\},$$

where the infimum is taken over all geodesics $[\xi_i, \xi_j]$ joining ξ_i and ξ_j. Since the Hausdorff distance between two infinite geodesics with the same endpoints is at most 2δ, we have

$$\operatorname{sd}(Q) \le h([\xi_1, \xi_4], [\xi_2, \xi_3]) - h([\xi_1, \xi_3], [\xi_2, \xi_4]) \le \operatorname{sd}(Q) + 8\delta \tag{5.1}$$

for all geodesics $[\xi_i, \xi_j]$ joining ξ_i and ξ_j. For $w \in Y$, the *cross difference* of Q based at w is:

$$\operatorname{cd}_w(Q) = (\xi_1|\xi_4)_w + (\xi_2|\xi_3)_w - (\xi_1|\xi_3)_w - (\xi_2|\xi_4)_w.$$

Proposition 3.2 implies that for all quadruples Q, each $w \in Y$, and every $0 < \varepsilon \le \varepsilon_0(\delta)$,

$$e^{\varepsilon \operatorname{cd}_w(Q)}/4 \le \operatorname{cr}(Q, h_{w,\varepsilon}) \le 4e^{\varepsilon \operatorname{cd}_w(Q)}. \tag{5.2}$$

Notice that if Y is a tree, then $\operatorname{sd}(Q) = \operatorname{cd}_w(Q)$ for all $w \in Y$ and all Q. The following result shows that in a general δ-hyperbolic geodesic space, $\operatorname{sd}(Q)$ and $\operatorname{cd}_w(Q)$ differ by at most a fixed multiple of δ. Recall that geodesic triangles in $Y \cup \partial^* Y$ are 24δ-thin.

Lemma 5.1. *The inequality $|\operatorname{cd}_w(Q) - \operatorname{sd}(Q)| \le 430\delta$ holds for all $w \in Y$ and all Q.*

Proof. Fix $w \in Y$. For $i, j \in \{1, 2, 3, 4\}$ with $i \ne j$, choose geodesic rays $[w, \xi_i]$ and geodesic lines $[\xi_i, \xi_j]$. Put $X = \bigcup_i [w, \xi_i]$. By Theorem 3.5 there is a tree $T(X)$ and a map $u : X \to T(X)$ with the properties stated in Theorem 3.5. Let $w' = u(w)$ and $\xi_i' \in \partial^* T(X)$ be such that $u|_{[w,\xi_i]}$ is an isometry onto $[w', \xi_i']$. Let $x_{ij}' \in T(X)$ be the unique point with $[w', x_{ij}'] = [w', \xi_i'] \cap [w', \xi_j']$ ($x_{ij}' = x_{ji}'$), and let $x_{ij} \in [w, \xi_i]$ be such that $u(x_{ij}) = x_{ij}'$ (x_{ij} may not equal x_{ji}).

Let $Q' = (\xi_1', \xi_2', \xi_3', \xi_4')$. We will obtain estimates connecting $\operatorname{cd}_{w'}(Q')$ to $\operatorname{cd}_w(Q)$, and estimates connecting $\operatorname{sd}(Q')$ to $\operatorname{sd}(Q)$, where $\operatorname{cd}_{w'}(Q')$ and $\operatorname{sd}(Q')$

are the quantities related to the tree $T(X)$. Since $T(X)$ is a tree and hence $\mathrm{cd}_{w'}(Q') = \mathrm{sd}(Q')$, these estimates will help us prove the lemma.

We can find sequences $x_k \in [w, \xi_i]$ converging to ξ_i and $y_k \in [w, \xi_j]$ converging to ξ_j. Since $T(X)$ is a tree and hence $(u(x_k)|u(y_l))_{w'} = d(w', x'_{ij})$, the properties of u from Theorem 3.5 imply that

$$d(w', x'_{ij}) - 3\delta \le (x_k|y_l)_w \le d(w', x'_{ij}).$$

Using $(\xi_i|\xi_j)_w - 2\delta \le \liminf_{k,l\to\infty}(x_k|y_l)_w \le (\xi_i|\xi_j)_w$, we obtain

$$d(w', x'_{ij}) - 3\delta \le (\xi_i|\xi_j)_w \le d(w', x'_{ij}) + 2\delta.$$

It follows that

$$\mathrm{cd}_{w'}(Q') - 10\delta \le \mathrm{cd}_w(Q) \le \mathrm{cd}_{w'}(Q') + 10\delta. \tag{5.3}$$

We next show that $HD([\xi_i, \xi_j], [x_{ij}, \xi_i] \cup [x_{ji}, \xi_j]) \le 100\delta$. To this end, let $y_{ij} \in [x_{ij}, \xi_i]$ be such that $h(x_{ij}, y_{ij}) = 25\delta$. The properties of u imply that $d(y_{ij}, [w, \xi_j]) \ge 25\delta$. Since the triangle $[w, \xi_i] \cup [w, \xi_j] \cup [\xi_i, \xi_j]$ is 24δ-thin, there is some point $z_{ij} \in [\xi_i, \xi_j]$ such that $h(y_{ij}, z_{ij}) \le 24\delta$. Consequently, the fact that $[y_{ij}, z_{ij}] \cup [y_{ij}, \xi_i] \cup [z_{ij}, \xi_i]$ is 24δ-thin implies that $HD([y_{ij}, \xi_i], [z_{ij}, \xi_i]) \le 48\delta$. Similarly $HD([y_{ji}, \xi_j], [z_{ji}, \xi_j]) \le 48\delta$. Since $h(x_{ij}, x_{ji}) \le 6\delta$, the triangle inequality implies that $h(z_{ij}, z_{ji}) \le 104\delta$. It follows that

$$HD([\xi_i, \xi_j], [x_{ij}, \xi_i] \cup [x_{ji}, \xi_j]) \le 48\delta + 52\delta = 100\delta. \tag{5.4}$$

For $\{i, j, k, l\} = \{1, 2, 3, 4\}$, we choose $p_{ij} \in [\xi_i, \xi_j]$ and $p_{kl} \in [\xi_k, \xi_l]$ with

$$h(p_{ij}, p_{kl}) = h([\xi_i, \xi_j], [\xi_k, \xi_l]).$$

Inequality (5.4) implies that there are $q_{ij} \in [x_{ij}, \xi_i] \cup [x_{ji}, \xi_j]$ and $q_{kl} \in [x_{kl}, \xi_k] \cup [x_{lk}, \xi_l]$ such that $|h(p_{ij}, p_{kl}) - h(q_{ij}, q_{kl})| \le 200\delta$. By the properties of u,

$$h(q_{ij}, q_{kl}) - 6\delta \le d(u(q_{ij}), u(q_{kl})) \le h(q_{ij}, q_{kl}).$$

Since $u(q_{ij}) \in [\xi'_i, \xi'_j]$ and $u(q_{kl}) \in [\xi'_k, \xi'_l]$, we have

$$d(u(q_{ij}), u(q_{kl})) \ge d([\xi'_i, \xi'_j], [\xi'_k, \xi'_l]).$$

Combining the above inequalities we obtain

$$h([\xi_i, \xi_j], [\xi_k, \xi_l]) \ge d([\xi'_i, \xi'_j], [\xi'_k, \xi'_l]) - 200\delta. \tag{5.5}$$

On the other hand, there exist points $r'_{ij} \in [\xi'_i, \xi'_j]$ and $r'_{kl} \in [\xi'_k, \xi'_l]$ such that

$$d(r'_{ij}, r'_{kl}) = d([\xi'_i, \xi'_j], [\xi'_k, \xi'_l]).$$

Observe that $u([x_{ij}, \xi_i] \cup [x_{ji}, \xi_j]) = [\xi'_i, \xi'_j]$. Hence there exists $r_{ij} \in [x_{ij}, \xi_i] \cup [x_{ji}, \xi_j]$ with $u(r_{ij}) = r'_{ij}$. Similarly there is a point $r_{kl} \in [x_{kl}, \xi_k] \cup [x_{lk}, \xi_l]$ with $u(r_{kl}) = r'_{kl}$. The properties of u gives $h(r_{ij}, r_{kl}) - 6\delta \le d(r'_{ij}, r'_{kl}) \le h(r_{ij}, r_{kl})$.

By inequality (5.4), there is a point $w_{ij} \in [\xi_i, \xi_j]$ with $h(w_{ij}, r_{ij}) \leq 100\delta$. Similarly there is some $w_{kl} \in [\xi_k, \xi_l]$ with $h(w_{kl}, r_{kl}) \leq 100\delta$. Thus,

$$\begin{aligned} h([\xi_i, \xi_j], [\xi_k, \xi_l]) &\leq h(w_{ij}, w_{kl}) \leq h(r_{ij}, r_{kl}) + 200\delta \\ &\leq d(r'_{ij}, r'_{kl}) + 206\delta = d([\xi'_i, \xi'_j], [\xi'_k, \xi'_l]) + 206\delta, \end{aligned}$$

whence by inequality (5.5),

$$|h([\xi_i, \xi_j], [\xi_k, \xi_l]) - d([\xi'_i, \xi'_j], [\xi'_k, \xi'_l])| \leq 206\,\delta. \tag{5.6}$$

It follows from inequality (5.1) that $|\operatorname{sd}(Q) - \operatorname{sd}(Q')| \leq 420\delta$. Since $T(X)$ is a tree, we have $\operatorname{cd}_{w'}(Q') = \operatorname{sd}(Q')$. Therefore by inequality (5.3), we have $|\operatorname{cd}_w(Q) - \operatorname{sd}(Q)| \leq 430\delta$. □

Remark 5.1. Since $T(X)$ is a tree, one of $d([\xi'_1, \xi'_4], [\xi'_2, \xi'_3])$, $d([\xi'_1, \xi'_3], [\xi'_2, \xi'_4])$ is 0. Hence it follows from inequality (5.6) that

$$\min\left\{ h([\xi_1, \xi_4], [\xi_2, \xi_3]),\ h([\xi_1, \xi_3], [\xi_2, \xi_4]) \right\} \leq 206\delta. \tag{5.7}$$

Corollary 5.2. *Set $c_0 = 4e^{86}$. The following holds for each $w \in Y$, all $0 < \varepsilon \leq \varepsilon_0(\delta)$, and all quadruples Q:*

$$\frac{e^{\varepsilon \operatorname{sd}(Q)}}{c_0} \leq cr(Q, h_{w,\varepsilon}) \leq c_0\, e^{\varepsilon \operatorname{sd}(Q)}.$$

Proof. Recall $\varepsilon_0(\delta) = \min\{1, \frac{1}{5\delta}\}$. The corollary now follows from Lemma 5.1 and (5.2). □

Corollary 5.3. *Let $c > 0$, $w \in Y$, $0 < \varepsilon \leq \varepsilon_0(\delta)$, and $Q = (\xi_1, \xi_2, \xi_3, \xi_4)$ a quadruple.*

(i) *If $cr(Q, h_{w,\varepsilon}) \leq c$, then for all geodesics $[\xi_i, \xi_j]$ joining ξ_i and ξ_j, $1 \leq i, j \leq 4$,*
$$h([\xi_1, \xi_4], [\xi_2, \xi_3]) \leq c' = c'(c, \varepsilon, \delta).$$
(ii) *If $h([\xi_1, \xi_4], [\xi_2, \xi_3]) \leq c$ for some geodesics $[\xi_i, \xi_j]$ joining ξ_i and ξ_j, $1 \leq i, j \leq 4$, then $cr(Q, h_{w,\varepsilon}) \leq c' = c'(c)$.*

Proof. We first prove (i). By hypothesis $\operatorname{cr}(Q, h_{w,\varepsilon}) \leq c$. Therefore, by Corollary 5.2 we have $\operatorname{sd}(Q) \leq \log(c_0 c)/\varepsilon$. Let $[\xi_i, \xi_j]$ be a geodesic with endpoints ξ_i and ξ_j for $1 \leq i, j \leq 4$. Now the claim follows from inequalities (5.1) and (5.7).

Now we prove (ii). Suppose that $h([\xi_1, \xi_4], [\xi_2, \xi_3]) \leq c$. Then $\operatorname{sd}(Q) \leq c$. Since $\varepsilon \leq 1$, by Corollary 5.2 we have $\operatorname{cr}(Q, h_{w,\varepsilon}) \leq c_0 e^{\varepsilon c} \leq c_0 e^c$. □

Corollary 5.4. *For all $w_1, w_2 \in Y$ and any $0 < \varepsilon \leq \varepsilon_0(\delta)$, the identity map $(\partial^* Y, h_{w_1,\varepsilon}) \to (\partial^* Y, h_{w_2,\varepsilon})$ is η-quasimöbius with $\eta(t) = 16e^{172}\, t = c_0^2\, t$.*

Proof. By Corollary 5.2,

$$\mathrm{cr}(Q, h_{w_2,\varepsilon}) \le c_0\, e^{\varepsilon\ \mathrm{sd}(Q)} \le c_0^2\, \mathrm{cr}(Q, h_{w_1,\varepsilon}).$$

□

Corollary 5.5. *For all $0 < \varepsilon_1, \varepsilon_2 \le \varepsilon_0(\delta)$ and any $w \in Y$, the identity map $(\partial^* Y, h_{w,\varepsilon_1}) \to (\partial^* Y, h_{w,\varepsilon_2})$ is η-quasimöbius with $\eta(t) = 4^{\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)} t^{\frac{\varepsilon_2}{\varepsilon_1}}$.*

Proof. Set $t = \mathrm{cr}(Q, h_{w,\varepsilon_1})$. Then by inequality (5.2)

$$\mathrm{cr}(Q, h_{w,\varepsilon_2}) \le 4e^{\varepsilon_2\, \mathrm{cd}_w(Q)} = 4(e^{\varepsilon_1\, \mathrm{cd}_w(Q)})^{\frac{\varepsilon_2}{\varepsilon_1}} \le 4(4t)^{\frac{\varepsilon_2}{\varepsilon_1}} = \eta(t).$$

□

Proposition 5.6. *Suppose (X, d) is c-quasiconvex and c-annular convex, $\Omega \subset X$ a domain, and $p \in \partial\Omega$. Suppose also that $d_0 = diam(\Omega, d) < \infty$ and $d_0' = diam(\partial\Omega, d) > 0$. Set $k' := k_p$. If (Ω, k) is δ-hyperbolic, then the boundary map $\partial f : (\partial_k^*\Omega, k_{w,\varepsilon}) \to (\partial_{k'}^*\Omega, k'_{w',\varepsilon})$ of the identity map $f : (\Omega, k) \to (\Omega, k')$ is η-quasimöbius with $\eta = \eta(\delta, c)$.*

By Proposition 4.2, (Ω, k') is δ'-hyperbolic with $\delta' = \delta'(\delta, c)$. Theorem 4.1 implies that the identity map $f : (\Omega, k) \to (\Omega, k')$ is bilipschitz, hence the boundary map is well-defined and is a homeomorphism. Abusing notation we denote $\partial f(\xi)$ $(\xi \in \partial_k^*\Omega)$ by ξ. Let K be as in Section 4 if $d_0 > 20c^2 d_0'$, and $K = \emptyset$ if $d_0 \le 20c^2 d_0'$.

The proof of Proposition 5.6 is achieved by combining Lemmas 5.7 through 5.9. Intuitively Proposition 5.6 follows from the geometric pictures of (Ω, k) and (Ω, k_p) described before Lemma 4.3.

Lemma 5.7. *There exists a constant $A' = A'(\delta, c)$ such that*

$$\frac{1}{L}\, k([\xi_i, \xi_j],]\xi_k, \xi_l]) - A' \le k'(\widehat{[\xi_i, \xi_j]}, \widehat{[\xi_k, \xi_l]}) \le L\, k([\xi_i, \xi_j], [\xi_k \xi_l]) + A'$$

for every quadruple $Q = (\xi_1, \xi_2, \xi_3, \xi_4)$ of distinct points in $\partial_k^*\Omega$, all geodesics $[\xi_i, \xi_j]$ in (Ω, k) joining ξ_i and ξ_j, and all geodesics $\widehat{[\xi_i, \xi_j]}$ in (Ω, k') joining ξ_i and ξ_j. Here L is the constant given by Lemma 4.3.

Proof. Let $q_{ij} \in [\xi_i, \xi_j]$ and $q_{kl} \in [\xi_k, \xi_l]$ with $k(q_{ij}, q_{kl}) = k([\xi_i, \xi_j], [\xi_k, \xi_l])$. If the geodesic $[\xi_i, \xi_j]$ intersects K then it passes through S. Let x and y respectively be the first and last points on $[\xi_i, \xi_j]$ that lie on S. Since $\mathrm{diam}(S, k) = D_1$, the subsegment $[x, y]$ of $[\xi_i, \xi_j]$ has length at most D_1. Hence there exists a point $w_{ij} \in [\xi_i, \xi_j] \backslash K$ such that $k(q_{ij}, w_{ij}) \le D_1$. If $[\xi_i, \xi_j]$ does not intersect K, then we can choose $w_{ij} = q_{ij}$. Similarly, there is a point $w_{kl} \in [\xi_k, \xi_l] \backslash K$ such that $k(q_{kl}, w_{kl}) \le D_1$. For the geodesics $\beta_{ij} = [\xi_i, \xi_j]$, consider the maps $\tilde\beta_{ij}$ given in the paragraph before Lemma 4.5. By Lemma 4.5 both $\tilde\beta_{ij}$ and $\tilde\beta_{kl}$ are (L, A)-quasigeodesics in (Ω, k'), with L and A depending solely on c. It follows from the stability lemma (Lemma 3.4) that $HD_{k'}(|\tilde\beta_{ij}|, \widehat{[\xi_i, \xi_j]}) \le b_1$ and

$HD_{k'}(|\tilde{\beta}_{kl}|, \widehat{[\xi_k, \xi_l]}) \le b_1$, where $b_1 = b_1(\delta', L, A) = b_1(\delta, c)$. Hence we find two points $z_{ij} \in \widehat{[\xi_i, \xi_j]}$ and $z_{kl} \in \widehat{[\xi_k, \xi_l]}$ with $k'(w_{ij}, z_{ij}), k'(w_{kl}, z_{kl}) \le b_1$. Now we have

$$\begin{aligned} k'(\widehat{[\xi_i, \xi_j]}, \widehat{[\xi_k, \xi_l]}) \le k'(z_{ij}, z_{kl}) &\le k'(w_{ij}, w_{kl}) + 2b_1 \\ &\le L\, k(w_{ij}, w_{kl}) + D_2 + 2b_1 \\ &\le L\,\{k(q_{ij}, q_{kl}) + 2D_1\} + D_2 + 2b_1 \\ &= L\, k([\xi_i, \xi_j], [\xi_k, \xi_l]) + (D_2 + 2b_1 + 2LD_1). \end{aligned}$$

The second inequality can be proven in a similar manner. $\square$

Denote by $Q' = (\partial f(\xi_1), \partial f(\xi_2), \partial f(\xi_3), \partial f(\xi_4))$ for every quadruple $Q = (\xi_1, \xi_2, \xi_3, \xi_4)$ of distinct points in $\partial_k^* \Omega$.

Lemma 5.8. *There exists a constant $b_2 = b_2(\delta, c)$ with the property that for every quadruple $Q = (\xi_1, \xi_2, \xi_3, \xi_4)$ of distinct points in $\partial_k^* \Omega$,*

(i) *if $\mathrm{sd}(Q) \ge 0$, then $\mathrm{sd}(Q') \le L\, \mathrm{sd}(Q) + b_2$;*

(ii) *if $\mathrm{sd}(Q) \le 0$, then $\mathrm{sd}(Q') \le \mathrm{sd}(Q)/L + b_2$.*

Proof. We first prove (i). Assume that $\mathrm{sd}(Q) \ge 0$. Recall

$$\mathrm{sd}(Q) \le k([\xi_1, \xi_4], [\xi_2, \xi_3]) - k([\xi_1, \xi_3], [\xi_2, \xi_4]) \le \mathrm{sd}(Q) + 8\delta.$$

Now inequality (5.7) and $\mathrm{sd}(Q) \ge 0$ imply $k([\xi_1, \xi_3], [\xi_2, \xi_4]) \le 206\delta$. By Lemma 5.7

$$\begin{aligned} \mathrm{sd}(Q') &\le k'(\widehat{[\xi_1, \xi_4]}, \widehat{[\xi_2, \xi_3]}) - k'(\widehat{[\xi_1, \xi_3]}, \widehat{[\xi_2, \xi_4]}) \\ &\le k'(\widehat{[\xi_1, \xi_4]}, \widehat{[\xi_2, \xi_3]}) \\ &\le L\, k([\xi_1, \xi_4], [\xi_2, \xi_3]) + A' \\ &\le L\,\{\mathrm{sd}(Q) + k([\xi_1, \xi_3], [\xi_2, \xi_4]) + 8\delta\} + A' \\ &\le L \cdot \mathrm{sd}(Q) + 214\, L\, \delta + A'. \end{aligned}$$

The proof of (ii) is similar to that of (i). Recall that by Proposition 4.2 (Ω, k') is δ'-hyperbolic for some $\delta' = \delta'(\delta, c)$. If $\mathrm{sd}(Q) \le 0$, then we have $k([\xi_1, \xi_4], [\xi_2, \xi_3]) \le 214\delta$, and therefore Lemma 5.7 implies

$$\begin{aligned} \mathrm{sd}(Q') &\le k'(\widehat{[\xi_1, \xi_4]}, \widehat{[\xi_2, \xi_3]}) - k'(\widehat{[\xi_1, \xi_3]}, \widehat{[\xi_2, \xi_4]}) + 8\delta' \\ &\le L\, k([\xi_1, \xi_4], [\xi_2, \xi_3]) + A' - \frac{1}{L}\, k([\xi_1, \xi_3], [\xi_2, \xi_4]) + A' + 8\delta' \\ &\le 214\, L\, \delta + 2A' + 8\delta' + \frac{1}{L}\,\{k([\xi_1, \xi_4], [\xi_2, \xi_3]) - k([\xi_1, \xi_3], [\xi_2, \xi_4])\} \\ &\le 214\, L\, \delta + 2A' + 8\delta' + \frac{1}{L}\,(\mathrm{sd}(Q) + 8\delta) \\ &= \frac{1}{L}\, \mathrm{sd}(Q) + 214 L\delta + 2A' + 8\delta' + 8\delta/L. \end{aligned}$$

□

Let $w, w' \in \Omega$ and $0 < \varepsilon \leq \min\{\varepsilon_0(\delta), \varepsilon_0(\delta')\}$. For a quadruple $Q = (\xi_1, \xi_2, \xi_3, \xi_4)$ of distinct points in $\partial_k^* \Omega$, set $\mathrm{cr}(Q) = \mathrm{cr}(Q, k_{w,\varepsilon})$ and $\mathrm{cr}(Q') = \mathrm{cr}(Q', k'_{w',\varepsilon})$.

Lemma 5.9. *The map $\partial f : (\partial_k^* \Omega, k_{w,\varepsilon}) \to (\partial_{k'}^* \Omega, k'_{w',\varepsilon})$ is η-quasimöbius for some $\eta = \eta(\delta, c)$.*

Proof. Let $Q = (\xi_1, \xi_2, \xi_3, \xi_4)$. Recall that $\varepsilon \leq \varepsilon_0(\delta) \leq 1$. If $\mathrm{sd}(Q) \geq 0$, then by Lemma 5.8, $\mathrm{sd}(Q') \leq L\, \mathrm{sd}(Q) + b_2$. Let $c_0 = 4e^{86}$. It follows from Corollary 5.2 that

$$\begin{aligned}\mathrm{cr}(Q') \leq c_0 e^{\varepsilon\, \mathrm{sd}(Q')} \leq c_0 e^{\varepsilon\, (L\, \mathrm{sd}(Q) + b_2)} = c_0 e^{\varepsilon b_2} (e^{\varepsilon\, \mathrm{sd}(Q)})^L &\leq c_0 e^{\varepsilon b_2} (c_0\, \mathrm{cr}(Q))^L \\ &= c_0^{L+1} e^{\varepsilon b_2} (\mathrm{cr}(Q))^L \\ &\leq c_0^{L+1} e^{b_2} (\mathrm{cr}(Q))^L.\end{aligned}$$

If $\mathrm{sd}(Q) \leq 0$, then $\mathrm{sd}(Q') \leq \mathrm{sd}(Q)/L + b_2$. It follows that

$$\begin{aligned}\mathrm{cr}(Q') \leq c_0 e^{\varepsilon\, \mathrm{sd}(Q')} \leq c_0 e^{\varepsilon\, (\mathrm{sd}(Q)/L + b_2)} &= c_0 e^{\varepsilon b_2} (e^{\varepsilon\, \mathrm{sd}(Q)})^{\frac{1}{L}} \\ &\leq c_0 e^{\varepsilon b_2} (c_0 \mathrm{cr}(Q))^{\frac{1}{L}} \\ &= c_0^{1+\frac{1}{L}} e^{\varepsilon b_2} (\mathrm{cr}(Q))^{\frac{1}{L}} \\ &\leq c_0^{1+\frac{1}{L}} e^{b_2} (\mathrm{cr}(Q))^{\frac{1}{L}}.\end{aligned}$$

Note that $\mathrm{cr}(Q) \to 0$ as $\mathrm{sd}(Q) \to -\infty$. □

The proof of Proposition 5.6 can be easily adapted to show the following:

Proposition 5.10. *Let $f : X \to Y$ be an (L, A) quasiisometry between two proper geodesic metric spaces. If X is δ-hyperbolic, then $\partial f : (\partial^* X, h_{x,\varepsilon}) \to (\partial^* Y, h_{y,\varepsilon})$ ($x \in X$, $y \in Y$) is η-quasimöbius with $\eta = \eta(L, A, \delta)$.*

Proposition 5.10 is contained in [V3, Theorem 5.38]. See also [BS, Theorem 6.5] for a related result.

6. Necessity

It is proved in [BHK] that a uniform domain is Gromov hyperbolic with respect to the quasihyperbolic metric and that when the domain is bounded the natural map exists and is quasimöbius. In this section we show that the natural map is quasimöbius for unbounded domains as well. These statements are quantitative. We first explain the notion of a natural map.

Let X' be the one point compactification $X \cup \{\infty\}$ of X if X is unbounded, and $X' = X$ if X is bounded. Let $Q = (x_1, x_2, x_3, x_4)$ be a quadruple of distinct points in X'. The cross ratio $\mathrm{cr}(Q, d)$ is defined as in Section 4 if all

$x_i \in X$, and if one of the x_i is ∞, then $\mathrm{cr}(Q,d)$ is obtained from the usual definition by canceling the terms involving ∞. For example, if $x_1 = \infty$, then

$$\mathrm{cr}(Q,d) = \frac{d(x_2,x_4)}{d(x_2,x_3)}.$$

Notice that for any $p \in X$, the metric space $(S_p(X), \hat{d}_p)$ is homeomorphic to X'. By using the properties of the metric $\hat{d}_p$ it is easy to check that $\mathrm{cr}(Q,\hat{d}_p)/16 \le \mathrm{cr}(Q,d) \le 16\mathrm{cr}(Q,\hat{d}_p)$ for any quadruple Q of distinct points in X'. This statement justifies the above definition of cross ratio on X', although there is no canonical metric on X'.

Let $\Omega \subset X$ be a rectifiably connected open subset with $\partial\Omega \neq \emptyset$. Denote by $\partial'\Omega$ the topological boundary of Ω in X'. Then $\partial'\Omega = \partial\Omega$ if Ω is bounded, and $\partial'\Omega = \partial\Omega \cup \{\infty\}$ if Ω is unbounded. Suppose (Ω, k) is Gromov hyperbolic. If the identity map $(\Omega,k) \to (\Omega,d)$ has a continuous extension to the Gromov closure $\Omega \cup \partial^*\Omega$ of (Ω,k), then the restriction of this extension to the Gromov boundary, $\partial^*\Omega \to \partial'\Omega$, is called a natural map of Ω. Since Ω is dense in the Gromov closure, the natural map is unique if it exists.

Suppose (X,d) is unbounded. Then for any $p \in X$, the metric space $(S_p(X), \hat{d}_p)$ is homeomorphic to the one point compactification X'. Let $\hat{\partial}_p\Omega$ be the boundary of Ω in $(S_p(X), \hat{d}_p)$. Notice that $\hat{\partial}_p\Omega = \partial'\Omega$ as sets and the identity maps $\hat{\partial}_p\Omega \to \partial'\Omega$ and $(\Omega, \hat{d}_p) \to (\Omega, d)$ are homeomorphisms. So by a natural map $(\partial^*\Omega, k_{x,\varepsilon}) \to (\hat{\partial}_p\Omega, \hat{d}_p)$ we mean the continuous extension to the Gromov boundary of the identity map $(\Omega, k) \to (\Omega, \hat{d}_p)$. It follows from the remark in the second paragraph of this section that the natural map $(\partial^*\Omega, k_{x,\varepsilon}) \to (\partial'\Omega, d)$ is η-quasimöbius if and only if the natural map $(\partial^*\Omega, k_{x,\varepsilon}) \to (\hat{\partial}_p\Omega, \hat{d}_p)$ is η'-quasimöbius, with η and η' depending only on each other.

Suppose (Y,ρ) is a metric space and $\Omega \subset Y$ is a bounded rectifiably connected open subset with $\partial_\rho\Omega$ (the boundary of Ω in (Y,ρ)) containing at least two points, and $p \in \partial_\rho\Omega$ is such that $X = I_p(Y)$ with $d = \rho_p$; that is, the metric space (X,d) is the "inversion" of (Y,ρ) at p. Then Ω is unbounded in (X,d). We note that X' and Y' are homeomorphic. If (Ω,k) is Gromov hyperbolic and $\phi : (\partial^*\Omega, k_{x,\varepsilon}) \to (\partial'\Omega, d)$ is a natural map, then by composing ϕ with the identification of X' and Y', we obtain another natural map $(\partial^*\Omega, k_{x,\varepsilon}) \to (\partial_\rho\Omega, \rho)$. By using the properties of the metric ρ_p, we see that the natural map $\phi : (\partial^*\Omega, k_{x,\varepsilon}) \to (\partial'\Omega, d)$ is η-quasimöbius if and only if the natural map $(\partial^*\Omega, k_{x,\varepsilon}) \to (\partial_\rho\Omega, \rho)$ is η'-quasimöbius, with η and η' depending only on each other.

Theorem 6.1 (Theorem 3.6 of [BHK]). *Let (X,d) be a proper metric space and $\Omega \subset X$ a c-uniform domain. Then (Ω,k) is a geodesic δ-hyperbolic space with $\delta = \delta(c)$. If Ω is bounded, then for each $w \in \Omega$ and all $0 < \varepsilon \le \varepsilon_0(\delta)$*

the natural map $\phi : (\partial^*\Omega, k_{w,\varepsilon}) \to (\partial\Omega, d)$ *exists and is* η*-quasimöbius with* $\eta = \eta(c, \varepsilon)$.

If we choose $\varepsilon = \varepsilon_0(\delta) = \varepsilon_0(c)$ for the visual metric $k_{w,\varepsilon}$, then the homeomorphism η in Theorem 6.1 depends only on c.

Theorem 6.2. *Let* X *be a proper metric space and* $\Omega \subset X$ *a* c*-uniform domain. There exists a constant* $\varepsilon_1(c) > 0$ *such that for every* $w \in \Omega$ *and* $0 < \varepsilon \leq \varepsilon_1(c)$*, the natural map* $\phi : (\partial^*\Omega, k_{w,\varepsilon}) \to (\partial'\Omega, d)$ *exists and is* η*-quasimöbius with* $\eta = \eta(c, \varepsilon)$.

Proof. By Theorem 6.1, it only remains to consider the case when Ω is unbounded. Suppose that Ω is an unbounded c-uniform domain. Fix $p \in \partial\Omega$ and consider the compact metric space $(S_p(X), \hat{d}_p)$. By Theorem 4.6 (ii), $(\Omega, \hat{d}_p)$ is c_1-uniform with $c_1 = c_1(c)$. Let k'' be the quasihyperbolic metric on Ω with respect to $\hat{d}_p$. By Theorem 6.1, (Ω, k'') is δ_1-hyperbolic with $\delta_1 = \delta_1(c_1) = \delta_1(c)$, and for any $w \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0(\delta_1)$, the natural map $\phi_1 : (\partial^*_{k''}\Omega, k''_{w,\varepsilon}) \to (\hat{\partial}_p\Omega, \hat{d}_p)$ exists and is η_1-quasimöbius with $\eta_1 = \eta_1(c_1, \varepsilon) = \eta_1(c, \varepsilon)$. On the other hand, Theorem 4.6 (i) implies that the identity map $f : (\Omega, k) \to (\Omega, k'')$ is $80c$-bilipschitz. By Proposition 5.10, for any $w \in \Omega$ and any ε satisfying $0 < \varepsilon \leq \varepsilon_1(c) := \min\{\varepsilon_0(\delta), \varepsilon_0(\delta_1)\}$, the boundary map $\partial f : (\partial^*\Omega, k_{w,\varepsilon}) \to (\partial^*_{k''}\Omega, k''_{w,\varepsilon})$ is η_2-quasimöbius with $\eta_2 = \eta_2(\delta, 80c) = \eta_2(c)$. Hence there is an η-quasimöbius natural map

$$\phi = \phi_1 \circ \partial f : (\partial^*\Omega, k_{w,\varepsilon}) \to (\hat{\partial}_p\Omega, \hat{d}_p)$$

with $\eta = \eta_1 \circ \eta_2$. □

Again if we choose $\varepsilon = \varepsilon_1(c)$, then the homeomorphism η in Theorem 6.2 depends only on c.

7. Annulus points, arc points and starlikeness

In this section we recall the notion of annulus points and arc points, and show that each arc point lies on an anchor (Lemma 7.3) and that domains with large boundaries are starlike (Theorem 7.4).

The following definitions are from Chapter 7 of [BHK].

Definition 7.1. *Let* $0 < \lambda \leq 1/2$*. A point* $x \in \Omega$ *is said to be a* λ-annulus point *if there is a point* $a \in \partial\Omega$ *with* $\delta_\Omega(x) = d(x, a)$ *so that* $B(a, \delta_\Omega(x)/\lambda) \setminus B(a, \lambda\delta_\Omega(x)) \subset \Omega$*. If* $x \in \Omega$ *is not a* λ*-annulus point, then it is said to be a* λ-arc point.

Definition 7.2. *Let* $x_0 \in \Omega$ *and* $c \geq 1$*. A path* $\gamma : a \frown b$ *in* $\overline{\Omega}$ *is a* c-anchor *of* x_0 *if*

(1) $x_0 \in |\gamma|$,
(2) $\ell_d(\gamma) \leq c\, d(a, b)$,

(3) *for every* $x \in |\gamma[a, x_0]|$ *we have* $\ell_d(\gamma[a, x]) \leq c\,\delta_\Omega(x)$,
(4) *for every* $x \in |\gamma[x_0, b]|$ *we have* $\ell_d(\gamma[x, b]) \leq c\,\delta_\Omega(x)$,
(5) $|\gamma| \cap \partial\Omega = \{a, b\}$,
(6) γ *is a continuous* (c, c)*-quasigeodesic in* (Ω, k)*:* $\ell_k(\gamma[x, y]) \leq c\,k(x, y) + c$ *for all* $x, y \in |\gamma| \backslash \{a, b\}$.

If x_0 is a λ-arc point, then whenever q is a point in the boundary of the domain closest to x_0, then the boundary of the domain near q is large at scale λ; hence we may find two boundary points and a uniform curve connecting these two points and passing through x_0 such that x_0 plays the role of a midpoint of the curve. This is the content of the following lemma, which is an analog of the anchor lemma 3.18 in [V1].

Lemma 7.3. *Suppose* (X, d) *is a* C_a*-annular convex geodesic space. If* $0 < \lambda \leq 1/(2C_a^2)$*, then every* λ*-arc point* $x_0 \in \Omega$ *has a* c*-anchor with* $c = c(\lambda, C_a)$.

Proof. In this proof, C and C' denote constants that depend only on λ and C_a, and their values may change from one occurance to another as they represent all such constants occuring in this proof that we do not need to keep track of.

Let x_0 be a λ-arc point. Choose $a \in \partial\Omega$ with $\delta_\Omega(x_0) = d(x_0, a)$. Since x_0 is a λ-arc point, there is a point $y \in X\backslash\Omega$ such that $\lambda\,\delta_\Omega(x_0) \leq d(a, y) < \delta_\Omega(x_0)/\lambda$. Let γ_1 be a d-geodesic connecting a to x_0. We break up the construction of the anchor into two cases.

Figure 1. The two cases in Lemma 7.3.

Case 1. Suppose $d(a, y) \geq \delta_\Omega(x_0)$. Let β_0 be a d-geodesic joining y to a; β_0 intersects the sphere $S(a, \delta_\Omega(x_0))$ at exactly one point w. By the annular convexity of X there is a rectifiable path β_1 joining x_0 and w in the annulus $B(a, C_\mathrm{a}\delta_\Omega(x_0)) \backslash B(a, \delta_\Omega(x_0)/C_\mathrm{a})$ with $\ell_d(\beta_1) \leq C_\mathrm{a} d(x_0, w)$. Let $\widehat{\beta_0} = \beta_0[y, w]$ and $\widehat{\gamma} = \widehat{\beta_0} * \beta_1 * \gamma_1$ be the concatenation of the three paths $\widehat{\beta_0}, \beta_1$, and γ_1.
Case 2. Suppose $d(a, y) < \delta_\Omega(x_0)$. In this case let $z \in |\gamma_1|$ be the unique point with $d(z, a) = d(a, y)$. Let $\widehat{\beta}_0 = \gamma_1[x_0, z]$ oriented from x_0 to z, and by the annular convexity, let β_1 be a rectifiable path in the annulus $B(a, C_\mathrm{a} d(y, a)) \setminus B(a, d(y, a)/C_\mathrm{a})$ joining z and y with $\ell_d(\beta_1) \leq C_\mathrm{a} d(z, y)$. Now set $\widehat{\gamma} = \beta_1 * \widehat{\beta}_0 * \gamma_1$.

Once such a path $\widehat{\gamma}$ has been constructed from the above cases, we modify this path further. Since $y \notin \Omega$ and $\delta_\Omega(x_0) > \dfrac{\lambda}{3C_\mathrm{a}}\delta_\Omega(x_0)$, there is a point $x_1 \in \Omega \cap (|\beta_1| \cup |\widehat{\beta}_0|)$ at which $|\beta_1| \cup |\widehat{\beta}_0|$, beginning from the point x_0, first achieves $\delta_\Omega(x_1) = \dfrac{\lambda}{3C_\mathrm{a}}\delta_\Omega(x_0)$. Let $\gamma_2 = \widehat{\gamma}[x_0, x_1]$. Then for all $x \in |\gamma_2| \setminus \{x_1\}$

we have

$$\delta_\Omega(x) > \frac{\lambda}{3C_\mathrm{a}}\delta_\Omega(x_0). \tag{7.1}$$

Let $b \in \partial\Omega$ with

$$\delta_\Omega(x_1) = d(x_1, b) = \frac{\lambda}{3C_\mathrm{a}}\delta_\Omega(x_0). \tag{7.2}$$

Now the choice of x_1 implies that x_1 is the point on γ_2 nearest to b. Let β_2 be a d-geodesic from x_1 to b, and let $\gamma = \beta_2 * \gamma_2 * \gamma_1$.

We next verify that γ satisfies conditions (1)–(6) of the definition of a c-anchor. By construction $x_0 \in |\gamma|$, so condition (1) is satisfied. Condition (5) is also clear.

Note that by equation (7.2),

$$\begin{aligned}\ell_d(\gamma) = \ell_d(\gamma_1) + \ell_d(\gamma_2) + \ell_d(\beta_2) &= d(x_0, a) + \ell_d(\gamma_2) + d(x_1, b)\\ &= \left(1 + \frac{\lambda}{3C_\mathrm{a}}\right)\delta_\Omega(x_0) + \ell_d(\gamma_2).\end{aligned}$$

In the situation of Case 1 above, we have

$$\ell_d(\gamma_2) \le \ell_d(\beta_1) + \ell_d(\widehat{\beta}_0) \le C_\mathrm{a} d(x_0, w) + d(w, y) \le 2C_\mathrm{a}\delta_\Omega(x_0) + \frac{1}{\lambda}\delta_\Omega(x_0),$$

and hence

$$\ell_d(\gamma) \le \left(1 + \frac{\lambda}{3C_\mathrm{a}} + 2C_\mathrm{a} + \frac{1}{\lambda}\right)\delta_\Omega(x_0). \tag{7.3}$$

In the situation of Case 2 above,

$$\ell_d(\gamma_2) \le \ell_d(\beta_1) + \ell_d(\widehat{\beta}_0) \le C_\mathrm{a} d(y, z) + \delta_\Omega(x_0) \le 2C_\mathrm{a}\delta_\Omega(x_0) + \delta_\Omega(x_0),$$

and we obtain inequality (7.3) again. Since γ_2 does not intersect the ball $B(a, \lambda\delta_\Omega(x_0)/C_\mathrm{a})$, and by equation (7.2) $\ell_d(\beta_2) = \lambda\delta_\Omega(x_0)/(3C_\mathrm{a})$, we see that $d(a, b) \ge \lambda\delta_\Omega(x_0)/C_\mathrm{a} - \lambda\delta_\Omega(x_0)/(3C_\mathrm{a}) = 2\lambda\delta_\Omega(x_0)/(3C_\mathrm{a})$. Thus by inequality (7.3), $\ell_d(\gamma) \le Cd(a, b)$, and hence condition (2) is also satisfied.

By Lemma 2.4, as $\gamma[a, x_0] = \gamma_1$, condition (3) holds as well.

We now prove condition (4). Recall that $\gamma[x_0, b] = \beta_2 * \gamma_2$. Again by Lemma 2.4, for all $x \in |\beta_2|$ we have $\ell_d(\gamma[x, b]) = \delta_\Omega(x)$. Let $x \in |\gamma_2|$. Then $\delta_\Omega(x) \ge \lambda\delta_\Omega(x_0)/(3C_\mathrm{a})$. Since $\ell_d(\gamma[x, b]) \le \ell_d(\gamma) \le C\delta_\Omega(x_0)$, we have $\ell_d(\gamma[x, b]) \le C\delta_\Omega(x)$, and condition (4) is satisfied.

It remains to prove condition (6). Note that inequality (7.3) and inequality (7.1) imply

$$\ell_k(\gamma_2) \le \frac{3C_\mathrm{a}}{\lambda\delta_\Omega(x_0)}\ell_d(\gamma_2) \le C. \tag{7.4}$$

Let $x, x' \in |\gamma|\backslash\{a, b\}$. We consider several cases.
Case (i): $x, x' \in |\gamma_1|$ or $x, x' \in |\beta_2|$. By Proposition 2.5, both γ_1 and β_2 are geodesic rays in (Ω, k). Hence we have $\ell_k(\gamma[x, x']) = k(x, x')$.

Case (ii): If both x, x' are in $|\gamma_2|$, then by inequality (7.4),

$$\ell_k(\gamma[x,x']) \le \ell_k(\gamma_2) \le C \le C + k(x,x'). \tag{7.5}$$

Case (iii): Suppose $x \notin |\gamma_2|$ and $x' \in |\gamma_2|$. If $x \in |\gamma_1|$, then by Case (i) and (7.5) (with $x_0, x' \in |\gamma_2|$),

$$\begin{aligned}\ell_k(\gamma[x,x']) \le \ell_k(\gamma[x,x_0]) + \ell_k(\gamma_2) &\le k(x,x_0) + C\\ &\le k(x,x') + k(x',x_0) + C\\ &\le k(x,x') + C. \end{aligned}\tag{7.6}$$

Similarly, (7.6) holds if $x \in |\beta_2|$.
Case (iv): Finally, if $x \in |\gamma_1|$ and $x' \in |\beta_2|$, then

$$\ell_k(\gamma[x,x']) = \ell_k(\gamma[x,x_0]) + \ell_k(\gamma_2) + \ell_k(\gamma[x_1,x']) \le k(x,x_0) + C + k(x_1,x'). \tag{7.7}$$

Let $\tau' = \dfrac{\lambda}{6C_{\mathrm{a}}}$.
Subcase 1: If $d(x,a) \ge \tau'\delta_\Omega(x_0)$, then the proof of Proposition 2.5 shows that

$$k(x,x_0) = \log\left(\frac{d(a,x_0)}{d(a,x)}\right) \le \log\left(\frac{1}{\tau'}\right) = C',$$

and in this case, by inequality (7.7),

$$\begin{aligned}\ell_k(\gamma[x,x']) &\le k(x,x_0) + C + k(x',x) + k(x,x_0) + k(x_0,x_1)\\ &\le C' + C + k(x,x') + C' + C = k(x,x') + C.\end{aligned}\tag{7.8}$$

Subcase 2: $d(x,a) < \tau'\delta_\Omega(x_0)$. By the choice of x_1, we have $d(a,x_1) \ge \dfrac{\lambda}{C_{\mathrm{a}}}\delta_\Omega(x_0)$. Therefore,

$$d(b,a) \ge d(a,x_1) - d(x_1,b) \ge \frac{\lambda}{C_{\mathrm{a}}}\delta_\Omega(x_0) - \frac{\lambda}{3C_{\mathrm{a}}}\delta_\Omega(x_0) = \frac{2\lambda}{3C_{\mathrm{a}}}\delta_\Omega(x_0).$$

Hence for all $z \in |\beta_2|$,

$$\begin{aligned}d(z,a) \ge d(a,b) - d(z,b) \ge d(a,b) - d(x_1,b) &\ge \frac{2\lambda}{3C_{\mathrm{a}}}\delta_\Omega(x_0) - \frac{\lambda}{3C_{\mathrm{a}}}\delta_\Omega(x_0)\\ &= \frac{\lambda}{3C_{\mathrm{a}}}\delta_\Omega(x_0).\end{aligned}$$

In particular, the above estimate holds for x'. Therefore by $\ell_d(\gamma[x,x']) \le \ell_d(\gamma) \le C\delta_\Omega(x_0)$,

$$\begin{aligned}d(x,x') \ge d(x',a) - d(x,a) \ge \frac{\lambda}{3C_{\mathrm{a}}}\delta_\Omega(x_0) - \tau'\delta_\Omega(x_0) &= \frac{\lambda}{6C_{\mathrm{a}}}\delta_\Omega(x_0)\\ &\ge C^{-1}\ell_d(\gamma[x,x']).\end{aligned}$$

Since γ satisfies conditions (3) and (4) of the definition of a c-anchor, so does $\gamma[x,x']$. Now Lemma 2.2 implies

$$
\begin{aligned}
\ell_k(\gamma[x,x']) &\le C' \log\left(1+\frac{\ell_d(\gamma[x,x'])}{\delta_\Omega(x)\wedge\delta_\Omega(x')}\right) \qquad (7.9)\\
&\le C' \log\left(1+\frac{Cd(x,x')}{\delta_\Omega(x)\wedge\delta_\Omega(x')}\right)\\
&\le C' \log\left(1+\frac{d(x,x')}{\delta_\Omega(x)\wedge\delta_\Omega(x')}\right)+C'\log C\\
&\le C'k(x,x')+C'\log C. \qquad (7.10)
\end{aligned}
$$

This completes the proof. □

The following result is an analog of Theorem 2.4 of [V1], and provides a starlikeness condition for the space (Ω,k).

Theorem 7.4. *Let (X,d) be a C_{a}-annular convex proper geodesic space and $\Omega\subset X$ a rectifiably connected open subset with $\partial\Omega\ne\emptyset$. Suppose that Ω is unbounded or* $\operatorname{diam}(\Omega)\le\operatorname{diam}(\partial\Omega)/\tau$ *for some $0<\tau<1$. If (Ω,k) is δ-hyperbolic, then there is a constant $C=C(\delta,\tau,C_{\mathrm{a}})$ such that for all $a\in\partial^*\Omega$ and all $x_0\in\Omega$, there exists a quasihyperbolic geodesic line α with a as one endpoint and $k(x_0,|\alpha|)\le C$.*

Proof. Let $\lambda=\tau/(3C_{\mathrm{a}}^2)$, and fix $a\in\partial^*\Omega$, $x_0\in\Omega$. We divide the proof into two cases.

Case 1: x_0 is a λ-arc point. Then by Lemma 7.3, there is a c-anchor γ with $x_0\in|\gamma|$, where $c=c(\lambda,C_{\mathrm{a}})=c(\tau,C_{\mathrm{a}})$. Since c-anchors are (c,c)-quasigeodesics in (Ω,k), by Lemma 3.4 there is a quasihyperbolic geodesic line β with endpoints $\xi,\eta\in\partial^*\Omega$ such that $HD_k(|\beta|,|\gamma|)\le M$, where $M=M(\delta,c,c)=M(\delta,\tau,C_{\mathrm{a}})$. Therefore there is a point $x_1\in|\beta|$ such that $k(x_1,x_0)\le M$. If $a=\xi$ or $a=\eta$, we are done. Suppose $a\notin\{\xi,\eta\}$. Let $\beta_1: a\frown\xi$, $\beta_2: a\frown\eta$ be two quasihyperbolic geodesic lines. Since geodesic triangles in $\Omega\cup\partial^*\Omega$ are $24\,\delta$-thin, we have $k(x_1,|\beta_1|\cup|\beta_2|)\le 24\delta$. Thus $k(x_0,|\beta_1|\cup|\beta_2|)\le M+24\,\delta$, and hence $k(x_0,|\beta_i|)\le M+24\,\delta$ for some $i\in\{1,2\}$; we choose $\alpha=\beta_i$ for this particular i.

Case 2: x_0 is a λ-annulus point. Then there is a point $b\in\partial\Omega$ such that $\delta_\Omega(x_0)=d(x_0,b)$ and $B(b,\delta_\Omega(x_0)/\lambda)\setminus B(b,\lambda\delta_\Omega(x_0))\subset\Omega$.

First, we prove that there is a quasihyperbolic geodesic line β intersecting the sphere $S(b,\delta_\Omega(x_0))$. If $\operatorname{diam}(\Omega)=\infty$, pick $x_n,y_n\in\Omega$ with $d(x_0,x_n)\to\infty$ and $d(y_n,b)\to 0$ and fix a quasihyperbolic geodesic $[x_n,y_n]$. Since $[x_n,y_n]$ intersects the compact set $S(b,\delta_\Omega(x_0))$ for all sufficiently large n, a subsequence of $\{[x_n,y_n]\}$ converges to a geodesic line intersecting $S(b,\delta_\Omega(x_0))$. Now assume $\operatorname{diam}(\Omega)<\infty$. Since $\operatorname{diam}(\Omega)\le\operatorname{diam}(\partial\Omega)/\tau$, we have by the choice of λ,

$$\lambda\delta_\Omega(x_0)\le\tau\delta_\Omega(x_0)/3\le\tau\operatorname{diam}(\Omega)/3\le\operatorname{diam}(\partial\Omega)/3;$$

hence there is a point $c \in \partial\Omega$ such that $d(c,b) \geq \lambda\delta_\Omega(x_0)$. The fact that the annulus $B(b,\delta_\Omega(x_0)/\lambda) \setminus B(b,\lambda\delta_\Omega(x_0)) \subset \Omega$ implies $c \notin B(b,\delta(x_0)/\lambda)$. As before, we can again obtain a quasihyperbolic geodesic line with end points b and c, intersecting the sphere $S(b,\delta_\Omega(x_0))$.

Given any $x, y \in S(b,\delta_\Omega(x_0))$, the annular convexity of X implies that there is a path $\gamma : x \frown y$ with $\ell_d(\gamma) \leq C_\mathrm{a} d(x,y)$ and $|\gamma| \subset B(b, C_\mathrm{a}\delta_\Omega(x_0)) \setminus B(b,\delta_\Omega(x_0)/C_\mathrm{a})$. Since $B(b,\delta_\Omega(x_0)/\lambda)\setminus B(b,\lambda\delta_\Omega(x_0)) \subset \Omega$ and $\lambda = \tau/(3C_\mathrm{a}^2) \leq 1/(2C_\mathrm{a})$, we have $\delta_\Omega(z) \geq \delta_\Omega(x_0)/(2C_\mathrm{a})$ for all $z \in \gamma$. It follows that $k(x,y) \leq \ell_k(\gamma) \leq 4C_\mathrm{a}^2$. Hence $\mathrm{diam}_k(S(b,\delta_\Omega(x_0))) \leq 4C_\mathrm{a}^2$, and so $k(x_0,x_1) \leq 4C_\mathrm{a}^2$ for $x_1 \in |\beta| \cap S(b,\delta_\Omega(x_0))$. Now one repeats the argument at the end of Case 1 and concludes the proof.

□

The following is a consequence of the proof of Theorem 7.4.

Corollary 7.5. *Suppose that (Ω,k) is δ-hyperbolic,* $\mathrm{diam}(\partial\Omega) > 0$*, and* $0 < \tau < 1$*. For any $a \in \partial^*\Omega$ and any $x_0 \in \Omega$,*

(i) *if x_0 is a $\tau/(3C_\mathrm{a}^2)$-arc point, or*
(ii) *if x_0 is a $\tau/(3C_\mathrm{a}^2)$-annulus point with $\delta_\Omega(x_0) \leq \mathrm{diam}(\partial\Omega)/\tau$,*

then there is a quasihyperbolic geodesic line γ with one endpoint a and satisfying $k(x_0,|\gamma|) \leq C = C(\delta,\tau,C_\mathrm{a})$.

The next result follows from the fact that triangles in $\Omega\cup\partial^*\Omega$ are 24δ-thin.

Lemma 7.6 (Lemma 6.35 of [V3])**.** *Let (Ω,k) be δ-hyperbolic, $a \in \partial^*\Omega$ and C_0 be a constant. Suppose that for each $x \in \Omega$, there is a quasihyperbolic geodesic line γ with one endpoint a and $k(x,|\gamma|) \leq C_0$. Then for any $x_1, x_2 \in \Omega$ there is a quasihyperbolic geodesic line α such that $k(x_i,|\alpha|) \leq C = C(C_0,\delta)$ for $i = 1, 2$.*

8. A "carrot" lemma for quasihyperbolic geodesics

In this section we show that, under the assumptions of Theorem 9.1, quasihyperbolic geodesic lines in Ω have properties very similar to the defining conditions for uniform paths. The proof of Theorem 9.1 will essentially be reduced to this situation.

Throughout this section, (X,d) is a C_a-annular convex proper geodesic space, and $\Omega \subset X$ is an unbounded rectifiably connected open subset with $\partial\Omega \neq \emptyset$. We suppose that (Ω,k) is δ-hyperbolic, that there is a natural map $\phi : (\partial^*\Omega, k_{w,\epsilon_0}) \to (\partial'\Omega, d)$ (for some $w \in \Omega$ and $\epsilon_0 = \epsilon_0(\delta) = \min\{1, \frac{1}{5\delta}\}$) and that ϕ is η-quasimöbius for some η. Recall that $\partial'\Omega = \partial\Omega \cup \{\infty\}$ and that the cross ratio in $(\partial'\Omega, d)$ is defined in the second paragraph of Section 6. By Corollary 5.4, we may assume that for each $x \in \Omega$, $\phi : (\partial^*\Omega, k_{x,\epsilon_0}) \to (\partial'\Omega, d)$ is η-quasimöbius. With a (slight) abuse of notation, for $\xi \in \partial^*\Omega$ we denote $\phi(\xi)$ also as ξ, and for $\xi \in \partial'\Omega$, we denote $\phi^{-1}(\xi)$ also by ξ.

A quasihyperbolic geodesic line from a point $b \in \partial\Omega$ to ∞ tries to avoid the boundary of Ω. In other words, a carrot-shaped region with the tip of the carrot at the base point b, must lie in the domain. This is the content of the next lemma, which generalizes [V1, Lemma 3.36]. The proof of this fact plays out differently for λ-arc points than for λ-annulus points.

Lemma 8.1. *If α is a quasihyperbolic geodesic line with endpoints $b, \infty \in \partial'\Omega$, then for all $x \in |\alpha|$ we have*

$$d(x,b) \le c_0\, \delta_\Omega(x),$$

where $c_0 = c_0(\delta, C_a, \eta)$.

Proof. Let $x \in |\alpha|$ and $\lambda = e^{-4C_{\mathrm{a}}^3}/(32C_{\mathrm{a}}^2)$. As before, we break the proof up into two cases.

Case 1: x is a λ-annulus point. Then there is a point $a \in \partial\Omega$ such that $\delta_\Omega(x) = d(x,a)$ and $B(a, \delta_\Omega(x)/\lambda) \setminus B(a, \lambda\delta_\Omega(x)) \subset \Omega$. Since $\alpha : b \frown \infty$, we can find a sequence $\{v_n\}$ from $|\alpha|$ with $v_n = \alpha(t_n)$, $t_n \to \infty$, such that $v_n \notin B(a, \delta_\Omega(x)/\lambda)$ for all n. Suppose $b \notin B(a, \lambda\delta_\Omega(x))$. Then $b \notin B(a, \delta_\Omega(x)/\lambda)$. As $x \in |\alpha[b, v_n]|$, Lemma 2.3 (i) applied to the quasihyperbolic geodesic $\alpha[b, v_n]$ with $t = 2e^{4C_{\mathrm{a}}^3}\delta_\Omega(x)$ shows that $d(x,a) \ge 2\delta_\Omega(x)$, contradicting $\delta_\Omega(x) = d(x,a)$. Hence $b \in B(a, \lambda\delta_\Omega(x))$. Therefore,

$$d(x,b) \le d(x,a) + d(a,b) \le \delta_\Omega(x) + \lambda\delta_\Omega(x) \le 2\delta_\Omega(x).$$

Case 2: x is a λ-arc point. Then by Lemma 7.3 there is a c-anchor $\tau : a_1 \frown a_2$ with $a_1, a_2 \in \partial\Omega$ and $x \in |\tau|$, where $c = c(\lambda, C_{\mathrm{a}}) = c(C_{\mathrm{a}})$. Let $\beta : a_1 \frown a_2$ be a quasihyperbolic geodesic line. Since τ is a (c,c)-quasigeodesic in (Ω, k) and $x \in |\alpha| \cap |\tau|$, we have

$$k(|\alpha|, |\beta|) \le k(x, |\beta|) \le HD_k(|\tau|, |\beta|) \le C(\delta, c, c) = C(\delta, C_{\mathrm{a}}).$$

Set $Q = (a_1, \infty, b, a_2)$. Then $\mathrm{sd}(Q) \le k(|\alpha|, |\beta|) \le C$. Since $\varepsilon \le \varepsilon_0(\delta) \le 1$, by Corollary 5.2 we have $\mathrm{cr}(Q, k_{x,\varepsilon}) \le c_0 e^{\varepsilon\, \mathrm{sd}(Q)} \le c_0 e^{\varepsilon C} \le c_0 e^{C} = C$, where $c_0 = 4e^{86}$.

Therefore by the quasimöbius property of the natural map ϕ,

$$\frac{d(a_1, b)}{d(a_1, a_2)} = \mathrm{cr}(Q, d) \le \eta(\mathrm{cr}(Q, k_{x,\varepsilon})) \le \eta(C) = C(\delta, C_{\mathrm{a}}, \eta);$$

that is, $d(a_1, b) \le C d(a_1, a_2)$. Since τ is a c-anchor of x with endpoints a_1 and a_2, by properties (3) and (4) of Definition 7.2 (with $x_0 = x$ here),

$$\ell_d(\tau_{a_1 x}) \le C\delta_\Omega(x) \quad \text{and} \quad \ell_d(\tau_{x a_2}) \le C\delta_\Omega(x),$$

and therefore $d(a_1, a_2) \le \ell_d(\tau) \le 2C\delta_\Omega(x)$; hence $d(a_1, b) \le C\delta_\Omega(x)$. Finally,

$$d(x,b) \le d(x, a_1) + d(a_1, b) \le \ell_d(\tau_{x a_1}) + C\delta_\Omega(x) \le C\delta_\Omega(x),$$

where we used property (3) of Definition 7.2 again. $\square$

Lemma 8.2. *Let $x_0 \in \Omega$ and $\tau : a_1 \frown a_2$ be a c-anchor for x_0 (for some $a_1, a_2 \in \partial\Omega$). Let $\alpha : a_1 \frown a_2$ and $\alpha_i : a_i \frown \infty$ ($i = 1, 2$) be quasihyperbolic geodesic lines. Let $x \in |\alpha|$ be such that $k(x, |\alpha_i|) \leq 24\delta$ for $i = 1, 2$. Then $k(x, x_0) \leq c' = c'(\delta, c, C_a, \eta)$.*

Proof. Since τ is a (c, c)-quasigeodesic in (Ω, k) and τ and α have the same endpoints, we have $HD_k(|\tau|, |\alpha|) \leq c_1 = c_1(\delta, c)$. Fix $y \in |\tau|$ with $k(x, y) \leq c_1$. We claim that there is a constant $c_2 = c_2(\delta, c, C_{\mathrm{a}}, \eta)$ such that $\delta_\Omega(y) \geq \delta_\Omega(x_0)/c_2$.

Assuming the claim, we proceed as follows. Since $\tau[x_0, y]$ satisfies the assumptions of Lemma 2.2 (this is because τ is an anchor), Lemma 2.2 together with definition 7.2 (3) or (4) applied to the point x_0, implies $k(x_0, y) \leq c_3 = c_3(\delta, c, C_{\mathrm{a}}, \eta)$, and hence $k(x, x_0) \leq k(x, y) + k(y, x_0) \leq c_1 + c_3$.

We next prove the claim. Let $c_2 = 2c[c - 1 + (c_0 + 1)e^{c_1+24\delta}]$, where $c_0 = c_0(\delta, C_{\mathrm{a}}, \eta)$ is the constant from Lemma 8.1. Suppose $\delta_\Omega(y) < \delta_\Omega(x_0)/c_2$. We may assume $y \in |\tau[a_2, x_0]|$. Then Definition 7.2 (4) implies $d(a_2, y) \leq \ell_d(\tau[a_2, y]) \leq c\delta_\Omega(y) \leq c\delta_\Omega(x_0)/c_2$. Let $y_1 \in |\alpha_1|$ with $k(y_1, x) \leq 24\delta$. Then $k(y_1, y) \leq c_1 + 24\delta$. By Lemma 2.1 we have $d(y_1, y) \leq (e^{c_1+24\delta} - 1)\delta_\Omega(y) \leq c_2^{-1}(e^{c_1+24\delta} - 1)\delta_\Omega(x_0)$ and so

$$\delta_\Omega(y_1) \leq \delta_\Omega(y) + d(y, y_1) \leq c_2^{-1} e^{c_1+24\delta} \delta_\Omega(x_0).$$

On the other hand, Lemma 8.1 applied to α_1 and y_1 implies that

$$d(a_1, y_1) \leq c_0\, \delta_\Omega(y_1) \leq \frac{c_0\, e^{c_1+24\delta}}{c_2}\, \delta_\Omega(x_0).$$

Now the triangle inequality gives

$$d(a_1, a_2) \leq d(a_1, y_1) + d(y_1, y) + d(y, a_2) \leq \frac{\delta_\Omega(x_0)}{2c} < \frac{\delta_\Omega(x_0)}{c}.$$

This is impossible since by Definition 7.2 (2),

$$d(a_1, a_2) \geq \ell_d(\tau)/c \geq d(x_0, a_1)/c \geq \delta_\Omega(x_0)/c.$$

□

The following is our analog of the length carrot Lemma 3.40 of [V1]. It improves the lower bound in the estimate of Lemma 8.1. Recall that by assumption the natural map $\phi : (\partial^*\Omega, k_{w,\epsilon_0}) \to (\partial'\Omega, d)$ exists and is η-quasimöbius.

Lemma 8.3. *If α is a quasihyperbolic geodesic line with endpoints $b, \infty \in \partial'\Omega$, then there is a constant $C = C(\delta, C_a, \eta)$ such that for all $x \in |\alpha|$,*

$$\ell_d(\alpha[b, x]) \leq C\, \delta_\Omega(x).$$

Proof. Let $\alpha : \mathbb{R} \to \Omega$ be the k-arclength parametrization of α such that $\lim_{t\to-\infty} \alpha(t) = b$ and $\lim_{t\to\infty} \alpha(t) = \infty$. For each $n \in \mathbb{Z}$ let $t_n = \sup\{t \in \mathbb{R} : \delta_\Omega(\alpha(t)) \leq 2^n\}$. Since $\lim_{t\to\infty} \alpha(t) = \infty$, Lemma 8.1 implies that for each n we have $t_n < \infty$, $\delta_\Omega(\alpha(t_n)) = 2^n$, and $t_n < t_{n+1}$.

Fix $x \in |\alpha|$. Then there exists $n \in \mathbb{Z}$ for which $x \in |\alpha|_{(t_n,t_{n+1}]}|$. We have

$$\ell_d(\alpha[b,x]) \le \ell_d(\alpha|_{(-\infty,t_{n+1}]}) = \sum_{-\infty}^{n} \ell_d(\alpha|_{(t_j,t_{j+1}]}).$$

By Lemma 2.1,

$$\ell_d(\alpha|_{(t_j,t_{j+1}]}) \le \delta_\Omega(\alpha(t_j)) \left[e^{\ell_k(\alpha|_{(t_j,t_{j+1}]})} - 1\right].$$

Hence,

$$\ell_d(\alpha[b,x]) \le \sum_{-\infty}^{n} 2^j \left[e^{\ell_k(\alpha|_{(t_j,t_{j+1}]})} - 1\right].$$

It suffices to show that there is a constant $K = K(\delta, \eta, C_{\mathrm{a}})$ such that for all $j \in \mathbb{Z}$,

$$\ell_k(\alpha|_{(t_j,t_{j+1}]}) \le K, \tag{8.1}$$

for then,

$$\ell_d(\alpha[b,x]) \le (e^K - 1) \sum_{-\infty}^{n} 2^j \le e^K\, 2^n \sum_{-\infty}^{0} 2^j = 2e^K\, 2^n.$$

On the other hand, as $x \in |\alpha|_{(t_n,t_{n+1}]}|$, we have $\delta_\Omega(x) \ge 2^n$. Thus we can infer that

$$\ell_d(\alpha[b,x]) \le 2e^K\, \delta_\Omega(x),$$

concluding the proof of the lemma.

Thus it remains to prove inequality (8.1).

Let $\lambda = (40C^3 e^{4C^3})^{-1}$, where $C = \max\{C_{\mathrm{a}}, 2c_0\}$ with c_0 the constant from the conclusion of Lemma 8.1.

Case 1: Both $x_1 := \alpha(t_j)$ and $x_2 := \alpha(t_{j+1})$ are λ-arc points. By the choice of t_{j+1}, we know that $\delta_\Omega(x_2) = 2^{j+1}$ and so $\delta_\Omega(x_2) = 2 \cdot 2^j \le 2\delta_\Omega(x)$ for every $x \in |\alpha|_{[t_j,t_{j+1}]}|$. By Lemma 7.3 there are c-anchors $\tau_i : a_i \frown c_i$ for $i = 1, 2$, with $x_i \in |\tau_i|$ and $c = c(\lambda, C_{\mathrm{a}}) = c(\delta, \eta, C_{\mathrm{a}})$. Without loss of generality, we may assume that $d(a_i, b) \ge d(c_i, b)$ for $i = 1, 2$.

Fix $y_0 = \alpha(t_0)$ with t_0 sufficiently large. For $i = 1, 2$, let α_i be a quasihyperbolic geodesic ray connecting y_0 to a_i and β_i a quasihyperbolic geodesic ray from y_0 to c_i. Set $X = |\alpha[y_0,\infty]| \cup |\alpha[y_0,b]| \cup |\alpha_1| \cup |\alpha_2| \cup |\beta_1| \cup |\beta_2|$. There is a tree $T(X)$ and a map $u : X \to T(X)$ with the properties stated in Theorem 3.5. We denote the metric on $T(X)$ by d_T. Let $a_i' \in \partial T(X)$ be such that u is an isometry from $|\alpha_i|$ onto the geodesic $[u(y_0), a_i']$ in $T(X)$. We similarly define $c_i', \infty', b' \in \partial T(X)$.

Let $y_i' \in T(X)$ be the branch point of $[u(y_0), a_i']$ and $[u(y_0), c_i']$, that is, $[u(y_0), a_i'] \cap [u(y_0), c_i'] = [u(y_0), y_i']$. Choose $y_{ia} \in |\alpha_i|$ and $y_{ic} \in |\beta_i|$ with

$u(y_{ia}) = u(y_{ic}) = y_i'$. By Theorem 3.5 (ii) $k(y_{ia}, y_{ic}) \le c(\delta)$. Fix a quasihyperbolic geodesic γ_i joining a_i and c_i. Then the argument in the proof of Lemma 5.1, inequality (5.4) shows that

$$HD_k(|\gamma_i|, |\alpha_i[y_{ia}, a_i]| \cup |\beta_i[y_{ic}, c_i]|) \le c(\delta).$$

Pick $y_i \in |\gamma_i|$ with $k(y_i, y_{ia}) \le c(\delta)$. Then $k(y_i, y_{ic}) \le 2c(\delta)$. The proof of Lemma 8.2 shows that $k(y_i, x_i) \le c_3 = c_3(\delta, C_{\mathrm{a}}, \eta)$. Hence $k(x_i, y_{ia}) \le c_3 + c(\delta)$. Set $C_1 = c_3 + c(\delta)$. We have $d_T(u(x_i), y_i') \le k(x_i, y_{ia}) \le C_1$.

FIGURE 2. three configurations of $Y_i' \cup [b', \infty']$

Consider the following subtrees of $T(X)$: $Y_i' = [\infty', c_i'] \cup [\infty', a_i']$, $Z_i' = [y_i', c_i'] \cup [y_i', a_i']$. Notice that Y_i' is a tripod and that Z_i' is a geodesic line. Let $z_i' \in Y_i'$ be the point where $[\infty', b']$ branches off from Y_i': $[\infty', b'] \cap Y_i' = [z_i', \infty']$. Since $u(x_i) \in u(|\alpha|) = [\infty', b']$ and $y_i' \in Z_i'$, the inequality $d_T(u(x_i), y_i') \le C_1$ implies that $d_T([\infty', b'], Z_i') \le C_1$. It follows that the branch point z_i' has to be close to Z_i', specifically, $d_T(z_i', Z_i') \le C_1$.

Let $Q_i = (a_i, \infty, c_i, b)$ for $i = 1, 2$; then

$$\mathrm{cr}(Q_i, d) = \frac{d(a_i, c_i)}{d(a_i, b)} \le \frac{d(a_i, b) + d(c_i, b)}{d(a_i, b)} \le \frac{2d(a_i, b)}{d(a_i, b)} = 2.$$

As $\phi : (\partial^*\Omega, k_{w,\varepsilon_0}) \to (\partial'\Omega, d)$ is η-quasimöbius, $\phi^{-1} : (\partial'\Omega, d) \to (\partial^*\Omega, k_{w,\varepsilon_0})$ is η'-quasimöbius with $\eta'(t) = 1/\eta^{-1}(1/t)$. Therefore,

$$\mathrm{cr}(Q_i, k_{w,\varepsilon_0}) \le \eta'(\mathrm{cr}(Q_i, d)) \le \eta'(2) = C = C(\eta).$$

By Corollary 5.3 we have $k([a_i, b], [c_i, \infty]) \le c_4 = c_4(\eta, \varepsilon_0, \delta) = c_4(\eta, \delta)$ for any geodesic $[a_i, b]$ joining a_i and b and any geodesic $[c_i, \infty]$ joining c_i and ∞. Now the property of u implies $d_T([a_i', b'], [c_i', \infty']) \le c_4 + c(\delta)$. Let $w_i' \in [\infty', b']$ be the branch point of $[\infty', b']$ and $[\infty', a_i']$: $[\infty', b'] \cap [\infty', a_i'] = [w_i', \infty']$. If $w_i' \in [y_i', \infty'] \setminus \{y_i'\}$, then $w_i' = z_i'$ and hence

$$d_T(y_i', w_i') = d_T(Z_i', z_i') \le C_1;$$

on the other hand, if $w_i' \in [y_i', a_i']$, then the inequality $d_T([a_i', b'], [c_i', \infty']) \le c_4 + c(\delta)$ implies that the branch point w_i' has to be close to y_i', that is,

$$d_T(w_i', y_i') = d_T([a_i', b'], [c_i', \infty']) \le c_4 + c(\delta).$$

In either case, we have $d_T(y_i', w_i') \le C_2 := \max\{C_1, c_4 + c(\delta)\}$. It follows that

$$d_T(u(x_i), w_i') \le C_1 + C_2. \tag{8.2}$$

As τ_i is a c-anchor of x_i, by Definition 7.2 (3),

$$d(a_i, x_i) \le \ell_d(\tau_i[a_i, x_i]) \le c\,\delta_\Omega(x_i),$$

and by Definition 7.2 (2),

$$\delta_\Omega(x_i) \le d(x_i, a_i) \le \ell_d(\tau_i) \le c\, d(a_i, c_i) \le 2c\, d(a_i, b).$$

By Lemma 8.1, $d(x_i, b) \le c_0\, \delta_\Omega(x_i)$. From the above group of inequalities,

$$\delta_\Omega(x_2) = 2\delta_\Omega(x_1) \le 4c\, d(a_1, b),$$

and hence

$$d(a_2, b) \le d(a_2, x_2) + d(x_2, b) \le c\delta_\Omega(x_2) + c_0\, \delta_\Omega(x_2) \le 4c(c + c_0)\, d(a_1, b).$$

Thus, considering the quadruple $Q_3 = (b, \infty, a_2, a_1)$, we obtain

$$\operatorname{cr}(Q_3, d) = \frac{d(a_2, b)}{d(a_1, b)} \le 4c(c + C) = C = C(\delta, C_{\mathrm{a}}, \eta).$$

It follows that $\operatorname{cr}(Q_3, k_{w,\varepsilon_0}) \le \eta'(C)$. By Corollary 5.3 we have

$$k([b, a_1], [a_2, \infty]) \le c_5 = c_5(\delta, C_{\mathrm{a}}, \eta, \varepsilon_0) = c_5(\delta, C_{\mathrm{a}}, \eta)$$

for any geodesic $[b, a_1]$ connecting b to a_1 and any geodesic $[a_2, \infty]$ connecting a_2 to ∞. Now the property of u implies $d_T([b', a_1'], [a_2', \infty']) \le C_3 := c_5 + c(\delta)$.

If $k(x_1, x_2) \le 3C_1 + 3C_2 + C_3$, then we are done. If $k(x_1, x_2) \ge 3C_1 + 3C_2 + C_3$, then $d_T(u(x_1), u(x_2)) = k(x_1, x_2) \ge 3C_1 + 3C_2 + C_3$. Then, since we have $w_i' \in [b', \infty']$, $d_T(w_i', u(x_i)) \le C_1 + C_2$ (by inequality (8.2)), and $u(x_1) \in [u(x_2), b']$, we have $w_1' \in [w_2', b']$ and $d_T(w_1', w_2') \ge C_1 + C_2 + C_3$. It follows that

$$d_T([b', a_1'], [a_2', \infty']) = d_T(w_1', w_2') \ge C_1 + C_2 + C_3 > C_3,$$

contradicting the inequality $d_T([b', a_1'], [a_2', \infty']) \le C_3$ from the preceding paragraph.

Case 2: At least one of $x_1 = \alpha(t_j)$, $x_2 = \alpha(t_{j+1})$ is a λ-annulus point. Then for some $i \in \{1, 2\}$, there exists $a \in \partial\Omega$ such that $\delta_\Omega(x_i) = 2^{j-1+i} = d(a, x_i)$ and $B(a, \delta_\Omega(x_i)/\lambda) \setminus B(a, \lambda\delta_\Omega(x_i)) \subset \Omega$. Since $\lambda = e^{-4C^3}/(40C^3)$ with C at least as large as the constant in the conclusion of Lemma 8.1, we see that

$$d(a, b) \le d(b, x_i) + d(x_i, a) \le (c_0 + 1)\delta_\Omega(x_i).$$

We break the rest of the proof up into two subcases.

Subcase 2(a): We consider the case when x_2 is a λ-annulus point. As $\delta_\Omega(x_2) = 2\, \delta_\Omega(x_1)$,

$$d(x_1, a) \ge \delta_\Omega(x_1) = \frac{1}{2}\delta_\Omega(x_2) > 2C_{\mathrm{a}}\lambda\, \delta_\Omega(x_2).$$

On the other hand, by Lemma 8.1 $\delta_\Omega(x_1) = 2^j \ge d(x_1, b)/c_0$, and we have

$$\begin{aligned} d(x_1, a) \le d(x_1, b) + d(b, a) \le c_0\, \delta_\Omega(x_2) + (c_0 + 1)\delta_\Omega(x_2) &= (2c_0 + 1)\, \delta_\Omega(x_2) \\ &< \frac{1}{2C_{\mathrm{a}}\lambda}\, \delta_\Omega(x_2). \end{aligned}$$

It follows that $x_1 \in B(a, \delta_\Omega(x_2)/(2C_\mathrm{a}\lambda)) \setminus B(a, 2C_\mathrm{a}\lambda\, \delta_\Omega(x_2))$. Hence by the annular convexity of X, there is a path β joining x_1 and x_2 in $B(a, \delta_\Omega(x_2)/(2\lambda)) \setminus B(a, 2\lambda\delta_\Omega(x_2)) \subset \Omega$ with

$$\begin{aligned}\ell_d(\beta) \le C_\mathrm{a}\, d(x_1, x_2) \le C_\mathrm{a}[d(x_1, b) + d(x_2, b)] &\le C_\mathrm{a}[c_0\, \delta_\Omega(x_1) + c_0\, \delta_\Omega(x_2)] \\ &\le 3C^2\, \delta_\Omega(x_1).\end{aligned}$$

For all $w \in |\beta|$, we have

$$\delta_\Omega(w) \ge \min\left\{ \left(\frac{1}{\lambda} - \frac{1}{2\lambda}\right) \delta_\Omega(x_2), (2\lambda - \lambda)\, \delta_\Omega(x_2) \right\} = \lambda\, \delta_\Omega(x_2).$$

Thus,

$$k(x_1, x_2) \le \ell_k(\beta) = \int_\beta \frac{1}{\delta_\Omega(z)}\, ds(z) \le \frac{1}{\lambda\, \delta_\Omega(x_2)}\, \ell_d(\beta) \le \frac{3C^2\delta_\Omega(x_1)}{\lambda\delta_\Omega(x_2)} = \frac{3C^2}{2\lambda},$$

proving inequality (8.1) in this subcase.

Subcase 2(b): x_1 is a λ-annulus point. The proof of this subcase is similar to the proof of Subcase 2(a), and is left to the reader. □

Lemma 8.4. *Let $a_1, a_2 \in \partial\Omega$ and $\alpha : a_1 \frown a_2$ be a quasihyperbolic geodesic line. Then for all $z \in |\alpha|$,*

$$\delta_\Omega(z) \le K\, d(a_1, a_2)$$

where $K = K(\delta, C_a, \eta)$ is independent of a_1, a_2, α.

Proof. Let $z \in |\alpha|$ and $\lambda = e^{-4C_\mathrm{a}^3}/(65C_\mathrm{a}^2)$. Two possibilities arise.

Case 1: z is a λ-annulus point. Then there exists $a \in \partial\Omega$ such that $\delta_\Omega(z) = d(z, a)$ and $B(a, \delta_\Omega(z)/\lambda) \setminus B(a, \lambda\delta_\Omega(z)) \subset \Omega$. Lemma 2.3 implies that exactly one of a_1, a_2 lies in $B(a, \lambda\delta_\Omega(z))$ with the other one in $X \setminus B(a, \delta_\Omega(z)/\lambda)$. Hence

$$d(a_1, a_2) \ge |d(a_2, a) - d(a_1, a)| \ge \left(\frac{1}{\lambda} - \lambda\right) \delta_\Omega(z).$$

Case 2: z is a λ-arc point. Then by Lemma 7.3 there is a c-anchor $\tau : b_1 \frown b_2$ for z with $c = c(\lambda, C_\mathrm{a}) = c(C_\mathrm{a})$. Let $\beta : b_1 \frown b_2$ and $\alpha_i : b_i \frown \infty$ $(i = 1, 2)$ be quasihyperbolic geodesic lines. By Lemma 3.1 there is some $x \in |\beta|$ such that $k(x, |\alpha_i|) \le 24\delta$ for $i = 1, 2$. By Lemma 8.2, $k(x, z) \le c' = c'(\delta, c, C_\mathrm{a}, \eta) = c'(\delta, C_\mathrm{a}, \eta)$. It follows that

$$k(|\alpha|, |\alpha_i|) \le k(z, |\alpha_i|) \le k(z, x) + k(x, |\alpha_i|) \le c' + 24\delta.$$

For $i = 1, 2$, set $P_i = (a_1, \infty, b_i, a_2)$. Corollary 5.3 implies that

$$\mathrm{cr}(P_i, k_{w,\varepsilon}) \le C = C(c' + 24\delta) = C(\delta, C_\mathrm{a}, \eta).$$

Since the natural map is η-quasimöbius, we have

$$\mathrm{cr}(P_i, d) = \frac{d(a_1, b_i)}{d(a_1, a_2)} \le \eta(C).$$

By the definition of a c-anchor, we have

$$\begin{aligned}\delta_\Omega(z) \le d(b_1, z) \le \ell_d(\tau) \le c\, d(b_1, b_2) &\le c\,[d(b_1, a_1) + d(a_1, b_2)]\\ &\le 2c\,\eta(C)\, d(a_1, a_2),\end{aligned}$$

which is the desired estimate. □

The following is an analog of Lemma 8.3 for quasihyperbolic geodesic lines that do not have ∞ as one of the endpoints. It says that there is a "banana"-shaped region with respect to the metric d around such a line in Ω. The proof in [V1] holds in our case, and we skip the details.

Lemma 8.5 (Lemma 3.54 of [V1])**.** *Suppose $a_1, a_2 \in \partial\Omega$ and $\alpha : a_1 \frown a_2$ is a quasihyperbolic geodesic line.*

(i) *There exists $\xi_\alpha \in |\alpha|$ such that if $x_1, x_2 \in |\alpha[a_1, \xi_\alpha]|$ with $k(x_2, \xi_\alpha) \le k(x_1, \xi_\alpha)$ or if $x_1, x_2 \in |\alpha[\xi_\alpha, a_2]|$ with $k(x_2, \xi_\alpha) \le k(x_1, \xi_\alpha)$, then $\ell_d(\alpha[x_1, x_2]) \le C\,\delta_\Omega(x_2)$ for some $C = C(\delta, \eta, C_a)$.*

(ii) *If $y_1, y_2 \in |\alpha|$ are such that $\max\{\delta_\Omega(y_1), \delta_\Omega(y_2)\} \le 2\,d(y_1, y_2)$, then $\ell_d(\alpha[y_1, y_2]) \le C\,d(y_1, y_2)$, where $C = C(\delta, \eta, C_a)$.*

9. Sufficiency

In this section we prove the main result of the paper. This result (Theorem 9.1), together with Theorem 6.2, provides a characterization of uniform domains among Gromov hyperbolic domains in annular convex metric spaces in terms of the quasiconformal structure on the Gromov boundary.

The reader is advised to review the three paragraphs before Theorem 6.1 for the notation $\partial'\Omega$ and the notion of natural maps.

Theorem 9.1. *Let (X, d) be a c-quasiconvex and c-annular convex proper metric space, and $\Omega \subset X$ a rectifiably connected open subset with $\partial\Omega \ne \emptyset$. Suppose that (Ω, k) is δ-hyperbolic and that the natural map $\phi : (\partial^*\Omega, k_{w,\varepsilon_0}) \to (\partial'\Omega, d)$ exists (for some $w \in \Omega$ and $\varepsilon_0 = \varepsilon_0(\delta) = \min\{1, \frac{1}{5\delta}\}$) and is η-quasimöbius. Then (Ω, d) is c_1-uniform with $c_1 = c_1(c, \delta, \eta)$.*

The following lemma reduces Theorem 9.1 to the case of geodesic metric spaces.

Lemma 9.2. *Let (X, d) be a proper c-quasiconvex metric space, and $\Omega \subset X$ a rectifiably connected open subset with $\partial\Omega \ne \emptyset$. Let d' be the length metric on X associated with d, and k' the quasihyperbolic metric on $\Omega \subset (X, d')$.*

(i) *For all $x, y \in X$, we have $d(x, y) \le d'(x, y) \le c\,d(x, y)$; in particular, (X, d') is a proper geodesic space;*

(ii) *If (X, d) is C_a-annular convex, then (X, d') is c'-annular convex with $c' = c'(c, C_a)$;*

(iii) *For all $x, y \in \Omega$, we have $k(x, y)/c \le k'(x, y) \le c\,k(x, y)$;*

(iv) *If (Ω, d') is c'-uniform, then (Ω, d) is c''-uniform with $c'' = c\,c'$;*

(v) *If* (Ω, k) *is* δ*-hyperbolic, then* (Ω, k') *is* δ'*-hyperbolic with* $\delta' = \delta'(\delta, c)$*;*

(vi) *Suppose* (Ω, k) *is* δ*-hyperbolic and there exists a natural map* $\phi : (\partial^*\Omega, k_{x,\varepsilon}) \to (\partial'\Omega, d)$ *for some* $x \in \Omega$ *and* $0 < \varepsilon \le \varepsilon_0(\delta)$ *and* ϕ *is* η*-quasimöbius. Then for* $0 < \varepsilon' \le \varepsilon_0(\delta')$ *there is a natural map* $\phi' : (\partial^*_{k'}\Omega, k'_{x,\varepsilon'}) \to (\partial'\Omega, d')$ *such that* ϕ' *is* η'*-quasimöbius with* $\eta' = \eta'(\eta, \delta, c, \varepsilon'/\varepsilon)$*. Here* δ' *is the constant from (v).*

Proof. (i): For any $x, y \in X$, there is a path $\gamma : x \frown y$ with $\ell_d(\gamma) \le c\, d(x, y)$. Hence $d'(x, y) \le c\, d(x, y)$. The inequality $d(x, y) \le d'(x, y)$ is clear. Since (X, d) is proper, it now follows that (X, d') is also proper. Being a proper length space, (X, d') has to be geodesic.

(ii): This follows easily from (i) and the annular convexity of (X, d).

(iii): For any $x \in \Omega$, let $\delta'_\Omega(x) = d'(x, \partial\Omega)$. It can be verified that $\ell_d(\gamma) \le \ell_{d'}(\gamma) \le c\, \ell_d(\gamma)$ for any path $\gamma \subset X$, and that $\delta_\Omega(x) \le \delta'_\Omega(x) \le c\, \delta_\Omega(x)$ for all $x \in \Omega$. Let $x, y \in \Omega$, γ a geodesic in (Ω, k) connecting x to y, and γ' a geodesic in (Ω, k') joining x and y. Then

$$k'(x, y) \le \int_\gamma \frac{1}{\delta'_\Omega(z)} d's(z) \le \int_\gamma \frac{1}{\delta_\Omega(z)} c\, ds(z) = c\, k(x, y),$$

and

$$k(x, y) \le \int_{\gamma'} \frac{1}{\delta_\Omega(z)}\, ds(z) \le \int_{\gamma'} \frac{c}{\delta'_\Omega(z)} d's(z) = c\, k'(x, y).$$

(iv): Let $x, y \in \Omega$ and $\gamma : x \frown y$ a c'-uniform path in (Ω, d'). Then

$$\ell_d(\gamma) \le \ell_{d'}(\gamma) \le c'\, d'(x, y) \le c' c\, d(x, y),$$

and for any $z \in \gamma$,

$$\begin{aligned} \delta_\Omega(z) \ge \frac{1}{c}\delta'_\Omega(z) &\ge \frac{1}{c}\,\frac{1}{c'}\ \min\{\ell_{d'}(\gamma[x, z]),\, \ell_{d'}(\gamma[z, y])\} \\ &\ge \frac{1}{c'c}\ \min\{\ell_d(\gamma[x, z]),\, \ell_d(\gamma[z, y])\}. \end{aligned}$$

Hence γ is $(c'c)$-uniform in (Ω, d).

(v): By (iii) the identity map $(\Omega, k) \to (\Omega, k')$ is c-bilipschitz. Recall that (Ω, k) and (Ω, k') are geodesic metric spaces, see [BHK, Proposition 2.8]. So by [V3, Theorem 3.18] if (Ω, k) is δ-hyperbolic, then (Ω, k') is δ'-hyperbolic with $\delta' = \delta'(\delta, c)$.

(vi): We claim that the identity map $(\Omega, k) \to (\Omega, k')$ induces an η_1-quasimöbius map $f : (\partial^*\Omega, k_{x,\varepsilon}) \to (\partial^*_{k'}\Omega, k'_{x,\varepsilon'})$ with $\eta_1 = \eta_1(\delta, c, \varepsilon'/\varepsilon)$. With this claim, the identity map $g : (\partial'\Omega, d) \to (\partial'\Omega, d')$ is used to construct the desired natural map $\phi' := g \circ \phi \circ f^{-1}$. Now we prove the claim. First assume $\varepsilon' \ge \varepsilon$. Let $i : (\Omega, k) \to (\Omega, k')$ be the identity map. Since by (iii) i is c-bilipschitz, Proposition 5.10 implies that the boundary map $\partial i : (\partial^*\Omega, k_{x,\varepsilon}) \to (\partial^*_{k'}\Omega, k'_{x,\varepsilon})$ is η_2-quasimöbius with $\eta_2 = \eta_2(\delta, c)$. By Lemma 5.5 the identity map $p : (\partial^*_{k'}\Omega, k'_{x,\varepsilon}) \to (\partial^*_{k'}\Omega, k'_{x,\varepsilon'})$ is η_3-quasimöbius with $\eta_3 = \eta_3(\varepsilon'/\varepsilon)$. It follows that $f = p \circ \partial i$ is $\eta_1 := \eta_3 \circ \eta_2$-quasimöbius with $\eta_1 = \eta_1(\delta, c, \varepsilon'/\varepsilon)$. The claim is similarly proved when $\varepsilon' < \varepsilon$: f in this case is the composition of the identity map $(\partial^*\Omega, k_{x,\varepsilon}) \to (\partial^*\Omega, k_{x,\varepsilon'})$ and the boundary map $(\partial^*\Omega, k_{x,\varepsilon'}) \to (\partial^*_{k'}\Omega, k'_{x,\varepsilon'})$ of the identity map $(\Omega, k) \to (\Omega, k')$. $\square$

Theorem 9.1 follows from Lemma 9.2, Theorem 9.3 and Theorem 9.5.

Theorem 9.3. *Let (X, d) be a C_{a}-annular convex proper geodesic metric space, and $\Omega \subset X$ an* **unbounded** *rectifiably connected open subset with $\partial\Omega \neq \emptyset$. Suppose (Ω, k) is δ-hyperbolic and there is an η-quasimöbius natural map $\phi : (\partial^*\Omega, k_{w,\varepsilon_0}) \to (\partial'\Omega, d)$ for some $w \in \Omega$ and $\varepsilon_0 = \varepsilon_0(\delta)$. Then (Ω, d) is c_1-uniform with $c_1 = c_1(C_{\mathrm{a}}, \delta, \eta)$.*

Proof. Let $x_1, x_2 \in \Omega$, and $\gamma : x_1 \frown x_2$ be a quasihyperbolic geodesic. By Theorem 7.4, Lemma 7.6, and the existence of a natural map there is a quasihyperbolic geodesic line $\alpha : a_1 \frown a_2$ with $a_1, a_2 \in \partial'\Omega$ such that for $i = 1, 2$, $k(x_i, |\alpha|) \leq C = C(\delta, C_{\mathrm{a}})$; there are points $w_1, w_2 \in |\alpha|$ satisfying $k(x_i, w_i) \leq C$. Let $f : |\gamma| \to |\alpha|$ be a length map with $f(x_1) = w_1$. Then by Lemma 3.6, for every $x \in |\gamma|$ we have $k(f(x), x) \leq C$. We will show that γ is a uniform path. By Lemma 2.4, we may assume that

$$d(x_1, x_2) \geq \max\{\delta_\Omega(x_1), \delta_\Omega(x_2)\}. \tag{9.1}$$

We first demonstrate that $\ell_d(\gamma[x_1, x]) \wedge \ell_d(\gamma[x_2, x]) \leq C\,\delta_\Omega(x)$ for all $x \in |\gamma|$. If $a_2 = \infty$, then by Lemma 8.3, $\ell_d(\alpha[f(x_1), f(x)]) \leq \ell_d(\alpha[a_1, f(x)]) \leq C\,\delta_\Omega(f(x))$. Hence by Lemma 2.6, as $k(f(z), z) \leq C$ for all $z \in |\gamma|$, we have

$$\begin{aligned} \ell_d(\gamma[x_1, x]) \leq e^C\,\ell_d(\alpha[f(x_1), f(x)]) &\leq C\,e^C\,\delta_\Omega(f(x)) \\ &\leq C\,e^C\,e^C\delta_\Omega(x) = C\,\delta_\Omega(x). \end{aligned}$$

The last inequality follows from inequality (2.1). We obtain a similar inequality if $a_1 = \infty$. Now we assume that $a_1 \neq \infty \neq a_2$, and let $\xi_\alpha \in |\alpha|$ be the point given by Lemma 8.5. After switching a_1 and a_2 if necessary, we may assume $f(x) \in |\alpha[a_1, \xi_\alpha]|$. We have $f(x_i) \in |\alpha[a_1, f(x)]|$ for some $i \in \{1, 2\}$. By Lemma 8.5 (i) and inequality (2.1),

$$\ell_d(\alpha[f(x_i), f(x)]) \leq C\,\delta_\Omega(f(x)) \leq C\,e^C\,\delta_\Omega(x).$$

Again by Lemma 2.6, we have $\ell_d(\gamma[x_i, x]) \leq C\,\delta_\Omega(x)$. This completes the proof that γ satisfies the second condition for a uniform path.

Finally, we need to prove that $\ell_d(\gamma) \leq C\,d(x_1, x_2)$. We break the proof up into two cases.

Case 1: We first assume $2d(f(x_1), f(x_2)) \geq \max\{\delta_\Omega(f(x_1)), \delta_\Omega(f(x_2))\}$. Note that by Lemma 2.1, as $k(f(x_i), x_i) \leq C$, we have $d(f(x_i), x_i) \leq e^C\,\delta_\Omega(x_i)$. If $a_1 = \infty$ or if $a_2 = \infty$, then Lemma 2.6 together with Lemma 8.3 now implies that

$$\begin{aligned}\frac{1}{C}\,\ell_d(\gamma) \le \ell_d(\alpha[f(x_1), f(x_2)]) &\le C\,\max\{\delta_\Omega(f(x_1)), \delta_\Omega(f(x_2))\}\\ &\le 2C\,d(f(x_1), f(x_2))\\ &\le 2C\,[d(f(x_1), x_1) + d(x_1, x_2) + d(f(x_2), x_2)]\\ &\le 2C\,\left[e^C\delta_\Omega(x_1) + e^C\delta_\Omega(x_2) + d(x_1, x_2)\right].\end{aligned}$$

By the basic assumption of (9.1) we made at the beginning of the proof, we now get

$$\ell_d(\gamma) \le 2C^2\left[2e^C\,d(x_1, x_2) + d(x_1, x_2)\right] = C\,d(x_1, x_2),$$

and we are done. If $a_1 \ne \infty \ne a_2$, then Lemma 2.6 and Lemma 8.5 (ii) show that

$$\frac{1}{C}\,\ell_d(\gamma) \le \ell_d(\alpha[f(x_1), f(x_2)]) \le C\,d(f(x_1), f(x_2)).$$

Now we repeat the above argument and again obtain $\ell_d(\gamma) \le C\,d(x_1, x_2)$.
Case 2: We now assume $2\,d(f(x_1), f(x_2)) < \max\{\delta_\Omega(f(x_1)), \delta_\Omega(f(x_2))\} = \delta_\Omega(f(x_2))$. Then $f(x_1) \in \overline{B}(f(x_2), \delta_\Omega(f(x_2))/2)$. Hence a geodesic β with respect to the metric d joining $f(x_1)$ and $f(x_2)$ is a 1-uniform path (see Lemma 2.4), and

$$\begin{aligned}k(f(x_1), f(x_2)) \le \int_\beta \frac{1}{\delta_\Omega(x)}\,ds(x) &\le \frac{2}{\delta_\Omega(f(x_2))}\,\ell_d(\beta)\\ &= \frac{2}{\delta_\Omega(f(x_2))}\,d(f(x_1), f(x_2)) < 1.\end{aligned}$$

So $k(x_1, x_2) = k(f(x_1), f(x_2)) \le 1$. Hence $\ell_k(\gamma) \le 1$, and by Lemma 2.1,

$$\ell_d(\gamma) \le e\,\delta_\Omega(x_1) \le C\,d(x_1, x_2),$$

where we again used the assumption (9.1) at the end. □

Lemma 9.4. *Let (X, d) be a c-quasiconvex c-annular convex metric space, and $\Omega \subset X$ be an open subset. If $\partial\Omega = \{p\}$, then (Ω, d) is $6c^2$-uniform.*

Proof. Let $x, y \in \Omega$. We may assume $d(x, p) \le d(y, p)$. Set $t = d(x, p)$. First assume that $d(y, p) \le 2t$. The annular convexity of (X, d) implies that there is a path $|\gamma| \subset B(p, 2ct) \setminus B(p, 2t/c)$ connecting x and y such that its length $\ell(\gamma) \le c\,d(x, y) \le 3ct$. Observe that $|\gamma| \subset \Omega$. For any $z \in |\gamma|$ we have

$$\delta_\Omega(z) = d(z, p) \ge \frac{2t}{c} \ge \frac{2\ell(\gamma)}{3c^2},$$

and hence γ is a $3c^2$-uniform path.

Now assume $d(y, p) > 2t$. Then $d(y, p)/2 \le d(x, y) \le 2d(y, p)$. Let $n \ge 2$ be the integer with $2^{n-1}t < d(y, p) \le 2^n t$. Take any path γ from y to x, let x_i, $1 \le i \le n-1$, be the first point on γ with $d(x_i, p) = 2^i t$. Set $x_0 = x$ and $x_n = y$. Then $x_i \in \Omega$. Let $\gamma_i \subset B(p, c2^i t) \setminus B(p, 2^i t/c)$ be a path from x_{i-1}

to x_i with $\ell(\gamma_i) \le c\, d(x_{i-1}, x_i) \le 3c2^{i-1}t$. Let γ' be the concatenation of the γ_i. Then

$$\ell(\gamma') = \sum_{i=1}^{n} \ell(\gamma_i) \le \sum_{i=1}^{n} 3c\, 2^{i-1}t \le 3c\, t\, 2^n \le 6c\, d(y,p) \le 12c\, d(x,y).$$

Similarly, $\sum_{i=1}^{k} \ell(\gamma_i) \le 3c\, t\, 2^k$. Let $z \in |\gamma_k|$. Then

$$\delta_\Omega(z) = d(z,p) \ge \frac{2^k t}{c} \ge \frac{1}{3c^2} \sum_{i=1}^{k} \ell(\gamma_i) \ge \frac{1}{3c^2}\, \ell(\gamma'[x,z]).$$

□

Theorem 9.5. *Let (X,d) be a $C_{\rm a}$-annular convex proper geodesic space, and $\Omega \subset X$ a* **bounded** *rectifiably connected open subset with $\partial\Omega \ne \emptyset$. Suppose that (Ω,k) is δ-hyperbolic, and that the natural map $\phi : (\partial^*\Omega, k_{w,\varepsilon_0}) \to (\partial\Omega, d)$ exists (for some $w \in \Omega$ and $\varepsilon_0 = \varepsilon_0(\delta)$) and is η-quasimöbius. Then (Ω,d) is c_1-uniform with $c_1 = c_1(C_{\rm a}, \delta, \eta)$.*

Proof. By Lemma 9.4, if $\partial\Omega$ consists of a single point, then Ω is $6C_{\rm a}^2$-uniform. Hence we may assume $\partial\Omega$ contains at least two points. Fix some $p \in \partial\Omega$ and consider $(I_p(X), d_p)$. By Theorem 4.1, $(I_p(X), d_p)$ is c'-quasiconvex and c'-annular convex with $c' = c'(C_{\rm a})$. Since (Ω,k) is δ-hyperbolic, Proposition 4.2 implies that (Ω, k_p) is δ'-hyperbolic with $\delta' = \delta'(\delta, C_{\rm a})$. Set $k' := k_p$ and $\epsilon' = \epsilon'(\delta, C_{\rm a}) := \min\{\epsilon_0(\delta), \epsilon_0(\delta')\}$. Proposition 5.6 implies that the boundary map $\partial f : (\partial^*_{k'}\Omega, k'_{w,\epsilon'}) \to (\partial^*_k\Omega, k_{w,\epsilon'})$ of the identity map $f : (\Omega,k') \to (\Omega,k)$ is η_1-quasimöbius with $\eta_1 = \eta_1(\delta, C_{\rm a})$. By Lemma 5.5 the identity map $g_1 : (\partial^*_k\Omega, k_{w,\epsilon'}) \to (\partial^*_k\Omega, k_{w,\epsilon_0})$ is η_2-quasimöbius with $\eta_2(t) = 4^{1+\frac{\epsilon_0}{\epsilon'}}\, t^{\frac{\epsilon_0}{\epsilon'}}$. Similarly the identity map $g_2 : (\partial^*_{k'}\Omega, k'_{w,\epsilon_1}) \to (\partial^*_{k'}\Omega, k'_{w,\epsilon'})$ is η_3-quasimöbius with $\eta_3(t) = 4^{1+\frac{\epsilon'}{\epsilon_1}}\, t^{\frac{\epsilon'}{\epsilon_1}}$, where $\epsilon_1 = \epsilon_0(\delta')$. Set $\phi' = \phi \circ g_1 \circ \partial f \circ g_2$. Then ϕ' is a natural map for (Ω, d_p) and is η'-quasimöbius with $\eta' = \eta'(\delta, C_{\rm a}, \eta) := \eta \circ \eta_2 \circ \eta_1 \circ \eta_3$. Since (Ω, d_p) is unbounded, Theorem 9.1 in the unbounded case now implies that (Ω, d_p) is c''-uniform with $c'' = c''(\delta', c', \eta') = c''(\delta, C_{\rm a}, \eta)$. Now the theorem follows from Theorem 4.1 (iii).

□

10. An application to quasimöbius maps

In this section we show that quasimöbius maps between domains in annular convex metric spaces preserve uniformity. This result is quantitative.

Theorem 10.1. *For $i = 1,2$ let (X_i, d_i) be a proper metric space and $\Omega_i \subset X_i$ an open subset with $\partial\Omega_i \ne \emptyset$. Let $h : \Omega_1 \to \Omega_2$ be an η-quasimöbius homeomorphism. If Ω_1 is c_1-uniform and (X_2, d_2) is c_2-quasiconvex and c_2-annular convex, then Ω_2 is c-uniform with $c = c(c_1, c_2, \eta)$.*

Theorem 10.1 has been generalized to the more general case where (X_2, d_2) is not assumed to be annular convex, [X].

The remainder of this section is devoted to the proof of Theorem 10.1. Some segments of the proof have been highlighted as lemmas.

Let $i \in \{1, 2\}$. If Ω_i is bounded, set $X_i' = X_i$ and $d_i' = d_i$; if Ω_i is unbounded, then fix any base point $p_i \in \partial\Omega_i$ and set $X_i' = S_{p_i}(X_i)$ and $d_i' = \hat{d}_{ip_i}$. Let Ω_i' be the image of Ω_i in X_i', and denote by $\partial\Omega_i'$ the boundary of Ω_i' in (X_i', d_i') and $\overline{\Omega}_i'$ the closure of Ω_i' in (X_i', d_i'). Let $f_i : (\Omega_i, d_i) \to (\Omega_i', d_i')$ be the identity map, and set $h' := f_2 \circ h \circ f_1^{-1} : (\Omega_1', d_1') \to (\Omega_2', d_2')$. Let $\eta_0(t) = 16t$.

Lemma 10.2. *The map h' extends to a η'-quasimöbius homeomorphism $\overline{\Omega}_1' \to \overline{\Omega}_2'$, which is still denoted by h'. Here $\eta' = \eta_0 \circ \eta \circ \eta_0$. In particular, there exist $a_1 \in \partial\Omega_1', a_2 \in \partial\Omega_2'$ such that for any $\{x_i\} \subset \Omega_1'$ with $x_i \to a_1$ we have $h'(x_i) \to a_2$.*

Proof. The fact that f_i is η_0-quasimöbius implies that the map $h' : (\Omega_1', d_1') \to (\Omega_2', d_2')$ is η'-quasimöbius. Since $\operatorname{diam}(\Omega_i', d_i') < \infty$, the map h' is a quasisymmetric map. Notice that (X_i', d_i') is proper. By Theorem 6.12 of [V2], h' extends to a quasisymmetric map between the closures of the domains. The continuity implies that the extension is also η'-quasimöbius. Now the lemma follows. □

If $\partial\Omega_1'$ is a single point, then $\partial\Omega_2'$ and $\partial\Omega_2$ are also single points. By Lemma 9.4 (Ω_2, d_2) is $6c_2^2$-uniform. From now on, we assume that $\partial\Omega_1'$ has at least two points.

Now we fix $a_1 \in \partial\Omega_1', a_2 \in \partial\Omega_2'$ with the property stated in Lemma 10.2. Let $X_i'' = I_{a_i}(X_i') = X_i' \backslash \{a_i\}$ and $d_i'' = (d_i')_{a_i}$. Let Ω_i'' be the image of Ω_i' in X_i'', and denote by $\partial\Omega_i''$ the boundary of Ω_i'' in (X_i'', d_i'') and $\overline{\Omega}_i''$ the closure of Ω_i'' in (X_i'', d_i''). It is readily seen that $\partial\Omega_i'' = \partial\Omega_i' \backslash \{a_i\}$ and $\overline{\Omega}_i'' = \overline{\Omega}_i' \backslash \{a_i\}$ as sets. Let

$$g_i : (X_i' \backslash \{a_i\}, d_i') \to (X_i'', d_i'')$$

be the identity map and set

$$h'' := g_2 \circ h' \circ g_1^{-1} : (\overline{\Omega}_1'', d_1'') \to (\overline{\Omega}_2'', d_2'').$$

Since g_i is η_0-quasimöbius, h'' is η''-quasimöbius, where $\eta'' := \eta_0 \circ \eta' \circ \eta_0$. The choice of a_1 and a_2 implies that for any $x \in \Omega_1''$ and $\{x_i\} \subset \Omega_1''$ with $d_1''(x_i, x) \to \infty$ we have $d_2''(h''(x_i), h''(x)) \to \infty$. It follows that h'' is η''-quasisymmetric.

Since (Ω_1, d_1) is c_1-uniform, Theorem 4.6 (ii) implies that (Ω_1', d_1') is c_1'-uniform with $c_1' = c_1'(c_1)$. Since $\partial\Omega_1'$ contains at least two points and $a_1 \in \partial\Omega_1'$, it follows from Theorem 4.1 (iv) that (Ω_1'', d_1'') is c_1''-uniform with $c_1'' = c_1''(c_1') = c_1''(c_1)$. Let k_i be the quasihyperbolic metric on (Ω_i'', d_i''). By Theorem 6.1, (Ω_1'', k_1) is δ_1-hyperbolic with $\delta_1 = \delta_1(c_1'') = \delta_1(c_1)$. Let $\varepsilon_1 = \varepsilon_1(c_1'')$

be as in Theorem 6.2. Then for any ε satisfying $0 < \varepsilon \le \varepsilon_1$, Theorem 6.2 implies that there is a natural map $\phi_1 : (\partial^*_{k_1}\Omega''_1, k_{1w_1,\varepsilon}) \to (\partial'\Omega''_1, d''_1)$ and the natural map is η_1-quasimöbius with $\eta_1 = \eta_1(c''_1, \varepsilon) = \eta_1(c_1, \varepsilon)$.

The fact that (Ω''_1, d''_1) is c''_1-uniform implies that $(\overline{\Omega}''_1, d''_1)$ is c''_1-quasiconvex. On the other hand, since (X_2, d_2) is c_2-quasiconvex and c_2-annular convex, Theorem 4.6 (iii) and Theorem 4.1 (i) together imply that (X''_2, d''_2) is both c''_2-quasiconvex and c''_2-annular convex with $c''_2 = c''_2(c_2)$.

Lemma 10.3 (Lemma 2.3 of [V2])**.** *Suppose X is λ_1-quasiconvex, $q > 0$, $\lambda_2 \ge 0$, and $f : X \to Y$ is a map such that $d(f(x), f(y)) \le \lambda_2$ whenever $d(x, y) \le q$. Then $d(f(x), f(y)) \le (\lambda_1\lambda_2/q)d(x, y) + \lambda_2$ for all $x, y \in X$.*

Lemma 10.4. *For $i = 1, 2$ let (Y_i, d_i) be a proper metric space and $\Omega_i \subset Y_i$ a rectifiably connected open subset with $\partial\Omega_i \neq \emptyset$. Suppose that Y_i is c''_i-quasiconvex and that there is an η''-quasisymmetric homeomorphism $g : (\Omega_1, d_1) \to (\Omega_2, d_2)$. Let k_i be the quasihyperbolic metric on (Ω_i, d_i). Then the map $g : (\Omega_1, k_1) \to (\Omega_2, k_2)$ is an (L, A)-quasi-isometry with L and A depending only on c''_1, c''_2 and η''.*

Proof. By symmetry we only need to show that there exist constants L and A depending only on η'' and c''_2 such that $k_2(g(x), g(y)) \le Lk_1(x, y) + A$ for all $x, y \in \Omega_1$. Since (Ω_1, k_1) is a geodesic space, by Lemma 10.3 it suffices to find a constant q depending only on η'' and c''_2 such that $k_2(g(x), g(y)) \le 1$ whenever $k_1(x, y) \le q$. We choose q to be the number

$$q = \log\left(1 + (\eta'')^{-1}((2c''_2)^{-1})\right).$$

Notice that q depends only on η'' and c''_2. We next show q has the required property.

As Y_1 and Y_2 are proper and $g : (\Omega_1, d_1) \to (\Omega_2, d_2)$ is an η''-quasisymmetric map, Theorem 6.12 of [V2] implies that g extends to an η''-quasisymmetric map $(\overline{\Omega}_1, d_1) \to (\overline{\Omega}_2, d_2)$, which is also denoted by g. Let $x, y \in \Omega_1$ with $k_1(x, y) \le q$. Then Lemma 2.1 implies $d_1(x, y) \le (e^q - 1)\,\delta_1(x)$, where $\delta_i(z) = d_i(z, \partial\Omega_i)$ for any $z \in \Omega_i$. Let $z \in \partial\Omega_1$ with $\delta_2(g(x)) = d_2(g(x), g(z))$. Since g is η''-quasisymmetric, we have

$$\begin{aligned}\frac{d_2(g(x), g(y))}{\delta_2(g(x))} = \frac{d_2(g(x), g(y))}{d_2(g(x), g(z))} \le \eta''\Big(\frac{d_1(x, y)}{d_1(x, z)}\Big) \le \eta''\Big(\frac{d_1(x, y)}{\delta_1(x)}\Big)& \\ \le \eta''(e^q - 1) = \frac{1}{2c''_2}&.\end{aligned}$$

Since (Y_2, d_2) is c''_2-quasiconvex, we can find a path γ connecting $g(x)$ and $g(y)$ such that $\ell(\gamma) \le c''_2 d_2(g(x), g(y))$. It follows that $\ell(\gamma) \le c''_2 d_2(g(x), g(y)) \le \delta_2(g(x))/2$, and hence $\delta_2(z) \ge \delta_2(g(x))/2$ for all $z \in \gamma$. Therefore,

$$k_2(g(x), g(y)) \le \int_\gamma \frac{1}{\delta_2(z)} d_2 s(z) \le \frac{2}{\delta_2(g(x))}\ell(\gamma) \le 1.$$

□

Lemma 10.4 implies that $h'' : (\Omega_1'', k_1) \to (\Omega_2'', k_2)$ is a (L, A)-quasi-isometry with L and A depending only on c_1'', c_2'' and η''. Since (Ω_1'', k_1) is δ_1-hyperbolic, (Ω_2'', k_2) is δ_2-hyperbolic with

$$\delta_2 = \delta_2(\delta_1, L, A) = \delta_2(\delta_1, c_1'', c_2'', \eta'') = \delta_2(c_1, c_2, \eta).$$

Set $\varepsilon_2 = \varepsilon_2(c_1, c_2, \eta) := \min\{\varepsilon_1, \varepsilon_0(\delta_2)\}$. By Proposition 5.10, the boundary map $\partial h'' : (\partial^*_{k_1}\Omega_1'', k_{1\,w_1,\varepsilon_2}) \to (\partial^*_{k_2}\Omega_2'', k_{2\,w_2,\varepsilon_2})$ of the map $h'' : (\Omega_1'', k_1) \to (\Omega_2'', k_2)$ is η'''-quasimöbius with $\eta''' = \eta'''(\delta_1, L, A) = \eta'''(c_1, c_2, \eta)$. By Lemma 5.5 the identity map $g : (\partial^*_{k_2}\Omega_2'', k_{2\,w_2,\varepsilon_0(\delta_2)}) \to (\partial^*_{k_2}\Omega_2'', k_{2\,w_2,\varepsilon_2})$ is η_4-quasimöbius with $\eta_4 = \eta_4(\varepsilon_0(\delta_2), \varepsilon_2) = \eta_4(c_1, c_2, \eta)$. It follows that $\phi_2 := h''\circ\phi_1\circ(\partial h'')^{-1}\circ g$ is a natural map of (Ω_2'', d_2'') that is η_2-quasimöbius for $\eta_2 = \eta_2(c_1, c_2, \eta) := \eta''\circ\eta_1\circ\eta_3\circ\eta_4$, where η_3 depends only on η'''. Now Theorem 9.1 implies that (Ω_2'', d_2'') is c'-uniform with $c' = c'(c_2'', \delta_2, \eta_2) = c'(c_1, c_2, \eta)$. Now the result follows from Theorem 4.1 (iii) and Theorem 4.6 (iv).

The proof of Theorem 10.1 is now complete.

11. Two examples and one question

We give two examples that show the conclusion of Theorem 9.1 may fail if the space X is not quasiconvex and annular convex.

Example 11.1.

The space X is a subset of $\mathbb{R}^2$. Let B_1 be the graph of $y = x\sin(\frac{1}{x})$, $-1 \le x < 0$, B_2 the graph of $y = (x-1)\sin(\frac{1}{x-1})$, $1 < x \le 2$, $B_3 = \{(x,y) : x = -1,\ \sin(1) \le y \le 2\}$, $B_4 = \{(x,y) : x = 2,\ \sin(1) \le y \le 2\}$ and $B_5 = \{(x,y) : -1 \le x \le 2,\ y = 2\}$. Let $\Omega = \bigcup_{i=1}^5 B_i$ and $X = \Omega \cup \{(0,0),(1,0)\}$. We equip X with the Euclidean metric. We notice that X is homeomorphic to $[0,1]$, Ω is homeomorphic to $(0,1)$ and $\partial\Omega = \{(0,0),(1,0)\}$. The space (Ω, k) is isometric to the real line and hence is hyperbolic; $\partial^*\Omega$ consists of two points. The natural map $(\partial^*\Omega, k_{x,\varepsilon}) \to (\partial\Omega, d)$ exists and is trivially quasimöbius, but (Ω, d) is not uniform. Indeed, for $x, y \in \Omega$, let γ_{xy} be the (unique) arc in Ω connecting x to y; then $\ell(\gamma_{xy}) \to \infty$ as $x \to (0,0)$ and $y \to (1,0)$ while $d(x,y) \le 2$. The metric space (X, d) is not quasiconvex.

Example 11.2.

Let $n \ge 1$ be an integer and set

$$X = ([-n,n]\times\{0\}) \cup (\{0\}\times[-1,n]) \subset \mathbb{R}^2.$$

Let $p_1 = (n,0)$, $p_2 = (-n,0)$, $p_3 = (0,n)$ and $p_4 = (0,-1)$. Let X be equipped with the path metric, and $\Omega = X\backslash\{p_1,p_2,p_3,p_4\}$. Then $\partial\Omega = \{p_1,p_2,p_3,p_4\}$. The space (Ω, k) is a tree with 4 rays glued at a common vertex, hence it is 0-hyperbolic. Let $w = (0,0)$ be the origin. The natural map $\phi : (\partial^*\Omega, k_{w,1}) \to$

$(\partial\Omega, d)$ exists and is a bijection. Notice that for any quadruple Q of distinct points in $\partial^*\Omega$, we have $\mathrm{cr}(Q, d_{w,1}) = 1$ and $\mathrm{cr}(Q, d) = 1$. It follows that ϕ is η-quasimöbius with $\eta(t) = t$. The domain (Ω, d) is n-uniform, but is not c-uniform for any $c < n$: any path from p_1 to p_2 has to pass through w, which is at distance 1 from p_4. Therefore the quantitative statement fails for $\Omega \subset (X, d)$. Observe that the metric space (X, d) is geodesic but is not annular convex.

Given the main theorem of the paper and the above two examples, it is natural to ask the following question:

Question 11.1. Let (X, d) be a quasiconvex proper metric space and $\Omega \subset X$ a rectifiably connected open subset with $\partial\Omega \neq \emptyset$. Suppose (Ω, k) is Gromov hyperbolic, and the natural map exists and is quasimöbius. Is (Ω, d) uniform ?

Example 11.2 shows that one can not expect to control the uniformity constant even if the answer to the above question is yes.

References

[Aik04] H. Aikawa, *Potential-theoretic characterizations of nonsmooth domains*, Bull. London Math. Soc. **36** (2004), no. 4, 469–482.

[Aik06] H. Aikawa, *Characterization of a uniform domain by the boundary Harnack principle*, Harmonic analysis and its applications, Yokohama Publ., Yokohama, 2006, pp. 1–17.

[BHK] M. Bonk, J. Heinonen, P. Koskela, *Uniformizing Gromov hyperbolic spaces,* Astérisque **270** (2001).

[BHX] S. Buckley, D. Herron, X. Xie, *Metric space inversions, quasihyperbolic distance, and uniform spaces,* Indiana Univ. Math. J. **57** (2008), no. 2, 837–890.

[BK] M. Bonk, B. Kleiner, *Rigidity for quasi-Möbius group actions,* J. Differential Geom. **61** (2002), 81–106.

[BS] M. Bonk, O. Schramm, *Embeddings of Gromov hyperbolic spaces,* Geom. Funct. Anal.**10** (2000), no. 2, 266–306.

[BSh] J. Björn and N. Shanmugalingam, *Poincaré inequalities, uniform domains and extension properties for Newton-Sobolev functions in metric spaces*, J. Math. Anal. Appl. **332** (2007), no. 1, 190–208.

[CDP] M. Coornaert, T. Delzant, A. Papadopoulos, *Géométrie et théorie des groupes,* Lecture Notes in Mathematics **1441** (1990).

[CGN00] L. Capogna, N. Garofalo, and D-M Nhieu, *Examples of uniform and NTA domains in Carnot groups*, Proceedings on Analysis and Geometry (Novosibirsk), Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., 2000, (Novosibirsk Akad., 1999), pp. 103–121.

[CT95] L. Capogna and P. Tang, *Uniform domains and quasiconformal mappings on the Heisenberg group*, Manuscripta Math. **86** (1995), no. 3, 267–281.

[GdlH] E. Ghys, P. de la Harpe(editors), *Sur les groupes hyperboliques d'aprés Mikhael Gromov,* Progress in Mathematics **83** (1990).

[Geh87] F.W. Gehring, *Uniform domains and the ubiquitous quasidisk*, Jahresber. Deutsch. Math.-Verein **89** (1987), 88–103.

[Gre01] A.V. Greshnov, *On uniform and NTA-domains on Carnot groups*, Sibirsk. Mat. Zh. **42** (2001), no. 5, 1018–1035.

[GP] F. Gehring, B. Palka, *Quasiconformally homogeneous domains,* J. Analyse Math. **30** (1976), 172–199.

[Jon81] P.W. Jones, *Quasiconformal mappings and extendability of functions in Sobolev spaces*, Acta Math. **147** (1981), 71–88.

[K] R. Korte, *Geometric implications of the Poincaré inequality,* Results Math. **50** (2007), 93–107.

[P] F. Paulin, *Un groupe hyperbolique est déterminé par son bord,* J. London Math. Soc. (2) **54** (1996), no. 1, 50–74.

[V1] J. Väisälä, *Hyperbolic and uniform domains in Banach spaces,* Ann. Acad. Sci. Fenn. Math. **30** (2005), 261–302.

[V2] J. Väisälä, *The free quasiworld. Freely quasiconformal and related maps in Banach spaces,* Quasiconformal geometry and dynamics, Banach Center Publ. (Lublin) **48** (1996), 55–118.

[V3] J. Väisälä, *Gromov hyperbolic spaces,* Expo. Math. **23** (2005), 187–231.

[V4] J. Väisälä, *Uniform domains*, Tôhoku Math. J. **40** (1988), 101–118.

[X] X. Xie, *Quasimöbius maps preserve uniform domains,* Ann. Acad. Sci. Fenn. Math. **32** (2007), no. 2, 481–495.

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