Newtonian Spaces: An Extension of Sobolev Spaces to Metric Measure Spaces

by

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Dedicated to my Ammah, Iyyah, Annai, and Periyammah

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CHAPTER I

INTRODUCTION

1.1 Background.

The theory of Sobolev spaces was originally developed for domains Ω in $\mathbb{R}^n$ and was based on the notion of distributional derivatives. For $1 \leq p < \infty$ the Sobolev space $W^{1,p}(\Omega)$ is defined to be the collection of all functions u in $L^p(\Omega)$ such that the distributional derivatives $\partial_i u$, $i = 1, ..., n$, are in $L^p(\Omega)$, and is equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \sum_{i=1}^{n} \|\partial_i u\|_{L^p}.$$

See [EG], [M], and [Z] for details of Sobolev spaces for domains in $\mathbb{R}^n$. The theory of quasiconformal maps on metric spaces and the theory of subelliptic operators for example indicate a need to study Sobolev type spaces on metric measure spaces. Since distributional derivatives are defined in terms of an action on smooth functions via integration by parts, an alternate way of defining Sobolev spaces needs to be found for general metric spaces.

Papers such as [HeK1], [HeK2], [KM2], [O], [S1], and [V] have studied the concepts of weak upper gradients used in this thesis, but these concepts had only recently been collected together to define a Sobolev type function space on general metric measure spaces; see [Sh] and [C]. This thesis studies such an alternate definition of Sobolev spaces on general metric measure spaces. A part of this thesis has been compiled in the paper [Sh].

The rest of this chapter introduces alternate characterizations of Sobolev spaces and develops the tools of moduli of path families necessary for the subject. The second and third chapters explore classical properties of one such alternate definition of Sobolev spaces such as Banach property and embedding theorems, and the fourth chapter explores spaces where this definition of Sobolev spaces yields interesting function spaces, and gives some examples. In the fifth chapter an analogue of the classical Dirichlet problem is explored.

1.2 Characterizations of Sobolev functions on domains in $\mathbb{R}^n$.

It has been shown by Hajłasz in [H1] that for $p > 1$ a p-integrable function u in $\mathbb{R}^n$ is in the Sobolev class $W^{1,p}(\mathbb{R}^n)$ if and only if there exists a non-negative p-integrable function g such that for almost all points x and y in $\mathbb{R}^n$,

$$|u(x) - u(y)| \leq |x - y|(g(x) + g(y)).$$

In fact, for functions u in $W^{1,p}(\mathbb{R}^n)$, the Hardy-Littlewood maximal function of $|\nabla u|$ is a candidate for the role of g in the above inequality. This inequality can be stated on any metric measure space X if the term $|x - y|$ is interpreted to be the metric distance between the points x and y, and therefore can be used to define Sobolev type spaces refered to in this thesis as Hajłasz spaces. In the following discussion a function f is said to be p-integrable if it is measurable and the integral $\int_X |f|^p$ is finite.

Definition 1.2.1. Let X be a metric space with a metric d and a measure μ. For $1 \leq p < \infty$ the *Hajłasz space* $M^{1,p}(X)$ is the collection of L^p-equivalence classes of functions u that together with some p-integrable non-negative function g, called a *Hajłasz gradient* of u, satisfy the inequality

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y)) \tag{1.1}$$

for μ-almost all x, y in X. The corresponding norm for functions u in $M^{1,p}(X)$ is given by

$$\|u\|_{M^{1,p}} = \|u\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over all Hajłasz gradients g of u. With this norm, $M^{1,p}(X)$ is a Banach space.

See [H1], [H2], [HK1], [HK2], and Section 4 below for properties of Hajłasz spaces.

There is another equivalent definition of Sobolev functions for domains in $\mathbb{R}^n$ due to Ohtsuka, based on the notion of primitives of vector fields. Ohtsuka showed that a p-integrable function u is in the Sobolev space $W^{1,p}(\Omega)$ if and only if u is a generalized primitive of a p-integrable vector field, that is, there is a vector field V on the domain Ω such that $\rho(x) = |V(x)|$ is a p-integrable function and for p-modulus almost all rectifiable compact paths one has the equality

$$u(x) - u(y) = \int_\gamma V \cdot \frac{d\gamma}{ds} ds,$$

where x and y are the end point and the starting point of γ respectively. See [O, Section 4.3, Theorem 4.21] for details. See Definition 1.3.1 below for the definition of moduli of path families and the notion of a property holding for p-modulus almost all paths.

Definition 1.2.2. Let u be a real-valued function on a metric space X. A non-negative Borel-measurable function ρ is said to be an *upper gradient* of u if for all compact rectifiable paths γ the following inequality holds:

$$|u(x) - u(y)| \leq \int_\gamma \rho ds, \tag{1.2}$$

where x and y are the endpoints of the path.

See [KM2] and [HeK1, Section 2.9] for a discussion on upper gradients; [HeK1] uses the term *very weak gradients* for this concept.

In the case of domains Ω of $\mathbb{R}^n$, it is easy to see in the light of [KM2, Lemma 2.4], Proposition 2.1.1, and [O, Theorem 4.16] that the existence of a p-integrable upper gradient is a necessary and sufficient condition for a p-integrable function to be a generalized primitive of a p-integrable vector field. Since the concept of upper gradient is definable on any metric space, for $1 \leq p < \infty$ one can define a Sobolev type space on a metric measure space X to be the collection of p-integrable functions with p-integrable upper gradients. See Definitions 1.3.6 and 1.3.7. In the event that the p-modulus of the family of all compact rectifiable paths in the space is zero, for example if the metric measure space has no rectifiable curves, by Lemma 1.3.4 the corresponding definition of Sobolev type space would yield the space $L^p(X)$. If the metric space has an abundance of rectifiable curves an interesting theory of Sobolev spaces develops. In contrast, the Hajłasz space can be strictly smaller than $L^p(X)$ even when the space has no rectifiable curves.

The Sobolev type spaces obtained by using the above definition are referred to in this thesis as Newtonian spaces in recognition of the fact that the idea behind their definition is a generalization of the fundamental theorem of calculus.

Yet another Sobolev-type definition is given in the paper [C], using L^p-convergence of function-upper gradient pairs; see Section 2.3. In Section 2.3 it will be shown that this definition of [C] yields the same Newtonian space whenever p is a finite number greater than 1.

1.3 Notations and Definitions.

This section lists the notations and definitions used throughout. The main focus of this section is to define the Newtonian spaces.

Throughout this thesis (X, d, μ) is a metric, Borel measure space. A measure is said to be a Borel measure if all Borel sets are measurable under the measure. Constants are labeled C, and the value of C might change even from one line of the same proof to the next. If E is a subset of X, then χ_E denotes the characteristic function of E. It is also assumed that p is a real number satisfying $1 \leq p < \infty$ unless specifically stated otherwise.

Paths γ in X are continuous maps $\gamma : I \to X$, where I is some interval in $\mathbb{R}$; by an abuse of terminology, the image $\gamma(I) =: |\gamma|$ of γ is also called a path. Let Γ_{rect} be the collection of all non-constant compact (that is, I is compact) rectifiable paths in the space X. For a discussion of rectifiable paths and path integration see [HeK1, Section 2] or [V, Chapter 1]. If A is a subset of X, then Γ_A is the family of all paths in Γ_{rect} that intersect the set A and Γ_A^+ is the family of all paths γ in Γ_{rect} such that the Hausdorff one-dimensional measure $\mathcal{H}_1(|\gamma| \cap A)$ is positive.

Modulus of a Collection of Paths.

The following definition is applicable to all families of paths, not necessarily only to collections of compact rectifiable paths. The rest of the thesis however will only consider families of non-constant compact rectifiable paths.

Definition 1.3.1. Let Γ be a collection of paths in X. The *p-modulus* of the family Γ, denoted $\mathrm{Mod}_p\Gamma$, is defined to be the (possibly infinite) number

$$\inf_\rho \|\rho\|_{L^p}^p,$$

where the infimum is taken over the set of all non-negative Borel-measurable functions ρ such that for each rectifiable path γ in Γ the path integral $\int_\gamma \rho ds$ is not smaller than 1.

It is known from [Fu1] that p-modulus is an outer measure on the collection of all paths in X. From the above definition it is clear that the p-modulus of the family of all non-rectifiable paths is zero. For additional information about p-moduli see [V], [Fu1, Chapter 1], and [AO].

A property relevant to paths in X is said to hold for *p-almost all paths* if the family of rectifiable compact paths on which the property does not hold has p-modulus zero. This is a slightly different definition from the standard definition used in other papers; the standard definition requires that the family of all non-compact locally rectifiable paths as well as compact rectifiable paths on which the property in question does not hold has zero p-modulus. The difference between these two definitions is immaterial in practice; for instance, all non-compact rectifiable paths can be completed to be compact rectifiable paths in the event that X is complete, and under some mild geometric conditions on X the family of all locally rectifiable paths that are not rectifiable has p-modulus zero.

For any path $\gamma \in \Gamma_{rect}$ and for distinct points x and y in $|\gamma|$,denote γ_{xy} to be a subpath $\gamma|_{[t_x,t_y]}$, where the two distinct numbers t_x and t_y are chosen from the domain of γ such that $\gamma(t_x) = x$ and $\gamma(t_y) = y$. The subpath γ_{xy} is not a well-defined notion as there can be more than one choice of the related numbers t_x and t_y. Because of this ambiguity any property that is required for one choice of subpath γ_{xy}, is also required for all such choices of subpaths.

Definition 1.3.2. Let $l(\gamma)$ denote the length of γ. A function u is said to be ACC_p or *absolutely continuous on p-almost every curve* if $u \circ \gamma$ is absolutely continuous on $[0, l(\gamma)]$ for p-almost every rectifiable arc-length parametrized path γ in X. If X is a domain in $\mathbb{R}^n$ a function u is said to have the ACL property, or *absolute continuity on almost every line*, if on almost every line parallel to the coordinate axes with respect to the Hausdorff $(n-1)$-measure the function is absolutely continuous. An ACL function therefore has directional derivatives almost everywhere. An ACL function is said to have the property ACL_p if its directional derivatives are p-integrable.

The notation here is a slight modification of the notation used in [V], where an ACL_p-function is required to be continuous. It is shown in [Fu1, Theorem 11] and [V, Theorem 28.2] that for functions on domains in $\mathbb{R}^n$ the ACL_p property is equivalent to the ACC_p property together with the property of having a p-integrable upper gradient. Recall that for domains Ω in $\mathbb{R}^n$, functions in $W^{1,p}(\Omega)$ have ACL_p representatives, and that conversely every p-integrable ACL_p-function is in $W^{1,p}(\Omega)$. See [EG] and [Z].

The following definition is due to [KM2], and is a weakening of the concept of upper gradient defined in Definition 1.2.2.

Definition 1.3.3. Let u be an arbitrary real-valued function on X, and let ρ be a non-negative Borel function on X. If there exists a family $\Gamma \subset \Gamma_{rect}$ such that $\mathrm{Mod}_p\Gamma = 0$ and inequality (1.2) is true for all paths γ in $\Gamma_{rect} \setminus \Gamma$, then ρ is said to be a *p-weak upper gradient* of u. If inequality (1.2) holds true for p-modulus almost all paths in a set $A \subset X$, then ρ is said to be a *p-weak upper gradient of* u *on* A. As the exponent p is usually fixed, in both cases ρ is simply called a weak upper gradient of u.

While the notion of upper gradients does not involve measures or the notion of p-modulus (and hence is independent of the index p), the notion of p-weak upper gradient is strongly dependent on the measure and the concept of p-modulus. By [KM2, Lemma 2.4], the existence of a p-integrable p-weak upper gradient implies the existence of a p-integrable upper gradient which approximates the given weak upper gradient to any desired accuracy in the L^p-norm. This statement follows easily from the following lemma, which is a direct generalization of a theorem of Fuglede, [Fu1, Theorem 2], to metric measure spaces. The proof given in [Fu1] remains true even in this generality, and is included here for the sake of completeness.

Lemma 1.3.4. *Let Γ be a collection of paths in X. Then $Mod_p\Gamma = 0$ if and only if there is a non-negative p-integrable Borel function ρ on X such that for all paths γ in Γ,*

$$\int_\gamma \rho ds = \infty.$$

Proof. First assume that $\mathrm{Mod}_p\Gamma = 0$. Then for each integer $n > 0$ there is a non-negative Borel-measurable p-integrable function ρ_n which is admissible for calculating the p-modulus of Γ and $\|\rho_n\|_{L^p} \leq 2^{-n}$. It is easy to see that the infinite sum $\rho = \sum_{i=1}^{\infty} \rho_n$ is a Borel measurable non-negative p-integrable function so that for all paths γ in Γ, the integral $\int_\gamma \rho ds$ is infinite. Now assume that there is a non-negative p-integrable Borel function ρ so that for each path γ in Γ the integral $\int_\gamma \rho ds$ is infinite. Then for each $n > 0$ the function $2^{-n}\rho$ is admissible for calculating the p-modulus of Γ. Hence the p-modulus of the family Γ is zero. □

The following lemma provides a connection between the modulus of the collection of paths through a set and its measure.

Lemma 1.3.5. *If E is a subset of X such that $\mu(E) = 0$, then $Mod_p(\Gamma_E^+)$ is zero.*

Proof. This lemma follows by Lemma 1.3.4 and the fact that $\infty\chi_E$ is an admissible function for calculating $\mathrm{Mod}_p(\Gamma_E^+)$. □

Formal Definition of Newtonian Spaces.

Definition 1.3.6. Let $\tilde{N}^{1,p}(X, d, \mu)$ be the collection of all real-valued p-integrable functions u on X that have a p-integrable weak upper gradient.

Note that $\tilde{N}^{1,p}$ is a collection of functions and is also a vector space, since if α, β are real numbers and $u_1, u_2 \in \tilde{N}^{1,p}$ with respective weak upper gradients ρ_1, ρ_2, then $|\alpha|\rho_1 + |\beta|\rho_2$ is a weak upper gradient of $\alpha u_1 + \beta u_2$. If u is a function in $\tilde{N}^{1,p}$, let

$$\|u\|_{\tilde{N}^{1,p}} = \|u\|_{L^p} + \inf_\rho \|\rho\|_{L^p},$$

where the infimum is taken over all p-integrable weak upper gradients of u. Again by [KM2, Lemma 2.4], the infimum could just as well be taken over all p-integrable upper gradients of u. However, weak upper gradients have the property that if a sequence of functions in $\tilde{N}^{1,p}$ converge in L^p to a function f in L^p and a corresponding choice of weak upper gradients converge in L^p to another function g in L^p, then g is a weak upper gradient of f; see Lemma 2.3.3. This property does not in general hold true for upper gradients. Since the function f in $\tilde{N}^{1,p}$ can be discontinuous on some p-modulus zero collection of curves, on some of these curves the upper gradient inequality (1.2) may not hold true.

It is easily seen that $\|\cdot\|_{\tilde{N}^{1,p}}$ satisfies the triangle inequality:

$$\|u+v\|_{\tilde{N}^{1,p}} \leq \|u\|_{\tilde{N}^{1,p}} + \|v\|_{\tilde{N}^{1,p}}.$$

If u, v are functions in $\tilde{N}^{1,p}$, let $u \sim v$ if $\|u-v\|_{\tilde{N}^{1,p}} = 0$. It can be easily seen that $\sim$ is an equivalence relation, partitioning $\tilde{N}^{1,p}$ into equivalence classes. This collection of equivalence classes, under the norm of Definition 1.3.6, is a normed vector space.

Definition 1.3.7. The *Newtonian space* corresponding to the index p, $1 \leq p < \infty$, denoted $N^{1,p}(X)$, is defined to be the normed space $\tilde{N}^{1,p}(X, d, \mu)/\sim$, with norm $\|u\|_{N^{1,p}} := \|u\|_{\tilde{N}^{1,p}}$.

If u, v are functions in $\tilde{N}^{1,p}$, then it is easily verified that the functions $|u|$, $\min\{u, v\}$, and $\max\{u, v\}$ are also in $\tilde{N}^{1,p}$. It is easily seen that if $\lambda \geq 0$, then $\min\{u, \lambda\}$ is in $\tilde{N}^{1,p}$, and that if $\lambda \leq 0$, then $\max\{u, \lambda\}$ is also in $\tilde{N}^{1,p}$. Thus $N^{1,p}(X)$ enjoys all the lattice properties found in classical first order Sobolev spaces.

CHAPTER II

NEWTONIAN SPACES ARE BANACH SPACES

This chapter explores some properties of Newtonian spaces, with the primary focus on proving in Theorem 2.2.8 that $N^{1,p}(X)$ is a Banach space. The difficulty in proving that the Cauchy sequences in $N^{1,p}(X)$ converge in its norm lies in taking care of the term involving upper gradients in the norm estimates. The problem lies in the fact that a difference of two $N^{1,p}$-functions does not necessarily have the difference of the respective weak upper gradients as a weak upper gradient.

2.1 Continuity Properties of Newtonian Functions.

Proposition 2.1.1. *If u is a function in $\tilde{N}^{1,p}$, then u is ACC_p.*

Proof. By the definition of $\tilde{N}^{1,p}$, u has a p-integrable weak upper gradient ρ. Let Γ be the collection of all paths in Γ_{rect} for which inequality (1.2) does not hold. Then by the definition of weak upper gradients, $\text{Mod}_p\Gamma = 0$. Let Γ_1 be the collection of all paths in Γ_{rect} that have some sub-path belonging to Γ. Then any admissible function used to estimate the modulus of Γ is an admissible function for Γ_1, and hence

$$\text{Mod}_p\Gamma_1 \leq \ \text{Mod}_p\Gamma = 0.$$

Let Γ_2 be the collection of all paths γ in Γ_{rect} such that $\int_\gamma \rho ds = \infty$. As ρ is p-integrable, $\text{Mod}_p\Gamma_2$ is zero. Hence $\text{Mod}_p(\Gamma_1 \cup \Gamma_2)$ is zero. If γ is a path in Γ_{rect} that is not in $\Gamma_1 \cup \Gamma_2$, then γ has no sub-path in Γ_1, and hence for all x, y in $|\gamma|$

$$|u(x) - u(y)| \leq \int_{\gamma_{xy}} \rho ds \ < \infty.$$

Therefore u is absolutely continuous on each path γ in $\Gamma_{rect} \backslash (\Gamma_1 \cup \Gamma_2)$.

$\square$

Note that the above proposition remains valid even if the function u is required only to have a p-integrable upper gradient, without itself being p-integrable.

Lemma 2.1.2. *Suppose u is a function in $\tilde{N}^{1,p}$ such that $\|u\|_{L^p} = 0$. Then the family*

$$\Gamma = \{\gamma \in \Gamma_{rect} \;:\; u(x) \neq 0 \text{ for some } x \in |\gamma|\}$$

has zero p-modulus.

Proof. Since $\|u\|_{L^p} = 0$, the set $E = \{x \in X \;:\; u(x) \neq 0\}$ has measure zero. With the notation introduced in Section 1.3, one has $\Gamma = \Gamma_E$ and

$$\Gamma = \Gamma_E^+ \cup (\Gamma_E \backslash \Gamma_E^+).$$

The subfamily Γ_E^+ can be disregarded since

$$\mathrm{Mod}_p \Gamma_E^+ \leq \|\infty \cdot \chi_E\|_{L^p} = 0.$$

The paths γ in $\Gamma_E \backslash \Gamma_E^+$ intersect E only on a set of linear measure zero, and hence with respect to linear measure almost everywhere on γ the function u takes on the value of zero. By the fact that γ also intersects E therefore, u is not absolutely continuous on γ. By Proposition 2.1.1,

$$\mathrm{Mod}_p(\Gamma_E \backslash \Gamma_E^+) = 0,$$

yielding that $\mathrm{Mod}_p \Gamma = 0$.

$\square$

This lemma indicates that functions in $\tilde{N}^{1,p}$ are well-defined outside a small set. For example, in $\mathbb{R}^n$ not all sets E of zero measure have the property that the p-modulus of the family Γ_E is zero, and hence unlike L^p-functions, Newtonian functions on $\mathbb{R}^n$ cannot be arbitrarily changed on sets of measure zero. The above lemma yields the following:

Corollary 2.1.3. *If u_1, u_2 are two functions in $\tilde{N}^{1,p}(X)$ such that $\|u_1 - u_2\|_{L^p} = 0$, then u_1 and u_2 belong to the same equivalence class in $N^{1,p}(X)$.*

The rest of the thesis will not explicitly distinguish between the functions in $\tilde{N}^{1,p}$ and their equivalence classes in $N^{1,p}$.

Remark 2.1.4. The following three lemmata show that upper gradients mimic the behaviour of derivatives of functions on Euclidean spaces. Lemmata 2.1.5 and 2.1.7 remain true if the requirement of openness for the set O is replaced by the requirement that for p-modulus almost every compact rectifiable curve γ the set $\gamma^{-1}(O)$ is open. Any set satisfying this requirement is defined to be *p-path open.* This property is closely related to the property of p-quasi open set property defined by Fuglede; see [Fu2]. A set $O \subset X$ is said to be *p-quasi open* if for each positive ϵ there is a set $F_\epsilon \subset X$ so that the p-capacity of F_ϵ is no more than ϵ and the set $O \cup F_\epsilon$ is open in the metric topology of X. Every p-quasi open set is p-path open. To see this, suppose $O \subset X$ is p-quasi open. Then, by definition, for each positive integer n there is a set F_n with $\mathrm{Cap}_p(F_n) \le 2^{-n}$ so that $O \cup F_n$ is open. It can be assumed that $F_{n_1} \subset F_{n_2}$ if $n_1 \ge n_2$. To see this, note that if $n_2 > n_1$ and F_{n_1} and $\tilde{F}_{n_2}$ are two sets with the properties that $\mathrm{Cap}_p(F_{n_1}) \le 2^{-n_1}$ and $\mathrm{Cap}_p(\tilde{F}_{n_2}) \le 2^{-n_2}$, and $O \cup F_{n_1}$, $O \cup \tilde{F}_{n_2}$ are open in the metric topology of X, then defining the set $F_{n_2} = \tilde{F}_{n_2} \cap F_{n_1}$, the p-capacity of F_{n_2} is no more than 2^{-n_2} and $O \cup F_{n_2} = (O \cup \tilde{F}_{n_2}) \cap (O \cup F_{n_1})$ is open in the metric topology of X. Let $u_n \in N^{1,p}(X)$ so that $u_n\big|_{F_n} = 1$, $0 \le u_n \le 1$, and $\|u_n\|^p_{N^{1,p}} \le 2^{-n+1}$. Let $v_i = \sum_{n=1}^i u_n$. Then the sequence v_i is a Cauchy sequence in $N^{1,p}(X)$, and hence by Theorem 2.2.8 (proved in the next section), this sequence converges to a function v in the $N^{1,p}(X)$ norm, with $v \in N^{1,p}(X)$. Suppose γ is a rectifiable curve. Clearly $\gamma^{-1}(F_n \cup O)$ is open in the domain of γ for each n. If there is an integer n_0 so that $\gamma^{-1}(F_{n_0} \cup O) = \gamma^{-1}(O)$, then $\gamma^{-1}(O)$ is open in the domain of γ. If there is no such integer n_0, then γ intersects F_n for each n. Let x_n be a point in this intersection. Then, as $F_m \subset F_n$ for each $m \le n$ and $u_m\big|_{F_m} = 1$, and since by the proof of Lemma 2.2.5 the function v can be taken to be the function $v(x) = \lim_{i\to\infty} v_i(x)$, therefore $v(x_n) \ge n$. Hence $v \circ \gamma$ is not bounded on the compact domain of γ, and hence v is not absolutely continuous on γ. Since $v \in N^{1,p}(X)$, by Proposition 2.1.1 the function v is not absolutely continuous only on a p-modulus zero family of curves, and therefore the collection of above such γ is of zero p-modulus.

If X has no rectifiable curves, every set is p-path open, but not every set is p-quasi open. The author does not know whether in the Euclidean setting p-path open sets are p-quasi open. Lemma 2.1.8 remains true if the condition of the set F being closed is replaced by the corresponding property that for p-modulus almost every compact rectifiable curve γ the set $\gamma^{-1}(F)$ is closed.

Lemma 2.1.5. *Let u_1 and u_2 be ACC_p functions on X, with weak upper gradients g_1 and g_2 respectively. If u is another ACC_p function in X such that there is an open set $O \subset X$ with the property that $u = u_1$ on O and $u = u_2$ on $X \backslash O$, then $g_1\chi_O + g_2$ and $g_1 + g_2\chi_{X\backslash O}$ are weak upper gradients of u.*

Proof. Let Γ be the collection of paths γ on which at least one of u, u_1, and u_2 is not absolutely continuous or on some subpath of γ the upper gradient inequality (1.2) is not satisfied by at least one of (u_1, g_1) and (u_2, g_2). Then $\mathrm{Mod}_p\Gamma = 0$. First consider $h = g_1\chi_O + g_2$. Suppose γ is a path in $\Gamma_{rect}\backslash\Gamma$ connecting points x, y in X. If $|\gamma| \subset O$ or if $x, y \in X\backslash O$, then clearly $|u(x) - u(y)| \le \int_\gamma h$. Suppose $|\gamma| \not\subset O$ and that x is in O. If y also belongs to O, since $|\gamma| \not\subset O$, it is possible to break γ into two pieces at a point $z \in |\gamma| \cap (X \backslash O)$, and consider the two pieces separately. Hence without loss of generality it can be assumed that y is not in O. Since O is open, $\gamma^{-1}O$ is an open subset of $I = [a, b]$, where $\gamma : I \to X$. Therefore $\gamma^{-1}O$ is a countable disjoint union of relatively open intervals in I. Thus there is a number $a_1 \in I$ such that x is in $\gamma([a, a_1)) \subset O$ and $\gamma(a_1) \in X \backslash O$, $a < a_1 \leq b$. On $\gamma([a, a_1))$, by hypothesis $u = u_1$. Since u, u_1, and u_2 are continuous on γ and $\gamma(a_1) \in X \backslash O$, and on $X \backslash O$ $u = u_2$, it is true that $u(\gamma(a_1)) = u_1(\gamma(a_1)) = u_2(\gamma(a_1))$. Thus

$$|u(\gamma(a)) - u(\gamma(a_1))| \leq \int_{\gamma([a,a_1))} g_1 = \int_{\gamma([a,a_1))} g_1 \chi_O,$$

and

$$|u(\gamma(a_1)) - u(\gamma(b))| \leq \int_{\gamma([a_1,b])} g_2.$$

Hence

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(\gamma(a_1))| + |u(\gamma(a_1)) - u(y)| \\ &\leq \int_{\gamma([a,a_1))} g_1 \chi_O + \int_{\gamma([a_1,b])} g_2 \\ &\leq \int_\gamma (g_1 \chi_O + g_2) \leq \int_\gamma h. \end{aligned}$$

Therefore h is a weak upper gradient of u, since $\mathrm{Mod}_p \Gamma = 0$.

Now consider $g = g_1 + g_2 \chi_{X \backslash O}$. If either $|\gamma| \subset X \backslash O$ or if $x, y \in O$, then clearly $|u(x) - u(y)| \leq \int_\gamma g$. Suppose $|\gamma| \cap O$ is not empty and that $x \in X \backslash O$. As before, without loss of generality it can be assumed that y is in O. Let $\gamma : I = [a, b] \to X$. The set $\overline{\gamma^{-1}(O)} \backslash \gamma^{-1}(O)$ corresponds to some of the points on $|\gamma|$ that lie on the boundary of O. Being a closed subset of the compact set I, this set is compact, and hence has a point a_1 such that there does not exist any number $t \in [a, a_1)$ with the property that $t \in \overline{\gamma^{-1}(O)} \backslash \gamma^{-1}(O)$; that is, $\gamma([a, a_1])$ does not intersect O (if $a = a_1$, then a is on the boundary of O, and the rest of the proof remains valid in this case as well), but there is a sequence of points in $\gamma^{-1}(O)$ that converge to a_1. On $\gamma([a, a_1])$, by hypothesis $u = u_2$, and hence,

$$|u(x) - u(\gamma(a_1))| \leq \int_{\gamma([a,a_1])} g_2 = \int_{\gamma([a,a_1])} g_2 \chi_{X \backslash O}.$$

Furthermore, as a_1 is a limit point of a sequence of points in I whose images under γ lie in O, and on O by hypothesis $u = u_1$ and u is absolutely continuous on γ, it is true that $u(\gamma(a_1)) = u_1(\gamma(a_1))$. Hence

$$|u(\gamma(a_1)) - u(y)| \leq \int_{\gamma([a_1,b])} g_1.$$

Therefore

$$|u(x) - u(y)| \leq \int_\gamma (g_1 + g_2\chi_{X\backslash O}),$$

and hence g is also a weak upper gradient of u. $\square$

Remark 2.1.6. In the situation of the above lemma, if $g_2 \geq g_1$ almost everywhere on O, then g_2 is a weak upper gradient of u; and if $g_2 \leq g_1$ almost everywhere on $X\backslash O$, then g_1 is a weak upper gradient of the function u. This follows from the proof of the above lemma.

Lemma 2.1.7. *Suppose u is an ACC_p function on X such that there exists an open set $O \subset X$ with $u = 0$ μ-almost everywhere on $X\backslash O$. Then if g is an upper gradient of u, then $g\chi_O$ is also a weak upper gradient of u.*

Proof. Let $E = \{x \in X\backslash O \;:\; u(x) \neq 0\}$. Then by assumption $\mu(E) = 0$. Hence $\mathrm{Mod}_p(\Gamma_E^+)$ is also zero (since $\infty\chi_E$ is then an admissible function for this collection of paths). Let Γ_0 be the collection of paths on which u is not absolutely continuous. Let $\gamma \in \Gamma_{rect}\backslash(\Gamma_E^+ \cup \Gamma_0)$ connecting two points $x, y \in X$. If γ lies entirely in $O \cup E$, then clearly $|u(x) - u(y)| \leq \int_\gamma g = \int_\gamma g\chi_O$ because γ intersects E only on a set of Hausdorff 1-measure zero. If x and y are not in $O \cup E$, then $u(x) = u(y) = 0$, and hence again $|u(x)-u(y)| \leq \int_\gamma g\chi_O$. If x is a point in $O\cup E$ and γ does not lie entirely in $O \cup E$, noting that $(u \circ \gamma)^{-1}(0)$ is a compact subset of the domain $I = [a, b]$ of γ, the set $(u \circ \gamma)^{-1}(0)$ has a greatest lower bound a_0 and a least upper bound b_0 in I with $u \circ \gamma(a_0) = u \circ \gamma(b_0) = 0$ (it is possible that $b_0 = b$ or $a_0 = a$, but that does not create a problem here). Thus,

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u \circ \gamma(a_0)| + |u \circ \gamma(a_0) - u \circ \gamma(b_0)| + |u \circ \gamma(b_0) - u(y)| \\ &\leq \int_{\gamma|_{[a,a_0]}} g + \int_{\gamma|_{[b_0,b]}} g \\ &\leq \int_\gamma g\chi_O \end{aligned}$$

since the subpaths $\gamma\big|_{[a,a_0]}$ and $\gamma\big|_{[b_0,b]}$ intersect $E \cup (X\backslash O)$ only on a set of Hausdorff 1-measure zero. By the above three cases the result follows. $\square$

Lemma 2.1.8. *Let u be an ACC_p function in X, and suppose g and h are two p-integrable weak upper gradients of u. If F is a closed subset of X, then the function $\rho = g\chi_F + h\chi_{X\backslash F}$ is also a weak upper gradient of u.*

Proof. Since both g and h are p-integrable, the collection of paths γ on which u is not absolutely continuous or the integral $\int_\gamma (g + h) = \infty$ or the upper gradient relation does not hold for either g or h has p-modulus zero. Denote that collection as Γ_0. Let Γ be the collection of all rectifiable paths that have a subpath in Γ_0. The p-modulus of Γ_0, and hence the p-modulus of Γ, are zero. Let γ be a compact rectifiable path in X that does not belong to the family Γ. If γ lies entirely in F or entirely outside F, then inequality (1.2) is satisfied by ρ and u. Suppose that γ passes through F and goes outside F as well. Since $X\backslash F$ is open, $\gamma^{-1}(X\backslash F)$ is a countable disjoint union of open intervals in the domain of γ. Let the images of these disjoint intervals be denoted γ_i. Denoting the end points of γ_i by x_i and y_i, and noting that u is continuous on γ, by induction it can be shown that ρ satisfies inequality (1.2):

$$\begin{aligned}|u(x)-u(y)| \leq& |u(x)-u(x_1)|+|u(x_1)-u(y_1)|+|u(y_1)-u(y)|\\ \leq& \int_{\gamma\backslash\gamma_1} g+\int_{\gamma_1} h\\ &\cdots\\ \leq& \int_{\gamma\backslash\bigcup_{1\leq i\leq n}\gamma_i} g+\int_{\bigcup_{1\leq i\leq n}\gamma_i} h\end{aligned}$$

for each positive integer n. By Lebesgue dominated convergence theorem applied to the path-integral on γ the result follows. $\square$

2.2 $N^{1,p}(X)$ is a Banach space.

The proof that $N^{1,p}(X)$ is a Banach space is done in two parts. It is first proved that a Cauchy sequence in $N^{1,p}(X)$ has a subsequence that converges in the L^p-norm to a function in $N^{1,p}(X)$. Next it is proved that a subsequence does indeed converge in the $N^{1,p}$-norm.

The following lemma was first proved for $\mathbb{R}^n$ by Fuglede, [Fu1, Theorem 3(f)]. The proof extends easily to metric measure spaces, but is included here for the sake of completeness.

Lemma 2.2.1. *If $\{\rho_i\}_{i=1}^\infty$ is a sequence of non-negative Borel functions in $L^p(X)$ converging to zero in the L^p-norm, then there exists a subsequence $\{\rho_{i_k}\}_{k=1}^\infty$ and a zero p-modulus family $\Gamma\subset\Gamma_{rect}$ such that for all paths γ in $\Gamma_{rect}\backslash\Gamma$*

$$\lim_{k\to\infty}\int_\gamma \rho_{i_k}\,ds=0.$$

Proof. Choose any subsequence such that $\|\rho_{i_k}\|_{L^p} \leq 2^{-k}$. Then

$$\begin{aligned}
\Gamma &= \{\gamma \in \Gamma_{rect} | \limsup_{k\to\infty} \int_\gamma \rho_{i_k} ds > 0\} \\
&= \bigcup_{n\in\mathbb{N}} \{\gamma \in \Gamma_{rect} | \limsup_{k\to\infty} \int_\gamma \rho_{i_k} ds \geq 1/n\} \\
&= \bigcup_{n\in\mathbb{N}} \{\gamma \in \Gamma_{rect} | \text{ for infinitely many } k, \int_\gamma \rho_{i_k} ds \geq 1/2n\}.
\end{aligned}$$

Hence it suffices to show that for each positive integer n the family of paths

$$\Gamma_n = \{\gamma \in \Gamma_{rect} | \text{ for infinitely many } k \int_\gamma \rho_{i_k} ds \geq 1/2n\}$$

has p-modulus zero. Let

$$\rho = \sum_{k=1}^{\infty} \rho_{i_k}.$$

Then by the choice of the subsequence ρ is p-integrable. For all γ in Γ_n,

$$\int_\gamma \rho ds \geq \sum_{k\in\mathbb{N}} \int_\gamma \rho_{i_k} ds = \infty$$

because for infinitely many k the integral $\int_\gamma \rho_{i_k} ds$ is greater than or equal to $1/2n$. Hence $\mathrm{Mod}_p \Gamma_n = 0$.

□

Remark 2.2.2. By Lemma 2.2.1, if ρ_i is a Cauchy sequence of non-negative Borel functions in L^p converging to ρ in L^p, then there is a subsequence ρ_{i_k} such that for p-modulus almost every path γ in Γ_{rect},

$$\lim_{k\to\infty} \int_\gamma \rho_{i_k} ds = \int_\gamma \rho ds < \infty.$$

Measure is the natural gauge for defining $L^p(X)$ functions; two functions in $L^p(X)$ are in the same L^p class if and only if the set on which they disagree is of measure zero. However, a finer gauge is needed to explore $N^{1,p}(X)$ function classes. One of the possible such gauges is p-capacity. Different definitions for a capacity of a set can be found in literature. The definition of capacity used here is based on [KM1] and [AO].

Definition 2.2.3. The *p-capacity* of a set $E \subset X$ with respect to the space $N^{1,p}(X)$ is defined by

$$\mathrm{Cap}_p E = \inf_u \|u\|^p_{N^{1,p}}, \tag{2.1}$$

where the infimum is taken over all the functions u in $N^{1,p}(X)$ whose restriction to E is bounded below by 1.

In the light of the following lemma, the discussion in [KM1] proving that the Hajłasz capacity is an outer measure is easily adaptable to show that Cap_p is indeed an outer measure. The papers [KM1] and [HeK1] explore the characteristics of sets of zero Hajłasz capacity. In particular, they discuss the Hausdorff measure properties of zero Hajłasz capacity sets. One of the benefits of knowing that Cap_p is an outer measure is that it can now be easily seen that if u_1, u_2 are two finctions in $N^{1,p}(X)$ such that $\|u_1 - u_2\|_{L^p(X)} = 0$, then $\mathrm{Cap}_p(E) = 0$ where E is the set of all points in X on which u_1 differs from u_2. In fact, this observation follows from the following two lemmata.

Lemma 2.2.4. *Suppose $\{u_i\}_{i=1}^\infty$ is a Cauchy sequence in $N^{1,p}(X)$. Then there exists a function u in $N^{1,p}(X)$ and a subsequence $\{u_{i_k}\}_{k=1}^\infty$ such that u_{i_k} converges to u both in the sense of L^p and pointwise μ-almost everywhere.*

Proof. By the fact that the sequence is Cauchy in $N^{1,p}(X)$, the sequence is also Cauchy in $L^p(X)$. Hence by passing to a subsequence if necessary, there is a function $\tilde{u}$ in $L^p(X)$ to which the subsequence converges both pointwise μ-almost everywhere and in the L^p-norm. Choose a further subsequence, also denoted $\{u_i\}_{i=1}^\infty$ for the sake of simplicity in notation, such that

$$\|u_i - \tilde{u}\|_{L^p} \le 2^{-i}, \tag{2.2}$$

$$u_i \to \tilde{u} \text{ pointwise } \mu - a.e., \tag{2.3}$$

$$\|g_{i+1,i}\|_{L^p} \le 2^{-i}, \tag{2.4}$$

where g_{ij} is an appropriately chosen upper gradient of $u_i - u_j$. If g_1 is a p-integrable upper gradient of u_1, then $u_2 = u_1 + (u_2 - u_1)$ has as an upper gradient $g_2 = g_1 + g_{12}$. In general,

$$u_i = u_1 + \sum_{k=1}^{i-1} (u_{k+1} - u_k)$$

has as a p-integrable upper gradient

$$g_i = g_1 + \sum_{k=1}^{i-1} g_{k+1,k}.$$

For $j < i$,

$$\|g_i - g_j\|_{L^p} \le \sum_{k=j}^{i-1} \|g_{k+1,k}\|_{L^p} \le \sum_{k=j}^{i-1} 2^{-k} \le 2^{-j+1} \to 0 \text{ as } j \to \infty.$$

Therefore $\{g_i\}_{i=1}^\infty$ is also a Cauchy sequence in $L^p(X)$, and hence converges in the L^p-norm to a non-negative Borel function g.

Now let a function u be defined by

$$u(x) = \frac{1}{2}\{\limsup_{i\to\infty} u_i(x) + \liminf_{i\to\infty} u_i(x)\}$$

wherever the definition makes sense. Since $u_i \to \tilde{u}$ μ-almost everywhere by (2.3), $u(x) = \tilde{u}(x)$ μ-almost everywhere, and hence u is p-integrable. Let

$$E = \{x \;:\; \limsup_{i\to\infty} |u_i(x)| = \infty\}.$$

The function u is well-defined outside of E. In order for u to be in the space $N^{1,p}(X)$ the function u has to be well-defined on almost all paths by Proposition 2.1.1. To this end it is shown that the p-modulus of the family Γ_E is zero.

Let Γ_1 be the collection of all paths γ from Γ_{rect} such that either $\int_\gamma gds = \infty$ or

$$\limsup_{i\to\infty} \int_\gamma g_i ds \neq \int_\gamma gds.$$

Then by Lemma 2.2.1, $\mathrm{Mod}_p\Gamma_1 = 0$. Recall from Section 1.3 that

$$\Gamma_E^+ = \{\gamma \in \Gamma_{rect} \;:\; \mathcal{H}_1(|\gamma| \cap E) > 0\}.$$

Since $\mu(E) = 0$ by (2.3), the p-modulus of Γ_E^+ is zero by Lemma 1.3.5. Therefore $\mathrm{Mod}_p(\Gamma_1 \cup \Gamma_E^+) = 0$. For any path γ in the family $\Gamma_{rect}\backslash(\Gamma_1 \cup \Gamma_E^+)$, by the fact that the path is not in Γ_E^+ there exists a point y in $|\gamma|$ such that y is not in E. For any point x in $|\gamma|$, since g_i is an upper gradient of u_i,

$$|u_i(x)| - |u_i(y)| \leq |u_i(x) - u_i(y)| \leq \int_\gamma g_i ds.$$

Therefore,

$$|u_i(x)| \leq |u_i(y)| + \int_\gamma g_i ds.$$

Taking limit suprema on both sides and using the fact that γ is not in Γ_1,

$$\limsup_{i\to\infty} |u_i(x)| \leq \limsup_{i\to\infty} |u_i(y)| + \int_\gamma gds < \infty,$$

and therefore x is not in E. Thus $\Gamma_E \subset \Gamma_1 \cup \Gamma_E^+$ and hence $\mathrm{Mod}_p\Gamma_E = 0$.

Finally, if γ is a path in $\Gamma_{rect}\backslash(\Gamma_1 \cup \Gamma_E^+)$, denoting the end points of γ as x and y and noting by the above argument that x and y are not in E, one has that

$$\begin{aligned}
|u(x) - u(y)| &= \frac{1}{2}|\limsup_{i\to\infty} u_i(x) - \liminf_{i\to\infty} u_i(y) + \liminf_{i\to\infty} u_i(x) - \limsup_{i\to\infty} u_i(y)| \\
&\leq \limsup_{i\to\infty} |u_i(x) - u_i(y)| \\
&\leq \lim_{i\to\infty} \int_\gamma g_i ds = \int_\gamma gds \qquad \text{since } \gamma \notin \Gamma_1.
\end{aligned}$$

Therefore g is a weak upper gradient of u, and hence u is in $N^{1,p}(X)$, proving the lemma.

□

To prove that $N^{1,p}(X)$ is Banach, the following lemma is now needed. A corollary of the following lemma is that just as functions in $L^p(X)$ are determined only up to sets of measure zero, functions in $N^{1,p}(X)$ are determined only up to p-capacity zero sets. In other words, zero p-capacity sets are removable for functions in $N^{1,p}(X)$; see Corollary 2.2.6.

Lemma 2.2.5. *If $E \subset X$ such that $Cap_p E = 0$, then $Mod_p \Gamma_E = 0$.*

Proof. Since $\mathrm{Cap}_p E = 0$, for each positive integer i there exists a function u_i in $N^{1,p}(X)$ such that $\|u_i\|_{N^{1,p}} \leq 2^{-i}$ with $u_i|_E \geq 1$. Let

$$v_n = \sum_{i=1}^{n} |u_i|.$$

Then v_n is in $N^{1,p}(X)$ for each n, and

$$\|v_n - v_m\|_{N^{1,p}} \leq \sum_{i=m+1}^{n} \|u_i\|_{N^{1,p}} \leq 2^{-m} \to 0 \qquad \text{as } m \to \infty.$$

Therefore the sequence $\{v_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $N^{1,p}(X)$ and by Lemma 2.2.4, there exists a function v in $N^{1,p}(X)$ such that $\|v_n - v\|_{L^p} \to 0$. By the construction used in Lemma 2.2.4 and by the fact that for all x the sequence $\{v_n(x)\}_{n=1}^{\infty}$ is monotonic increasing, outside of a set F such that $\mathrm{Mod}_p \Gamma_F = 0$ one can write

$$v(x) = \lim_{n\to\infty} v_n(x)$$

with $v(x)$ finite. If $E \backslash F$ is non-empty, then

$$v|_{E\backslash F} \geq v_n|_{E\backslash F} = \sum_{i=1}^{n} |u_i|\big|_{E\backslash F} \geq n$$

for arbitrarily large n, yielding that the restriction of v to the set $E\backslash F$ is infinite, which is not possible as x is not in the set F. Therefore $E\backslash F$ is empty, and hence the family of paths Γ_E is a subset of the family of paths Γ_F, and the lemma follows. □

Corollary 2.2.6. *Let E be a subset of X. If u is a function in $N^{1,p}(X\backslash E)$ and the p-capacity of E is zero, then there is an extension of u to E that is in $N^{1,p}(X)$. Any two such extensions of u to all of X are in the same equivalence class of $N^{1,p}(X)$.*

Remark 2.2.7. The converse of Lemma 2.2.5 does not hold in general. For example, if the metric measure space X has no rectifiable curves, then the p-modulus of the family Γ_X is zero. However, if the measure of X is also positive, then the p-capacity of X is not zero. Under certain conditions on X, the converse of Lemma 2.2.5 holds; see Lemma 4.2.4.

Now the main theorem of this chapter is ready to be proven.

Theorem 2.2.8. *$N^{1,p}(X)$ is a Banach space.*

Proof. Let $\{u_i\}_{i=1}^{\infty}$ be a Cauchy sequence in $N^{1,p}(X)$. To show that this sequence is a convergent sequence in $N^{1,p}(X)$ it suffices to show that some subsequence is a convergent sequence in $N^{1,p}(X)$. By Lemma 2.2.4, there exists a function u in $N^{1,p}(X)$ and a subsequence, for simplicity in notation hereafter also denoted $\{u_i\}_{i=1}^{\infty}$, such that conditions (2.2) through (2.4) of Lemma 2.2.4 are satisfied with u replacing $\tilde{u}$, where by construction

$$u(x) = \frac{1}{2}\{\limsup_{i\to\infty} u_i(x) + \liminf_{i\to\infty} u_i(x)\}$$

whenever x is not in the set E defined in Lemma 2.2.4. Passing to a further subsequence if necessary, it can also be assumed that

$$\|u_k - u_{k+1}\|_{N^{1,p}} \leq 2^{-k(p+1)/p}. \tag{2.5}$$

Let

$$E_k = \{x \in X \ : \ |u_k(x) - u_{k+1}(x)| > 2^{-k}\}.$$

Then $2^k|u_k - u_{k+1}|$ is in $N^{1,p}(X)$ and $2^k|u_k - u_{k+1}|\big|_{E_k} \geq 1$, and hence by the above inequality (2.5)

$$\text{Cap}_p E_k \leq 2^{kp}\|u_k - u_{k+1}\|_{N^{1,p}}^p \leq 2^{kp}2^{-k(p+1)} \leq 2^{-k}.$$

Let $F_j = \bigcup_{k=j}^{\infty} E_k$. Then

$$\text{Cap}_p F_j \leq \sum_{k=j}^{\infty} \text{Cap}_p E_k \leq 2^{-j+1}.$$

Therefore the p-capacity of $F = \bigcap_{j\in\mathbb{N}} F_j$ is zero. If x is a point in $X\backslash F$, there exists a positive integer j such that x is not in $F_j = \bigcup_{k=j}^{\infty} E_k$. Hence for all k in $\mathbb{N}$ such that $k \geq j$, the point x is not in E_k; for all k larger than j therefore $|u_k(x) - u_{k+1}(x)| \leq 2^{-k}$. Therefore whenever $l \geq k \geq j$ it is true that

$$|u_k(x) - u_l(x)| \leq 2^{-k+1},$$

and thus the sequence $\{u_k(x)\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and therefore is convergent to a finite number. Hence if $x \in X \backslash F$, then

$$u(x) = \lim_{k\to\infty} u_k(x).$$

For $k < m$,

$$u_m = u_k + \sum_{n=k}^{m-1}(u_{n+1} - u_n).$$

Therefore for each x in $X\backslash F$,

$$\begin{aligned} u(x) &= u_k(x) + \sum_{n=k}^{\infty}(u_{n+1}(x) - u_n(x)), \\ u(x) - u_k(x) &= \sum_{n=k}^{\infty}(u_{n+1}(x) - u_n(x)). \end{aligned} \tag{2.6}$$

Noting by Lemma 2.2.5 that $\mathrm{Mod}_p\Gamma_F = 0$ and that for each path γ in $\Gamma_{rect}\backslash\Gamma_F$ and for all points x in $|\gamma|$ equation (2.6) holds, with the notation as in Lemma 2.2.4 conclude that $\sum_{n=k}^{\infty} g_{n+1,n}$ is a weak upper gradient of $u - u_k$. Therefore

$$\begin{aligned} \|u - u_k\|_{N^{1,p}} &\le \|u - u_k\|_{L^p} + \sum_{n=k}^{\infty} \|g_{n+1,n}\|_{L^p} \\ &\le \|u - u_k\|_{L^p} + \sum_{n=k}^{\infty} 2^{-n} && \text{by condition (2.4) in Lemma 2.2.4} \\ &\le \|u - u_k\|_{L^p} + 2^{-k+1} \to 0 && \text{as } k \to \infty. \end{aligned}$$

Therefore the subsequence converges in the norm of $N^{1,p}(X)$ to u. The proof of the theorem is now complete. □

Remark 2.2.9. The proof of the above theorem did not use the fact that the definition of (weak) upper gradients was based on all the compact rectifiable paths in X. One can therefore modify the definition of Newtonian spaces by modifying the definition of (weak) upper gradients by considering a particular family of compact rectifiable paths in X, and the modified spaces will also be Banach. This is useful in considering Example 4.4.3.

In the above proof it was shown that for each positive integer j there exists a set F_j of capacity no more than 2^{-j+1} such that the chosen subsequence converged uniformly outside of F_j. Hence the following corollary holds true:

Corollary 2.2.10. *Any Cauchy sequence $\{u_i\}_{i=1}^{\infty}$ in $N^{1,p}(X)$ has a subsequence that converges pointwise outside a set of zero p-capacity. Furthermore, the subsequence can be chosen so that there exist sets of arbitrarily small p-capacity with the property that the subsequence converges uniformly in the complement of each of these sets.*

The above corollary makes it possible to apply the machinery developed in [Fu2]. See also Corollary 3.1.7.

2.3 An Alternate Characterization of $N^{1,p}(X)$.

The paper [C] defines another characterization of Sobolev spaces on metric measure spaces using only upper gradients and bypassing the notions of moduli of path families and weak upper gradients. Theorem 2.3.2 shows that this characterization yields the same space $N^{1,p}(X)$ when $p > 1$. The definition below is from [C].

Definition 2.3.1. For $p \geq 1$, the Sobolev space $H_{1,p}(X)$ is the subspace of $L^p(X)$ consisting of functions f for which the norm

$$|f|_{1,p} = \|f\|_{L^p} + \inf_{\{g_i\}} \liminf_{i\to\infty} \|g_i\|_{L^p} \tag{2.7}$$

is finite. Here the limit infimum is taken over all upper gradients (or equivalently, weak upper gradients) g_i of the functions f_i, where the sequence f_i converges in the L^p-norm to the function f.

Theorem 2.3.2. *The above space $H_{1,p}(X)$ is isometrically equivalent to $N^{1,p}(X)$ when $p > 1$.*

The following lemma from [KSh] is needed in the proof of the above theorem.

The proof of Lemma 2.2.4 yields the following result: given a Cauchy sequence of functions in $N^{1,p}(X)$ and a corresponding Cauchy sequence in L^p of respective upper gradients, the two functions that the respective sequences converge in the L^p-norm to are related as a function-upper gradient pair. The following lemma from [KSh] proves a stronger version of this result. This result can be used most of the time in place of Mazur's lemma; see [Y, Chapter V, page 120] for a statement of Mazur's lemma.

Lemma 2.3.3. *Let Y be a metric measure space and $p > 1$. If $\{f_j\}_{j\in\mathbb{N}}$ is a sequence of functions in $L^p(Y)$ with upper gradients $\{g_j\}_{j\in\mathbb{N}}$ in $L^p(Y)$ such that f_j weakly converges to f in L^p and g_j weakly converges to g in L^p, then g is a weak upper gradient of f and there is a convex combination sequence $\tilde{f}_j = \sum_{k=j}^{n_j} \lambda_{kj} f_k$ and $\tilde{g}_j = \sum_{k=j}^{n_j} \lambda_{kj} g_k$ with $\sum_{k=j}^{n_j} \lambda_{kj} = 1$, $\lambda_{kj} > 0$, so that $\tilde{f}_j$ converges in L^p to f and $\tilde{g}_j$ converges in L^p to the function g.*

Proof. Applying Mazur's lemma (see [Y, Chapter V]) to each sequence $\{f_j\}_{j=k}^{\infty}$, for each $k \in \mathbb{N}$ a sequence of convex combinations of f_j's that converge in the $L^p(Y)$-norm to f can be formed. Extracting a member from each sequence of convex combinations corresponding to each $k \in \mathbb{N}$, a sequence $\{\overline{f}_k\}_{k\in\mathbb{N}}$ can be formed so that each $\overline{f}_k$ is a convex combination of the original sequence $\{f_j\}_{j\in\mathbb{N}}$, and $\{\overline{f}_k\}_{k\in\mathbb{N}}$ converges in the $L^p(Y)$-norm to f. It is easy to see that the corresponding convex combination $\{\overline{g}_k\}_{k\in\mathbb{N}}$ of the sequence $\{g_j\}_{j\in\mathbb{N}}$ is a sequence of upper gradients of $\{\overline{f}_k\}_{k\in\mathbb{N}}$, and because of the way $\{f_k\}_{k\in\mathbb{N}}$ was formed, the sequence $\{\overline{g}_k\}_{k\in\mathbb{N}}$ converges weakly in $L^p(Y)$ to g. Next, repeating this process to the pair of sequences $\{\overline{f}_k\}_{k\in\mathbb{N}}$ and $\{\overline{g}_k\}_{k\in\mathbb{N}}$, convex combination sequences $\{\tilde{f}_i\}_{i\in\mathbb{N}}$ and $\{\tilde{g}_i\}_{i\in\mathbb{N}}$, that converge in the $L^p(Y)$-norm to f and g respectively, can be obtained, with $\tilde{g}_i$ being an upper gradient of $\tilde{f}_i$. Being convex combinations of convex combinations, the final sequences are themselves convex combinations of the original sequences.

A slight modification of the proof of Lemma 2.2.4 shows that g is a weak upper gradient of u. □

One of the consequences of the above lemma is that every ACC_p function with a p-integrable weak upper gradient has a minimal p-integrable weak upper gradient. This fact has been proved in [C]. In [C] weak upper gradients are called generalized upper gradients. An alternate proof using the definition of weak upper gradients is given here.

Corollary 2.3.4. *Let X be a metric measure space equipped with a σ-finite measure. If u is a function with a p-integrable weak upper gradient, $p > 1$, then there exists a p-integrable weak upper gradient ρ_u of u with the following property: if ρ is another weak upper gradient of u, then $\rho_u \le \rho$ almost everywhere.*

Proof. Since $p > 1$, by the reflexivity property of the space $L^p(X)$ and by the above lemma there is a p-integrable weak upper gradient ρ_u with the smallest L^p-norm amongst all the p-integrable weak upper gradients of u. Let ρ be another weak upper gradient of u, and let E be the set on which ρ_u is larger than ρ. Suppose the measure of E is positive. Then there is a closed subset F of E of positive measure; see [Sr, Lemma 3.4.14], [Fe, Theorem 2.2.2], or the beginning of the next chapter for a statement of this lemma. By Lemma 2.1.8 the function $\rho_u\chi_{X\setminus F} + \rho\chi_F$ is a weak upper gradient, of strictly smaller L^p-norm than ρ_u. This contradicts the choice of ρ_u. Hence the measure of E must be zero. □

Proof of **Theorem 2.3.2** . Clearly functions in $N^{1,p}(X)$ satisfy the above definition: the sequence f_i could be taken to be the function itself. By Lemma 2.3.3, it is also clear that functions satisfying the above definition have a L^p-representative in $N^{1,p}(X)$. Moreover, the $N^{1,p}(X)$-norm is equal to the norm (2.7). □

From Theorem 2.3.2 and from [C] the following result is obtained:

Theorem 2.3.5. *If X is a doubling metric measure space that supports a $(1,p)$-Poincaré inequality; $p > 1$, then $N^{1,p}(X)$ is reflexive.*

For the definition of a doubling measure see Definition 3.1.1, and for the definition of $(1,p)$-Poincaré inequality see Definition 3.1.2.

CHAPTER III

$N^{1,p}(X)$, $M^{1,p}(X)$, AND THE POINCARÉ INEQUALITY

In the previous chapter it was shown that even in the most general setting of the metric measure space $N^{1,p}(X)$ is a Banach space. However, if X does not have enough rectifiable paths, then $N^{1,p}(X)$ reduces to the space $L^p(X)$. This chapter attempts to answer the question: when is $N^{1,p}(X)$ a reasonable space to consider.

Just as in the previous chapters, X is a metric measure space endowed with a Borel measure μ. Throughout the rest of the thesis the open ball of radius r centered at x is denoted $B(x,r)$. If B is an open ball in X, then for $t > 0$ the set tB is the open ball concentric with B but with t times the radius of B.

It is also assumed throughout the rest of the thesis that bounded sets have finite measure and that open sets have positive measure. It should be noted that under this additional assumption μ is also an inner and an outer measure, that is, μ is Borel regular; see [Sr, Lemma 3.4.14]. A measure μ is said to be Borel regular if for each Borel set B the following holds:

$$\mu(B) = \sup\{\mu(F) \; : \; F \subset B, \; F \text{ closed } \} = \inf\{\mu(U) \; : \; B \subset U, \; U \text{ open } \}.$$

For convenience [Sr, Lemma 3.4.14] is stated below.

Lemma 3.0.6 (Sr, Lemma 3.4.14). *If X is a metrizable topological space and μ is a finite Borel measure on X, then μ is Borel regular.*

Since sets of measure zero are measurable, by the above lemma, for every set $E \subset X$ the measure of E is positive if and only if there is a closed subset $F \subset E$ whose measure is positive.

The above lemma holds true if the finiteness of μ is replaced by σ-finiteness, and for the results in this thesis this is a sufficient assumption.

3.1 Poincaré Inequality and Quasicontinuity of Newtonian Functions.

Definition 3.1.1. A metric measure space X is said to be a *doubling space* if there exists a constant $C \geq 1$ so that for each open ball $B \subset X$,

$$\mu(2B) \leq C\mu(B).$$

Note that $\mathbb{R}^n$, together with Lebesgue measure and Euclidean distance, is a doubling space.

Definition 3.1.2. The space X is said to support a *weak* $(1, p)$*-Poincaré inequality* if there exist constants $C > 0$ and $\tau \geq 1$ such that for all open balls B in X and all pairs of functions u and ρ defined on τB, whenever ρ is an upper gradient of u on τB and u is integrable on B the following inequality holds true:

$$\fint_B |u - u_B| \leq C diam(B)(\fint_{\tau B} \rho^p)^{1/p}, \tag{3.1}$$

where, if f is a measurable function on X, then

$$f_B := \frac{1}{\mu(B)} \int_B f =: \fint_B f.$$

Recall that by assumption $0 < \mu(B) < \infty$. Hence f_B is well-defined if f is integrable over B.

Note by the Hölder inequality that if a space supports a weak $(1, p)$-Poincaré inequality then it satisfies a weak $(1, q)$-Poincaré inequality for all $q > p$. For more discussion and examples of spaces with a Poincaré ineqality, see [HeK1], [HK2], [KM2], [S1], and [He]. These papers have a definition similar to the above, but requiring only that the inequality of Definition 3.1.2 be satisfied by continuous functions together with their upper gradients. The two definitions coincide under certain conditions; see [HeK2]. The following definition too, restricted to bounded continuous functions, can be found in [HeK1] and [HeK2].

Definition 3.1.3. The space X is said to support a $(1, p)$*-Poincaré inequality* if X supports a weak $(1, p)$-Poincaré inequality with $\tau = 1$.

Hajłasz and Koskela show in [HK1, Theorem 1] that if X is doubling, bounded, and satisfies an (a, M)-chain condition for some $M \geq 1$ and $a > 1$ with respect to all open balls $B \subset X$, then X supports a weak $(1, p)$-Poincaré inequality if and only if it supports a $(1, p)$-Poincaré inequality. Here X is said to satisfy an (a, M)-chain condition with respect to a ball $B_0 \subset X$ if for each point x in X there exists a sequence of open balls $\{B_i\}_{i\in\mathbb{N}}$ such that the following three conditions hold:

i) B_i is centered at x for all sufficiently large i,

ii) if the radius of B_i is r_i, then $a^{-i}M^{-1}\text{diam}X \le r_i \le a^{-i}M\text{diam}X$ for all i, and

iii) for each i, there exists an open ball $B_i' \subset B_i \cap B_{i+1}$ so that $B_i \cup B_{i+1} \subset MB_i'$.

The condition of weak $(1,p)$-Poincaré inequality is strong enough to force some nice geometric structure on the space X. For example, if X supports a weak $(1,p)$-Poincaré inequality and bounded open sets in X have positive and finite measure, then X has to be connected, since if not, then X is a disjoint union of two non-empty open sets U, V, and the characteristic function of U has the zero function as an upper gradient, violating the Poincaré inequality. In fact, there is a point y on the sphere $S(x, sr) := \{z \in X \ : \ d(x,z) = sr\}$. To see this, note that if there is no point on the sphere $S(x, sr)$ for some r, s, x, then the ball $B(x, sr)$ is pathwise disconnected from the rest of the space, and by assumption there must be some point in $X \backslash B(x, sr)$. The function $\chi_{B(x,sr)}$ then is in $N^{1,p}(X)$ and has zero as a weak upper gradient, violating the Poincaré inequality. Hence there must be a point on the sphere $S(x, sr)$. This argument also shows that every ball in X that is not all of X, is connected to the exterior of that ball by a rectifiable curve. For uses of this fact, see [C]. This argument does not imply however that the space itself is path-connected (that is, every point is connected to every other point by a curve). However, if X is in addition doubling in measure and is proper (i.e. closed and bounded sets are compact), then supporting a weak $(1,p)$-Poincaré inequality implies that if X is rectifiably connected, then X is also quasiconvex (i.e. every pair of points in X can be connected by a rectifiable curve whose length is comparable to the distance between the two points); see [HK2].

It is a classical result that smooth functions form a dense set in $W^{1,p}(\Omega)$ whenever Ω is a domain in $\mathbb{R}^n$. The following theorem is an analogue of this result for metric measure spaces supporting a weak $(1,p)$-Poincaré inequality as in Definition 3.1.2. The proof of the theorem is a modification of an idea due to Semmes, [S2].

Theorem 3.1.4. *If X is a doubling space supporting a weak $(1,p)$-Poincaré inequality, then Lipschitz functions are dense in $N^{1,p}(X)$.*

The proof of this theorem uses the following lemmata.

Lemma 3.1.5. *Let X be a doubling space, and let M^* be the non-centered maximal operator defined by*

$$M^*f(x) := \sup_B \fint_B |f|\,d\mu, \tag{3.2}$$

where the supremum is taken over balls B in X containing the point x. Then if g is a function in $L^1(X)$, then

$$\lim_{\lambda\to\infty} \lambda\ \mu(\{x \in X\ :\ M^*g(x) > \lambda\}) = 0.$$

Proof. For $R > 0$ let M_R^* be the modified non-centered maximal opertator defined by

$$M_R^* f(x) := \sup_B \fint_B |f| d\mu,$$

where the supremum is taken over all balls B in X containing the point x such that $diam(B) \le 2R$. Let g be a function in $L^1(X)$, and let

$$E_\lambda^R = \{x \in X\ :\ M_R^* g(x) > \lambda\},$$

and

$$E_\lambda = \{x \in X\ :\ M^* g(x) > \lambda\}.$$

Note that $E_\lambda^R \subset E_\lambda$. If x is a point in E_λ^R, there exists y_x in X and $r_x > 0$ so that x lies in $B(y_x, r_x)$ and

$$\fint_{B(y_x,r_x)} |g| d\mu > \lambda.$$

Note that then $B(y_x, r_x)$ is a subset of E_λ^R. The set E_λ^R is covered by such balls $B(y_x, r_x)$ with radii r_x bounded by R, and by the basic covering theorem (see [Ma, Theorem 2.1 and the remark in page 23] or [He, Section 1]) there exists a countable disjoint subcollection $\{B(x_i, r_i)\}_{i=1}^\infty$ of this covering collection of balls with the property that the collection $\{B(x_i, 5r_i)\}_{i=1}^\infty$ covers E_λ^R. Hence

$$\begin{aligned}\mu(E_\lambda^R) &\le \sum_{i=1}^\infty \mu(B(x_i, 5r_i)) \le C \sum_{i=1}^\infty \mu(B(x_i, r_i)) \\ &\le (C/\lambda) \sum_{i=1}^\infty \int_{B(x_i,r_i)} |g| = (C/\lambda) \int_{\cup_i B(x_i,r_i)} |g| \le (C/\lambda) \int_{E_\lambda^R} |g|.\end{aligned}$$

Therefore

$$\mu(E_\lambda^R) \le \frac{C}{\lambda} \int_{E_\lambda^R} |g| d\mu.$$

For $0 < R_1 < R_2$ clearly $E_\lambda^{R_1} \subset E_\lambda^{R_2}$, with $E_\lambda = \bigcup_{R>0} E_\lambda^R$. Therefore

$$\mu(E_\lambda^R) \le \frac{C}{\lambda} \int_{E_\lambda} |g| d\mu,$$

and hence taking the limit as $R \to \infty$,

$$\mu(E_\lambda) \le \frac{C}{\lambda} \int_{E_\lambda} |g| d\mu. \tag{3.3}$$

Then

$$\mu(E_\lambda) \le \frac{C}{\lambda} \int_X |g| d\mu,$$

and hence $\lim_{\lambda\to\infty} \mu(E_\lambda) = 0$. Therefore one has that $\int_{E_\lambda} |g| \to 0$ as $\lambda \to \infty$ since g is a function in $L^1(X)$, and hence one has by inequality (3.3) that

$$\lambda\mu(E_\lambda) \to 0 \qquad \text{as} \qquad \lambda \to \infty,$$

and the lemma is proved. $\square$

Fix $x_0 \in X$, and for each positive integer j consider the following function:

$$\eta_j(x) = \begin{cases} 1 & \text{if } d(x_0, x) \le j-1 \\ j - d(x_0, x) & \text{if } j-1 < d(x_0, x) < j \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that this function is 1-Lipschitz.

Lemma 3.1.6. *Suppose u is a function in $N^{1,p}(X)$. Then the function $v_j = u\eta_j$ is also in $N^{1,p}(X)$, and furthermore, the sequence v_j converges to u in $N^{1,p}(X)$.*

Proof. Let x, y be two points in X. Then,

$$\begin{aligned} |v_j(x) - v_j(y)| &\le |\eta_j(x)u(x) - \eta_j(x)u(y)| + |\eta_j(x)u(y) - \eta_j(y)u(y)| \\ &\le |u(x) - u(y)| + |u(y)||\eta_j(x) - \eta_j(y)| \\ &\le |u(x) - u(y)| + d(x,y)(|u(x)| + |u(y)|). \end{aligned}$$

Hence by Lemma 3.2.5 (see next section), if g is an upper gradient of u, then $g+4|u|$ is also an upper gradient of v_j. Since v_j vanishes on the set $X\backslash B(x_0, j)$, by Lemma 2.1.7 the function $g_j := (g+4|u|)\chi_{B(x_0,j)}$ is a weak upper gradient of v_j. Since $u = v_j$ on the set $\overline{B(x_0, j-1)}$ and $g + g_j$ is a weak upper gradient of $u - v_j$, therefore $(g + g_j)\chi_{X\backslash B(x_0,j-1)}$ is an upper gradient of $u - v_j$ by Lemma 2.1.7. Now,

$$\|u - v_j\|_{L^p(X)} \le \left(\int_{X\backslash B(x_0,j-1)} |u - v_j|^p \right)^{1/p} \le 2 \left(\int_{X\backslash B(x_0,j-1)} |u|^p \right)^{1/p} \to 0,$$

since u is p-integrable. Moreover,

$$\left(\int_X [(g+g_j)\chi_{X\setminus B(x_0,j-1)}]^p\right)^{1/p} \le 2\left(\int_{X\setminus B(x_0,j-1)} g^p\right)^{1/p} +4\left(\int_{X\setminus B(x_0,j-1)} |u|^p\right)^{1/p} \to 0 \text{ as } j\to\infty$$

since the two functions g and $|u|$ are p-integrable. Now it can be concluded that $\|u-v_j\|_{N^{1,p}(X)} \to 0$ as j tends to infinity. □

Proof of **Theorem 3.1.4**. Let u be a function in $N^{1,p}(X)$. By the above lemma, without loss of generality it can be assumed that u is zero outside of a bounded set. Let

$$E_\lambda = \{x\in X \;:\; M^*g^p(x) > \lambda^p\},$$

where g is a p-integrable upper gradient of u. Then by Lemma 3.1.5,

$$\lambda^p\mu(E_\lambda)\to 0 \qquad \text{as} \qquad \lambda\to\infty. \tag{3.4}$$

If x is a point in $X\setminus E_\lambda$, then for all $r>0$ one has that

$$\fint_{B(x,r)} |u-u_{B(x,r)}| \le Cr(\fint_{\tau B(x,r)} g^p)^{1/p} \le Cr(M^*g^p(x))^{1/p} \le Cr\lambda.$$

Here τ is the constant relevant to the weak Poincaré inequality. Hence for $s\in[r/2,r]$ one has that whenever $x\in X\setminus E_\lambda$,

$$\begin{aligned}|u_{B(x,s)}-u_{B(x,r)}| &\le \fint_{B(x,s)} |u-u_{B(x,r)}| \\ &\le \frac{\mu(B(x,r))}{\mu(B(x,s))}\fint_{B(x,r)} |u-u_{B(x,r)}| \\ &\le C\lambda r.\end{aligned}$$

By a chaining argument for any positive $s<r$ (i.e. bounding s to be in an interval $[r/2^n, r/2^{n-1}]$ and then using triangle inequalities to move up to the radius r), for x in $X\setminus E_\lambda$ it is seen that

$$|u_{B(x,s)}-u_{B(x,r)}| \le C\lambda r$$

and hence any sequence $u_{B(x,r_i)}$ is a Cauchy sequence in $\mathbb{R}$ and hence is convergent. Therefore on $X\setminus E_\lambda$ the following function can be defined:

$$u_\lambda(x) := \lim_{r\to 0} u_{B(x,r)}.$$

Since the measure is doubling, almost every point in X is a Lebesgue point of u; see [Ma, Theorem 2.12], [Fe, Corollary 2.9.9], or the appendix. Note that at Lebesgue points of u in $X\backslash E_\lambda$ it is true that $u_\lambda = u$, and that E_λ is an open set. Hence $u - u_\lambda$ satisfies the hypotheses of Lemma 2.1.7. For $x, y \in X$ consider the chain of balls $\{B_i\}_{i=-\infty}^{\infty}$, where

$$B_1 = B(x, d(x,y)) \qquad \text{and} \qquad B_{-1} = B(y, d(x,y))$$

and inductively for $i > 1$ obtain

$$B_i = (1/2)B_{i-1} \qquad \text{and} \qquad B_{-i} = (1/2)B_{-i+1}.$$

If x and y are in $X\backslash E$, then by construction,

$$|u_\lambda(x) - u_\lambda(y)| \le \sum_{i=-\infty}^{\infty} |u_{B_i} - u_{B_{i+1}}| \le C\lambda d(x,y).$$

Hence u_λ is $C\lambda$-Lipschitz on $X\backslash E_\lambda$. Extend u_λ as a $C\lambda$-Lipschitz extension to the entire X; see [MS] for existence of such extensions. Choose an extension such that u_λ is bounded by $2C\lambda$. This can be done by truncating any Lipschitz extension at $C\lambda$. Such truncation will not affect the values of u_λ on the set $X\backslash E_\lambda$ as for large enough λ the original function u is bounded on its Lebesgue points of $X\backslash E_\lambda$, since u itself is zero outside a bounded set and hence the non-zero values of u_λ in $X\backslash E_\lambda$ lie within a bounded set which is independent of λ, and u_λ is $C\lambda$-Lipschitz.

Now,

$$\begin{aligned}\int_X |u - u_\lambda|^p = \int_{E_\lambda} |u - u_\lambda|^p &\le C\int_{E_\lambda} |u|^p + C\int_{E_\lambda} |u_\lambda|^p \\ &\le C\int_{E_\lambda} |u|^p + C\lambda^p \mu(E_\lambda).\end{aligned}$$

By the fact that $\mu(E_\lambda) \to 0$ as $\lambda \to \infty$, by the fact that u is p-integrable, and by (3.4), it can be observed that the two terms on the right hand side of the above inequality tends to zero as λ tends to infinity. Hence u_λ converges to u as λ tends to infinity, the convergence occuring in the L^p-norm. The non-zero values of $u - u_\lambda$ are obtained only at points in the open set E_λ and on the set L whose measure is zero, and therefore by Lemma 2.1.7 the function $(C\lambda + g)\chi_{E_\lambda}$ is a weak upper gradient of $u - u_\lambda$. Hence $u - u_\lambda$ is in $N^{1,p}(X)$, and therefore so is u_λ. Since by (3.4),

$$\int_X |\lambda\chi_{E_\lambda}|^p = \lambda^p \mu(E_\lambda) \to 0$$

and

$$\int_X |g\chi_{E_\lambda}|^p = \int_{E_\lambda} |g|^p \to 0$$

as $\lambda \to \infty$, the sequence u_λ converges to u in $N^{1,p}(X)$.

$\square$

The following corollary is obtained by Corollary 2.2.10 and the above theorem.

Corollary 3.1.7. *If X is a doubling metric measure space that supports a $(1,p)$-Poincarè inequality, then for each function u in $N^{1,p}(X)$ there are open sets of arbitrarily small capacity such that u is continuous in the complement of these sets.*

Such continuity property is called *quasicontinuity.* The classical Sobolev spaces and Hajłasz spaces are composed of L^p-equivalence classes of functions, with each equivalence class containing a quasicontinuous function; see [EG, Section 4.2.1]. Due to the approach taken in this thesis in defining Newtonian spaces, the equivalence classes in the Newtonian spaces consist solely of quasicontinuous functions whenever X supports a $(1,p)$-Poincaré inequality and is not too thin in the sense of the above theorem. In other words, if X is not too thin with respect to μ and supports a $(1,p)$-Poincaré inequality, then functions in $\tilde{N}^{1,p}(X)$ are automatically quasicontinuous. The uses of the quasicontinuity properties are many. For example, [KSh] uses this fact to prove that in a wide class of metric measure spaces the capacity estimate and the modulus estimate are equal. See also Chapter V for an additional use of quasicontinuity.

3.2 $N^{1,p}(X)$ and $M^{1,p}(X)$.

Theorem 3.2.1. *If $X = \Omega$ is a domain in $\mathbb{R}^n$, $d(x,y) = |x-y|$, and μ is Lebesgue n-measure, then as Banach spaces $N^{1,p}(X)$ and $W^{1,p}(\Omega)$ are isometrically equivalent.*

Proof. Fuglede proved in [Fu1] that $W^{1,p}(\Omega) \subset N^{1,p}(\Omega)$. See also [O] and [V].

Suppose $u \in N^{1,p}(X)$. Then by Proposition 2.1.1, u has property ACC_p and has a p-integrable weak upper gradient ρ in L^p. Therefore u is ACL with principal directional gradient matrix ∇u. By applying the fundamental theorem of calculus and a Lebesgue point argument one easily sees that $|\nabla u| \le \rho$ *a.e.* Hence u has property ACL_p and hence by [Z, Theorem 2.1.4], $u \in W^{1,p}(\Omega)$.

$\square$

The following Lemma has an easily verifiable proof.

Lemma 3.2.2. *If $u \in M^{1,p}(X)$, $p \ge 1$, with a Hajłasz gradient g, then there exists two functions $\tilde{u}, \tilde{g}$ such that $u = \tilde{u}$ a.e., and $\|g\|_p = \|\tilde{g}\|_p$, and for all points x, y in X*

$$|\tilde{u}(x) - \tilde{u}(y)| \le d(x,y)(\tilde{g}(x) + \tilde{g}(y)).$$

Furthermore, if u is a continuous function in $M^{1,p}(X)$, then it is possible to choose $\tilde{u} = u$ everywhere.

Inequality (1.1) defining the space $M^{1,p}(X)$ in Definition 1.2.1 is required to hold only almost everywhere. Hence $M^{1,p}(X)$ is a collection of equivalence classes of functions, with two functions belonging to the same equivalence class if and only if they differ only on a set of measure zero.

The idea for the proof of the following lemma is from [H2, Proposition 1].

Lemma 3.2.3. *The set of all equivalence classes of continuous functions u in the space $M^{1,p}(X)$ embeds into $N^{1,p}(X)$, with*

$$\|u\|_{N^{1,p}(X)} \leq 4\|u\|_{M^{1,p}(X)}.$$

Proof. Suppose u is a continuous representative of its equivalence class in $M^{1,p}(X)$. Then by Lemma 3.2.2, for each Hajłasz gradient of u there exists a function g in L^p with the same L^p-norm such that inequality (1.1) holds true everywhere. Let $x, y \in X$ and γ be an arc-length parametrizing rectifiable path connecting x to y. If $\int_\gamma g = \infty$, then it is clear that $|u(x) - u(y)| \leq \int_\gamma g$. So suppose the integral of g over γ is finite. For each number n in $\mathbb{N}$ let σ_n be the partition of the domain of γ into n pieces of equal length. On each partition $\gamma_i = \gamma\big|_{\sigma_n(i),\sigma_n(i+1)}$, $0 \leq i \leq n-1$, there exists x_i in $|\gamma_i|$ such that

$$g(x_i) \leq \fint_{\gamma_i} g\,ds.$$

Note that $d(x_i, x_{i+1}) \leq 2l(\gamma_i)$. Using these points x_i, one has that

$$\begin{aligned}
|u(x_0) - u(x_n)| \leq \sum_{i=0}^{n-1} |u(x_i) - u(x_{i+1})| &\leq \sum_{i=0}^{n-1} d(x_i, x_{i+1})(g(x_i) + g(x_{i+1})) \\
&\leq 4\sum_{i=0}^{n} l(\gamma_i) \fint_{\gamma_i} g\,ds \\
&\leq 4\sum_{i=0}^{n} \int_{\gamma_i} g\,ds = 4\int_\gamma g\,ds.
\end{aligned}$$

Now, as u is continuous, by letting $n \to \infty$ the following inequality is obtained:

$$|u(x) - u(y)| \leq 4\int_\gamma g\,ds.$$

Therefore the continuous representative u of its equivalence class in $M^{1,p}(X)$ belongs to an equivalence class in $N^{1,p}(X)$, with $\|u\|_{N^{1,p}} \leq 4\|u\|_{M^{1,p}}$. By Corollary 2.1.3, if any representative in the equivalence class of u in $M^{1,p}(X)$ belongs to an equivalence class in $N^{1,p}(X)$, then it belongs to the same equivalence class as u in $N^{1,p}(X)$. Hence the embedding is well-defined.

□

Theorem 3.2.4. *The space $M^{1,p}(X)$ continuously embeds into the space $N^{1,p}(X)$.*

Proof. By Theorem 2.2.8 the space $N^{1,p}(X)$ is a Banach space. Hence the closure of the subspace of equivalence classes of continuous functions in $M^{1,p}(X)$ in the norm of $M^{1,p}(X)$ yields a subspace of $N^{1,p}(X)$ by Lemma 3.2.3. By [H1] Lipschitz functions and therefore continuous functions are dense in $M^{1,p}(X)$ and hence such closure is the Hajłasz space $M^{1,p}(X)$, yielding the required result.

□

It would be interesting to know whether the embedding norm 4 in the above theorem can be improved.

The proof of Lemma 3.2.3 also yields the following lemma. Its proof is similar to the proof of Lemma 3.2.3 and hence will be omitted here.

Lemma 3.2.5. *If u is a function on X such that there exist non-negative Borel-measurable functions g, h on X with the property that*

$$|u(x) - u(y)| \leq \int_\gamma g ds + d(x,y)(h(x) + h(y))$$

whenever γ is a compact rectifiable path in X with end points x, y, then $g + 4h$ is an upper gradient of u.

There are metric measure spaces X for which the spaces $M^{1,p}(X)$ and $N^{1,p}(X)$ do not agree. For example, if X is a unit disc in $\mathbb{R}^2$ with the radius $[0,1) \times \{0\}$ removed, then $N^{1,p}(X)$ does not embed into $M^{1,p}(X)$. For example, representing points in X by polar coordinates; $x = (r, \theta)$, $0 < r < 1$ and $0 < \theta < 2\pi$; the function $u((r,\theta)) = r\theta$ is in $N^{1,p}(X)$ for all finite $p \geq 1$ since the function is bounded with bounded derivatives in X, but u is not in $M^{1,p}(X)$ when $p \geq 1$ as $M^{1,p}(X) = M^{1,p}(B(0,1)) \subset N^{1,p}(B(0,1))$ and u is not ACL on the unit ball $B(0,1)$.

For more examples of spaces X where the Hajłasz spaces do not coincide with the Newtonian spaces, see Examples 4.4.1 and 4.4.2.

The following theorem gives a criterion under which $N^{1,p}(X)$ agrees with $M^{1,p}(X)$. The proof of the following theorem is obtained by Theorem 3.2.4, [KM2, Theorem 4.5], and the fact that if X supports a weak $(1,q)$-Poincaré inequality for some q in $[1,p)$ then $N^{1,p}(X) \subset P^{1,p}(X) \cap L^p(X)$. Here $P^{1,p}(X)$ is the vector space of p-integrable functions u that together with some p-integrable non-negative function ρ, not necessarily an upper gradient of u, satisfy the weak $(1,p)$-Poincaré inequality (3.1) on each open ball B. The better Poincaré inequality ($q < p$) is required in order to apply [KM2, Theorem 4.5]. While [KM2] assumes X to be proper (that is, closed balls are compact), their proof of Theorem 4.5 does not need this assumption, for they consider a modified version of the Korevaar and Schoen space, [KS].

Theorem 3.2.6. *If X is a metric measure space equipped with a doubling measure, and X supports a weak $(1,q)$- Poincaré inequality for some $q \in (1,p)$, then as sets*

$$M^{1,p}(X) = N^{1,p}(X) = P^{1,p}(X) \cap L^p(X).$$

Moreover, $N^{1,p}(X)$ and $M^{1,p}(X)$ are isomorphically equivalent as Banach spaces.

3.3 Classical Sobolev Embedding Theorem

When $X = \mathbb{R}^n$, d the Euclidean metric, and μ the Lebesgue n-measure, the following classical embeddings are well known:

$$\begin{aligned} W^{1,p}(X) &\hookrightarrow L^{\frac{np}{n-p}}(X) \qquad \text{if } p < n, \\ W^{1,p}(X) &\hookrightarrow C^{0,1-n/p}(X) \qquad \text{if } p > n, \end{aligned}$$

where $C^{0,\alpha}(X)$ is the collection of all real-valued functions u on X for which there exists a constant $C > 0$, depending only on the function u, such that for each pair of points x, y in X the following inequality holds:

$$|u(x) - u(y)| \leq Cd(x,y)^{\alpha}.$$

The functions in $C^{0,\alpha}(X)$ are said to be Hölder continuous with exponent α. Under certain conditions on the space X this section looks at the possiblity of obtaining similar embedding theorems. See [HK1] and [HK2] for similar results for the Hajłasz spaces.

Theorem 3.3.1. *Let $Q > 0$. If X is a doubling space satisfying $\mu(B(x,r)) \geq Cr^Q$, with C independent of $x \in X$ and $0 < r < 2 diam X$, and supporting a weak $(1,p)$-Poincaré inequality for some $p > Q$, then $N^{1,p}(X)$ continuously embeds into the space $C^{0,1-Q/p}(X)$.*

Proof. Let u be a function in $N^{1,p}(X)$. For $x, y \in X$ consider the chain of balls $\{B_i\}_{i=-\infty}^{\infty}$, where

$$B_1 = B(x, d(x,y)) \qquad \text{and} \qquad B_{-1} = B(y, d(x,y))$$

and inductively for $i > 1$ obtain

$$B_i = (1/2)B_{i-1} \qquad \text{and} \qquad B_{-i} = (1/2)B_{-i+1}.$$

If x and y are also Lebesgue points of u, the following is obtained:

$$\begin{aligned}|u(x)-u(y)| &\leq \sum_{i=-\infty}^{\infty}|u_{B_i}-u_{B_{i+1}}| \\ \text{with } |u_{B_i}-u_{B_{i+1}}| &\leq C\ diam(2B_i)(⨍_{2\tau B_i}\rho^p)^{1/p} \\ &\leq C(diam(2B_i))^{1-Q/p}(\int_{2\tau B_i}\rho^p)^{1/p} \\ &\leq C(diam(2B_i))^{1-Q/p}\|\rho\|_{L^p}.\end{aligned}$$

Therefore

$$\begin{aligned}|u(x)-u(y)| &\leq Cd(x,y)^{1-Q/p}(\sum_{i=-\infty}^{\infty}2^{-|i|(1-Q/p)})\|\rho\|_{L^p} \\ &\leq C(Q,p)\|\rho\|_{L^p}d(x,y)^{1-Q/p}.\end{aligned}$$

Let L be the set of non-Lebesgue points of u. Since X is doubling, $\mu(L)=0$; see the appendix. By the above argument $u\big|_{X\setminus L}$ is Hölder continuous with index $1-Q/p$, and hence by [MS], can be extended as a Hölder continuous function $\tilde{u}$ to all of X. Note that by Lemma 1.3.5 the p-modulus of the collection Γ_L^+ is zero. If Γ_0 is the collection of curves on which u is not absolutely continuous, then $\text{Mod}_p\Gamma_0=0$. If γ is a path in the collection $\Gamma_{rect}\setminus(\Gamma_L^+\cup\Gamma_0)$, then on $|\gamma|$ almost everywhere with respect to the one dimensional Hausdorff measure it is true that $\tilde{u}=u$. As u and $\tilde{u}$ are both continuous on $|\gamma|$, the two functions u and $\tilde{u}$ must agree on all of $|\gamma|$. Therefore if E is the collection of all points on which the two functions do not agree, then the p-modulus of the collection Γ_E is zero. Hence $\tilde{u}$ is in $N^{1,p}(X)$ and belongs to the same equivalence class as u. □

Definition 3.3.2. Let $Q>0$. A metric measure space X is said to be *Ahlfors Q-regular* (or Q-regular) if there exists a constant $C\geq 1$ so that for each point x in X and for each positive $r<2\ \text{diam}X$,

$$\frac{1}{C}r^Q \leq \mu(B(x,r)) \leq Cr^Q.$$

Theorem 3.3.3. *If X is bounded, Q-regular with $Q>1$, and supports a weak $(1,q)$-Poincaré inequality for some q such that $1<q<p$ and $1<p/q<Q$, then if u is in $N^{1,p}(X)$ and ρ is an upper gradient of u, then*

$$\left(\int_X |u(x)-u_X|^t d\mu(x)\right)^{1/t} \leq C\,diam X^{1-1/q}\|\rho\|_{L^p},$$

where $t = Qpq/(Qq-p)$ and C is a constant dependent only on the constant in the formulation of the $(1,q)$-Poincaré inequality, the numbers p and q, and the doubling constant.

The condition $p/q < Q$ is easy to satisfy as one can always increase q while keeping the validity of weak $(1,q)$-Poincaré inequality. It is important that $q > 1$, since the constant C calculated in the following proof tends to infinity as $q \to 1$. However, this condition too is easy to satisfy for the same reason as above.

Proof. For Lebesgue points x in the space X consider the collection of balls $\{B_i\}_{i=0}^{\infty}$ such that $X = B_0 = B(x, \mathrm{diam}X)$ and for each $i > 0$ the ball $B_i = B(x, 2^{-i}\mathrm{diam}X)$. Then,

$$\begin{aligned}|u(x) - u_X| \le \sum_{i=0}^{\infty} |u_{B_i} - u_{B_{i+1}}| &\le C\sum_{i=0}^{\infty} 2 \fint_{B_i} |u(z) - u_{B_i}| dz \\ &\le \sum_{i=0}^{\infty} C diam(B_i) (\fint_{\tau B_i} \rho^q)^{1/q} \\ &\le \sum_{i=0}^{\infty} C 2^{-i(1-1/q)} \mathrm{diam}X^{1-1/q} (\int_{\tau B_i} \frac{\rho(z)^q}{d(x,z)^{Q-1}} dz)^{1/q} \\ &\le C \mathrm{diam}X^{1-1/q} (\int_X \frac{\rho^q(z)}{d(x,z)^{Q-1}} dz)^{1/q}.\end{aligned}$$

The last integral is the Riesz potential $I_1(\rho^q)(x)$, and since in Q-regular spaces the operator I_1 is a bounded operator from L^s to $L^{Qs/(Q-s)}$ for $s < Q$, the last integral yields a function in $L^{Qpq/(Qq-p)}$; see [Z] and [He] for properties of Riesz potentials. Thus the theorem is proved.

□

Corollary 3.3.4. *If X satisfies the hypotheses of Theorem 3.3.3 with $p = Q$ and $1 < q < Q$, then there exists a constant $C > 0$, dependent only on the constant in the formulation of the Poincaré inequality, the doubling constant, and q, such that for each function $u \in N^{1,p}(X)$ and its p-integrable upper gradient ρ,*

$$\left(\fint_X |u(x) - u_X|^t d\mu \right)^{1/t} \le C \|\rho\|_{L^p},$$

where $t = \frac{Qq}{q-1}$.

Proof. Note that as X is Q-regular,

$$\mathrm{diam}X^{1-1/q} = \mathrm{diam}X^{Q/t} \le C(\mu(X))^{1/t}. \tag{3.5}$$

Now the corollary follows from Theorem 3.3.3. □

If X is a Q-regular metric measure space supporting a weak $(1,Q)$-Poincaré Inequality, then the functions in $N^{1,Q}(X)$ are in the BMO space, since by the Poincaré inequality, if B is an open ball in X of radius R, then

$$\fint_B |u-u_B| \le CR\left(\fint_B \rho^Q\right)^{1/Q} \le CR\frac{1}{\mu(B)^{1/Q}}\left(\int_B \rho^Q\right)^{1/Q} \le C\|\rho\|_{L^Q}$$

whenever u is in $N^{1,Q}(X)$ with ρ a Q-integrable weak upper gradient. The BMO space is the space of real-valued functions f on X for which there is a constant $C_f > 0$ so that the integral average $\fint_B |u-u_B|$ is no more than C_f for each ball B in X. The smallest possible such constant C_f is called the BMO norm of the function f. In the above situation, $N^{1,Q}(X)$ embeds continuously into the BMO space.

CHAPTER IV

MEC_p SPACES AND MORE PROPERTIES OF NEWTONIAN SPACES

Just as in the previous chapter, here too it is assumed that μ is a Borel measure on a metric measure space X such that open sets have positive measure and bounded sets have finite measure.

This chapter explores a property, called an MEC_p criterion, under which the Newtonian spaces are non-trivial, that is, the Newtonian spaces are strictly smaller than the Lebesgue spaces $L^p(X)$.

4.1 Capacity Definitions

Definition 4.1.1. An alternative definition for p-capacities of subsets E of X is as follows:

$$\mathrm{Cap}_p^* E = \inf_u \|u\|_{N^{1,p}}^p,$$

where the infimum is taken over all functions u in $N^{1,p}(X)$ such that for p-almost all paths γ intersecting E the limit of $u \circ \gamma(t)$ along γ as $\gamma(t)$ and converges to any intersecting point in E exists and is not smaller than 1. This definition in Euclidean spaces was used in [AO].

Another definition of capacity, $\mathrm{Cap}_p^{**} E$, is obtained when the corresponding infimum is taken over all the functions u in $N^{1,p}$ that are bounded below by 1 in a neighbourhood of E.

Aikawa and Ohtsuka show in [AO, Theorem 5] that under certain conditions on the measure these two definitions of capacity agree for subsets of bounded domains in $\mathbb{R}^n$. Recall the easily provable fact that if $\lambda \in \mathbb{R}$ and $u \in N^{1,p}(X)$ then the function $v = \min\{u, \lambda\}$ is also in $N^{1,p}(X)$ with any weak upper gradient of u also being a weak upper gradient of v. Hence the condition "≥ 1" can be replaced with the condition "$= 1$" in the above definitions of capacity. By the same reasoning, it can also be assumed that these test functions u are non-negative.

Lemma 4.1.2. *If $E \subset X$, then $Cap_p^{**}E \geq Cap_pE \geq Cap_p^*E$.*

Proof. Let u be any function in $N^{1,p}(X)$ such that $u|_E \geq 1$. Then as u is ACC_p by Proposition 2.1.1, it is also an admissible test function in determining Cap_p^*E. The first inequality $\mathrm{Cap}_p^{**}E \geq \mathrm{Cap}_pE$ is obvious. □

The proof of the following lemma is similar to the arguments found in [KSh].

Proposition 4.1.3. *If X satisfies the conclusions of Theorem 3.1.4 and E is a subset of X, then $Cap_p^{**}E = Cap_pE$.*

Proof. In light of the above lemma, it suffices to show that $\mathrm{Cap}_p^{**}E \leq \mathrm{Cap}_pE$. Let u be a function in $N^{1,p}(X)$ such that $0 \leq u \leq 1$, $u\big|_E = 1$, and $\mathrm{Cap}_pE \geq \|u\|_{N^{1,p}}^p - \epsilon$. By Corollary 3.1.7, the function u is quasi-continuous. Therefore there exists an open set $F_\epsilon \subset X$ such that $\mathrm{Cap}_pF_\epsilon \leq \epsilon$ and u is continuous on $X \setminus F_\epsilon$. Hence there exists an open set $U_\epsilon \subset X$ such that the restriction of u to $U_\epsilon \setminus F_\epsilon$ is not less than $1-\epsilon$ and $E \subset U_\epsilon$. Let w_ϵ be a function in $N^{1,p}(X)$ such that $w_\epsilon\big|_{F_\epsilon} = 1$, $0 \leq w_\epsilon \leq 1$, and $\|w_\epsilon\|_{N^{1,p}}^p \leq 2\epsilon$. Let

$$v_\epsilon = \frac{1}{1-\epsilon}u + w_\epsilon.$$

Then $v_\epsilon\big|_{U_\epsilon} \geq 1$ and it is in $N^{1,p}(X)$. Therefore

$$(\mathrm{Cap}_p^{**}E)^{1/p} \leq \|v_\epsilon\|_{N^{1,p}} \leq \frac{1}{1-\epsilon}\|u\|_{N^{1,p}} + \|w_\epsilon\|_{N^{1,p}},$$

and hence

$$(\mathrm{Cap}_p^{**}E)^{1/p} \leq \frac{1}{1-\epsilon}(\mathrm{Cap}_pE + \epsilon)^{1/p} + (2\epsilon)^{1/p}.$$

Letting $\epsilon \to 0$ completes the proof of the lemma. □

4.2 MEC_p Spaces.

The following definition for functions in $\mathbb{R}^n$ is due to Ohtsuka, [O].

Definition 4.2.1. Let ρ be a p-integrable non-negative Borel function in X. Such a function defines an equivalence relation $\sim_\rho$ as follows: For $x, y \in X$, $x \sim_\rho y$ if either $y = x$ or there exists a path γ in Γ_{rect} connecting x to y such that $\int_\gamma \rho\, ds < \infty$. It is easy to see that this is indeed an equivalence relation, and $\sim_\rho$ partitions X into equivalence classes.

A metric measure space X is said to admit the *Main Equivalence Class* property with respect to p, or MEC_p, if each p-integrable non-negative Borel function ρ generates an equivalence class G_ρ, hereafter referred to as the *main equivalence class* of ρ, such that $\mu(X \setminus G_\rho) = 0$.

It has been shown in [O] that $\mathbb{R}^n$ has the MEC_p-property for all p.

Proposition 4.2.2. *If X is an MEC_p space, then $Cap_p E = Cap_p^* E$.*

Proof. Suppose u is a function in $N^{1,p}(X)$ such that for p-almost every path γ in Γ_E

$$\lim_{\gamma(t)\to|\gamma|\cap E}(u\circ\gamma(t)) \geq 1.$$

Let E_1 be the set of all points x in E such that $u(x)$ is strictly less than 1. Then for each γ in Γ_{E_1} either

$$\lim_{\gamma(t)\to|\gamma|\cap E_1}(u\circ\gamma(t)) < 1$$

or else, either the limit does not exist or the limit exists and is greater than or equal to 1 while u is less than 1 at the limiting point in $|\gamma|\cap E_1$; that is, u is not absolutely continuous on γ. By the choice of u and by Proposition 2.1.1 therefore $\text{Mod}_p\Gamma_{E_1} = 0$. By Lemma 1.3.4 and by the fact that X is an MEC_p space it follows that the measure of E_1 is zero. Therefore the value of u can be adjusted on E_1 to be greater than or equal to 1 to obtain a $N^{1,p}(X)$-function in the same $N^{1,p}(X)$-equivalence class as u but with the property of being greater than or equal to 1 on all of E. Hence $\text{Cap}_p E \leq \text{Cap}_p^* E$. By Lemma 4.1.2 the result follows. □

Corollary 4.2.3. *If X is an MEC_p space and Lipschitz functions are dense in $N^{1,p}(X)$, then*

$$Cap_p E = Cap_p^* E = Cap_p^{**} E.$$

Proof. This follows by the above proposition and by Proposition 4.1.3. □

Note that if the measure on X is doubling and supports a weak $(1,p)$-Poincaré inequality, then Lipschitz functions are dense in $N^{1,p}(X)$ (see Theorem 2.2.4), and hence if X is in addition path-connected and hence is MEC_p (see Theorem 4.2.14 and [HK2]), then the conclusion of the above corollary holds.

Lemma 4.2.4. *Let X be an MEC_p-space containing at least two distinct points. If E is a subset of X, then $Mod_p\Gamma_E = 0$ if and only if $Cap_p E = 0$.*

Proof. Suppose $E \subset X$ such that $\text{Mod}_p\Gamma_E = 0$. Then by Lemma 1.3.4 there exists a p-integrable non-negative Borel function ρ such that for all γ in Γ_E the integral $\int_\gamma \rho ds$ is infinite. By the MEC_p property of X, ρ has a main equivalence class G_ρ. By the assumption made at the beginning of this chapter that open sets have positive measure and by the assumption that X contains two distinct points and therefore two disjoint open sets, G_ρ has more than one point. If x is in E and $y \neq x$ is in

G_ρ, then any path connecting x to y is in Γ_E and therefore by the choice of ρ it is easy to see that $x \not\sim_\rho y$. Hence E is a subset of $X \backslash G_\rho$. Thus $\mu(E) = 0$. Therefore the function $u = \chi_E$ is in L^p and is absolutely continuous on all the paths in Γ_{rect} that are not in Γ_E. In addition, the zero function is a weak upper gradient of u, and hence u is in $N^{1,p}(X)$. Hence

$$\mathrm{Cap}_p E \leq \|u\|^p_{N^{1,p}} = 0.$$

Now suppose that $E \subset X$ such that $\mathrm{Cap}_p E = 0$. Then by Lemma 2.2.5 the p-modulus of Γ_E is zero.

□

Corollary 4.2.5. *If X is an MEC_p space containing at least two distinct points, then $Mod_p \Gamma_{rect}$ is strictly positive.*

Proof. Since the measure of X is positive, the p-capacity of X is not zero; for each admissible function u chosen in calculating $\mathrm{Cap}_p X$,

$$\|u\|^p_{N^{1,p}} \geq \mu(X),$$

and hence taking the infimum over all such admissible functions,

$$\mathrm{Cap}_p X \geq \mu(X) > 0.$$

Hence by Lemma 4.2.4 the p-modulus of Γ_{rect} is not zero. □

The following lemma gives a criterion on X under which the Newtonian spaces are strictly smaller than the Lebesgue spaces.

Lemma 4.2.6. *If X is an MEC_p-space containing at least two distinct points, then there exists an equivalence class of functions in L^p such that no function u in this equivalence class is in an equivalence class of $N^{1,p}(X)$. In particular, $N^{1,p}(X)$ is strictly smaller than the space $L^p(X)$.*

Proof. Note that under the hypothesis above, there exists an open ball B in X so that $\mu(B) > 0$ (by the standing assumption made at the beginning of this chapter) and $\mu(X \backslash B) > 0$. Let $\tilde{u} = \chi_B$ and $[\tilde{u}]$ its equivalence class in L^p; here $\|\tilde{u}\|_{L^p} = (\mu(B))^{1/p} < \infty$. Suppose u is a function in this equivalence class that also belongs to $\tilde{N}^{1,p}$. Then $u(x) = 1$ for almost all x in B and $u(x) = 0$ for almost all x in $X \backslash B$. Let

$$E = \{x \in X \ : \ u(x) \neq \tilde{u}(x)\}.$$

As u is in the same L^p-equivalence class as $\tilde{u}$ one can conclude that $\mu(E) = 0$. Hence

$$\mathrm{Mod}_p(\Gamma_E^+ \cup \Gamma_u) = 0$$

where Γ_u is the collection of paths on which u is not absolutely continuous (Proposition 2.1.1), and so by Lemma 1.3.4 there exists a non-negative Borel-measurable p-integrable function ρ so that for all paths γ in $\Gamma_E^+ \cup \Gamma_u$ the integral $\int_\gamma \rho ds$ is infinite. As X is an MEC_p space ρ has a main equivalence class $G : \mu(X \backslash G) = 0$. Thus there is a point x in B and a point y in $X \backslash B$ so that x and y are both in G: there is a rectifiable path γ connecting x to y so that $\int_\gamma \rho ds < \infty$. By the choice of ρ one then has that γ is in neither Γ_E^+ nor in Γ_u, and hence u is absolutely continuous on γ and

$$\mathcal{H}_1(|\gamma| \cap B \cap E) = 0 = \mathcal{H}_1(|\gamma| \cap (X \backslash B) \cap E).$$

Let x_0 be the point in $|\gamma|$ at which γ first leaves the closed set $X \backslash B$ (such a point exists since $|\gamma|$ is a compact set). The function u however cannot be continuous at x_0 as every neighbourhood in $|\gamma|$ of x_0 contains points at which u is zero and also points at which u is 1. Thus u cannot be in $\tilde{N}^{1,p}$. $\square$

The above discussion shows that the MEC_p property is an ideal criterion on X in order to consider $N^{1,p}(X)$ as a useful function space. The following discussion gives criteria under which X is an MEC_p space.

Definition 4.2.7. Let $Q \geq 1$. The space X is said to be a *Q-Loewner space* if X is path-connected and there is a decreasing function $\varphi : (0, \infty) \to (0, \infty)$ such that for all disjoint non-degenerate continua E and F the family $\Gamma(E, F)$ of all paths connecting E to F in X satisfies:

$$\mathrm{Mod}_Q(\Gamma(E, F)) \geq \varphi(\Delta(E, F)),$$

where

$$\Delta(E, F) = \frac{\mathrm{dist}(E, F)}{\min\{\mathrm{diam}(E), \mathrm{diam}(F)\}}.$$

See [HeK1, Section 3] for details.

The Euclidean spaces $\mathbb{R}^n$ are n-regular, support a $(1, 1)$-Poincaré inequality, and are n-Loewner; see [V]. The Carnot group $\mathbb{H}^3$ supports a $(1, 1)$-Poincaré inequality and is 4-Loewner, and is a 4-regular space. For a definition of $\mathbb{H}^3$ see [St, Chapter 12]. A question asked by Heinonen and Semmes was whether there are Q-regular Q-Loewner spaces for non-integral values of Q; see [HeS]. This question was answered in the affirmative by [BP] and [L]. Their spaces also support a $(1, 1)$-Poincaré inequality and are also geodesic spaces. In fact, for any $Q > 1$, there are spaces that are Q-regular, Q-Loewner, and support a $(1,1)$-Poincaré inequality; see [L].

The following definition was suggested by Juha Heinonen.

Definition 4.2.8. A metric measure space Y is said to be a *Vitali space* if for each set $A \subset Y$ and each collection $\mathcal{B}$ of closed balls satisfying the following two properties

1. $A \subset \cup_{B \in \mathcal{B}} B$,

2. for each point $x \in A$ there is a sequence of balls in the collection, centered at x, so that the infimum of the radii of these balls is zero,

there is a countable subcollection $\{B_i\}$ so that the measure of the set $A \backslash \cup_i B_i$ is zero and $B_i \cap B_j$ is empty whenever $i \neq j$.

Euclidean spaces equipped with Radon measures are Vitali spaces (see [Ma]), and so are metric measure spaces equipped with doubling measures and Riemannian manifolds; see the appendix. It is shown in the appendix that Lebesgue differentiation theorem holds for Vitali spaces.

Theorem 4.2.9. *If X is both a Q-Loewner space and a Vitali space such that for almost all points x in X the mass density* $\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^Q} < \infty$*, then X is a MEC_Q-space.*

Note that the mass density condition is weaker than the regularity condition. For example, the Hyperbolic n-space posesses this mass density condition with $Q = n$, but is not n-regular. Hyperbolic spaces are also Vitali spaces; see the appendix.

Proof. Fix $\rho \in L^Q(X)$ and let $\Gamma_0 = \{\gamma \in \Gamma_{rect} \ : \ \int_\gamma \rho ds = \infty\}$. Then $\mathrm{Mod}_Q \Gamma_0 = 0$. Let

$$G = \bigcup_{\gamma \in \Gamma_{rect} \backslash \Gamma_0} |\gamma|.$$

Clearly if $x, y \in |\gamma|$ for some $\gamma \in \Gamma_{rect} \backslash \Gamma_0$, then $x \sim_\rho y$. If $x \in |\gamma_1|$, $y \in |\gamma_2|$, with $\gamma_1, \gamma_2 \in \Gamma_{rect} \backslash \Gamma_0$ and γ_1 and γ_2 do not intersect, then as γ_1 and γ_2 are compact sets, by the Loewner property there exists $\gamma_3 \in \Gamma_{rect} \backslash \Gamma_0$ intersecting both γ_1 and γ_2, and hence $x \sim_\rho y$. Therefore all elements of G belong to the same equivalence class with respect to ρ. Furthermore, if $x \in G$ and $y \notin G$, then there does not exist $\gamma \in \Gamma_{rect} \backslash \Gamma_0$ such that $x \in |\gamma|$ and $y \in |\gamma|$ and therefore $x \not\sim_\rho y$. Thus G is an equivalence class with respect to ρ.

Let $A_0 = X \backslash G$. It remains to show that $\mu(A_0) = 0$. Suppose not. As bounded sets have finite measure, μ is an inner measure; if $E \subset X$, then $\mu(E)$ is equal to the supremum of $\mu(C)$ over all closed sets $C \subset E$ (see 3.0.6). Hence there exists a closed subset A of A_0 whose measure is positive. By supposition $\mu(A) > 0$. Therefore A has a point x_0 such that $\limsup_{r\to 0} \mu(B(x_0,r))/r^Q \leq C_{x_0}/2$ and such that x_0 is a point of density of A:

$$\liminf_{r\to 0} \frac{\mu(B(x_0,r)\bigcap A)}{\mu(B(x_0,r))} = 1.$$

Such x_0 exists because by the Lebesgue differentiation theorem (see the appendix) almost every point in A is a point of density of A. Therefore for each positive number ϵ there exists a positive number r_ϵ such that

$$\frac{\mu(B(x_0,r_\epsilon)\backslash A)}{\mu(B(x_0,r_\epsilon))} \leq \epsilon.$$

Consider $E, F \subset B(x_0, r_\epsilon/2)$ such that E and F are non-degenerate continua with the relative distance

$$\Delta(E,F) = \frac{\mathrm{dist}(E,F)}{\min\{\mathrm{diam}(E),\mathrm{diam}(F)\}}$$

comparable to a constant, and $\mathrm{dist}(E,F) \geq kr_\epsilon$ where $k \leq 1/2$ is some positive constant independent of ϵ. Such E, F exist because X is path-connected. For example, take E to be a path connecting the boundary of $B(x_0, r_\epsilon/16)$ to the boundary of the ball $B(x_0, r_\epsilon/8)$ without going outside the closure of $B(x_0, r_\epsilon/8)$, and take F to be a path connecting the boundary of $B(x_0, r_\epsilon/4)$ to the boundary of $B(x_0, r_\epsilon/3)$ without going into the ball $B(x_0, r_\epsilon/4)$ nor outside the closure of the ball $B(x_0, r_\epsilon/3)$. By the Loewner property of X the Q-modulus of all the paths joining E to F is bounded away from zero by a constant C independent of ϵ. Denote the collection of all such paths Γ_ϵ. Then

$$\Gamma_\epsilon \subset \Gamma_A \bigcup (\Gamma_\epsilon \backslash \Gamma_A),$$

where, recall that Γ_A is the collection of rectifiable paths intersecting the set A. If the path $\gamma \in \Gamma_A$, then $\gamma \in \Gamma_0$, and hence $\mathrm{Mod}_Q \Gamma_A = 0$ by Lemma 1.3.4. Therefore,

$$\mathrm{Mod}_Q \Gamma_\epsilon = \mathrm{Mod}_Q(\Gamma_\epsilon \backslash \Gamma_A).$$

However, the function

$$\rho = \frac{1}{kr_\epsilon}\chi_{B(x_0,r_\epsilon)\backslash A}$$

is an admissible test function for $\Gamma_\epsilon\backslash\Gamma_A$, and hence the Q-modulus of Γ_ϵ is less than or equal to $C_{x_0}\epsilon$ for some constant C_{x_0} independent of ϵ. This term converges to zero as ϵ tends to zero, contradicting the Loewner property. Therefore $\mu(A) = 0$ whenever A is a closed subset of A_0. Hence $\mu(X\backslash G) = 0$.

□

The following lemma shows that the MEC_p criterion is more of a local phenomenon than indicated by the above theorem. This lemma remains true if the condition of path-connectedness is replaced by the condition of connectedness, and can be proved in this generality by a slight modification of the following proof.

Lemma 4.2.10. *If X is a path connected metric measure space that is covered by a countable collection $\{U_\alpha\}_{\alpha\in A}$ of open sets that are themselves MEC_p spaces under the metric and measure inherited from X, then X is an MEC_p space.*

Proof. Let ρ be a non-negative Borel measurable p-integrable function on X. Then $\rho\big|_{U_\alpha} \in L^p(U_\alpha)$ for each $\alpha \in A$. Let G_α be the corresponding main equivalence class induced by ρ in U_α. Note that then G_α is a subset of a ρ-equivalence class in X. Since $\mu(U_\alpha \backslash G_\alpha) = 0$, if $G = \bigcup_{\alpha\in A} G_\alpha$, then $\mu(X\backslash G) = 0$. It remains to show that G is a subset of a ρ-equivalence class in X.

Let x, y be two points in G. If the two points belong to the same G_α, then clearly $x \sim_\rho y$. So suppose that x is in G_α and y is in G_β, with $\alpha \neq \beta$. As X is path-connected, there is a path $\gamma : [a, b] \to X$ connecting x to y. Since $|\gamma|$ is compact, there exists a finite subcollection $\{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_n}\}$ of the open cover that covers the set $|\gamma|$. Without loss of generality x lies in U_{α_1}. Let $a_2 = \sup\{t \in [a, b] \,:\, \gamma(t) \in U_{\alpha_1}\}$. Without loss of generality $\gamma(a_2)$ lies in U_{α_2}. Inductively, let a_i be the supremum of the set of numbers $t \in [a, b]$ for which $\gamma(t)$ lies in $U_{\alpha_{i-1}}$. Note that then $a_1 < a_2 < ....$ Since the cover of $|\gamma|$ is finite, there exists an integer $j \leq n$ such that $\gamma(a_j)$ lies in U_{α_j} and y also lies in U_{α_j}. Now consider the subcollection $\{U_{\alpha_i}\}_{i=1}^{j}$. Clearly $U_{\alpha_i} \cap U_{\alpha_{i+1}}$ is not empty and is open, and hence has a positive measure by the assumption made at the beginning of this chapter. Therefore there exists a point x_i in $U_{\alpha_i} \cap U_{\alpha_{i+1}}$ that also lies in G_{α_i} and in $G_{\alpha_{i+1}}$. A chain of equivalences is now obtained: $x \sim_\rho x_1$, $x_1 \sim_\rho x_2$, ..., $x_j \sim_\rho y$. Hence both x and y are in the same equivalence class of ρ in X. Thus every pair of points in G are equivalent with respect to ρ. $\square$

Theorem 4.2.9 and the above lemma together yield the following corollary.

Corollary 4.2.11. *Let $Q \geq 1$. Suppose X is a path-connected metric measure space covered by a countable collection of open sets $\{U_\alpha\}_{\alpha\in A}$ such that for each α, U_α is a Q-Loewner space under the metric and measure inherited from X. Suppose also that for almost every point x in X,*

$$\limsup_{r\to 0} \frac{\mu(B(x, r))}{r^Q} < \infty,$$

and that the metric measure subspace U_α, equipped with the metric inherited from X and the measure $\mu\big|_{U_\alpha}$, is a Vitali space for each α. Then X is a MEC_Q space.

By the corollary above, all Riemannian manifolds of dimension n are MEC_n-spaces.

While the above two results show that locally MEC_p spaces satisfy the global MEC_p criterion, the converse does not in general hold true. The following example shows that there are MEC_p spaces containing domains that are not MEC_p.

Example 4.2.12. Let $X = \mathbb{R}^2 \setminus (B((1,0),1) \cup B((-1,0),1))$ be equipped with Euclidean metric and Lebesgue measure. It is easy to see that X is an MEC_p space for each $p \geq 1$. Note that the origin $(0,0)$ is in X. The set $B((0,0),1) \cap X$ is a path-connected open subset of X, and hence is a domain in X. Since the p-modulus of the collection of all curves through $(0,0)$ is zero when $1 \leq p \leq 2$, it is clear that this subset does not satisfy the MEC_p property when $1 \leq p \leq 2$.

A similar result to Lemma 4.2.4 holds true for Q-Loewner spaces whose almost every point x satisfies the condition $\limsup_{r\to 0} \mu(B(x,r))/r^Q < \infty$. In such a space X, null sets E are characterized by the condition that the Q-modulus of Γ_E^+ is zero. This can be proved by an argument similar to the proof of Theorem 4.2.9. Moreover, here it is sufficient to require the space X to be "locally Loewner" in the sense of Corollary 4.2.11.

Proposition 4.2.13. *Let $Q \geq 1$. Suppose X is a path-connected metric measure space covered by a countable collection of open sets $\{U_\alpha\}_{\alpha \in A}$ such that U_α is a Q-Loewner Vitali subspace of X for each α, and that for μ-almost all points x in X the number $\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^Q}$ is finite. Then for each subset E of X, $\mu(E) = 0$ if and only if $Mod_Q(\Gamma_E^+) = 0$.*

Proof. By Lemma 1.3.5, if $\mu(E) = 0$ then $\mathrm{Mod}_Q(\Gamma_E^+) = 0$.

Now suppose that $\mathrm{Mod}_Q(\Gamma_E^+) = 0$ and that $\mu(E) > 0$. Then as μ-almost all points of E are points of density of E, there exists a point x in E such that

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^Q} =: C_x/2 < \infty,$$

and by the Lebesgue differentiation theorem (see the appendix),

$$\lim_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} = 1.$$

For this point x there exists $r_0 > 0$ such that whenever $0 < r < r_0$,

$$\frac{\mu(B(x,r))}{r^Q} \leq C_x, \qquad B(x,r_0) \subset U_{\alpha_0},$$

and there exists a sequence of positive numbers $r_\epsilon \to 0$ as $\epsilon \to 0$ such that $r_\epsilon < r_0$ for each $0 < \epsilon < 1$, and

$$\frac{\mu(B(x,r_\epsilon)\backslash E)}{\mu(B(x,r_\epsilon))} \leq \epsilon.$$

As in the proof of Theorem 4.2.9, obtain disjoint non-trivial continua H, F in $B(x, r_\epsilon)$ for which $\Delta(H,F)$ is comparable to a positive constant and such that, by the Q-Loewner property of U_{α_0}, satisfies

$$\mathrm{Mod}_Q(\Gamma_\epsilon\backslash\Gamma_\epsilon^+) = \mathrm{Mod}_Q\Gamma_\epsilon \geq C,$$

where Γ_ϵ is the collection of paths connecting H and F, and Γ_ϵ^+ is the collection of paths γ in Γ_ϵ such that $\mathcal{H}_1(|\gamma|\cap E)$ is positive. But then, as in the proof of Theorem 4.2.9, the function $\rho = \chi_{B(x_0,r_\epsilon)\backslash A}1/(kr_\epsilon)$ is an admissible function for calculating the Q-modulus of $\Gamma_\epsilon\backslash\Gamma_\epsilon^+$, which implies that the Q modulus of Γ_ϵ is bounded above by $C_x\epsilon$, which goes to zero as ϵ tends to zero. This is a contradiction of the Loewner property of the open set U_{α_0}. Hence it must be true that $\mu(E) = 0$. $\square$

A metric space X is said to be *φ-convex* if there is a cover of X by open sets $\{U_\alpha\}$ together with homeomorphisms $\{\varphi_\alpha : [0,\infty) \to [0,\infty)\}$ such that each pair of distinct points x and y in U_α can be joined by a curve whose length does not exceed $\varphi_\alpha(d(x,y))$. The paper [HeK1] showed that if X is proper, φ-convex, and Q-regular, then X is Q-Loewner if and only if X supports a weak $(1,Q)$-Poincaré inequality. Hence by Theorem 4.2.9, if X is proper, φ-convex, Q-regular, and supports a weak $(1,Q)$-Poincaré inequality, then X is an MEC_Q space. The following theorem is a generalization of this result.

Theorem 4.2.14. *Let X be a proper, φ-convex metric measure space supporting a weak $(1,p)$-Poincaré inequality. Then X is an MEC_p space.*

Proof. Since X is proper and φ-convex, by [KSh, Lemma 3.2 and Remark 3.3] it is true that if E and F are disjoint compact subsets of X, then $\mathrm{Mod}_p(E,F) = N^{1,p}_{loc} - \mathrm{Cap}_p(E,F)$. Here $\mathrm{Mod}_p(E,F)$ is the p-modulus of the collection of curves connecting E to F, and $N^{1,p}_{loc} - \mathrm{Cap}_p(E,F)$ is the infimum of $\|\rho\|^p_{L^p}$ over all functions ρ that are p-integrable upper gradients of functions $u \in L^p_{loc}(X)$ such that $u\big|_E = 1$ and $u\big|_F = 0$.

Suppose X is not an MEC_p space. Then there exists a non-negative Borel measurable p-integrable function g on X and disjoint subsets A, B of X such that $X = A\cup B$, $\mu(A) > 0$, $\mu(B) > 0$, and for each point x in A and each point y in B the points x and y cannot be connected by a path γ such that $\int_\gamma \rho$ is finite. Hence the p-modulus of the collection of curves joining A to B is zero. Since the measure is an inner measure (see Lemma 3.0.6), there exist closed and bounded sets $E \subset A$ and $F \subset B$ whose measures are positive. As $\mathrm{Mod}_p(E,F) \le \mathrm{Mod}_p(A,B)$, the p-modulus of curves connecting the compact sets E and F is zero. Therefore $N^{1,p}_{loc} - \mathrm{Cap}_p(E,F) = 0$. Hence for each positive integer n there is a function u_n and its upper gradient ρ_n so that $\|\rho_n\|^p_{L^p} < 2^{-n}$ and $u_n \in L^p_{loc}(X)$ with $u_n\big|_E = 1$, $u_n\big|_F = 0$, and $0 \le u_n \le 1$. But there exists a ball B containing $E \cup F$, and on this ball,

$$0 < \frac{\mu(E)}{\mu(B)} \le u_{nB} = \fint_B u_n \le \frac{\mu(B) - \mu(F)}{\mu(B)} < 1.$$

Therefore

$$\fint_B |u_n - u_{nB}| \ge \frac{\mu(F)\mu(E)}{\mu(B)^2} > 0,$$

but

$$\left(\fint_{\tau B} \rho_n^p\right)^{1/p} \le \frac{2^{-n/p}}{\mu(\tau B)^{1/p}} \to 0 \quad \text{as } n \to \infty.$$

Therefore X cannot support a weak $(1,p)$-Poincaré inequality. $\square$

Remark 4.2.15. In the above theorem properness and φ-convexity were used only to get the result that $\mathrm{Mod}_p(E,F) = N^{1,p}_{loc} - \mathrm{Cap}_p(E,F)$. Hence in the above lemma these two hypotheses can be replaced by the hypothesis that if E and F are disjoint compact subsets of X, then $\mathrm{Mod}_p(E,F) = N^{1,p}_{loc} - \mathrm{Cap}_p(E,F)$. As mentioned in [HeK1] and in [KSh], it is not currently known whether this hypothesis holds for all metric measure spaces. However, the above lemma holds true even when the condition of φ-convexity is replaced by the more easily verifiable conditions of path-connectedness and doubling in measure, since doubling, proper, and path-connected metric measure spaces supporting a weak $(1,p)$-Poincaré inequality are quasiconvex; see [HK2].

The converse of the above theorem does not hold true. The following examples show that the MEC_p condition does not imply the support of a weak $(1,p)$-Poincaré inequality.

Example 4.2.16. Let X be the open unit ball in $\mathbb{R}^2$ with the radial segment $(0,1) \times \{0\}$ removed. Then X, equipped with the Euclidean metric and Lebesgue measure, is a domain in $\mathbb{R}^2$ and hence by Lemma 4.2.10, is an MEC_p space for all finite $p \ge 1$. However, X does not support a weak $(1,p)$-Poincaré inequality; for example, let

$\tau \geq 1$ and let $y = (1/2, \epsilon)$, $B = B(y, 3\epsilon/\tau) \cap X$. For sufficiently small ϵ the ball B is in two disjoint connected pieces. Let u be a function defined by

$$u(x) = \begin{cases} 1 & \text{if } \pi_2(x) > 0, \\ 0 & \text{if } \pi_2(x) < 0 \end{cases}$$

where π_2 is the projection onto the second coordinate axis. Zero is a weak upper gradient of u on τB, but $⨍_B |u - u_B| > 0$. Hence the Poincaré inequality is not satisfied on B.

A metric space Y is said to be *quasiconvex* if there exists a constant $C \geq 1$ so that whenever x and y are two distinct points in X there is a rectifiable path joining the two points and whose length is no more than $Cd(x, y)$. In the above example X is not quasiconvex. It can be modified to be quasiconvex by including back into X the rational points on the radius $[0, 1) \times \{0\}$. This modified space is still MEC_p since the collection of new points have measure zero. For $1 \leq p < 2$ the collection of paths in $\mathbb{R}^2$ that pass through any given point has p-modulus zero. Hence the collection of paths in X that pass through these rational points on the radius has modulus zero. Hence the ball mentioned in the above example does not satisfy Poincaré inequality for the same reason as above.

Lemma 4.2.10 might lead one to ask whether the MEC_p condition implies at least a local Poincaré inequality almost everywhere. The next example shows that this is not the case.

Definition 4.2.17. A metric measure space X is said to support a *local weak* $(1, p)$-*Poincaré inequality* at a point x in X if there is a neighbourhood U of x and a constant $\tau \geq 1$ such that whenever B is a ball in X such that τB is contained in U, and u is an integrable function on τB with ρ as its weak upper gradient in τB, then

$$⨍_B |u - u_B| \leq C \mathrm{diam}(B) \left(⨍_{\tau B} \rho^p \right)^{1/p}$$

with $C > 0$ independent of ρ, u, and B.

Example 4.2.18. Let $E \subset [0, 1]$ be a Cantor set of positive Lebesgue 1-measure, and let E_1 be the collection of accessible points of E. Let $X = ([-1, 2] \times [-1, 2]) \backslash ([0, 1] \times E_1) \cup E_1 \times E_1$ together with Euclidean metric and Lebesgue 2-measure.

Let $1 \leq p \leq 2$. As $E_1 \times E_1$ is countable, the collection of paths in X passing through $E_1 \times E_1$ has p-modulus zero. For the same reason that the previous example does not support a $(1, p)$-Poincaré inequality, X does not support any $(1, p)$-Poincaré inequality even locally at each of the points in $[0,1]\times(E\backslash E_1)$. Note that the Lebesgue 2-measure of this set is positive. Therefore there is a positive measure collection of points at which X does not support a weak $(1,p)$-Poincaré inequality.

Let ρ be a non-negative Borel measurable p-integrable function on X. Then on p-modulus almost all lines of the form $[-1,2]\times\{y\}$; $-1\leq y\leq 2$ and $y\notin E_1$; the integral $\int_\gamma \rho$ is finite. If E_0 is a subset of $[-1,2]\backslash E_1$ of positive Lebesgue 1-measure, then the p-modulus of the collection of curves of form $[-1,2]\times\{y\}$; $y\in E_0$ is positive (see [V]). Hence for almost all y in $[-1,2]\backslash E_1$ the integral $\int_{[-1,2]\times\{y\}}\rho$ is finite. Since on $X\backslash([0,1]\times[0,1])$ the MEC_p criterion is satisfied, on this set there is a corresponding main equivalence class G_ρ. The projection of this equivalence class into the y-axis is of full measure by the above discussion. Hence for Lebesgue 1-measure almost all points y in $[-1,2]\backslash E_1$ the points on the horizontal line $[-1,2]\times\{y\}$ are in the same equivalence class as points in G_ρ. Hence by Fubini's theorem, almost all points in X are in the same equivalence class with the points of G_ρ. Therefore X has MEC_p property, even though it does not support a local weak Poincaré inequality.

The following lemma reflects the decreasing property of L^p_{loc}, that is, if $q>p$, then $L^q_{loc}\subset L^p_{loc}$.

Lemma 4.2.19. *Let X be an MEC_p space. Then for each $q>p$ the space X is a MEC_q space.*

Proof. Let ρ be a non-negative Borel measurable q-integrable function on X. Then $\rho^{q/p}$ is a p-integrable Borel measurable non-negative function on X. Thus X has a corresponding main equivalence class G with respect to $\rho^{q/p}$; $\mu(X\backslash G)=0$. It now suffices to show that G is a subset of an equivalence class of ρ in X. Suppose x,y are two distinct points in G. Then there exists a compact rectifiable path γ connecting x to y such that the integral $\int_\gamma \rho^{q/p}$ is finite. But then

$$\begin{aligned}\int_\gamma \rho =& \int_{\{x\in\gamma\,:\,\rho(x)\leq 1\}}\rho + \int_{\{x\in\gamma\,:\,\rho(x)>1\}}\rho\\ \leq& \int_\gamma 1 + \int_{\{x\in\gamma\,:\,\rho(x)>1\}}\rho^{q/p}\\ \leq& l(\gamma)+\int_\gamma \rho^{q/p} \;<\; \infty.\end{aligned}$$

Hence $x\sim_\rho y$. $\square$

Remark 4.2.20. Another reason for looking at the MEC_p condition is that any quasisymmetric map between two Q-regular metric measure spaces preserves the MEC_Q-property provided the map has property (N) (that is, images of sets of measure zero are of measure zero: see [V] and [T2]). A homeomorphism f between two metric spaces is said to be *quasisymmetric* if there exists a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that for all points x, y, z in the domain of the function the following inequality holds:

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \leq \eta\left(\frac{d(x,y)}{d(x,z)}\right).$$

The proof of the above claim is given below.

Suppose $f : X \to Y$ is a quasisymmetric map satisfying property (N), with X and Y Q-regular spaces, and X an MEC$_Q$-space. Let ρ be a non-negative Borel-measurable Q-integrable function on Y. For points x in X the following quantity was defined in [HeK1] and in various other literature:

$$L_f(x) := \lim_{r\to 0} \frac{\sup_{|x-y|\leq r} d_Y(f(x), f(y))}{r}. \tag{4.1}$$

Let $\tilde{\rho}(x) := \rho \circ f(x) L_f(x)$. Then as f is quasisymmetric, $\tilde{\rho}$ is also a non-negative Borel-measurable Q-integrable function; see [HeK1, Section 7]. By [T1, Theorem 1.4] and by [T2], the family Γ_0 of all compact rectifiable paths γ in X on which f is not absolutely continuous or $f \circ \gamma$ is not rectifiable in Y has zero Q-modulus. Thus there exists a non-negative Borel-measurable Q-integrable function ρ_1 such that for all paths γ in Γ_0 the integral $\int_\gamma \rho_1 ds = \infty$. Let

$$\Gamma_1 = \{\gamma \in \Gamma_{rect}(X) \ : \ \int_\gamma \tilde{\rho} ds = \infty\}.$$

Then the Q-modulus of Γ_1 is also zero. Let $\rho_0 = \tilde{\rho} + \rho_1$ and G be its main equivalence class in the MEC_Q-space X, and let $G_Y = fG$. For points $w_1 = f(x)$ and $w_2 = f(y)$ in G_Y, x, y are points in G and hence there exists a rectifiable path γ connecting x to y in X so that $\int_\gamma \rho_0 ds < \infty$. Therefore $\int_\gamma \rho_1 ds < \infty$ and hence γ is not in Γ_0. Thus $f\gamma$ is a rectifiable path in Y connecting w_1 to w_2. By the choice of γ the function f is absolutely continuous on γ, and hence by [HeK1, Section 7],

$$\int_{f\gamma} \rho ds \leq \int_\gamma \rho \circ f L_f ds = \int_\gamma \tilde{\rho} ds < \infty$$

since by choice of ρ_0 the path γ is not in Γ_1 either. Hence w_1 and w_2 belong to the same ρ-equivalence class whenever they are both in G_Y. As $\mu_X(X \backslash G) = 0$ and f has condition (N), $\mu_Y(Y \backslash G_Y) = 0$. Therefore G_Y is a subset of a main equivalence class of ρ in Y.

4.3 Examples of MEC_p Spaces.

As noted before, for each $n > 1$ the Euclidean space $\mathbb{R}^n$ is an MEC_p space for each finite $p \geq 1$. This follows from Theorem 4.2.14. Another proof of this can be found in [O]. The space $\mathbb{R}^n$, equipped with the snow flake metric $|x-y|^\epsilon$ with $0<\epsilon<1$, does not have any rectifiable path, and hence is not an MEC_p space for any p. Thefollowing example gives a space that is MEC_p for some values of p but not others.

Example 4.3.1. Let T be a non-degenerate Euclidean triangular region in $\mathbb{R}^2$, and let T' be an identical copy of T. Let X be the metric measure space obtained by identifying a vertex V of T and a vertex V' of T', equipped with the standard Lebesgue measure and metric d defined by: $d(x,y)=|x-y|$ if both x and y are in T or both in T', and if x is in T and y is in T', then $d(x,y)=|x-V|+|y-V'|$. The space X is a bow-tie shaped subset of $\mathbb{R}^2$. For $p>2$ the collection of all rectifiable curves through the identified vertices $V=V'$ is positive. Since the triangles are MEC_p spaces, if ρ is any non-negative Borel-measurable p-integrable function on $X\subset\mathbb{R}^2$, T and T' have corresponding main equivalence classes G and G' respectively. Now, the collection of all rectifiable paths connecting G to G' is of positive p-modulus, and as $\mathrm{Mod}_p(\{\gamma\ :\ \int_\gamma\rho=\infty\})=0$, there exist points $x\in G$ and $y\in G'$ and a rectifiable path γ connecting x to y through the identified vertex $V=V'$ so that the integral $\int_\gamma\rho$ is finite. Hence $G_0=G\cup G'\cup\{V=V'\}$ is a subset of an equivalence class corresponding to ρ in X. Since the measure on X is the 2-dimensional Lebesgue measure, G_0 is of full measure in X (in fact, $G_0=X$). Hence X is an MEC_p space whenever $p>2$. However, if $1\le p\le 2$, then the collection of curves through $V=V'$ has zero p-modulus, and hence X is not an MEC_p space for these values of p. In fact, the function $\rho(z)=1/|z|$ has no main equivalence class when $1\le p\le 2$ and $V=V'=(0,0)$.

The same type of construction in higher dimensions yield spaces that are MEC_p spaces for p larger than the dimension, and not MEC_p for p smaller than the dimension. The paper [HeK1] has another example of spaces that satisfy the MEC_p condition for some values of p but not for all values of p. For each $p>1$ and $1\le q<p$ their construction glues two copies of $\mathbb{R}^n$ along a suitable Cantor set so that the resulting space X supports a $(1,p)$-Poincaré inequality and hence satisfies the MEC_p condition, but with the q-modulus of the family of all curves through the afore-mentioned Cantor set zero so that X is not an MEC_q-space; see [HeK1, Remark 6.19(b)].

As mentioned before, the space $(\mathbb{R}^n,|.|^\epsilon)$ is not MEC_p for any value of p because it has no rectifiable curves. The following example gives a space that is rectifiably connected (that is, every pair of distinct points are connected by a rectifiable path) but is not an MEC_p space for any p.

Example 4.3.2. Let I be the closed unit interval in $\mathbb{R}^2$, and let $X=I\times I$. Let the measure on X be 2-dimensional Lebesgue measure. Let Γ_1 be the collection of paths (continuous maps from intervals to X in the topology of $\mathbb{R}^2$) in X that are finite unions of horizontal and vertical lines. Let $\epsilon : [0,1] \to [1/2,1]$ be a function given by .

$$\epsilon(s) = \begin{cases} s & \text{if } s \in [1/2,1] \\ 1-s & \text{if } s \in [0,1/2]. \end{cases}$$

Define a function d_0 as follows: if x and y are two distinct points in X that are on the same vertical line, then $d_0(x,y) = |x-y|$, and if the two points are on the same horizontal line, then $d_0(x,y) = |x-y|^{\epsilon(\pi_2(y))}$, where π_2 is the projection onto the second coordinate axis. Now let X be equipped with the following metric. Let x, y be two distinct points in X, and let Γ_{xy} be the collection of all paths from Γ_1 that have x and y as their endpoints. The metric distance between x and y is the Carathéodory metric

$$d(x,y) = \inf_{\gamma \in \Gamma_{xy}} l_0(\gamma),$$

where

$$l_0(\gamma) := \sum_i d_0(x_i, y_i),$$

with x_i and y_i being the endpoints of the horizontal and vertical segments of γ. It is seen by the following that d is a metric:

Clearly $d(x,y) = d(y,x)$. If x, y, z are three points in X, then the concatenation of any path from Γ_{xz} and any path from Γ_{zy} yields a path in Γ_{xy}, and hence $d(x,y) \leq d(x,z) + d(z,y)$. It is also clear that $d(x,x) = 0$. Let π_1 be the projection onto the first coordinate axis. Then if x, y are in X, then $d(x,y) \geq \max\{|\pi_1(x) - \pi_1(y)|,\ |\pi_2(x) - \pi_2(y)|\}$ (note that if x and y are on the same vertical or horizontal line, then $|x-y| \leq 1$, and hence $d_0(x,y) \geq |x-y|$). Hence if $d(x,y) = 0$, then $x = y$.

With this metric, the only rectifiable paths in X are horizontal lines lying on the boundary of X and all vertical lines and concaternations thereof; that is, curves of form $\gamma : I \to X$ such that for some constant $a \in [0,1]$, $|\gamma| \subset \{(a, x_2) \in X \ : \ x_2 \in [0,1]\} \cup \{(x_1, 0), (x_1, 1) \ : \ x_1 \in [0,1]\}$ and concaternation of such curves. To see this, note that if $\gamma : I \to X$ is a vertical curve connecting two points x and y, then

$$l(\gamma) = |\pi_2(x) - \pi_2(y)| = |x-y| < \infty.$$

If γ is a horizontal line connecting two points $x = (x_1, a)$ and $y = (y_1, a)$ with $a \in \{0,1\}$, then

$$l(\gamma) = |\pi_1(x) - \pi_1(y)| = |x_1 - y_1| = |x-y| < \infty.$$

Hence the above-mentioned types of paths are rectifiable in the metric d. Now suppose $\gamma : I \to X$ is a curve that is not in the above category. Then there is a subcurve γ_p of γ so that γ_p connects two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ such that $x_1 \neq y_1$ and $x_1, x_2, y_1, y_2 \in (0,1)$. Without loss of generality it can also be assumed that $x_2 \leq \pi_2(\gamma_1(t)) \leq y_2$. Let $\epsilon_0 = \max\{\epsilon(t) \ : \ x_2 \leq t \leq y_2\}$. Then $\epsilon_0 < 1$. Let σ be a partition of γ_p into n segments, and let the corresponding cut points be denoted $t_1, t_2, ..., t_n \in I_0$ where $\gamma_1 : I_0 \to X$. Then

$$l(\gamma) \geq \sum_{i=1}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})).$$

The distance $d(\gamma(t_i), \gamma(t_{i+1}))$ is measured by paths $\gamma_i \in \Gamma_{\gamma(t_i),\gamma(t_{i+1})}$ to within an error of no more than $2^{-i}\epsilon$. Let γ_0 be a path obtained by concaternating the finite family of paths $\{\gamma_i\}_{i=1}^{n-1}$. Then $\gamma_0 \in \Gamma_{xy}$, and

$$l(\gamma_p) \geq l_0(\gamma_0) - \sum_{i=1}^{n} 2^{-i}\epsilon \geq l_0(\gamma_0) - \epsilon.$$

The collection $\{\gamma_i\}_{i=1}^{n-1}$ can be separated into two categories: say γ_i is of category A if γ_i leaves the set $\{x \in X \ : \ \pi_2(x) \in [x_2/2, y_2 + (1-y_2)/2]\}$, and is of category B if it is not of category A. Then

$$\begin{aligned} l(\gamma_p) &\geq \sum_{\gamma_i \in \mathrm{cat} A} l_0(\gamma_i) + \sum_{\gamma_i \in \mathrm{cat} B} l_0(\gamma_i) - \epsilon \\ &\geq \sum_{\gamma_i \in \mathrm{cat} A} 2\min\{(1-y_2)/2, x_2/2\} + \sum_{\gamma_i \in \mathrm{cat} B} |\pi_1(\gamma(t_i)) - \pi_1(\gamma(t_{i+1}))|^{\epsilon_0}. \end{aligned}$$

Choose the partition σ to be such that it cuts the set $|\gamma_p|$ in such a way so that the horizontal projection of each of the segments of $|\gamma_p|$ is of equal length $|x_1 - y_1|/n$. If more than $n/2$ of the paths γ_i are in category A, then

$$l(\gamma_p) \geq \frac{n}{2} 2\min\{(1-y_2)/2, x_2/2\} = n\min\{\frac{1-y_2}{2}, \frac{x_2}{2}\}$$

and if no more than $n/2$ of the paths γ_i are in category A, then

$$l(\gamma_p) \geq \frac{n}{2}\frac{1}{n^{\epsilon_0}}|x_1 - y_1|^{\epsilon_0}$$

Hence

$$l(\gamma_p) \geq \begin{cases} n\min\{\frac{1-y_2}{2}, \frac{x_2}{2}\} & \text{in the first case,} \\ n^{1-\epsilon_0}\frac{|x_1-y_1|^{\epsilon_0}}{2} & \text{in the second case.} \end{cases}$$

Since $l(\gamma_p)$ is approximated better as the number of partitions $n \to \infty$, and both $n \min\{\frac{1-y_2}{2}, \frac{x_2}{2}\}$ and $n^{1-\epsilon_0}|x_1 - y_1|^{\epsilon_0}/2$ tend to ∞ as $n \to \infty$, it is true that $l(\gamma_1) = \infty$. Thus the length of γ also is infinite.

With this metric d, the space X is rectifiably connected, since if x and y are two distinct points in X, then the curve obtained by connecting x by a vertical line to $(\pi_1(x), 0)$, then connecting $(\pi_1(x), 0)$ to $(\pi_1(y), 0)$ by a horizontal line, and then connecting $(\pi_1(y), 0)$ to y by a vertical line, is a rectifiable curve connecting the two points. However X is not an MEC_p space for any p, as the Borel function

$$\rho(x) = \begin{cases} 0 & \text{if } \pi_2(x) \notin \{0, 1\} \\ \infty & \text{otherwise.} \end{cases}$$

partitions X into equivalence classes that are vertical lines, none of which is of full Lebesgue 2-measure in X.

4.4 Further Examples.

The following example shows that it is not always the case that $N^{1,p}(X)$ embeds into $M^{1,p}(X)$.

Example 4.4.1. In [K] for every $q \in (1, n]$ Koskela has an example of a space $X = \mathbb{R}^n \backslash E$ with E a closed subset of $\mathbb{R}^{n-1} \times \{0\}$, so that X, together with the Euclidean metric inherited from $\mathbb{R}^n$ and the Lebesgue n-measure, supports a $(1, p)$-Poincaré inequality for every $p \geq q$ but does not support a $(1, p)$-Poincaré inequality for any $p < q$. In these spaces, by Theorem 3.1.4, one knows that Lipschitz functions are dense in $N^{1,p}(X)$ whenever $p \geq q$. Hence as Lipschitz functions are extendable uniquely (since $|E| = 0$) to all of $\mathbb{R}^n$, all $N^{1,p}(X)$ functions are extendable to all of $\mathbb{R}^n$:

$$N^{1,p}(X) = N^{1,p}(\mathbb{R}^n) = M^{1,p}(\mathbb{R}^n), \qquad p \geq q.$$

Since inequality (1.1) is needed to be satisfied only almost everywhere for $M^{1,p}$-functions and $|E| = 0$, it is true that $M^{1,p}(\mathbb{R}^n) = M^{1,p}(X)$ for all p, $1 \leq p < \infty$. Hence whenever $p \geq q$ one has that $N^{1,p}(X) = M^{1,p}(X)$. When $1 < p < q$, by [K, Theorem A] and Theorem 3.2.1 the space $N^{1,p}(X) \neq N^{1,p}(\mathbb{R}^n)$ and hence, as $M^{1,p}(\mathbb{R}^n) = M^{1,p}(X)$, in this case the space $N^{1,p}(X)$ does not embed into the space $M^{1,p}(X)$ if $p < q$.

Another question one could ask is whether in Theorem 3.2.6 one really needs $q < p$- i.e. does there exist an example of a space that supports a $(1, p)$-Poincaré inequality but does not support a $(1, q)$-Poincaré inequality for any $q < p$ and also

$N^{1,p}$ does not embed into $M^{1,p}$. The above example does not answer this. The above question does not make sense when $p = 1$, but for $p > 1$ the author does not know the answer to this question.

Example 4.4.2. Let $p = 1$ and X be a unit ball B in $\mathbb{R}^n$. Then $N^{1,p}(X) = W^{1,p}(X)$ is not the same space as $M^{1,p}(X)$ by the comments in [H2] and [HK2], and there is no number $q < p$ so that X supports a $(1,q)$-Poincaré inequality.

Example 4.4.3. Let Ω be a domain in $\mathbb{R}^n$, and $X = \{X_1, ..., X_k\}$ be a collection of vector fields in Ω with real-valued locally Lipschitz coefficients. Such X defines a differential operator on locally Lipschitz functions u on Ω:

$$Xu(x) = \sum_{j=1}^{k} X_j u(x) = \sum_{j=1}^{k} \langle X_j(x), \nabla u(x) \rangle \,,$$

where ∇u is defined almost everywhere on Ω by a theorem of Rademacher. Associated with such vector fields there is a Carnot-Carathéodory "metric" ρ: see [GN] and [HK2, Section 11]. In general this "metric" is not defined as a real-valued function on $\Omega \times \Omega$, and hence is not really a metric in general. For example, take $\Omega = \mathbb{R}^2$ and $X = \{\frac{\partial}{\partial x}\}$. If a and b are two points in Ω that are not on the same horizontal line, then the value of $\rho(a,b)$ is infinite.

Suppose the collection X satisfies the additional assumptions that the associated Carnot-Carathéodory metric ρ is indeed a metric on Ω, that the metric space (Ω, ρ) satisfies a $(1,p)$-Poincaré inequality, and that the identity map from Ω equipped with the Euclidean metric to Ω equipped with the Carnot-Carathéodory metric is a homeomorphism: that is, the two induced topologies are equivalent. Vector fields satisfying Hörmander's condition, in particular the vector fields generating the tangent planes of a Carnot group, satisfy these conditions; see [FHK], [HK2], and [GN]. Under these assumptions [HK2, Proposition 11.6] shows that if one restricts attention to the class of compact rectifiable paths γ whose tangent vectors are spanned by X, then $|Xu|$ is an upper gradient for each locally Lipschitz function u on Ω. In this structure, there is a natural definition of Newtonian spaces, namely the space $N^{1,p}_X(\Omega)$ of p-integrable functions u that have p-integrable upper gradients g, that is, for each compact rectifiable path γ whose tangent vectors are spanned by X inequality (1.2) is satisfied. From the papers [FHK, Theorems 10, 11, and 12] and [HK2, Section 11] it is seen that in this situation, if (Ω, ρ) supports a $(1,p)$-Poincaré inequality, then

$$N^{1,p}_X(\Omega) \subset P^{1,p}(\Omega) \cap L^p(\Omega) \subset H^{1,p}_X(\Omega),$$

where $H^{1,p}_X(\Omega)$ is the closure of the collection of all locally Lipschitz p-integrable functions on Ω such that $|Xu|$ is p-integrable, the closure being taken in the norm

$$\|u\| = \|u\|_{L^p} + \||Xu|\|_{L^p}.$$

Hence in this situation, by Remark 2.2.9, $N^{1,p}_X(\Omega) = H^{1,p}_X(\Omega) = P^{1,p}(\Omega) \cap L^p(\Omega)$. For more on Carnot-Carathéodory metric and Sobolev spaces generated by vector fields, see [GN], [HK2], and [H2]. The paper [HK2] contains further references to this topic.

CHAPTER V

NEWTONIAN SPACES WITH ZERO BOUNDARY VALUES

In order to solve a Dirichlet boundary value problem in a domain $\Omega \subset \mathbb{R}^n$, it should be possible to compare two Sobolev functions on the boundary $\partial\Omega$, see [HeKM]. Thus it became necessary to consider the space $W_0^{1,p}(\Omega)$ of Sobolev functions on Ω whose trace on the boundary of Ω vanishes.

As a next step in generalization, it is natural to consider the space $N_0^{1,p}(E)$ of Newtonian functions on a subset E of X whose trace on $X \backslash E$ is zero. The paper [KKM] defines and analyzes some properties of the space $M_0^{1,p}(E)$ of Hajłasz functions on $E \subset X$ whose trace on $X \backslash E$ vanishes. This chapter follows their approach to this generalization.

The results in this chapter that use a $(1,p)$-Poincaré inequality remain valid under weak $(1,p)$-Poincaré inequality assumption; the proofs remain valid after minor modifications.

As in the past two chapters, it is assumed that non-empty open sets have positive measure and that bounded sets have finite measure.

5.1 Definitions of $N_0^{1,p}(E)$ and $H_0^{1,p}(E)$.

As pointed out in [KKM], there are two approaches to defining Sobolev spaces of functions with zero boundary values. One is to consider the set of Lipschitz functions on X that vanish on $X \backslash E$, and close that set under an appropriate norm. Another way is to consider the space of all Newtonian functions u on X that are zero p-quasi everywhere in $X \backslash E$: that is, the set of points in $X \backslash E$ on which u is non-zero has zero p-capacity. In general the two approaches yield two different spaces, but for a broad class of spaces it will be shown that these two definitions agree. In this chapter a third definition of Sobolev spaces of functions with zero boundary values is also studied. This definition is similar to the first of the above two definitions, but instead of considering the class of all Lipschitz functions that vanish on $X \backslash E$, the subclass of such Lipschitz functions with compact support in E are considered. Again, for a broad class of metric measure spaces this definition agrees with the above two definitions. However, even in some Euclidean domains this definition yields the classical Sobolev space with zero boundary values while the above definitions yield larger function spaces; see Example 5.2.3.

Definition 5.1.1. Let E be a subset of a metric measure space X. Let $\tilde{N}_0^{1,p}(E)$ be the set of all functions u from E to $[-\infty, \infty]$ for which there exists a function $\tilde{u}$ in $\tilde{N}^{1,p}(X)$ (see Definition 1.3.6 for the definition of $\tilde{N}^{1,p}$) such that

1. $\tilde{u} = u$ μ-almost everywhere on E,
   and
2. $\mathrm{Cap}_p\left(\{x \in X\backslash E \ : \ \tilde{u}(x) \neq 0\}\right) = 0$.

If u and v are two functions in $\tilde{N}_0^{1,p}(E)$, say $u \sim v$ if $u = v$ μ-almost everywhere on E. It is easy to see that $\sim$ is an equivalence relation. Let $N_0^{1,p}(E) = \tilde{N}_0^{1,p}(E)/\sim$, equipped with the norm

$$\|u\|_{N_0^{1,p}(E)} = \|\tilde{u}\|_{N^{1,p}(X)}.$$

The norm on $N_0^{1,p}(E)$ is unambiguously defined. Since zero p-capacity sets (such as in criterion 2 of the above definition) have zero measure (for every $A \subset X$ it is easy to see that $\mu(A) \le \mathrm{Cap}_p(A)$), by Corollary 2.1.3, if $\tilde{u}$ and $\tilde{u}'$ both correspond to u as in the above definition, then $\|\tilde{u} - \tilde{u}'\|_{N^{1,p}(X)} = 0$.

In a similar definition in [KKM], $\tilde{u}$ is required to be p-quasicontinuous. Functions in Newtonian spaces are already p-quasicontinuous if X satisfies certain criteria: see Corollary 3.1.7.

Definition 5.1.2. Let $\mathrm{Lip}_0^{1,p}(E)$ be the collection of all Lipschitz functions in the function space $N^{1,p}(X)$ that vanish on $X\backslash E$, and let $\mathrm{Lip}_{C,0}^{1,p}(E)$ be the collection of functions in $\mathrm{Lip}_0^{1,p}(E)$ that have compact support. Let $H_0^{1,p}(E)$ be the closure of $\mathrm{Lip}_0^{1,p}(E)$ in the norm of $N^{1,p}(X)$, and let $H_{C,0}^{1,p}(E)$ be the closure of $\mathrm{Lip}_{C,0}^{1,p}(E)$ in the norm of $N^{1,p}(X)$.

Proposition 5.1.3. *The space $H_0^{1,p}(E)$ embeds isometrically into $N_0^{1,p}(E)$, and the space $H_{C,0}^{1,p}(E)$ embeds isometrically into $H_0^{1,p}(E)$.*

Proof. Let u be a function in $H_0^{1,p}(E)$. Then there is a sequence of Lipschitz functions $\{u_i\}_{i\in\mathbb{N}}$ from $N^{1,p}(X)$ so that u_i converge to u in $N^{1,p}(X)$, and for each integer i the function $u_i\big|_{X\backslash E} = 0$. As in the proof of Theorem 2.2.8, going through a subsequence if necessary, can write

$$\tilde{u} = \frac{1}{2}\left(\limsup_i u_i + \liminf_i u_i\right)$$

outside a set B of p-capacity zero. Then $\tilde{u}$ is a function in $N^{1,p}(X)$ and $u\big|_E = \tilde{u}\big|_E$ μ-almost everywhere. Furthermore, $\tilde{u}\big|_{(X\setminus E)\setminus B} = 0$. Hence $u\big|_E$ is in $N_0^{1,p}(E)$, with the two norms equal. Since $\mathrm{Lip}_{C,0}^{1,p}(E)$ is a subset of $\mathrm{Lip}_0^{1,p}(E)$, it is easy to see that $H_{C,0}^{1,p}(E)$ embeds into $H_0^{1,p}(E)$ isometrically. □

The definitions of $H_0^{1,p}(E)$ and $H_{C,0}^{1,p}(E)$ automatically guarantees that they are Banach spaces. The following theorem shows that $N_0^{1,p}(E)$ too is a Banach space.

Theorem 5.1.4. *The function space $N_0^{1,p}(E)$ is Banach.*

Proof. Let $\{u_i\}_{i\in\mathbb{N}}$ be a Cauchy sequence in $N_0^{1,p}(E)$. Then there is a corresponding Cauchy sequence $\{\tilde{u}_i\}_{i\in\mathbb{N}}$ in $N^{1,p}(X)$, where $\tilde{u}_i$ is the function in $N^{1,p}(X)$ corresponding to u_i as in the definition of $N_0^{1,p}(E)$. Then as $N^{1,p}(X)$ is a Banach space, there is a function $\tilde{u}$ in $N^{1,p}(X)$, and a subsequence, also denoted $\{\tilde{u}_i\}_{i\in\mathbb{N}}$ for the sake of brevity, so that as in the proof of Theorem 2.2.8 the sequence $\tilde{u}_i \to \tilde{u}$ pointwise outside a set B of zero p-capacity, and also in the norm of $N^{1,p}(X)$. Let A_i be the set of points in $X\setminus E$ on which $\tilde{u}_i$ is non-zero. Then the p-capacity of $\cup_i A_i$ is zero. Moreover, on $(X\setminus E)\setminus(\cup_i A_i \cup B)$,

$$\tilde{u}(x) = \lim_{i\to\infty} \tilde{u}_i(x) = 0.$$

Since the p-capacity of $B \cup (\bigcup_i A_i)$ is zero, the function $u = \tilde{u}\big|_E$ is in $N_0^{1,p}(E)$. Since

$$\|u_i - u\|_{N_0^{1,p}(E)} = \|\tilde{u}_i - \tilde{u}\|_{N^{1,p}(X)} \to 0 \text{ as } i \to \infty,$$

the space $N_0^{1,p}(E)$ is Banach. □

The rest of the thesis will make no distinction between the function u in $N_0^{1,p}(E)$ and its Newtonian extension $\tilde{u}$.

5.2 Sufficient Criteria for $N_0^{1,p}(E)$ to Coincide With $H_0^{1,p}(E)$ and $H_{C,0}^{1,p}(E)$.

There are examples of spaces X and $E \subset X$ for which the three function spaces $N_0^{1,p}(E)$, $H_0^{1,p}(E)$, and $H_{C,0}^{1,p}(E)$ are different. Some of the examples given below are modifications of the examples in [KKM].

Example 5.2.1. Let $X = (-1,0) \cup (0,1) \subset \mathbb{R}$, and let $E = (-1,0)$. Let u be the characteristic function of E. Since there are no rectifiable paths connecting points in E to points in X outside of E, this function has zero as an upper gradient. Hence u is in $N^{1,p}(X)$, and $u\big|_E$ is in $N_0^{1,p}(E)$. However, if f is a function in $\mathrm{Lip}_0^{1,p}(E)$, then it is in $N^{1,p}((-1,1))$, since $\lim_{t\to 0^-} f(t) = 0 = \lim_{t\to 0^+} f(t)$. The space $H_0^{1,p}(E)$ with respect to X is therefore a subspace of $N^{1,p}((-1,1))$, and hence functions in $H_0^{1,p}(E)$

are absolutely continuous on the interval $(-1,1)$. Hence $u=\chi_E$ is not in the space $H_0^{1,p}(E)$. It is easy to see that the function space $H_0^{1,p}(E)$, defined with respect to the ambient space X, is the same as the function space $W_0^{1,p}(E)$ defined with respect to the ambient space $(-1,1)$, and that the function space $H_{C,0}^{1,p}(E)$, defined with respect to X, too is the same as $W_0^{1,p}(E)$.

The example space above is not even connected. However, the following example is path connected.

Example 5.2.2. Let $X=B$ be the unit ball in $\mathbb{R}^2$, endowed with Euclidean metric and 2-dimensional Lebesgue measure, and let E be the set of all points in B whose at least one coordinate is an irrational number. Then $B\backslash E$ is countable and hence is of zero measure. Since $B\backslash E$ is dense in B, the collections $\mathrm{Lip}_0^{1,p}(E)$ and $\mathrm{Lip}_{C,0}^{1,p}(E)$ are the same collection containing only the zero function, and therefore $H_{C,0}^{1,p}(E)$ and $H_0^{1,p}(E)$ are the same trivial space $\{0\}$. Let u be a function defined by

$$u(x)=\mathrm{dist}(x,\partial B)=1-|x|.$$

It is easy to see that this function is in $N^{1,p}(B)=W^{1,p}(B)$ when $1<p\leq 2$. For these values of p, the p-capacity of $B\backslash E$ is zero (as it is countable, and points have zero p-capacity; see [V] or [HKM]), and hence u is in $N_0^{1,p}(E)$.

Example 5.2.3. Let X be the open unit ball $B(0,1)$ in the Euclidean space $\mathbb{R}^n$, endowed with the Euclidean metric and n-dimensional Lebesgue measure. Let $E=X$. Then $H_0^{1,p}(E)$, $N_0^{1,p}(E)$, $N^{1,p}(B(0,1))$, and $W^{1,p}(B(0,1))$ are all the same function space, since Lipschitz functions are dense in $W^{1,p}(B(0,1))$. However, the function space $H_{C,0}^{1,p}(E)$ is the same space as the classical function space $W_0^{1,p}(B(0,1))$, which is smaller than $W^{1,p}(B(0,1))$.

The above example shows that the spaces $H_0^{1,p}(E)$ and $N_0^{1,p}(E)$ depend on the ambient space X more than $H_{C,0}^{1,p}(E)$ depends on X.

The following example shows that it is possible for all three formulations of Sobolev functions with zero boundary values to be different.

Example 5.2.4. Let $X=\mathbb{R}^2\backslash[0,2]\times\{0\}$ and $E=B(0,1)\cap X$. Then functions in $\mathrm{Lip}_{C,0}^{1,p}(E)$ with respect to the ambient space X are also in $\mathrm{Lip}_0^{1,p}(E)$ with respect to $\mathbb{R}^2$, since these functions can be extended to be zero on $[0,1]\times\{0\}$. Hence the function space $H_{C,0}^{1,p}(E)$ with respect to X is the same as the function space $H_0^{1,p}(E)$ with respect to $\mathbb{R}^2$. However, there are functions in $\mathrm{Lip}_0^{1,p}(E)$ with respect to X that have only non-zero Lipschitz extension to $[0,1]\times\{0\}$, and in fact $H_0^{1,p}(E)$ with respect to X is the same as $H_0^{1,p}(B(0,1))$ with respect to $\mathbb{R}^2$. Hence in this case $H_0^{1,p}(E)$ is larger than $H_{C,0}^{1,p}(E)$.

Fix $0 < \epsilon < 1/4$. Let $\varphi : [0,\infty) \to \mathbb{R}$ be a non-negative Lipschitz map so that $\varphi\big|_{[0,\epsilon]} = 1$ and $\varphi\big|_{[1-\epsilon,\infty)} = 0$. Then the function $u(x) = u((r,\theta)) = r\theta\varphi(r)$ is in $N_0^{1,p}(E)$, but since it is not an ACL function on $B(0,1)$, it is not in the function space $H_0^{1,p}(E) = W_0^{1,p}(B(0,1))$. Hence $N_0^{1,p}(E)$ is a larger space than $H_0^{1,p}(E)$.

The following theorem gives a sufficient condition under which $N_0^{1,p}(E)$ and $H_0^{1,p}(E)$ agree.

Theorem 5.2.5. *Let X be a proper (that is, closed and bounded subsets of X are compact) doubling metric measure space supporting a $(1,p)$-Poincaré inequality. If E is an open subset of X, then $H_{C,0}^{1,p}(E) = H_0^{1,p}(E) = N_0^{1,p}(E)$.*

The proof of this theorem uses the following lemmata.

Lemma 5.2.6. *Let Y be a metric measure space equipped with a Borel measure that is finite on bounded sets. If u is a non-negative function in $N^{1,p}(Y)$, then the sequence of functions $u_k = \min\{u,k\}$; $k \in \mathbb{N}$, converges in the norm of $N^{1,p}(Y)$ to u.*

Proof. Let $E_k = \{x \in Y \ : \ u(x) > k\}$. If the measure of this set is zero, then $u_k = u$ almost everywhere and as u_k is also in $N^{1,p}(Y)$, by Lemma 2.1.3 the $N^{1,p}(Y)$-norm of $u - u_k$ is zero for sufficiently large k. So suppose that the measure of E_k is positive. Since the measure is finite on bounded sets, it is an outer measure (see Lemma 3.0.6), and hence there is an open set O_k with $E_k \subset O_k$ such that $\mu(O_k) \le \mu(E_k) + 2^{-k}$. Since

$$\mu(E_k) \le \frac{1}{k^p}\|u\|^p_{L^p(E_k)}, \tag{5.1}$$

it is true that

$$\mu(O_k) \le \mu(E_k) + 2^{-k} \le \frac{1}{k^p}\|u\|^p_{L^p(Y)} + 2^{-k}.$$

Hence the measure of O_k tends to zero as k tends to ∞. Note that $u = u_k$ on $X\setminus O_k$. Thus $u - u_k$ has $2g\chi_{O_k}$ as a weak upper gradient whenever g is an upper gradient of u and hence of u_k as well; see Lemma 2.1.7. Hence u_k converges to u in $N^{1,p}(Y)$. $\square$

Lemma 5.2.7. *Let X satisfy the hypotheses of Theorem 5.2.5, and let u be a function in $N^{1,p}(X)$. Suppose in addition that $0 \le u \le M$ everywhere in X, and that the set A of the points in X at which u is not zero is a bounded subset of X. For each positive integer k, consider the function*

$$\varphi_k = (1 - w_k)\max\{u - \frac{1}{k}, 0\},$$

where w_k is a function in $N^{1,p}(X)$ such that $0 \le w_k \le 1$, $\|w_k\|_{N^{1,p}(X)} \le 2^{-k}$, and $w_k\big|_{F_k} = 1$, with F_k an open subset of X such that u is continuous on $X\setminus F_k$. Then $\varphi_k \to u$ in $N^{1,p}(X)$.

By Corollary 3.1.7, if u is in $N^{1,p}(X)$ there always exist sets F_k and functions w_k as above.

Proof. Let E_k be the set of points x in X such that $u(x) < 1/k$. Then by Corollary 3.1.7 and by the choice of F_k, there is an open set U_k such that $E_k \backslash F_k = U_k \backslash F_k$. Let $V_k = U_k \cup F_k$. Since the restriction of w_k to F_k is identically 1 and $u\big|_{E_k} < 1/k$, the set $\{x : \varphi_k(x) \neq 0\}$ is a subset of $A \backslash V_k \subset A$. Let $v_k = u - \varphi_k$. Then $0 \leq v_k \leq M$ since $0 \leq \varphi_k \leq u$. On the set $A \backslash V_k$ it is easy to see that $\varphi_k = (1 - w_k)(u - 1/k)$, and on V_k it is clear that $\varphi_k = 0$. Therefore on $A \backslash V_k$,

$$v_k = w_k u + (1 - w_k)/k, \tag{5.2}$$

and on V_k,

$$v_k = u. \tag{5.3}$$

If x and y are two points in X, then

$$\begin{aligned}|w_k(x)u(x) - w_k(y)u(y)| \leq& |w_k(x)u(x) - w_k(x)u(y)| + |w_k(x)u(y) - w_k(y)u(y)| \\ \leq& w_k(x)|u(x) - u(y)| + M|w_k(x) - w_k(y)|.\end{aligned}$$

Let ρ_k be an upper gradient of w_k such that $\|\rho_k\|_{L^p} \leq 2^{-k+1}$. Let ρ be a p-integrable upper gradient of u. If γ is a path connecting two points x and y, then

$$|w_k(x)u(x) - w_k(y)u(y)| \leq w_k(x) \int_\gamma \rho + M \int_\gamma \rho_k.$$

Hence, if z is a point in $|\gamma|$, then

$$\begin{aligned}|w_k(x)u(x) - w_k(y)u(y)| \leq& |w_k(x)u(x) - w_k(z)u(z)| + |w_k(z)u(z) - w_k(y)u(y)| \\ \leq& w_k(z) \int_{\gamma_{xz}} \rho + M \int_{\gamma_{xz}} \rho_k + w_k(z) \int_{\gamma_{zy}} \rho + M \int_{\gamma_{zy}} \rho_k \\ \leq& w_k(z) \int_\gamma \rho + M \int_\gamma \rho_k,\end{aligned}$$

where γ_{xz} and γ_{zy} are such that the concatenation of these two segments yields the original path γ back again. Therefore

$$|w_k(x)u(x) - w_k(y)u(y)| \leq \int_\gamma \left(\inf_{z \in |\gamma|} w_k(z) \rho + M \rho_k \right),$$

and hence

$$|w_k(x)u(x) - w_k(y)u(y)| \leq \int_\gamma (w_k \rho + M \rho_k).$$

Therefore $w_k\rho + M\rho_k$ is an upper gradient of $w_k u$. Clearly $M\rho_k$ is p-integrable, and as $\|w_k\|_{N^{1,p}(X)} \to 0$, the L^p-norm of $M\rho_k$ also tends to zero as k goes to infinity. Note that $w_k\rho \le \rho$ everywhere on X. Hence by the dominated convergence theorem and by the fact that since $\|w_k\|_{L^p} \le 2^{-k+1}$ it is true that $w_k \to 0$ almost everywhere,

$$\int_X (w_k\rho)^p \to 0 \qquad \text{as } k \to \infty.$$

By Lemma 2.1.5 and equations (5.2) and (5.3),

$$g_k := \left(w_k\rho + M\rho_k + \frac{1}{k}\rho_k\right)\chi_{O_k} + \rho\chi_{O_k\cap V_k}$$

is a weak upper gradient of v_k, where, with B a fixed bounded open set containing the bounded set A, $O_k = (A \cup F_k) \cap B$ is a bounded open subset of X such that $A = \{x \ : \ u(x) > 0\} \subset O_k$. Now,

$$\|g_k\|_{L^p(X)} \le \left(\int_{O_k} (w_k\rho)^p\right)^{1/p} + (M + 1/k)\left(\int_{O_k} \rho_k^p\right)^{1/p} + \left(\int_{O_k\cap V_k} \rho^p\right)^{1/p}.$$

As shown above, $\int_{O_k}(w_k\rho)^p \to 0$ as $k \to \infty$. Since $\|\rho_k\|_{L^p} \le 2^{-k+1}$, the integral $(M+1/k)^p \int_{O_k} \rho_k^p$ also tends to zero as k tends to infinity. Since $O_k \cap V_k$ is a subset of $(E_k \cap A) \cup F_k$, the measure of $O_k \cap V_k$ is bounded above by $\mu(E_k \cap A) + \mu(F_k)$, that is, by $\mu(\{x \in X \ : \ 0 < u(x) < 1/k\}) + \mathrm{Cap}_p(F_k)$, and hence tends to zero as k becomes larger, for bounded sets have finite measure and therefore the measure of $\{x \in X \ : \ 0 < u(x) < 1/k\}$ tends to $\mu(\emptyset) = 0$. Hence as ρ is p-integrable, the integral $\int_{O_k\cap V_k} \rho^p \to 0$ as $k \to \infty$. Hence,

$$\begin{aligned}
\|v_k\|_{L^p(X)} = \|u - \varphi_k\|_{L^p} &\le \left(\int_{A\setminus V_k} (w_k u)^p\right)^{1/p} + \frac{1}{k}\left(\int_{A\setminus V_k} |1 - w_k|^p\right)^{1/p} \\
&\qquad + \left(\int_{O_k\cap V_k} |u|^p\right)^{1/p} \\
&\le M\left(\int_{A\setminus V_k} w_k^p\right)^{1/p} + \frac{1}{k}\mu(A\setminus V_k)^{1/p} + \left(\int_{O_k\cap V_k} |u|^p\right)^{1/p} \\
&\le M\|w_k\|_{N^{1,p}(X)} + \frac{1}{k}\mu(A)^{1/p} + \left(\int_{O_k\cap V_k} |u|^p\right)^{1/p}.
\end{aligned}$$

The right hand side of the above inequality tends to zero as k tends to infinity. Hence φ_k converges to u in $N^{1,p}(X)$. ☐

Now Theorem 5.2.5 is ready to be proven.

Proof of **Theorem 5.2.5**. Since by Proposition 5.1.3 it is true that $H^{1,p}_{C,0}(E) \subset N^{1,p}_0(E)$, it suffices to prove the reverse inclusion. Let u be a function in $N^{1,p}_0(E)$. Identify u with its extension $\tilde{u}$. By the lattice properties of $N^{1,p}(X)$ itself, it is easy to see that then $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$ are both in $N^{1,p}_0(E)$, and $u = u^+ - u^-$. Hence it suffices to show that u^+ and u^- are in $H^{1,p}_0(E)$. Thus without loss of generality, $u \geq 0$. Since $N^{1,p}_0(E)$ is a Banach space that is isometrically embedded in $N^{1,p}(X)$, if $\{u_n\}_{n\in\mathbb{N}}$ is a sequence in $N^{1,p}_0(E)$ that is Cauchy in $N^{1,p}(X)$, then the function u that the sequence converges to really lies in $N^{1,p}_0(E)$. Hence by Lemma 5.2.6, it also suffices to consider u to be bounded in X: $0 \leq u \leq M$. By Lemma 3.1.6, it also suffices to consider u to be such that the set $A = \{x \in X \ : \ u(x) \neq 0\}$ is a bounded set. By Lemma 5.2.7, it suffices to show that for each positive integer k the function

$$\varphi_k = (1 - w_k) \max\{u - 1/k, 0\}$$

is in $H^{1,p}_0(E)$.

As in the proof of Lemma 5.2.7, get $A \cup F_k = O_k \cup F_k$ with O_k and F_k being open subsets of X and $\mathrm{Cap}_p F_k \leq 2^{-k}$. Moreover, it is possible to choose O_k to be bounded (since u has bounded support) and to be contained in E. By the choice of w_k, it is true that $w_k\big|_{F_k} = 1$. Hence $\varphi_k\big|_{F_k} = 0$. Again, as in the proof of Lemma 5.2.7, if E_k is the set of all points in E on which the value of u is less than $1/k$, there exists an open set $U_k \subset E$ such that $E_k = U_k \backslash F_k$. As $\varphi_k\big|_{E_k} = 0$, therefore $\varphi_k\big|_{U_k \cup F_k} = 0$. Therefore,

$$\begin{aligned}\overline{\{x \ : \ \varphi_k(x) \neq 0\}} &\subset \{x \in E \ : \ u(x) \geq 1/k\} \backslash F_k \\ &= O_k \backslash (E_k \cup F_k) \subset O_k \subset E.\end{aligned}$$

Since X is proper, the support of φ_k is a compact subset of E, and hence the distance between the support of φ_k and $X\backslash E$, $\mathrm{dist}(\mathrm{supt}\varphi_k, X\backslash E) =: \delta$ is a positive number. Now, as X supports a $(1,p)$-Poincaré inequality and has a doubling measure, by Theorem 3.1.4 the function φ_k is approximated by Lipschitz functions in $N^{1,p}(X)$. Furthermore, if x is a point in $X\backslash E$, then if g_k is an upper gradient of φ_k (without loss of generality assuming that $g_k\big|_{X\backslash O_k} = 0$ by Lemma 2.1.5), then

$$\begin{aligned}M^* g_k^p(x) =& \sup_{x\in B} \fint_B g_k^p \\ =& \sup_{x\in B, \mathrm{rad} B > \delta/2} \fint_B g_k^p \\ \leq& C_0 \frac{\|g_k\|^p_{L^p}}{(\delta/2)^s} < \infty,\end{aligned}$$

where $s = \log_2 C$, with C the doubling constant, and C_0 is a constant depending only on the doubling constant. Hence in the proof of Theorem 3.1.4, choosing the number $\lambda > C_0 \frac{\|g_k\|^p_{L^p}}{(\delta/2)^s}$, it is ensured that the corresponding Lipschitz approximations agree with the functions φ_k on $X \setminus E$. Hence these Lipschitz approximations are in $H^{1,p}_0(E)$, and hence so is φ_k. Moreover, these Lipschitz approximations have compact support in E, and hence φ_k is in $H^{1,p}_{C,0}(E)$, completing the proof of the theorem. $\square$

5.3 Existence of the Energy Integral Minimizer.

One approach to solving the classical Dirichlet problem is to find a minimizer for the energy integral within a certain function space. The energy integral to be minimized is

$$I(u) = \inf_g \|g\|^2_{L^2},$$

where the infimum is over all upper gradients (or equivalently over all weak upper gradients) of the function $u + w$ where u is in $W^{1,2}_0(\Omega)$ and w is a fixed function in $W^{1,2}(\mathbb{R}^n)$. The unique minimizer in this case turns out to be smooth and solves the problem

$$\begin{aligned} \triangle u = 0 \quad \text{on } \Omega, \\ u\big|_{\mathbb{R}^n \setminus \Omega} = w\big|_{\mathbb{R}^n \setminus \Omega}. \end{aligned} \tag{5.4}$$

See for example [E, Section 5.13] and [KiMa].

While smoothness is not definable in a general metric measure space and (5.4) is so far not defined for Newtonian functions, finding a minimizer for the energy integral is an interesting problem in the abstract setting.

The following theorem is well known in functional analysis. See for example [HeKM] and [H3].

Theorem 5.3.1. *Let B be a reflexive Banach space. If $I : B \to \mathbb{R}$ is a convex, lower semicontinuous, coercive operator, then there is an element $\tilde{u}$ in B so that*

$$I(\tilde{u}) = \inf_{u \in B} I(u). \tag{5.5}$$

Here I is said to be *convex* if for all t in the range $[0, 1]$ and for each pair u, v in B it is true that $I(tu + (1-t)v) \le tI(u) + (1-t)I(v)$. The operator I is *lower semicontinuous* if whenever u_n is a sequence of elements in B converging to u it is true that $I(u) \le \liminf_{n\to\infty} I(u_n)$, and is *coercive* if whenever u_n is a sequence of elements in B such that $\|u_n\|_B \to \infty$ it is also true that $I(u_n) \to \infty$.

Note that in general I need not be linear.

Let $B = N_0^{1,p}(E)$ and w be a function in $N^{1,p}(X)$. If u is in $N_0^{1,p}(E)$, then $(u+w)\big|_{X\backslash E} = w$. Actually, for specific representative functions u and w, it is only known that $(u+w)\big|_{X\backslash E} = w$ quasi-everywhere. However, since by Lemma 2.2.5 and by the fact that sets of zero p-capacity have zero measure it is true that sets of zero p-capacity are removable for Newtonian functions, the representative u can be adjusted to be so that $(u+w)\big|_{X\backslash E} = w$. Now the operator to be considered here is

$$I(u) = \inf_g \|g\|_{L^p}^p, \tag{5.6}$$

where the infimum is over weak upper gradients of $u + w$. The aim of this section is to show that under certain conditions on X and E, if $p > 1$ the minimizer exists and is unique.

Lemma 5.3.2. *Suppose X is a doubling metric measure space supporting a $(1, q)$-Poincaré inequality for some $1 \le q < p$. Let E be a bounded subset of X such that the interior of $X\backslash E$ is not empty. Then the energy operator defined in (5.6) is coercive on $N_0^{1,p}(E)$.*

Proof. Let u_n be a sequence of functions in $N_0^{1,p}(E)$ whose norms tend to infinity. Identifying u_n with its representative in $N^{1,p}(X)$, therefore $\|u_n + w\|_{N^{1,p}(X)} \to \infty$. For each integer n let g_n be an upper gradient of $u_n + w$. By the assumption on E, a point y can be chosen from $X\backslash E$ so that $w(y)$ is finite, $Mg_w^q(y)$ is finite, y is a Lebesgue point of $u_n + w$ for each integer n, $u_n(y) = 0$, and there exists a positive number δ so that $B(y,\delta) \cap E = \emptyset$. Here g_w is a p-integrable upper gradient of w, and Mg_w^q is defined by the Hardy-Littlewood maximal operator

$$Mg^q(x) = \sup_{r>0} \fint_{B(x,r)} g^q(z)\,d\mu(z).$$

Note that $w(y) + u_n(y) = w(y)$. Choose a Lebesgue point $x \in E$ of $u_n + w$. Let $B_i = B(a_i, 2^{-|i|}d(x,y))$, with $a_i = x$ if $i > 0$ and $a_i = y$ if $i < 0$. Let $B_0 = B(x, 2d(x,y))$. Using the $(1,q)$-Poincaré inequality,

$$\begin{aligned}
|u_n(x) + w(x) - w(y)| \le& \sum_{i\in\mathbb{Z}} |u_{n_{B_i}} + w_{B_i} - u_{n_{B_{i+1}}} - w_{B_{i+1}}| \\
\le& C \sum_i \operatorname{diam} B_i \left(\fint_{B_i} g_n^q \right)^{1/q} \\
\le& C d(x,y) \left(Mg_n^q(x)^{1/q} + Mg_n^q(y)^{1/q} \right).
\end{aligned}$$

Therefore, since $d(x,y) \le \operatorname{diam} E + \operatorname{dist}(E,y)$,

$$|u_n(x) + w(x)| \le C \left((Mg_n^q(x))^{1/q} + (Mg_n^q(y))^{1/q} \right) + |w(y)|,$$

where C is a constant that depends on E and y (and hence on g_w), but not on n. Since g_n is p-integrable, g_n^q is p/q-integrable. As X is doubling and the Hardy-Littlewood maximal operator M is a bounded linear map from $L^{p/q}$ to $L^{p/q}$, therefore Mg_n^q is p/q-integrable with its p/q-norm bounded by a constant times the p-norm of g_n.

Since on $B(y,\delta)$ it is true that $u_n(z)+w(z)=w(z)$, it can be assumed that $g_n\big|_{B(y,\delta)}=g_w$ by Lemma 2.1.5. Therefore,

$$\begin{aligned}(Mg_n^q(y))^{p/q} =&\sup_{r>0}\left(\fint_{B(y,r)} g_n^q\right)^{p/q}\\ =&\max\left\{\sup_{0<r\le\delta}\left(\fint_{B(y,r)} g_w^q\right)^{p/q},\ \sup_{r>\delta}\left(\fint_{B(y,r)} g_n^q\right)^{p/q}\right\}\\ \le&\max\left\{Mg_w^q(y)^{p/q},\ \sup_{r>\delta}\left(\fint_{B(y,r)} g_n^q\right)^{p/q}\right\}.\end{aligned}$$

If $r>\delta$, then by Hölder's inequality,

$$\left(\fint_{B(y,r)} g_n^q\right)^{p/q}\le\frac{1}{\mu B(y,\delta)}\int_X g_n^p.$$

Therefore,

$$(Mg_n^q(y))^{p/q}\le(Mg_w^q(y))^{p/q}+\frac{1}{\mu B(y,\delta)}\int_X g_n^p,$$

with $Mg_w^q(y)$ finite by the choice of y. Noting that the measure of E is finite since E is bounded and

$$|u_n(x)+w(x)|^p\le CMg_n^q(x)^{p/q}+CMg_n^q(y)^{p/q}+C|w(y)|^p,$$

therefore,

$$\begin{aligned}\int_E|u_n(x)+w(x)|^p d\mu(x)\le&C\int_E Mg_n^q(x)^{p/q}d\mu(x)+C\left(Mg_w^q(y)^{p/q}+|w(y)|^{p/q}\right)\mu(E)\\ \le&C\int_E g_n^p+C.\end{aligned}$$

Since u_n+w tends to ∞ in the norm of $N^{1,p}(X)$ and hence $\int_E|u+w|^p+\int_E g_n^p$ tends to ∞, it is necessary by above that $g_n\to\infty$ in the L^p norm as well. Thus $I(u_n)$ tends to infinity as $n\to\infty$. $\square$

Lemma 5.3.3. *If $1 \le p < \infty$, then I is convex.*

Proof. This follows from the fact that the function $F(x) = x^p$ is convex, and by the fact that if g_1 and g_2 are weak upper gradients of two functions u_1 and u_2, then $tg_1 + (1-t)g_2$ is a weak upper gradient of $tu_1 + (1-t)u_2$ whenever $0 \le t \le 1$. □

Lemma 5.3.4. *The operator I is lower semicontinuous.*

Proof. Suppose u_n is a sequence of functions in $N_0^{1,p}(E)$ that converge to u in the norm of $N_0^{1,p}(E)$. Choose upper gradients g_n of u_n+w so that $I(u_n)^{1/p} \ge \|g_n\|_{L^p} - 2^{-n}$. Since $\|u_n - u\|_{N^{1,p}(X)} \to 0$, there exist upper gradients ρ_n of $u_n - u$ such that $\|\rho_n\|_{L^p} \to 0$. It is easy to see that $\rho_n + g_n$ is an upper gradient of $u_n + w$. Hence,

$$\begin{aligned} I(u)^{1/p} \le & \|\rho_n + g_n\|_{L^p} \le \|\rho_n\|_{L^p} + \|g_n\|_{L^p} \\ \le & I(u_n)^{1/p} + \|\rho_n\|_{L^p} + 2^{-n}. \end{aligned}$$

Therefore, taking the limit infimum as n tends to infinity,

$$I(u)^{1/p} \le \liminf_{n\to\infty} I(u_n)^{1/p}.$$

□

Now the main theorem of this section can be proved using the above lemmata. The paper [C] has an alternate proof of this theorem.

Theorem 5.3.5. *Let $p > 1$, and let X be a proper doubling metric measure space supporting a $(1,q)$-Poincaré inequality for some $1 \le q < p$. Suppose also that E is a bounded subset of X such that the interior of $X \backslash E$ is not empty. Then for each function w in $N^{1,p}(X)$ there exists a function $\tilde{u}$ in $N_0^{1,p}(E)$ satisfying*

$$I(\tilde{u}) = \inf_{u \in N_0^{1,p}(E)} I(u),$$

where I is the energy operator associated with w. Furthermore, if X is also path-connected, such function is unique up to sets of p-capacity zero.

Proof. By [C], $N^{1,p}(X)$ is reflexive. Since $N_0^{1,p}(E)$ is an isometric copy of a closed subspace of $N^{1,p}(X)$, the reflexivity property holds true for $N_0^{1,p}(E)$ as well. Now Lemmata 5.3.2, 5.3.3, 5.3.4, and Theorem 5.3.1 together yield the existence of the minimizing function.

Now suppose that u_1 and u_2 satisfy such minimizing property. Then by the argument in [C, Theorem 7.15], $g_{u_1-u_2} = 0$, where $g_{u_1-u_2}$ is the minimal weak upper gradient of $u_1 - u_2$, shown to exist in Corollary 2.3.4. By Theorem 4.2.11 and by the fact that X is path-connected and hence, by the $(1,q)$-Poincaré inequality, is quasiconvex (see [HK2]), X satisfies the MEC_p property by Theorem 4.2.14. Hence for almost all points x, y in X,

$$|(u_1 - u_2)(x) - (u_1 - u_2)(y)| \leq \int_\gamma g_{u_1 - u_2} = 0.$$

Since by hypothesis there is a positive measure set B of points on which u_1 and u_2 agree, choosing one such possible point for y that also satisfies the above inequality yields that

$$|u_1(x) - u_2(x)| = 0.$$

Therefore $u_1 = u_2$ almost everywhere and hence quasi-everywhere by Lemma 2.1.2 and Lemma 4.2.4. Thus the minimizer is unique upto sets of p-capacity zero.

□

Remark 5.3.6. The Dirichlet type problem solved by the above theorem yields a solution $f = \tilde{u} + w$, where the boundary condition of the problem is $f\big|_{X\setminus E} = w\big|_{X\setminus E}$. If $w \in N^{1,p}(X)$ is the zero function $w = 0$, then $\tilde{u} = 0$, since then $I(u) = \inf_g \|g\|_{L^p}^p$, with the infimum being taken over weak upper gradients g of $u = u + w$ itself, and $I(0) = 0 \leq I(u)$ for each function u in $N_0^{1,p}(E)$.

Henceforth, $N_0^{1,p}(E) + w$ will denote the set of all functions of form $v + w$ with v in the space $N_0^{1,p}(E)$.

5.4 The Maximum Principle.

In the classical theory of harmonic functions, it is well-known that the minimizing functions found in the previous section satisfies a maximum principle. A natural question now is, does a maximum principle hold for the minimizing functions found in the previous section in such generality. It will be shown in this section that such a maximum principle does indeed hold.

Following the footsteps of [C], the previous section yields a natural definition of p-harmonic functions.

Definition 5.4.1. Let $1 < p < \infty$. A function u in $N^{1,p}(X)$ is said to be a *relaxed p-harmonic function* on a set $E \subset X$ if for each function v in $N_0^{1,p}(E) + u$ it is true that

$$\|g_u\|_{L^p}^p \leq \|g_v\|_{L^p}^p,$$

where g_u and g_v are the minimal weak upper gradients of u and v respectively; see Corollary 2.3.4.

Since it is true that if $F \subset E$ then $H_0^{1,p}(F) \subset N_0^{1,p}(E)$, it is also true that if u is a relaxed p-harmonic function on E then for every subset F of E that function is p-harmonic on F.

Definition 5.4.2. Let E be a subset of X. A function u on X for which there exists a function w in $N^{1,p}(X)$ with the properties that $u-w$ is in $N_0^{1,p}(E)$ and $u-w$ satisfies the conclusion of Theorem 5.3.5, is called a *relaxed solution of a Dirichlet problem on $E \subset X$ with boundary condition w*, where w is as in the definition (5.6).

Lemma 5.4.3. *Let $1 < p < \infty$. A function $u \in N^{1,p}(X)$ is a relaxed p-harmonic function on $E \subset X$ if and only if it is a relaxed solution to a Dirichlet problem on E.*

Proof. Suppose that u is a relaxed p-harmonic function on E. Then u satisfies Theorem 5.3.5 with $w = u$, and hence is a relaxed solution to a Dirichlet problem on E.

Conversely, if u is a relaxed solution to a Dirichlet problem on E, then u agrees with its boundary condition w on $X \setminus E$. If v is another function in $N_0^{1,p}(E)+u$, then it is in $N_0^{1,p}(E)+w$, and by the energy minimizing property of u it is true that

$$\|g_u\|_{L^p}^p \leq \|g_v\|_{L^p}^p.$$

Hence u is a relaxed p-harmonic function on E. $\square$

In the classical situation it is known that Dirichlet solutions are Hölder continuous; see for example [HeKM]. In the above definitions the word "relaxed" is included to indicate that it is not known whether such functions are continuous or not in the generality of the setting considered.

The following main theorem of this section establishes a maximum principle for such relaxed solutions. In the following theorem the two relaxed solutions u and v correspond to two different boundary conditions, since if the boundary conditions are the same, by uniqueness of the solution the maximum principle is trivial.

Theorem 5.4.4. *Suppose X is an MEC_p-space and that $p > 1$. Let $U \subset X$ and suppose u is a relaxed solution to a Dirichlet problem on U. If v is a relaxed solution to a Dirichlet problem on $V \subset U$ such that $v \leq u$ quasi-everywhere in $\overline{U \setminus V}$, then almost everywhere on V it is true that $v \leq u$.*

The proof of the above theorem requires the following lemma.

Lemma 5.4.5. *Let X be a metric measure space, and $1 \leq p < \infty$. If u and v are two functions in $N^{1,p}(X)$ and E is a subset of X such that $u \leq v$ quasi-everywhere on E and $u > v$ quasi-everywhere on $X \setminus E$, then the function*

$$w = u\chi_E + v\chi_{X\setminus E}$$

is also in $N^{1,p}(X)$.

Proof. Since both u and v are continuous on p-modulus almost every curve, it is clear that E is p-path closed; that is, the preimage of p-almost every curve is closed. Let g and h be weak upper gradients of u and v respectively. Let Γ_0 be the collection of curves γ on which at least one of u, v, is not absolutely continuous or at least one of the function-weak upper gradients pairs (u, g), (v, h) does not satisfy inequality (1.2) on some subpath of γ or γ passes through a point in E where $u > v$ or through a point in $X \backslash E$ where $u \leq v$. The p-modulus of this collection is zero by Proposition 2.1.1 and Lemma 2.2.5.

Let γ be a compact rectifiable curve that is not in Γ_0. Denote the two endpoints of γ by x and y. If both these points lie in E or if both lie in $X \backslash E$, then w satisfies inequality (1.2) together with the function $g + h$. Hence without loss of generality x is in E and y lies outside of E. Since $\gamma^{-1}(E)$ is a closed subset of a compact interval, there is a point x_0 on ∂E at which γ passes from E to outside of E for the last time. Now, as $w(x_0) = u(x_0) = v(x_0)$,

$$|w(x) - w(x_0)| = |u(x) - u(x_0)| \leq \int_\gamma g ds,$$

and

$$|w(x_0) - w(y)| = |v(x_0) - v(y)| \leq \int_\gamma h ds.$$

Hence $|w(x) - w(y)| \leq \int_\gamma (g + h) ds$. Therefore $g + h$ is a weak upper gradient of w, and hence w is in $N^{1,p}(X)$. □

proof of **Theorem 5.4.4** . Because X is an MEC_p space and $p > 1$, by the proof of uniqueness in Theorem 5.3.5, relaxed solutions to a specific Dirichlet problem are unique up to sets of zero p-capacity. Let E be the set of points x in V for which $v(x) > u(x)$. The aim is to show that this set has measure zero.

Let Γ_0 be the family of all compact rectifiable paths γ on which at least one of u, v is not absolutely continuous or at least one of the function-weak upper gradient pairs (u, g_u), (v, g_v) does not satisfy inequality (1.2). Here g_u and g_v are the minimal weak upper gradients of u and v respectively, courtesy of Corollary 2.3.4. It is clear that the p-modulus of Γ_0 is zero.

Define a function w on V by $w(x) = \max\{u(x), v(x)\}$, and extend w to all of X by

$$w(x) = \begin{cases} u(x) & \text{if } x \in X \backslash E \\ v(x) & \text{if } x \in E. \end{cases}$$

Since by assumption $v \leq u$ on $\overline{U\backslash V}$, therefore $\max\{u(x), v(x)\} = u(x)$ on $\overline{U\backslash V}$. Hence by Lemma 5.4.5 the function w is in $N^{1,p}(X)$. Now by Lemma 2.1.5, the functions $g_v\chi_E + g_u$ and $g_v + g_u\chi_{X\backslash E}$ are weak upper gradients of w. Hence by Lemma 2.1.8, the function $g_0 = g_v\chi_E + g_u\chi_{X\backslash E}$ is a weak upper gradient of w. Therefore, by the energy minimizing property of u and by the fact that w is a function in $N_0^{1,p}(U) + u$,

$$||g_v||^p_{L^p(E)} + ||g_u||^p_{L^p(X\backslash E)} = ||g_0||^p_{L^p} \geq ||g_u||^p_{L^p(X)},$$

and hence $||g_v||_{L^p(E)} \geq ||g_u||_{L^p(E)}$. Now by considering the fact that v is a Dirichlet solution on V and by looking at the function

$$f(x) = \begin{cases} \min\{u(x), v(x)\} & \text{if } x \in V \\ v(x) & \text{if otherwise ,} \end{cases}$$

a similar argument as above yields that $||g_v||_{L^p(E)} \leq ||g_u||_{L^p(E)}$, since the function f is in $N_0^{1,p}(V) + v$. Hence $||g_v||_{L^p(E)} = ||g_u||_{L^p(E)}$, and therefore, since $g_w \leq g_0$ almost everywhere, $||g_w||_{L^p(E)} = ||g_u||_{L^p(E)}$. Thus w is also a relaxed solution to the same Dirichlet problem as u. Hence by the uniqueness of Dirichlet solutions, $w = u$ quasi-everywhere on X; in particular, $\mu(E) = 0$. $\square$

APPENDIX

APPENDIX A

LEBESGUE DIFFERENTIATION THEOREM

The aim of this section is to prove the Lebesgue differentiation theorem in a general setting. The Euclidean spaces $\mathbb{R}^n$, together with the Euclidean distance and the n-dimensional Lebesgue measure, are known to support the Lebesgue differentiation theorem; see [Ma]. The Lebesgue differentiation theorem in this case states that for every locally integrable real-valued function f it is true that for almost every point x, the limit $\lim_{r\to 0} ⨍_{B(x,r)} f$ exists and equals the value of the function at x. The book [Ma] extends this result to all Radon measures on $\mathbb{R}^n$ that are absolutely continuous with respect to the n-dimensional Lebesgue measure; see [Ma, Theorem 2.12] and [He]. The proof for the theorem below as well as the proofs of other results in this appendix essentially follows the proof of [Ma, Theorem 2.12]. For a definition of Vitali space used in the following theorem, see Definition 4.2.8.

Theorem A.0.6. *If X is a Vitali space equipped with an outer measure so that for each ball $B \subset X$ the measure of B is positive and finite, then the following are true:*

1. *If f is a real-valued, locally integrable function on X, then for almost all points x in X,*

$$f(x) = \lim_{r\to 0} ⨍_{B(x,r)} f.$$

2. *If $A \subset X$ is a Borel set, then for almost every point x in A,*

$$\lim_{r\to 0} \frac{\mu(B(x,r)\cap A)}{\mu(B(x,r))} = 1.$$

The proof of the above theorem follows from the following lemma. This lemma is a generalization of the Radon-Nikodym theorem for Lebesgue measures in the Euclidean space setting.

Lemma A.0.7. *Under the hypotheses of the above theorem, if ν is another measure on X that is absolutely continuous with respect to the given measure μ, then the Radon-Nikodym derivative of ν with respect to μ,*

$$D(\nu,\mu,x) = \lim_{r\to 0^+} \frac{\nu(B(x,r))}{\mu(B(x,r))}$$

exists for μ-almost every x in X. Furthermore, for each μ-measurable set $S \subset X$,

$$\nu(S) = \int_S D(\nu,\mu,x)d\mu(x).$$

Proof. Define

$$D^+(\nu,\mu,x) = \limsup_{r\to 0^+} \frac{\nu(B(x,r))}{\mu(B(x,r))},$$

and

$$D^-(\nu,\mu,x) = \liminf_{r\to 0^+} \frac{\nu(B(x,r))}{\mu(B(x,r))}.$$

Fix x_0 in X. For non-negative rational numbers s and t, let

$$\begin{aligned}
A_{s,t,r_0} &= \{x \in B(x_0,r_0) \ : \ D^-(\nu,\mu,x) \le s < t \le D^+(\nu,\mu,x)\},\\
A_{t,r_0} &\subset \{x \in B(x_0,r_0) \ : \ D^+(\nu,\mu,x) \ge t\},\\
A^s_{r_0} &\subset \{x \in B(x_0,r_0) \ : \ D^-(\nu,\mu,x) \le s\}.
\end{aligned}$$

Let $\epsilon > 0$. Since μ and ν are outer measures, there are open sets U_ϵ containing A_{t,r_0} and V_ϵ containing $A^s_{r_0}$ so that $\nu(U_\epsilon) \le \nu(A_{t,r_0}) + \epsilon$ and $\mu(V_\epsilon) \le \mu(A^s_{r_0}) + \epsilon$. For each $x \in A_{t,r_0}$ there are balls $B(x,r_i) \subset U_\epsilon$ so that

$$\frac{\nu(B(x,r_i))}{\mu(B(x,r_i))} \ge t - \epsilon \tag{A.1}$$

for a sequence of radii $r_i \to 0$. Such balls constitute a Vitali covering collection of the set A_{t,r_0}. Hence as X is a Vitali space, there is a countable, pairwise disjoint subcollection $\{B_i\}_{i\in\mathbb{N}}$ so that

$$\mu(A_{t,r_0} \backslash \bigcup_{i\in\mathbb{N}} B_i) = 0, \tag{A.2}$$

and hence as ν is absolutely continuous with respect to μ,

$$\nu(A_{t,r_0} \backslash \bigcup_{i\in\mathbb{N}} B_i) = 0.$$

Hence

$$\nu(A_{t,r_0}) + \epsilon \geq \nu\left(\bigcup_{i\in\mathbb{N}} B_i\right) = \sum_i \nu(B_i)$$

by the disjointness of the subcollection. Therefore

$$\nu(A_{t,r_0}) + \epsilon \geq (t-\epsilon)\sum_i \mu(B_i)$$

by inequality (A.1) and by the fact that $B_i \subset U_\epsilon$. Hence by inequality (A.2),

$$\nu(A_{t,r_0}) \geq (t-\epsilon)\mu\left(\bigcup_{i\in\mathbb{N}} B_i\right) \geq (t-\epsilon)\mu(A_{t,r_0}) - \epsilon.$$

Similarly, for each $x \in A^s_{r_0}$ there are balls $B(x, r_j) \subset V_\epsilon$ so that

$$\frac{\nu(B(x,r_i))}{\mu(B(x,r_i))} \leq s + \epsilon \tag{A.3}$$

for a sequence of radii $r_j \to 0$. As before, obtain a countable disjoint subcollection $\{B_i\}_{i\in\mathbb{N}}$ so that

$$\mu(A^s_{r_0} \backslash \bigcup_{i\in\mathbb{N}} B_i) = 0, \tag{A.4}$$

and hence

$$\nu(A^s_{r_0} \backslash \bigcup_{i\in\mathbb{N}} B_i) = 0.$$

therefore by inequality (A.3),

$$\nu(A^s_{r_0}) \leq \nu\left(\bigcup_{i\in\mathbb{N}} B_i\right) \leq (s+\epsilon)\sum_i \mu(B_i),$$

and hence

$$\nu(A^s_{r_0}) \leq (s+\epsilon)\mu(V_\epsilon) \leq (s+\epsilon)\left(\mu(A^s_{r_0}) + \epsilon\right).$$

Now letting $\epsilon \to 0$,

$$\nu(A_{t,r_0}) \geq t\mu(A_{t,r_0}), \tag{A.5}$$

$$\text{and } \nu(A^s_{r_0}) \leq s\mu(A^s_{r_0}). \tag{A.6}$$

Since the set A_{s,t,r_0} satisfies the conditions of both sets $A^s_{r_0}$ and A_{t,r_0}, therefore

$$s\mu(A_{s,t,r_0}) \geq \nu(A_{s,t,r_0}) \geq t\mu(A_{s,t,r_0}).$$

As $s < t$, therefore it must be true that the μ-measure of A_{s,t,r_0} must be zero for each non-negative rational number s, t and each positive integer r_0. Hence the μ-measure of the set $\{x \in X \ : \ D^-(\nu, \mu, x) < D^+(\nu, \mu, x)\}$, which can be expressed as a countable union

$$\bigcup_{r_0 \in \mathbb{N},} \bigcup_{0<s<t \text{ rational}} \{x \in B(x_0, r_0) \ : \ D^-(\nu, \mu, x) \leq s < t \leq D^+(\nu, \mu, x)\}$$

is zero. Hence the derivative $D(\nu, \mu, x)$ exists for μ-almost every $x \in X$. To see that this derivative is finite almost everywhere, note that since $t\mu(A_{t,r_0}) \leq \nu(A_{t,r_0}) \leq \nu(B(x_0, r_0)) < \infty$, if

$$A_{r_0} = \{x \in B(x_0, r_0) \ : \ D^+(\nu, \mu, x) = \infty\},$$

then for each $n \in \mathbb{N}$

$$n\mu(A_{r_0}) \leq \nu(B(x_0, r_0)) < \infty.$$

Hence the μ-measure of A_{r_0} is zero.

Let $S \subset X$ be μ-measurable. Let $t > 1$, and for each integer n let B_n be the set of all points x in S for which $t^n \leq D(\nu, \mu, x) < t^{n+1}$. Now, by inequality (A.5),

$$\begin{aligned}
\int_S D(\nu, \mu, x) d\mu(x) &= \sum_{n \in \mathbb{Z}} \int_{B_n} D(\nu, \mu, x) d\mu(x) \\
&\leq \sum_{n \in \mathbb{Z}} t^{n+1} \mu(B_n) \\
&\leq \sum_{n \in \mathbb{Z}} t^{n+1} t^{-n} \nu(B_n) \leq t\nu(S).
\end{aligned}$$

Now letting $t \to 1$, it is clear that $\int_S D(\nu, \mu, x) d\mu(x) \leq \nu(S)$. However, by inequality (A.6),

$$\int_S D(\nu, \mu, x) d\mu(x) \geq \sum_{n \in \mathbb{Z}} t^n \mu(B_n) \geq \sum_{n \in \mathbb{Z}} \nu(B_n) \geq \nu(S).$$

This concludes the proof of the last part of the lemma. □

Now Theorem A.0.6 is ready to be proven.

Proof of **Theorem A.0.6** . Let f be a locally integrable function on X, and define a measure ν on X by

$$\nu(A) = \int_A f d\mu$$

for every μ-measurable subset A of X. By Lemma A.0.7, for every Borel set,

$$\int_A f d\mu = \nu(A) = \int_A D(\nu, \mu, x) d\mu(x).$$

Hence $D(\nu, \mu, x) = f(x)$ for μ-almost every point x. Therefore, for μ-almost every x in X,

$$f(x) = \lim_{r \to 0} \fint_{B(x,r)} f d\mu.$$

Such points are called Lebesgue points of f.

If $A \subset X$ is a Borel set, the function $f = \chi_A$ is a locally integrable function on X, and hence the above argument also proves the second part of the theorem. $\square$

Next, a criterion for a space to be a Vitali space is explored.

Definition A.0.8. A metric measure space X is said to have a *locally doubling measure* if there exists a constant $C > 0$ so that for μ-almost every $x \in X$ there exists $r_x > 0$ so that for each positive $r < r_x$, it is true that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$.

Riemannian manifolds and metric spaces equipped with a doubling measure are locally doubling. Hyperbolic spaces are locally doubling, but are not doubling.

Lemma A.0.9. *If X is a metric space equipped with a locally doubling measure, then X is a Vitali space.*

Proof. Let A be a bounded subset of X and $\{B_\alpha\}$ be a Vitali type cover of A. Without loss of generality, for all $x \in A$ there is a positive number r_x associated with the local doubling condition. Let U be an open neighbourhood of A such that $\mu(U) \leq (1 + \epsilon)\mu(A)$. By the general covering theorem (see [HeK1]), there is a countable disjoint sub collection $\{B_i\}_{i \in \mathbb{N}}$ so that $A \subset \cup_{i \in \mathbb{N}} B(x_i, 5r_i)$, $5r_i \leq r_{x_i}$, and $B(x_i, 5r_i) \subset U$. Then, by the local doubling property,

$$\mu(A) \leq \sum_i \mu(B(x_i, 5r_i)) \leq C \sum_i \mu(B(x_i, r_i)).$$

Therefore there exists a positive integer k_0 so that $\mu(A) \leq (C+1) \sum_{i=1}^{k_0} \mu(B(x_i, r_i))$. Hence

$$\frac{1}{C+1}\mu(A) \leq \mu\left(\bigcup_{i=1}^{k_0} B(x_i, r_i) \right).$$

Let $A_1 = A \setminus \cup_{i=1}^{k_0} B(x_i, r_i)$. Then A_1 is a subset of $U \setminus \cup_{i=1}^{k_0} B(x_i, r_i)$, and hence

$$\begin{aligned}\mu(A_1) \leq& \mu(U) - \mu\left(\bigcup_{i=1}^{k_0} B(x_i, r_i)\right)\\ \leq& (1+\epsilon)\mu(A) - \frac{1}{C+1}\mu(A) \leq (1+\epsilon - \frac{1}{C+1})\mu(A).\end{aligned}$$

Choosing $\epsilon < 1/(C+1)$ and letting $\delta = 1 + \epsilon - \frac{1}{C+1}$, get $\mu(A_1) \leq \delta\mu(A)$ with $0 < \delta < 1$. Repeating the argument as above with A_1 replacing the role of A and the original Vitali cover replaced by the collection of balls in that collection that do not intersect the balls B_i, $1 \leq i \leq k_0$, obtain $A_2 = A_1 \setminus \cup_{i=k_0+1}^{k_1} B(x_i, r_i)$ so that $\mu(A_2) \leq \delta\mu(A_1)$ and hence $\mu(A_2) \leq \delta^2\mu(A)$. Continuing inductively, obtain

$$A_i = a_{i-1} \setminus \bigcup_{j=k_{i-2}+1}^{k_i} B_j = A \setminus \bigcup_{j=1}^{k_i} B_j$$

and

$$\mu(A_i) \leq \delta^i \mu(A).$$

Since $\delta < 1$, the measure of A_i goes to zero. Hence the measure of $A \setminus \cup_{i\in\mathbb{N}} B_i$ is zero, and by construction the balls B_i are pairwise disjoint. $\square$

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ABSTRACT

Newtonian Spaces: An Extension of Sobolev Spaces to Metric Measure Spaces

by
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This thesis studies a definition of Sobolev spaces on metric measure spaces, using the notion of upper gradients. Let X be a space equipped with a metric d and a measure μ. If f is a real-valued function on X, then a non-negative Borel measurable function ρ on X is said to be an upper gradient of f if for each rectifiable curve γ in X the following inequality

$$|f(x) - f(y)| \leq \int_\gamma \rho ds$$

holds, where x and y denote the end-points of γ. The definition of Sobolev spaces studied in this thesis is as follows. Let $1 \leq p < \infty$. The space of all p-integrable real-valued functions on X with p-integrable upper gradients is called the Newtonian space of index p, in recognition of the fact that the above inequality defining upper gradients is a generalization of the fundamental theorem of calculus. It is proven in this thesis that even with no assumptions on X, the Newtonian spaces are Banach spaces. It is shown that for Euclidean domains the Newtonian spaces are indeed

the classical Sobolev spaces. It is also shown that real-valued functions on X that satisfy a Sobolev type definition of Hajłasz with index p are also in the Newtonian space of index p for all finite $p \geq 1$. Furthermore, it is shown that if X supports a $(1, q)$-Poincaré inequality with $1 \leq q < p$ and the measure on X is doubling, then functions in the Newtonian space of index p satisfy the Hajłasz definition. Other influences of $(1, p)$-Poincaré inequality are also explored. It is shown that if X supports a $(1, p)$-Poincaré inequality and the measure on X satisfies some geometric conditions, then Lipschitz functions are dense in the Newtonian space of index p, and furthermore, functions in this Newtonian space are continuous outside of sets of arbitrarily small p-capacity. Sobolev embedding theorems are also proven under assumptions of Poincaré inequalities and some geometric conditions on the measure.

This thesis also studies a criterion, called the MEC_p criterion, under which the Newtonian space of index p is not degenerate; that is, it is strictly smaller than the L^p class. The effect of Poincaré inequalities on this criterion is studied, and some examples are explored in this context. The effect of the criterion on p-capacity estimates are also studied. For domains in X, Newtonian spaces with zero boundary values outside the domain are defined, and it is shown that under certain conditions on X, Lipschitz functions are dense in these spaces. In such spaces, the existence of energy-minimizing functions is proved; that is, a Dirichlet-type problem is solved. The main tool used in this thesis is the concept of p-modulus of curve families in X. This concept was first developed by Fuglede and Ziemer.