A Rigidity Property of Some Negatively Curved Solvable Lie Groups

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Abstract

We show that for some negatively curved solvable Lie groups, all self quasiisometries are almost isometries. We prove this by showing that all self quasisymmetric maps of the ideal boundary (of the solvable Lie groups) are bilipschitz with respect to the visual metric. We also define parabolic visual metrics on the ideal boundary of Gromov hyperbolic spaces and relate them to visual metrics.

1 Introduction

In recent years, there have been a lot of interest in the large scale geometry of solvable Lie groups and finitely generated solvable groups ([D], [EFW1], [EFW2], [FM1], [FM2], [FM3], [Pe]). In particular, Eskin, Fisher and Whyte ([EFW1], [EFW2]) proved the quasiisometric rigidity of the 3-dimensional solvable Lie group Sol. In this paper, we use quasiconformal analysis to prove a rigidity property of some negatively curved solvable Lie groups.

Let A be an $n \times n$ diagonal matrix with real eigenvalues α_i with $\alpha_{i+1} > \alpha_i > 0$:

$$A = \begin{pmatrix} \alpha_1 I_{n_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \alpha_2 I_{n_2} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \alpha_r I_{n_r} \end{pmatrix},$$

where I_{n_i} is the $n_i \times n_i$ identity matrix and the $\mathbf{0}$'s are zero matrices (of various sizes). Let $\mathbb{R}$ act on $\mathbb{R}^n$ by the linear transformations e^{tA} ($t \in \mathbb{R}$) and we can form the semidirect product $G_A = \mathbb{R}^n \rtimes_A \mathbb{R}$. That is, $G_A = \mathbb{R}^n \times \mathbb{R}$ as a smooth manifold, and the group operation is given for all $(x,t), (y,s) \in \mathbb{R}^n \times \mathbb{R}$ by:

$$(x,t) \cdot (y,s) = (x + e^{tA}y, t+s).$$

The group G_A is a simply connected solvable Lie group and is the subject of study in this paper.

We endow G_A with the left invariant metric determined by taking the standard Euclidean metric at the identity of $G_A \approx \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$. With this metric G_A has sectional curvature $-\alpha_r^2 \leq K \leq -\alpha_1^2$ (and so is Gromov hyperbolic). Hence G_A has a well defined ideal boundary ∂G_A. There is a so-called cone topology on $\overline{G_A} = G_A \cup \partial G_A$, in which ∂G_A is homeomorphic to the n-dimensional sphere and $\overline{G_A}$ is homeomorphic to the closed $(n+1)$-ball in the Euclidean space. For each $x \in \mathbb{R}^n$, the map $\gamma_x : \mathbb{R} \to G_A$, $\gamma_x(t) = (x, t)$ is a geodesic. We call such a geodesic a vertical geodesic. It can be checked that all vertical geodesics are asymptotic as $t \to +\infty$. Hence they define a point ξ_0 in the ideal boundary ∂G_A.

Since G_A is Gromov hyperbolic, there is a family of visual metrics on ∂G_A. For each $\xi \in \partial G_A$, there is also the so-called parabolic visual metric on $\partial G_A \backslash \{\xi\}$. The relation between visual metrics and parabolic visual metrics is analogous to the relation between spherical metric (on the sphere) and the Euclidean metric (on the one point complement of the sphere). See Section 5 for a discussion of all these in the setting of Gromov hyperbolic spaces. We next recall the parabolic visual metric D on ∂G_A viewed from ξ_0.

The set $\partial G_A \backslash \{\xi_0\}$ can be naturally identified with $\mathbb{R}^n$ (see Section 2). Write $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r}$, where $\mathbb{R}^{n_i}$ is the eigenspace associated to the eigenvalue α_i of A. Each point $x \in \mathbb{R}^n$ can be written as $x = (x_1, \cdots, x_r)$ with $x_i \in \mathbb{R}^{n_i}$. The parabolic visual metric D on $\partial G_A \backslash \{\xi_0\} \approx \mathbb{R}^n$ is defined by:

$$D(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|^{\alpha_1/\alpha_2}, \cdots, |x_r - y_r|^{\alpha_1/\alpha_r}\},$$

for all $x = (x_1, \cdots, x_r), y = (y_1, \cdots, y_r) \in \mathbb{R}^n$.

Let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism. An embedding of metric spaces $f : X \to Y$ is an *η-quasisymmetric embedding* if for all distinct triples $x, y, z \in X$, we have

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \leq \eta\left(\frac{d(x, y)}{d(x, z)}\right).$$

If f is further assumed to be a homeomorphism, we say it is *η-quasisymmetric*. A map $f : X \to Y$ is quasisymmetric if it is η-quasisymmetric for some η.

When $r \geq 2$, Bruce Kleiner has proved that ([K]) every self quasisymmetry of ∂G_A (equipped with a visual metric) preserves the horizontal foliation (see Section 3) and fixes the point ξ_0. This is one of the main ingredients in the proof of our main result. Since Kleiner's proof is unpublished, we include a proof here for completeness. Notice that Kleiner's result implies that a self quasisymmetry of ∂G_A induces a self map of $(\mathbb{R}^n, D)$.

The following is the main result of this paper.

Theorem 1.1. *Let G_A and $\xi_0 \in \partial G_A$ be as above. If $r \geq 2$, then every self quasisymmetry of ∂G_A (equipped with a visual metric) is bilipschitz with respect to the parabolic visual metric D.*

One should compare this with quasiconformal maps on Euclidean spaces ([GV]) and Heisenberg groups ([B]), where there are non-bilipschitz quasiconformal maps. On the other hand, the conclusion of Theorem 1.1 is not as strong as in the cases of quaternionic hyperbolic spaces, Cayley plane ([P2]) and Fuchsian buildings ([BP], [X]), where every quasisymmetric map of the ideal boundary is actually a conformal map. In our case, there are many non-conformal quasisymmetric maps of the ideal boundary of G_A. We also remark that in [T2, Section 15] Tyson has previously classified (quasi)metric spaces of the form $(\mathbb{R}^n, D)$ up to quasisymmetry.

We list three consequences of Theorem 1.1.

Let $L \geq 1$ and $C \geq 0$. A (not necessarily continuous) map $f : X \to Y$ between two metric spaces is an (L, A)-*quasiisometry* if:
(1) $d(x_1, x_2)/L - C \leq d(f(x_1), f(x_2)) \leq L\, d(x_1, x_2) + C$ for all $x_1, x_2 \in X$;
(2) for any $y \in Y$, there is some $x \in X$ with $d(f(x), y) \leq C$.
In the case $L = 1$, we call f an *almost isometry.*

Corollary 1.2. *Assume that $r \geq 2$. Then every self quasiisometry of G_A is an almost isometry.*

Notice that an almost isometry is not necessarily a finite distance away from an isometry.

The following result was previously obtained by B. Kleiner [K].

Corollary 1.3. *If $r \geq 2$, then G_A is not quasiisometric to any finitely generated group.*

In the identification of G_A with $\mathbb{R}^n \times \mathbb{R}$, we view the map $h : \mathbb{R}^n \times \mathbb{R}$, $h(x, t) = t$ as the height function. A quasiisometry φ of G_A is hight-respecting if $|h(\varphi(x, t)) - t|$ is bounded independent of x, t.

Corollary 1.4. *Assume that $r \geq 2$. Then all self quasiisometries of G_A are height-respecting.*

The question of whether a quasiisometry of G_A is height-respecting is important for the following three reasons. First, Mosher and Farb ([FM1]) have classified a large class of solvable Lie groups (including groups of type G_A) up to height-respecting quasiisometries. Second, there is no known examples of non-height-respecting quasi-isometries except for rank one symmetric spaces of noncompact type. Finally, showing a quasiisometry is height-respecting is a main step in the proof of the quasiisometric rigidity of Sol ([EFW1], [EFW2]).

When $r = 1$, the group G_A is isometric to a rescaling of the real hyperbolic space. In this case, all the above results fail.

This paper is structured as follows. In Section 2 we review some basics about the group G_A. In Section 3 we prove that quasisymmetric self-maps of $\partial G_A \backslash \{\xi_0\}$ equipped with the parabolic visual metric preserve horizontal foliations, and in Section 4 we will prove that such maps are bilipschitz with respect to this metric. The main result of this paper, Theorem 1.1, is proven in Section 5, where a discussion of parabolic visual metrics on the ideal boundary and their connection to the visual metrics can also be found. In Section 6 we provide the proofs of the Corollaries stated in Section 1.

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2 The Solvable Lie Groups G_A

In this section we review some basic facts about the group G_A and define several parabolic visual (quasi)metrics on the ideal boundary.

Let A and G_A be as in the Introduction. We endow G_A with the left invariant metric determined by taking the standard Euclidean metric at the identity of $G_A \approx \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$. At a point $(x,t) \in \mathbb{R}^n \times \mathbb{R} \approx G_A$, the tangent space is identified with $\mathbb{R}^n \times \mathbb{R}$, and the Riemannian metric is given by the symmetric matrix

$$\begin{pmatrix} e^{-2tA} & 0 \\ 0 & 1 \end{pmatrix}.$$

With this metric G_A has sectional curvature $-\alpha_r^2 \le K \le -\alpha_1^2$. Hence G_A has a well defined ideal boundary ∂G_A. All vertical geodesics γ_x ($x \in \mathbb{R}^n$) are asymptotic as $t \to +\infty$. Hence they define a point ξ_0 in the ideal boundary ∂G_A.

The sets $\mathbb{R}^n \times \{t\}$ ($t \in \mathbb{R}$) are horospheres centered at ξ_0. For each $t \in \mathbb{R}$, the induced metric on the horosphere $\mathbb{R}^n \times \{t\} \subset G_A$ is determined by the quadratic form e^{-2tA}. This metric has distance formula $d_{\mathbb{R}^n\times\{t\}}((x,t),(y,t)) = |e^{-tA}(x-y)|$. Here $|\cdot|$ denotes the Euclidean norm. The distance between two horospheres, corresponding to $t = t_1$ and $t = t_2$, is $|t_1 - t_2|$. It follows that for $(x_1,t_1),(x_2,t_2) \in G_A = \mathbb{R}^n \times \mathbb{R}$,

$$d((x_1,t_1),(x_2,t_2)) \ge |t_1 - t_2|. \tag{2.1}$$

Each geodesic ray in G_A is asymptotic to either an upward oriented vertical geodesic or a downward oriented vertical geodesic. The upward oriented geodesics are asymptotic to ξ_0 and the downward oriented vertical geodesics are in 1-to-1 correspondence with $\mathbb{R}^n$. Hence $\partial G_A \backslash \{\xi_0\}$ can be naturally identified with $\mathbb{R}^n$.

Given $x, y \in \mathbb{R}^n \approx \partial G_A \backslash \{\xi_0\}$, the parabolic visual quasimetric $D_e(x,y)$ is defined as follows: $D_e(x,y) = e^t$, where t is the unique real number such that at height t the two vertical geodesics γ_x and γ_y are at distance one apart in the horosphere; that is, $d_{\mathbb{R}^n\times\{t\}}((x,t),(y,t)) = |e^{-tA}(x-y)| = 1$. Here the subscript e in D_e means it corresponds to the Euclidean norm. This definition of parabolic visual quasimetric is very natural, but D_e does not have a simple formula. Next we describe another parabolic visual quasimetric which is bilipschitz equivalent with D_e and admits a simple formula. Recall that a quasimetric on a set A is a function $\rho : A \times A \to [0, \infty)$ satisfying: (1) $\rho(x, y) = \rho(y, x)$ for all $x, y \in A$; (2) $\rho(x, y) = 0$ only when $x = y$; (3) there is a constant $L \geq 1$ such that $\rho(x, z) \leq L(\rho(x, y) + \rho(y, z))$ for all $x, y, z \in A$.

In addition to the Euclidean norm, there is another norm on $\mathbb{R}^n$ that is naturally associated to G_A. Write $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r}$, where $\mathbb{R}^{n_i}$ is the eigenspace associated to the eigenvalue α_i of A. Each point $x \in \mathbb{R}^n$ can be written as $x = (x_1, \cdots, x_r)$ with $x_i \in \mathbb{R}^{n_i}$. The block supernorm is given by: $|x|_s = \max\{|x_1|, \cdots, |x_r|\}$ for $x = (x_1, \cdots, x_r)$. Using this norm one can define another parabolic visual quasimetric on $\partial G_A \backslash \{\xi_0\}$ as follows: $D_s(x, y) = e^t$, where t is the unique real number such that at height t the two vertical geodesics γ_x and γ_y are at distance one apart with respect to the norm $|\cdot|_s$; that is, $|e^{-tA}(x - y)|_s = 1$. Here the subscript s in D_s means it corresponds to the block supernorm $|\cdot|_s$. Then D_s is given by [D, Lemma 7]:

$$D_s(x, y) = \max\{|x_1 - y_1|^{\frac{1}{\alpha_1}}, \cdots, |x_r - y_r|^{\frac{1}{\alpha_r}}\},$$

for all $x = (x_1, \cdots, x_r), y = (y_1, \cdots, y_r) \in \mathbb{R}^n$.

Notice that $|x|_s \leq |x| \leq \sqrt{r}\, |x|_s$ for all $x \in \mathbb{R}^n$. Using this, one can verify the following elementary lemma, whose proof is left to the reader.

Lemma 2.1. *For all $x, y \in \mathbb{R}^n$ we have $D_s(x, y) \leq D_e(x, y) \leq r^{1/2\alpha_1} D_s(x, y)$.*

In general, D_s does not satisfy the triangle inequality. However, for each $0 < \epsilon \leq \alpha_1$, the function D_s^ϵ is always a metric, called a parabolic visual metric. In this paper we consider the following parabolic visual metric

$$D(x, y) = D_s^{\alpha_1}(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|^{\alpha_1/\alpha_2}, \cdots, |x_r - y_r|^{\alpha_1/\alpha_r}\}.$$

With respect to this metric the rectifiable curves in $\mathbb{R}^n \approx \partial G_A \setminus \{\xi_0\}$ are necessarily curves of the form $\gamma : I \to \mathbb{R}^n$ with $\gamma(t) = (\gamma_1(t), c_2, \cdots, c_r)$ where $c_i \in \mathbb{R}^{n_i}$, $2 \leq i \leq r$, are constant vectors. This follows from the fact that the directions corresponding to $\mathbb{R}^{n_i}$, $i \geq 2$, have their Euclidean distance components "snowflaked" by the power $\alpha_1/\alpha_i < 1$.

3 Quasisymmetric maps preserve horizontal foliations

In this section we show that every self-quasisymmetry of ∂G_A fixes the point $\xi_0 \in \partial G_A$ and preserves a natural foliation on $\partial G_A \backslash \{\xi_0\}$.

Recall that a metric space X endowed with a Borel measure μ is an *Ahlfors Regular space of dimension Q* (for short, a Q-regular space) if there exists a constant $C_0 \geq 1$ so that

$$C_0^{-1} r^Q \leq \mu(B_r) \leq C_0 r^Q$$

for every ball B_r with radius $r < \mathrm{diam}(X)$.

We need the following result; see [T1] for the definition of the modulus Mod_Q of a family of curves.

Theorem 3.1 ([T1, Theorem 1.4])**.** *Let X and Y be locally compact, connected, Q-regular metric spaces ($Q > 1$) and let $f : X \to Y$ be an η-quasisymmetric homeomorphism. Then there is a constant C depending only on η, Q and the regularity constants of X and Y so that*

$$\frac{1}{C}\, Mod_Q\Gamma \leq Mod_Q f(\Gamma) \leq C\, Mod_Q\Gamma$$

for every curve family Γ in X.

Recall that we write $\mathbb{R}^n$ as $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r}$. Set $Y = \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r}$ and write $\mathbb{R}^n = \mathbb{R}^{n_1} \times Y$. Since we assume $r \geq 2$, the set Y is nontrivial. The subsets $\{\mathbb{R}^{n_1} \times \{y\} : y \in Y\}$ form a foliation of $\mathbb{R}^n$. We call this foliation the horizontal foliation and each leaf $\mathbb{R}^{n_1} \times \{y\}$ a horizontal leaf. Since $\frac{\alpha_1}{\alpha_i} < 1$ for all $2 \leq i \leq r$, we notice that a curve in $(\mathbb{R}^n, D)$ is not rectifiable if it is not contained in a horizontal leaf.

Observe that $(\mathbb{R}^{n_i}, |\cdot|^{\alpha_1/\alpha_i})$ with the Hausdorff measure (which is comparable to the n_i-dimensional Lebesgue measure) is $n_i\alpha_i/\alpha_1$-regular. Let μ be the product of the Hausdorff measures on the factors $(\mathbb{R}^{n_i}, |\cdot|^{\alpha_1/\alpha_i})$. Then it is easy to see that $(\mathbb{R}^n, D)$ with the measure μ is Q-regular with $Q = \Sigma_{i=1}^r n_i \frac{\alpha_i}{\alpha_1}$. It follows that Theorem 3.1 applies to the metric space $(\mathbb{R}^n, D)$. We also point out here that the Hausdorff measure μ is comparable to the canonical n-dimensional Lebesgue measure on $\mathbb{R}^n$.

Theorem 3.2. *If $r \geq 2$, then every quasisymmetry $F : (\mathbb{R}^n, D) \to (\mathbb{R}^n, D)$ preserves the horizontal foliation on $\mathbb{R}^n$.*

Proof. Suppose F does not preserve the horizontal foliation. Then there are two points p and q in some $\mathbb{R}^{n_1} \times \{y\}$ such that $f(p)$ and $f(q)$ are not in the same horizontal leaf. Let γ be the Euclidean line segment from p to q and Γ be the family of straight segments parallel to γ in $\mathbb{R}^n$ whose union is an n-dimensional circular cylinder with γ as the central axis. The curves in Γ are rectifiable with respect to the metric D. Since f is a homeomorphism, by choosing the radius of the circular cylinder to be sufficiently small (by a compactness argument) we may assume that no curve in Γ is mapped into a horizontal leaf. It follows that $f(\Gamma)$ has no locally rectifiable curve and so $\mathrm{Mod}_Q f(\Gamma) = 0$. On the other hand, [V1], 7.2 (page 21) shows that $\mathrm{Mod}_Q\Gamma > 0$ (the Euclidean length element on each $\beta \in \Gamma$ is the same as the length element on β obtained from the metric D). Since $Q = \Sigma_{i=1}^r n_i \frac{\alpha_i}{\alpha_1} > 1$, this contradicts Theorem 3.1. Hence each horizontal leaf is mapped to a horizontal leaf. □

4 Quasisymmetry implies Bilipschitz

In this section we show that each self quasisymmetry of $(\mathbb{R}^n, D)$ is actually a bilipschitz map. One should contrast this with the case of Euclidean spaces and Heisenberg groups, where there are non-bilipschitz quasisymmetric maps ([GV], [B]). On the other hand, $(\mathbb{R}^n, D)$ is not as rigid as the ideal boundary of a quaternionic hyperbolic space or a Cayley plane ([P2]) or a Fuchsian building ([BP], [X]), where each self quasisymmetry is a conformal map.

Let $K \geq 1$ and $C > 0$. A bijection $F : X_1 \to X_2$ between two metric spaces is called a K-quasisimilarity (with constant C) if

$$\frac{C}{K}\, d(x,y) \leq d(F(x), F(y)) \leq C\, K\, d(x,y)$$

for all $x, y \in X_1$. It is clear that a map is a quasisimilarity if and only if it is a bilipschitz map. The point of using the notion of quasisimilarity is that sometimes there is control on K but not on C.

Theorem 4.1. *Let $F : (\mathbb{R}^n, D) \to (\mathbb{R}^n, D)$ be an η-quasisymmetry. Then F is a K-quasisimilarity with $K = (\eta(1)/\eta^{-1}(1))^{2r+2}$.*

In this section, we first develop some intermediate results, and then use these results to provide a proof of this theorem. We first recall some definitions.

Let $g : X_1 \to X_2$ be a homeomorphism between two metric spaces. We define for every $x \in X_1$ and $r > 0$,

$$L_g(x,r) = \sup\{d(g(x), g(x')) : d(x,x') \leq r\},$$
$$l_g(x,r) = \inf\{d(g(x), g(x')) : d(x,x') \geq r\},$$

and set

$$L_g(x) = \limsup_{r\to 0} \frac{L_g(x,r)}{r}, \quad l_g(x) = \liminf_{r\to 0} \frac{l_g(x,r)}{r}.$$

Then

$$L_{g^{-1}}(g(x)) = \frac{1}{l_g(x)} \quad \text{and} \quad l_{g^{-1}}(g(x)) = \frac{1}{L_g(x)}$$

for any $x \in X_1$. If g is an η-quasisymmetry, then $L_g(x,r) \leq \eta(1) l_g(x,r)$ for all $x \in X_1$ and $r > 0$. Hence if in addition

$$\lim_{r\to 0} \frac{L_g(x,r)}{r} \quad \text{or} \quad \lim_{r\to 0} \frac{l_g(x,r)}{r}$$

exists, then

$$0 \leq l_g(x) \leq L_g(x) \leq \eta(1) l_g(x) \leq \infty.$$

Recall the decomposition $\mathbb{R}^n = \mathbb{R}^{n_1} \times Y$. Given points $y = (x_2, \cdots, x_r)$ and $y' = (x'_2, \cdots, x'_r) \in Y$ with $x_i, x'_i \in \mathbb{R}^{n_i}$, set

$$D_Y(y, y') = \max\{|x_2 - x'_2|^{\frac{\alpha_1}{\alpha_2}}, |x_3 - x'_3|^{\frac{\alpha_1}{\alpha_3}}, \cdots, |x_r - x'_r|^{\frac{\alpha_1}{\alpha_r}}\}.$$

For $p = (x_1, y), p' = (x'_1, y') \in \mathbb{R}^{n_1} \times Y$, we have $D(p, p') = \max\{|x_1 - x'_1|, D_Y(y, y')\}$. We notice that for every $y_1, y_2 \in Y$, the Hausdorff distance in the metric D of the two horizontal leaves,

$$HD(\mathbb{R}^{n_1} \times \{y_1\}, \mathbb{R}^{n_1} \times \{y_2\}) = D_Y(y_1, y_2). \tag{4.1}$$

Also, for any $p = (x_1, y_1) \in \mathbb{R}^{n_1} \times Y$ and any $y_2 \in Y$,

$$D(p, \mathbb{R}^{n_1} \times \{y_2\}) = D_Y(y_1, y_2). \tag{4.2}$$

By Theorem 3.2 the quasisymmetry F preserves the horizontal foliation. Hence it induces a map $G : Y \to Y$ such that for any $y \in Y$, $F(\mathbb{R}^{n_1} \times \{y\}) = \mathbb{R}^{n_1} \times \{G(y)\}$. For each $y \in Y$, let $H(\cdot, y) : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ be the map such that $F(x, y) = (H(x, y), G(y))$ for all $x \in \mathbb{R}^{n_1}$. Because $F : (\mathbb{R}^n, D) \to (\mathbb{R}^n, D)$ is an η-quasisymmetry, it follows that for each fixed $y \in Y$, the map $H(\cdot, y) : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ is an η-quasisymmetry with respect to the Euclidean metric on $\mathbb{R}^{n_1}$. The following lemma together with equations (4.1) and (4.2) imply that $G : (Y, D_Y) \to (Y, D_Y)$ is also an η-quasisymmetry.

Lemma 4.2. ([T2, Lemma 15.9]) *Let $g : X_1 \to X_2$ be an η-quasisymmetry and $A, B, C \subset X_1$. If $HD(A, B) \leq t\, HD(A, C)$ for some $t \geq 0$, then there is some $a \in A$ such that*

$$HD(g(A), g(B)) \leq \eta(t) d(g(a), g(C)).$$

We recall that if $g : X_1 \to X_2$ is an η-quasisymmetry, then $g^{-1} : X_2 \to X_1$ is an η_2-quasisymmetry, where $\eta_2(t) = (\eta^{-1}(t^{-1}))^{-1}$, see [V2, Theorem 6.3]. Note that $\eta_2(1) = 1/\eta^{-1}(1)$ and $\eta_2^{-1}(1) = 1/\eta(1)$.

In the proofs of the following lemmas, the quantities $l_G, L_G, l_{G^{-1}}, L_{G^{-1}}$ are defined with respect to the metric D_Y. Similarly, $l_{H(\cdot,y)}$, $L_{H(\cdot,y)}$, l_{I_y} and L_{I_y} are defined with respect to the Euclidean metric on $\mathbb{R}^{n_1}$, where $I_y := H(\cdot, y)^{-1} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$. Lemmas 4.6 and 4.7 together verify Theorem 4.1 for the case $r = 2$. At the end of this section we will use induction to then complete the proof of Theorem 4.1 for the general case $r \geq 2$.

Lemma 4.3. *The following holds for all $y \in Y$ and $x \in \mathbb{R}^{n_1}$:*
(1) $L_G(y, r) \leq \eta(1)\, l_{H(\cdot,y)}(x, r)$ for $r > 0$;
(2) $\eta^{-1}(1)\, l_{H(\cdot,y)}(x) \leq l_G(y) \leq \eta(1)\, l_{H(\cdot,y)}(x)$;
(3) $\eta^{-1}(1)\, L_{H(\cdot,y)}(x) \leq L_G(y) \leq \eta(1)\, L_{H(\cdot,y)}(x)$.

Proof. To prove (1), let $y \in Y$, $x \in \mathbb{R}^{n_1}$ and $r > 0$. Let $y' \in Y$ be an arbitrary point with $D_Y(y, y') \leq r$ and $x' \in \mathbb{R}^{n_1}$ an arbitrary point with $|x - x'| \geq r$. Then $D((x, y), (x, y')) \leq r \leq D((x, y), (x', y))$. Since F is η-quasisymmetric, we have

$$\begin{aligned} D_Y(G(y), G(y')) \leq D(F(x, y), F(x, y')) &\leq \eta(1)\, D(F(x, y), F(x', y)) \\ &= \eta(1)\, |H(x, y) - H(x', y)|. \end{aligned}$$

Since y' and x' are chosen arbitrarily and are independent of each other, the inequality follows.

Next we prove (2) and (3). The second inequality of (2) follows directly from (1) and the fact that $l_G(y, r) \leq L_G(y, r)$, while the second inequality of (3) follows from (1) and the fact that $l_{H(\cdot,y)}(x, r) \leq L_{H(\cdot,y)}(x, r)$.

To prove the first inequalities in (2) and (3), observe that the inverse map $F^{-1} : (\mathbb{R}^n, D) \to (\mathbb{R}^n, D)$ is an η_2-quasisymmetry, with

$$F^{-1}(x, y) = (H(\cdot, G^{-1}(y))^{-1}(x), G^{-1}(y)) = (I_{G^{-1}(y)}(x), G^{-1}(y)).$$

Applying the second inequality of (2) proven above to I_y and G^{-1}, we obtain:

$$\frac{1}{L_G(y)} = l_{G^{-1}}(G(y)) \leq \eta_2(1) \cdot l_{I_y}(H(x, y)) = \frac{1}{\eta^{-1}(1)} \cdot \frac{1}{L_{H(\cdot,y)}(x)},$$

hence $L_G(y) \geq \eta^{-1}(1) L_{H(\cdot,y)}(x)$, which is the first inequality of (3). Similarly, using the second inequality of (3) we obtain the first inequality of (2). $\square$

When $r = 2$, we have $Y = \mathbb{R}^{n_2}$ and $D_Y = |\cdot|^{\frac{\alpha_1}{\alpha_2}}$.

Lemma 4.4. *Assume that $r = 2$. Then $0 < l_G(y) \leq L_G(y) \leq \eta(1) l_G(y) < \infty$ for a.e. $y \in Y$ with respect to the Lebesgue measure on $Y = \mathbb{R}^{n_2}$.*

Proof. Observe in this case that $D_Y(y, y') = |y - y'|^{\alpha_1/\alpha_2}$ for $y, y' \in Y = \mathbb{R}^{n_2}$. Because G is an η-quasisymmetry with respect to the metric D_Y, it is η_1-quasisymmetric with respect to the Euclidean metric, where $\eta_1(t) = (\eta(t^{\alpha_1/\alpha_2}))^{\alpha_2/\alpha_1}$. Hence the map $G : (\mathbb{R}^{n_2}, |\cdot|) \to (\mathbb{R}^{n_2}, |\cdot|)$ is differentiable a.e. with respect to the Lebesgue measure. With L^e_G, l^e_G the distortion quantities of the map G with respect to the Euclidean metric, the differentiability property of G shows that $\lim_{r\to 0} \frac{L^e_G(y,r)}{r}$ and $\lim_{r\to 0} \frac{l^e_G(y,r)}{r}$ exist. Since $L_G(y, r) = L^e_G(y, r^{\alpha_2/\alpha_1})^{\alpha_1/\alpha_2}$ and $l_G(y, r) = l^e_G(y, r^{\alpha_2/\alpha_1})^{\alpha_1/\alpha_2}$, this implies that both $\lim_{r\to 0} \frac{L_G(y,r)}{r}$ and $\lim_{r\to 0} \frac{l_G(y,r)}{r}$ exist for a.e. $y \in Y$. It follows that

$$0 \leq l_G(y) \leq L_G(y) \leq \eta(1) l_G(y) \leq \infty.$$

Fix $y \in Y$ such that both $\lim_{r\to 0} \frac{L_G(y,r)}{r}$ and $\lim_{r\to 0} \frac{l_G(,r)}{r}$ exist. We next prove that $L_G(y) \neq 0, \infty$. Suppose that $L_G(y) = \infty$. Then $l_G(y) = \infty$ and so by Lemma 4.3 (2), $l_{H(\cdot,y)}(x) = \infty$ for all $x \in \mathbb{R}^{n_1}$. Hence $I_y = H(\cdot, y)^{-1} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ has the property that $L_{I_y}(x) = 0$ for all $x \in \mathbb{R}^{n_1}$. This implies that I_y is a constant map, contradicting the fact that it is a homeomorphism. Similarly we use Lemma 4.3 (3) to show that $L_G(y) \neq 0$. $\square$

In the next two lemmas we use the fact that $\eta(1) \geq 1$ and $0 < \eta^{-1}(1) \leq 1$.

Lemma 4.5. *Suppose that $r = 2$. Then, for a.e. $y \in Y$, the map $H(\cdot, y) : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ is an $\eta(1)/\eta^{-1}(1)$-quasisimilarity with constant $l_G(y) > 0$.*

Proof. By Lemma 4.3 (2) we have $l_{H(\cdot,y)}(x) \geq l_G(y)/\eta(1)$. Lemma 4.3 (3) and Lemma 4.4 imply that, for a.e. $y \in Y$, we have $l_G(y) > 0$ and

$$L_{H(\cdot,y)}(x) \leq L_G(y)/\eta^{-1}(1) \leq (\eta(1)/\eta^{-1}(1))\, l_G(y)$$

for all $x \in \mathbb{R}^{n_1}$. Because $\mathbb{R}^{n_1}$ is a geodesic space, for a.e. $y \in Y$ the map $H(\cdot, y)$ is an $\eta(1)/\eta^{-1}(1)$-quasisimilarity with constant $l_G(y)$. □

Lemma 4.6. *If $r = 2$, then there exists a constant $C > 0$ with the following properties:*

(1) For each $y \in Y$, $H(\cdot, y)$ is an $(\eta(1)/\eta^{-1}(1))^4$-quasisimilarity with constant C;

(2) $G : (Y, D_Y) \to (Y, D_Y)$ is an $(\eta(1)/\eta^{-1}(1))^5$-quasisimilarity with constant C.

Proof. (1) Fix any $y_0 \in Y$ that satisfies both Lemma 4.4 and Lemma 4.5. Set $C = l_G(y_0)$. Let $y \in Y$ be an arbitrary point satisfying both Lemma 4.4 and Lemma 4.5. Fix $x_0 \in \mathbb{R}^{n_1}$ and choose $x \in \mathbb{R}^{n_1}$ such that $|x - x_0| \geq D_Y(y, y_0)$. Then

$$D((x, y_0), (x_0, y)) = D((x, y), (x_0, y)) = |x - x_0|.$$

By choosing x so that in addition $|H(x, y_0) - H(x_0, y)| > D_Y(G(y_0), G(y))$, by the η-quasisymmetry of F we have

$$\begin{aligned} |H(x, y_0) - H(x_0, y)| &= D(F(x, y_0), F(x_0, y)) \\ &\leq \eta(1) D(F(x, y), F(x_0, y)) = \eta(1)|H(x, y) - H(x_0, y)|. \end{aligned}$$

By the choice of y and Lemma 4.5, we have

$$|H(x, y) - H(x_0, y)| \leq (\eta(1)/\eta^{-1}(1)) l_G(y)|x - x_0|.$$

On the other hand,

$$\begin{aligned} |H(x, y_0) - H(x_0, y)| &\geq |H(x, y_0) - H(x_0, y_0)| - |H(x_0, y_0) - H(x_0, y)| \\ &\geq \frac{l_G(y_0)}{\eta(1)/\eta^{-1}(1)}|x - x_0| - |H(x_0, y_0) - H(x_0, y)|. \end{aligned}$$

Combining the above inequalities and letting $|x - x_0| \to \infty$, we obtain

$$l_G(y) \geq \frac{1}{(\eta(1))^3(\eta^{-1}(1))^{-2}} l_G(y_0) = \frac{C}{(\eta(1))^3(\eta^{-1}(1))^{-2}}. \tag{4.3}$$

Switching the roles of y_0 and y, we obtain $l_G(y) \le (\eta(1))^3(\eta^{-1}(1))^{-2} l_G(y_0)$. By Lemma 4.4, we have

$$L_G(y) \le \eta(1) l_G(y) \le (\eta(1))^4(\eta^{-1}(1))^{-2} C. \tag{4.4}$$

Because $\mathbb{R}^{n_1}$ is a geodesic space, to show that $H(\cdot, y)$ is a quasisimilarity it suffices to gain control over $l_{H(\cdot,y)}$ and $L_{H(\cdot,y)}$. By (4.4) and Lemma 4.3 (3),

$$L_{H(\cdot,y)}(x) \le L_G(y)/\eta^{-1}(1) \le C(\eta(1))^4(\eta^{-1}(1))^{-3}$$

for all $x \in \mathbb{R}^{n_1}$, and by (4.3) and Lemma 4.3 (2),

$$l_{H(\cdot,y)}(x) \ge \frac{1}{\eta(1)} l_G(y) \ge \frac{C}{(\eta(1))^4(\eta^{-1}(1))^{-2}}.$$

for all $x \in \mathbb{R}^{n_1}$. Hence for a.e. y, $H(\cdot, y)$ is an $(\eta(1)/\eta^{-1}(1))^4$-quasisimilarity with constant C. A limiting argument shows this is true for all y. Hence (1) holds.

(2) Recall that when $r = 2$ we have $Y = \mathbb{R}^{n_2}$ and $D_Y = |\cdot|^{\alpha_1/\alpha_2}$. Hence to prove (2) it suffices to show that $G : (\mathbb{R}^{n_2}, |\cdot|) \to (\mathbb{R}^{n_2}, |\cdot|)$ is a K-quasisimilarity with $K = (\eta(1)/\eta^{-1}(1))^{5\alpha_2/\alpha_1}$. As observed before, G is η_1-quasisymmetric with respect to the Euclidean metric, where $\eta_1(t) = (\eta(t^{\alpha_1/\alpha_2}))^{\alpha_2/\alpha_1}$. Because $\mathbb{R}^{n_2}$ is a geodesic space, it suffices to gain control over l_G^e and L_G^e, where l_G^e and L_G^e are similar to l_G and L_G, but with Euclidean metric instead of the metric D_Y. Because $l_G^e(p) = l_G(p)^{\alpha_2/\alpha_1}$ and $L_G^e(p) = L_G(p)^{\alpha_2/\alpha_1}$, it suffices to gain control over the quantities l_G and L_G in terms of $(\eta(1)/\eta^{-1}(1))^5$.

Notice that (1) implies

$$\frac{C}{(\eta(1)/\eta^{-1}(1))^4} \le l_{H(\cdot,y)}(x) \le L_{H(\cdot,y)}(x) \le C(\eta(1)/\eta^{-1}(1))^4$$

for all $x \in \mathbb{R}^{n_1}$ and all $y \in Y$. By Lemma 4.3, for all $y \in Y$ we have

$$\frac{C}{(\eta(1)/\eta^{-1}(1))^5} \le l_G(y) \le L_G(y) \le C(\eta(1)/\eta^{-1}(1))^5.$$

Hence (2) holds.

$\square$

Lemma 4.7. *Suppose that $r \ge 2$ and there are constants $K \ge 1$ and $C > 0$ with the following properties:*

(1) $G : (Y, D_Y) \to (Y, D_Y)$ is a K-quasisimilarity with constant C;

(2) For each $y \in Y$, $H(\cdot, y)$ is a K-quasisimilarity with constant C.

Then F is an $(\eta(1)/\eta^{-1}(1))K$-quasisimilarity with constant C.

Proof. Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^{n_1} \times Y$. We shall first establish a lower bound for $D(F(x_1, y_1), F(x_2, y_2))$. If $|x_1 - x_2| \le D_Y(y_1, y_2)$, then $D((x_1, y_1), (x_2, y_2)) = D_Y(y_1, y_2)$ and by (1),

$$D(F(x_1, y_1), F(x_2, y_2)) \ge D_Y(G(y_1), G(y_2)) \ge \frac{C}{K} D_Y(y_1, y_2) = \frac{C}{K} D((x_1, y_1), (x_2, y_2)).$$

If $|x_1 - x_2| > D_Y(y_1, y_2)$, then

$$D((x_1, y_1), (x_2, y_2)) = D((x_1, y_2), (x_2, y_2)) = |x_1 - x_2|,$$

and since F is an η-quasisymmetry, by using (2),

$$\begin{aligned} D(F(x_1, y_1), F(x_2, y_2)) &\ge \frac{1}{\eta(1)} D(F(x_1, y_2), F(x_2, y_2)) \\ &= \frac{1}{\eta(1)} |H(x_1, y_2) - H(x_2, y_2)| \\ &\ge \frac{C}{\eta(1)K} |x_1 - x_2| \\ &= \frac{C}{\eta(1)K} D((x_1, y_1), (x_2, y_2)). \end{aligned}$$

Hence we have a lower bound for $D(F(x_1, y_1), F(x_2, y_2))$.

By (1), $G^{-1} : (Y, D_Y) \to (Y, D_Y)$ is an K-quasisimilarity with constant C^{-1}. Similarly, (2) implies that for each $y \in Y$, $(H(\cdot, y))^{-1}$ is a K-quasisimilarity with constant C^{-1}. Also recall that F^{-1} is an η_2-quasisymmetry and F is an η-quasisymmetry. Now the argument in the previous paragraph applied to F^{-1} implies

$$D(F^{-1}(x_1, y_1), F^{-1}(x_2, y_2)) \ge \frac{1}{CK\eta_2(1)} D((x_1, y_1), (x_2, y_2)).$$

It follows that

$$D(F(x_1, y_1), F(x_2, y_2)) \le CK\eta_2(1) D((x_1, y_1), (x_2, y_2)) = \frac{CK}{\eta^{-1}(1)} D((x_1, y_1), (x_2, y_2))$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^n$, completing the proof. □

Proof of Theorem 4.1. We induct on r. Lemmas 4.6 and 4.7 yield the desired result in the case $r = 2$. Now we assume that $r \ge 3$ and that the Theorem is true for $r - 1$. By Lemma 4.2, F induces an η-quasisymmetry $G : (Y, D_Y) \to (Y, D_Y)$. It follows that G is η_1-quasisymmetric with respect to the metric $D_Y^{\alpha_2/\alpha_1}$ (and it is easy to verify that this is indeed a metric), where $\eta_1(t) = [\eta(t^{\alpha_1/\alpha_2})]^{\alpha_2/\alpha_1}$. We point out here that for $(x_2, \cdots, x_r), (x_2', \cdots, x_r') \in Y$,

$$D_Y((x_2, \cdots, x_r), (x_2', \cdots, x_r'))^{\alpha_2/\alpha_1} = \max\{|x_2 - x_2'|, |x_3 - x_3'|^{\alpha_2/\alpha_3}, \cdots, |x_r - x_r'|^{\alpha_2/\alpha_r}\}.$$

Hence the induction hypothesis applied to $G : (Y, D_Y^{\alpha_2/\alpha_1}) \to (Y, D_Y^{\alpha_2/\alpha_1})$ shows that G is an $(\eta_1(1)/\eta_1^{-1}(1))^{2r}$-quasisimilarity with constant C. Therefore $G : (Y, D_Y) \to (Y, D_Y)$ is a K_1-quasisimilarity with constant C^{α_1/α_2}, where

$$K_1 = \left(\frac{\eta_1(1)}{\eta_1^{-1}(1)}\right)^{\frac{2r\alpha_1}{\alpha_2}} = \left(\frac{\eta(1)}{\eta^{-1}(1)}\right)^{2r}. \tag{4.5}$$

This implies that $C^{\alpha_1/\alpha_2}/K_1 \le l_G(y) \le L_G(y) \le C^{\alpha_1/\alpha_2}K_1$ for all $y \in Y$. Now Lemma 4.3 yields

$$C^{\alpha_1/\alpha_2}\ \frac{1}{K_1\eta(1)} \le l_{H(\cdot,y)}(x) \le L_{H(\cdot,y)}(x) \le C^{\alpha_1/\alpha_2}\ \frac{K_1}{\eta^{-1}(1)}$$

for all $y \in Y$ and all $x \in \mathbb{R}^{n_1}$. Since $\mathbb{R}^{n_1}$ is a geodesic space, for each $y \in Y$ the map $H(\cdot, y)$ is a $K_1\frac{\eta(1)}{\eta^{-1}(1)}$-quasisimilarity with constant C^{α_1/α_2}. By Lemma 4.7, the map F is a $K_1(\frac{\eta(1)}{\eta^{-1}(1)})^2$-quasisimilarity with constant C^{α_1/α_2}. Here K_1 is as in (4.5). $\square$

5 Parabolic Visual Metrics

In this section we introduce parabolic visual metrics, discuss their relation with the visual metrics and give a sufficient condition for them to be doubling. We then use these results to complete the proof of Theorem 1.1.

Parabolic visual metrics have been defined by Bonk-Kleiner ([BK]) for CAT(-1) spaces. Here we formally construct parabolic visual metrics in the setting of Gromov hyperbolic spaces. Since G_A is Gromov hyperbolic, the theory developed here is applicable to ∂G_A as well. The metric D (on $\mathbb{R}^n = \partial G_A \setminus \{\xi_0\}$) used in the previous sections is bilipschitz equivalent with a parabolic visual metric constructed in this section, see the discussion after Proposition 5.1.

Parabolic visual metric is defined on the one-point complement of the ideal boundary. The relationship between visual metric and parabolic visual metric is similar to the relationship between the spherical metric (on the sphere) and the Euclidean metric (on the one point complement of the sphere). See Proposition 5.4 for the precise statement.

Let X be a δ-hyperbolic proper geodesic metric space for some $\delta \ge 0$. Let $\xi \in \partial X$ and $p \in X$. Then there exists a ray from p to ξ. Let $\gamma : [0, \infty) \to X$ be such a ray. Define $B_\gamma : X \to \mathbb{R}$ by $B_\gamma(x) = \lim_{t\to+\infty}(d(\gamma(t), x) - t)$. The triangle inequality implies that the limit exists and that $|B_\gamma(x) - B_\gamma(y)| \le d(x, y)$ for all $x, y \in X$. Note that $B_\gamma(\gamma(t_0)) = -t_0$ for all $t_0 \ge 0$. Since any two rays γ_1 and γ_2 from p to ξ are at Hausdorff distance at most δ from each other, we have $|B_{\gamma_1}(x) - B_{\gamma_2}(x)| \le \delta$ for all $x \in X$.

The *Buseman function* $B_{\xi,p} : X \to \mathbb{R}$ centered at ξ with base point p is:

$$B_{\xi,p}(x) = \sup\{B_\gamma(x) : \gamma \text{ is a geodesic ray from } p \text{ to } \xi\}.$$

Because B_γ is 1-Lipschitz, $B_{\xi,p}$ is 1-Lipschitz. The above discussion shows that $B_\gamma(x) \leq B_{\xi,p}(x) \leq B_\gamma(x) + \delta$ for all $x \in X$ and every ray γ from p to ξ. By Proposition 8.2 of [GdlH], there exists a constant $c = c(\delta)$ such that for any two points $p_1, p_2 \in X$, any $\xi \in \partial X$ and all $x \in X$ we have

$$|B_{\xi,p_1}(x) - B_{\xi,p_2}(x) - B_{\xi,p_1}(p_2)| \leq c. \tag{5.1}$$

Let $\epsilon > 0$, $p \in X$, $\xi \in \partial X$, and $\eta_1 \neq \eta_2 \in \partial X \backslash \{\xi\}$. Given a complete geodesic σ from η_1 to η_2, let $H_{\xi,p}(\sigma) = \inf\{B_{\xi,p}(x) : x \in \sigma\}$. Define

$$D_{\xi,p,\epsilon}(\eta_1, \eta_2) = e^{-\epsilon\, H_{\xi,p}(\eta_1,\eta_2)},$$

where

$$H_{\xi,p}(\eta_1, \eta_2) = \inf\{H_{\xi,p}(\sigma) :\ \sigma \text{ is a complete geodesic from } \ \eta_1 \text{ to } \eta_2\}.$$

Since any two complete geodesics from η_1 to η_2 are at most Hausdorff distance 2δ apart, we have $H_{\xi,p}(\sigma) - 2\delta \leq H_{\xi,p}(\eta_1, \eta_2) \leq H_{\xi,p}(\sigma)$ for any complete geodesic σ from η_1 to η_2.

An argument similar to that found in [CDP, p.124] shows the following:

Proposition 5.1. *There exists a constant ϵ_0, depending only on δ, with the following property. If X is a δ-hyperbolic proper geodesic metric space, for each $0 < \epsilon \leq \epsilon_0$, each $p \in X$ and each $\xi \in \partial X$ there exists a metric $d_{\xi,p,\epsilon}$ on $\partial X \backslash \{\xi\}$ such that $\frac{1}{2} D_{\xi,p,\epsilon}(\eta_1, \eta_2) \leq d_{\xi,p,\epsilon}(\eta_1, \eta_2) \leq D_{\xi,p,\epsilon}(\eta_1, \eta_2)$ for all $\eta_1, \eta_2 \in \partial X \backslash \{\xi\}$.*

The metric $d_{\xi,p,\epsilon}$ is called a *parabolic visual metric*. With $X = G_A$, $p = (0, 0)$, by using Lemmas 6.1 and 6.2 one can see that $D_{\xi_0,p,1}$ is bilipschitz equivalent with D_e. It follows from Lemma 2.1 and Proposition 5.1 that d_{ξ_0,p,α_1} is bilipschitz equivalent with the metric D considered in the previous sections.

We next discuss how $d_{\xi,p,\epsilon}$ varies with p and ϵ.

Proposition 5.2. *Suppose X is a δ-hyperbolic proper geodesic metric space. Then*
(1) For any $p_1, p_2 \in X$, the identity map $id : (\partial X \backslash \{\xi\}, d_{\xi,p_1,\epsilon}) \to (\partial X \backslash \{\xi\}, d_{\xi,p_2,\epsilon})$ is a K-quasisimilarity, where K depends only on δ;
(2) For $0 < \epsilon_1, \epsilon_2 \leq \epsilon_0$, the identity map $id : (\partial X \backslash \{\xi\}, d_{\xi,p,\epsilon_1}) \to (\partial X \backslash \{\xi\}, d_{\xi,p,\epsilon_2})$ is η-quasisymmetric with $\eta(t) = 2^{1+\frac{\epsilon_2}{\epsilon_1}} t^{\frac{\epsilon_2}{\epsilon_1}}$;
(3) For any $p_1, p_2 \in X$ and any $0 < \epsilon_1, \epsilon_2 \leq \epsilon_0$, the identity map $id : (\partial X \backslash \{\xi\}, d_{\xi,p_1,\epsilon_1}) \to (\partial X \backslash \{\xi\}, d_{\xi,p_2,\epsilon_2})$ is quasisymmetric.

Proof. To prove (1) let $\eta_1, \eta_2 \in \partial X \backslash \{\xi\}$. Then Proposition 5.1 and inequality (5.1) imply

$$\begin{aligned} d_{\xi,p_2,\epsilon}(\eta_1, \eta_2) \leq D_{\xi,p_2,\epsilon}(\eta_1, \eta_2) = e^{-\epsilon\, H_{\xi,p_2}(\eta_1,\eta_2)} &\leq e^{-\epsilon H_{\xi,p_1}(\eta_1,\eta_2) + \epsilon B_{\xi,p_1}(p_2) + c\epsilon} \\ &\leq 2\, e^{c\epsilon} \cdot e^{\epsilon B_{\xi,p_1}(p_2)} \cdot d_{\xi,p_1,\epsilon}(\eta_1, \eta_2). \end{aligned}$$

Similarly, we obtain $d_{\xi,p_2,\epsilon}(\eta_1,\eta_2) \geq \frac{1}{2\,e^{c\epsilon}} \cdot e^{\epsilon B_{\xi,p_1}(p_2)} \cdot d_{\xi,p_1,\epsilon}(\eta_1,\eta_2)$. The statement holds with $K = 2\,e^{c\epsilon_0}$ and constant $C = e^{\epsilon B_{\xi,p_1}(p_2)}$.

The claim (2) follows from Proposition 5.1, and (3) follows from (1) and (2). $\square$

We next discuss the relation between the parabolic visual metric and the visual metric. Recall that there is a constant ϵ_1 depending only on δ such that for any $p \in X$ and any $0 < \epsilon \leq \epsilon_1$, there is a visual metric $d_{p,\epsilon}$ on ∂X satisfying

$$\frac{1}{2} e^{-\epsilon(\eta_1|\eta_2)_p} \leq d_{p,\epsilon}(\eta_1,\eta_2) \leq e^{-\epsilon(\eta_1|\eta_2)_p} \tag{5.2}$$

for all $\eta_1, \eta_2 \in \partial X$. Here $(\xi|\eta)_p$ denotes the Gromov product of ξ and η based at p, and is defined by

$$(\xi|\eta)_p = \frac{1}{2} \sup \ \liminf_{i,j\to\infty} \left(d(p,x_i) + d(p,y_j) - d(x_i,y_j)\right)$$

where the supremum is taken over all sequences $\{x_i\} \to \xi$, $\{y_i\} \to \eta$. By the δ-hyperbolicity of X,

$$(\xi|\eta)_p - 2\delta \leq \liminf_{i,j\to\infty} (x_i|y_j)_p \leq (\xi|\eta)_p \tag{5.3}$$

for all $p \in X$, all $\xi, \eta \in \partial X$ and all sequences $\{x_i\} \to \xi$, $\{y_i\} \to \eta$; we refer the interested reader to Chapter 7 of [GdlH].

To formulate the relation between visual metric and parabolic visual metric, we need to recall the notion of metric inversion and sphericalization. The reader is referred to [BHX] for more details.

Given a metric space (X,d) and $p \in X$, there is a metric d_p on $X\backslash\{p\}$ satisfying

$$\frac{d(x,y)}{4d(x,p)\,d(y,p)} \leq d_p(x,y) \leq \frac{d(x,y)}{d(x,p)\,d(y,p)}$$

for all $x, y \in X\backslash\{p\}$. Furthermore, the identity map $(X\backslash\{p\}, d) \to (X\backslash\{p\}, d_p)$ is η-quasimöbius with $\eta(t) = 16t$. We call d_p the *metric inversion* of (X,d) at p.

Let X be an unbounded metric space and $p \in X$. Let $S_p(X) = X \cup \{\infty\}$, where ∞ is a point not in X. We define a function $s_p : S_p(X) \times S_p(X) \to [0,\infty)$ as follows:

$$s_p(x,y) = s_p(y,x) = \begin{cases} \frac{d(x,y)}{[1+d(x,p)][1+d(y,p)]} & \text{if } x, y \in X, \\ \frac{1}{1+d(x,p)} & \text{if } x \in X \text{ and } y = \infty, \\ 0 & \text{if } x = \infty = y. \end{cases}$$

It was shown in [BK] that there is a metric $\widehat{d}_p$ on $S_p(X)$ satisfying

$$\frac{1}{4} s_p(x,y) \leq \widehat{d}_p(x,y) \leq s_p(x,y) \quad \text{for all} \quad x, y \in S_p(X). \tag{5.4}$$

Furthermore, the identity map $(X,d) \to (X,\widehat{d_p})$ is η-quasimöbius with $\eta(t) = 16t$. We call $\widehat{d_p}$ the *sphericalization* of (X,d) at p.

If (Y,d) is a bounded metric space, and if a metric inversion is applied to Y, followed by an application of sphericalization, the resulting space is bilipschitz equivalent to (Y,d). To be more precise, let $p \neq q \in Y$ and $f : (Y,d) \to (S_q(Y\backslash\{p\}), \widehat{(d_p)_q})$ be the map that is identity on $Y\backslash\{p\}$ with $f(p) = \infty$. Then f is bilipschitz (see for example [BHX, Proposition 3.9]).

We need the following result for the proof of Proposition 5.4.

Theorem 5.3 ([CDP, Chapter 8]). *Let (Y,h) be a δ-hyperbolic space, $y_0 \in Y$, and $Y_0 = \{y_0, y_1, \cdots, y_n\}$ be a set of $n+1$ points in $Y \cup \partial Y$. For each $1 \le i \le n$, let $[y_0, y_i]$ be a fixed geodesic connecting y_0 and y_i. Let X denote the union of the geodesics $[y_0, y_i]$, and choose a positive integer k such that $2n \le 2^k + 1$. Then there exists a simplicial tree, denoted $T(X)$, and a continuous map $u : X \to T(X)$ which satisfies the following properties:*

(i) For each i, the restriction of u to the geodesic $[y_0, y_i]$ is an isometry;

(ii) For every x and y in X we have $h(x,y) - 2k\delta \le d(u(x), u(y)) \le h(x,y)$, where d is the metric on $T(X)$.

Proposition 5.4. *Let X be a δ-hyperbolic proper geodesic metric space, $\xi \in \partial X$, $p \in X$ and $0 < \epsilon \le \min\{\epsilon_0, \epsilon_1\}$.*
(1) The identity map

$$id : (\partial X\backslash\{\xi\}, d_{\xi,p,\epsilon}) \to (\partial X\backslash\{\xi\}, (d_{p,\epsilon})_\xi)$$

is L-bilipschitz, where L is a constant depending only on δ. In particular, the parabolic visual metric and the metric inversion of the visual metric about the point ξ are bilipschitz equivalent;
(2) Let $\eta \in \partial X\backslash\{\xi\}$ and

$$f : (\partial X, d_{p,\epsilon}) \to (S_\eta(\partial X\backslash\{\xi\}), \widehat{(d_{\xi,p,\epsilon})_\eta})$$

be the bijection that is identity on $\partial X\backslash\{\xi\}$ and maps ξ to ∞. Then f is bilipschitz. In particular, the visual metric and the sphericalization of the parabolic visual metric are bilipschitz equivalent.

Proof. Let $D = d_{p,\epsilon}$ denote the visual metric. Then $(d_{p,\epsilon})_\xi = D_\xi$.

We first prove (1). Let $\eta_1, \eta_2 \in \partial X\backslash\{\xi\}$. By Proposition 5.1 and inequality (5.2),

$$\begin{aligned}
\frac{D_\xi(\eta_1,\eta_2)}{d_{\xi,p,\epsilon}(\eta_1,\eta_2)} &\le \frac{d_{p,\epsilon}(\eta_1,\eta_2)}{d_{p,\epsilon}(\xi,\eta_1)d_{p,\epsilon}(\xi,\eta_2)}\ 2\,e^{\epsilon H_{\xi,p}(\eta_1,\eta_2)} \\
&\le e^{-\epsilon(\eta_1|\eta_2)_p}\ 2\,e^{\epsilon(\xi|\eta_1)_p}\ 2\,e^{\epsilon(\xi|\eta_2)_p}\ 2\,e^{\epsilon H_{\xi,p}(\eta_1,\eta_2)} \\
&= 8\,e^{\epsilon\{H_{\xi,p}(\eta_1,\eta_2)+(\xi|\eta_1)_p+(\xi|\eta_2)_p-(\eta_1|\eta_2)_p\}}.
\end{aligned}$$

Similarly,

$$\frac{D_\xi(\eta_1,\eta_2)}{d_{\xi,p,\epsilon}(\eta_1,\eta_2)} \geq \frac{1}{8}\, e^{\epsilon\{H_{\xi,p}(\eta_1,\eta_2)+(\xi|\eta_1)_p+(\xi|\eta_2)_p-(\eta_1|\eta_2)_p\}}.$$

Now (1) follows from the following claim.

Claim: There is a constant C depending only on δ such that if $\eta_1 \neq \eta_2$ then $|H_{\xi,p}(\eta_1,\eta_2)+(\xi|\eta_1)_p+(\xi|\eta_2)_p-(\eta_1|\eta_2)_p| \leq C$.

We now prove the claim. Let γ be a ray from p to ξ. Pick a point $y_0 \in \gamma$ that is far away from any complete geodesic joining η_1 and η_2. Let γ_i $(i=1,2)$ be a ray from y_0 to η_i. Set $X=\gamma\cup\gamma_1\cup\gamma_2$. By Theorem 5.3 (with the choice $k=3$) there is a tree $T:=T(X)$ and a map $u: X\to T$ with the properties stated in Theorem 5.3. Let $y_0', p' \in T$ and $\xi', \eta_1', \eta_2' \in \partial T$ be the points corresponding to y_0, p, ξ η_1 and η_2 respectively. Also let x' be the branch point of $\xi'\eta_1'$ and $\xi'\eta_2'$, and let y' be the projection of p' onto the tripod $Y:=x'\xi'\cup x'\eta_1'\cup x'\eta_2'$. Let $y\in\gamma$ be the point on γ that is mapped to y' by u (by choosing y_0 far away from p we may assume that y lies between p and y_0). Similarly let $x_i\in\gamma_i$ be the point mapped to x' by u. Let σ be a complete geodesic from η_1 to η_2. Because X is δ-hyperbolic, geodesic triangles in $X\cup\partial X$ are 24δ-thin. Also notice that the union $x'\eta_1'\cup x'\eta_2'$ is a complete geodesic in T. Now the properties of the map u given by Theorem 5.3 imply that the Hausdorff distance between σ and $x_1\eta_1\cup x_2\eta_2$ is bounded above by a constant $c_1=c_1(\delta)$.

Choose $z_j\in\gamma_1$ and $w_j\in\gamma_2$ with $z_j\to\eta_1$ and $w_j\to\eta_2$. Then the property of the map u and inequality (5.3) imply that $|(\eta_1|\eta_2)_p-(\eta_1'|\eta_2')_{p'}|\leq 11\delta$. Notice that on the tree T we have $(\eta_1'|\eta_2')_{p'}=d(p',\eta_1'\eta_2')$. Hence $|(\eta_1|\eta_2)_p-d(p',\eta_1'\eta_2')|\leq 11\delta$. Similar inequalities hold for $(\xi|\eta_1)_p$ and $(\xi|\eta_2)_p$.

Since the Hausdorff distance between σ and $x_1\eta_1\cup x_2\eta_2$ is at most c_1, the definition of $H_{\xi,p}(\sigma)$ and the property of the map u imply that $|H_{\xi,p}(\sigma)-H_{\xi',p'}(\eta_1',\eta_2')|\leq c_1+13\delta$. The discussion about $H_{\xi,p}(\eta_1,\eta_2)$ shows that $|H_{\xi,p}(\eta_1,\eta_2)-H_{\xi,p}(\sigma)|\leq 2\delta$. It follows that $|H_{\xi,p}(\eta_1,\eta_2)-H_{\xi',p'}(\eta_1',\eta_2')|\leq c_1+15\delta$. Now on the tree T, by considering three cases depending on whether $y'\in x'\xi'$, $y'\in x'\eta_1'$ or $y'\in x'\eta_2'$, we can verify that

$$H_{\xi',p'}(\eta_1',\eta_2')+d(p',\xi'\eta_1')+d(p',\xi'\eta_2')-d(p',\eta_1'\eta_2')=0.$$

Now the claim follows by combining the above estimates.

We now prove (2). By (1), the identity map

$$id:(\partial X\backslash\{\xi\}, d_{\xi,p,\epsilon})\to(\partial X\backslash\{\xi\}, D_\xi)$$

is bilipschitz. Pick $\eta\in\partial X\backslash\{\xi\}$. Then the map id extends to a map F between their sphericalizations

$$F:(S_\eta(\partial X\backslash\{\xi\}), \widehat{(d_{\xi,p,\epsilon})}_\eta)\to(S_\eta(\partial X\backslash\{\xi\}), \widehat{(D_\xi)}_\eta).$$

Since id is bilipschitz, inequality (5.4) can be used on $\widehat{(d_{\xi,p,\epsilon})}_\eta$ and $\widehat{(D_\xi)}_\eta$ to verify that F is also bilipschitz. On the other hand, the natural identification between $(\partial X, d_{p,\epsilon})$ and $(S_\eta(\partial X\backslash\{\xi\}), \widehat{(D_\xi)}_\eta)$ is bilipschitz. The statement now follows. $\square$

We next give a sufficient condition for the parabolic visual metric to be doubling. Recall that a metric space is *doubling* if there is a constant N such that every open ball with radius $R > 0$ can be covered by at most N open balls with radius $R/2$. By a theorem of Assouad ([A]), a metric space is doubling if and only if the metric space admits a quasisymmetric embedding into some Euclidean space.

A metric space X has bounded growth at some scale, if there are constants r, R with $R > r > 0$, and an integer $N \geq 1$ such that every open ball of radius R in X can be covered by N open balls of radius r.

The following is a consequence of Proposition 5.4, a result of Bonk-Schramm and Assouad's theorem.

Theorem 5.5. *Let X be a Gromov hyperbolic geodesic metric space with bounded growth at some scale. Then for any $\xi \in \partial X$, $p \in X$ and any $0 < \epsilon \leq \epsilon_0$, the metric space $(\partial X \backslash \{\xi\}, d_{\xi,p,\epsilon})$ is doubling.*

Proof. Under the assumption of the Theorem, Bonk-Schramm has proved that the ideal boundary with the visual metric is doubling ([BS, Theorem 9.2]). Hence there is a quasisymmetric embedding $f : (\partial X, d_{p,\epsilon}) \to \mathbb{R}^n$ for some $n \geq 1$. By Lemma 5.6 below, $f : (\partial X \backslash \{\xi\}, (d_{p,\epsilon})_\xi) \to (\mathbb{R}^n \backslash \{f(\xi)\}, |\cdot|_{f(\xi)})$ is also a quasisymmetric embedding, where $|\cdot|$ denotes the Euclidean metric. However, the metric inversion of the Euclidean space is still a Euclidean space (with one point removed). Hence $(\partial X \backslash \{\xi\}, (d_{p,\epsilon})_\xi)$ admits a quasisymmetric embedding into a Euclidean space, and so is doubling. Since doubling is invariant under bilipschitz map, the theorem now follows from Proposition 5.4 (1). $\square$

Recall that a homeomorphism $f : X \to Y$ between two metric spaces is η-quasimöbius for some homeomorphism $\eta : [0, \infty) \to [0, \infty)$, if for every four distinct points $x_1, x_2, x_3, x_4 \in X$, we have

$$\frac{d(f(x_1), f(x_3))\, d(f(x_2), f(x_4))}{d(f(x_1), f(x_4))\, d(f(x_2), f(x_3))} \leq \eta\left(\frac{d(x_1, x_3)\, d(x_2, x_4)}{d(x_1, x_4)\, d(x_2, x_3)}\right).$$

Lemma 5.6. *Suppose that $f : (X, d) \to (Y, d)$ is a quasisymmetric embedding. Then for any $p \in X$, $f : (X \backslash \{p\}, d_p) \to (Y \backslash \{f(p)\}, d_{f(p)})$ is also a quasisymmetric embedding.*

Proof. Suppose f is an η-quasisymmetric embedding for some η. Then f is an $\tilde{\eta}$-quasimöbius embedding for some $\tilde{\eta}$ depending only on η, see [V2, Theorem 6.25]. Now let $x, y, z \in X \backslash \{p\}$ be three distinct points. Set $q = f(p)$. We calculate

$$\begin{aligned}\frac{d_q(f(x), f(z))}{d_q(f(y), f(z))} &\leq \frac{d(f(x), f(z))}{d(f(x), f(p))\, d(f(z), f(p))} \; \frac{4\, d(f(y), f(p))\, d(f(z), f(p))}{d(f(y), f(z))} \\ &= 4\, \frac{d(f(x), f(z))\, d(f(y), f(p))}{d(f(x), f(p))\, d(f(y), f(z))}.\end{aligned}$$

Similarly,

$$\frac{d_p(x,z)}{d_p(y,z)} \geq \frac{1}{4}\ \frac{d(x,z)\, d(y,p)}{d(x,p)\, d(y,z)}.$$

It follows that

$$\frac{d_q(f(x),f(z))}{d_q(f(y),f(z))} \leq 4\ \tilde{\eta}\left(4\,\frac{d_p(x,z)}{d_p(y,z)}\right).$$

Hence $f : (X\backslash\{p\}, d_p) \to (Y\backslash\{f(p)\}, d_{f(p)})$ is η'-quasisymmetric with $\eta'(t) = 4\,\tilde{\eta}(4t)$. $\square$

Let (X,d) be an unbounded complete metric space with an Ahlfors Q-regular ($Q>1$) Borel measure μ, and $p \in X$. On the sphericalization $(S_p(X), \widehat{d_p})$ we define a measure μ' as follows: $\mu'(\{\infty\}) = 0$, and on $X = S_p(X)\backslash\{\infty\}$, μ' is absolutely continuous with respect to μ with Radon-Nikodym derivative

$$\frac{d\mu'}{d\mu}(x) = \frac{1}{(1+d(p,x))^{2Q}}$$

for $x \in X$. It can be shown that $(S_p(X), \widehat{d_p})$ with μ' is also Q-regular.

Proof of Theorem 1.1. Let $F : (\partial G_A, d_{p,\epsilon}) \to (\partial G_A, d_{p,\epsilon})$ be a quasisymmetric map, where $d_{p,\epsilon}$ is a visual metric ($p \in G_A$ and $\epsilon > 0$ is sufficiently small). We first prove that $F(\xi_0) = \xi_0$. Let D be the metric on $\partial G_A\backslash\{\xi_0\} = \mathbb{R}^n$ considered in the previous sections. We have observed that D is bilipschitz equivalent with a parabolic visual metric on $\partial G_A\backslash\{\xi_0\}$. Let $\theta \in \partial G_A\backslash\{\xi_0\} = \mathbb{R}^n$. Proposition 5.4 implies that the natural identification

$$(S_\theta(\mathbb{R}^n), \widehat{D}_\theta) = (S_\theta(\partial G_A\backslash\{\xi_0\}), \widehat{D}_\theta) \to (\partial G_A, d_{p,\epsilon})$$

is bilipschitz. It follows that (after the above natural identification)

$$F : (S_\theta(\mathbb{R}^n), \widehat{D}_\theta) \to (S_\theta(\mathbb{R}^n), \widehat{D}_\theta)$$

is quasisymmetric. Let μ be the product of the Hausdorff measures on the factors $(\mathbb{R}^{n_i}, |\cdot|^{\frac{\alpha_1}{\alpha_i}})$ of $\mathbb{R}^n$. Since the metric measure space $(\mathbb{R}^n, D, \mu)$ is Q-regular with $Q = \Sigma_{i=1}^r n_i \frac{\alpha_i}{\alpha_1}$, the remark preceding the proof shows that the metric measure space $(S_\theta(\mathbb{R}^n), \widehat{D}_\theta, \mu')$ is also Q-regular. Here μ' is obtained from μ as described in the remark preceding the proof. Hence Theorem 3.1 applies to the map $F : (S_\theta(\mathbb{R}^n), \widehat{D}_\theta) \to (S_\theta(\mathbb{R}^n), \widehat{D}_\theta)$ and the measure μ'.

Suppose $F(\xi_0) \neq \xi_0$. Under the above natural identification, this means that $F(\infty) \neq \infty$. Then $F^{-1}(\infty)$ lies in exactly one horizontal leaf. Fix some $y \in Y$ such that $\mathbb{R}^{n_1} \times \{y\}$ does not contain $F^{-1}(\infty)$. Notice that the subset $(\mathbb{R}^{n_1} \times \{y\}) \cup \{\infty\}$ of $S_\theta(\mathbb{R}^n)$ is an n_1-dimensional topological sphere. So $F(\mathbb{R}^{n_1} \times \{y\} \cup \{\infty\})$ is an n_1-dimensional topological sphere in $\mathbb{R}^n$. Since each horizontal leaf is an n_1-dimensional

Euclidean space, the set $F(\mathbb{R}^{n_1} \times \{y\} \cup \{\infty\})$ is not contained in any horizontal leaf. It follows that as a dense subset of $F(\mathbb{R}^{n_1} \times \{y\} \cup \{\infty\})$, the set $F(\mathbb{R}^{n_1} \times \{y\})$ is also not contained in any horizontal leaf. Hence there are two points p and q in $\mathbb{R}^{n_1} \times \{y\}$ such that $F(p)$ and $F(q)$ are not in the same horizontal leaf.

Let γ be the Euclidean line segment from p to q and Γ be the family of straight segments parallel to γ in $\mathbb{R}^n$ whose union is an n-dimensional circular cylinder C with γ as the central axis. The curves in Γ are rectifiable with respect to the metric D. Since F is a homeomorphism, by choosing the radius of the circular cylinder to be sufficiently small (by a compactness argument) we may assume that no curve in Γ is mapped into a horizontal leaf and that $F^{-1}(\infty)$ is not in this cylinder. It follows that $F(\Gamma)$ has no locally rectifiable curve with respect to D. Now notice that both C and $F(C)$ are compact subsets of $\mathbb{R}^n$. Hence the two metrics D and $\widehat{D}_\theta$ are bilipschitz equivalent on C, as well as on $F(C)$. It follows that $F(\Gamma)$ has no locally rectifiable curve with respect to $\widehat{D}_\theta$. Hence $\mathrm{Mod}_Q F(\Gamma) = 0$ in the metric measure space $(S_\theta(\mathbb{R}^n), \widehat{D}_\theta, \mu')$. Theorem 3.1 then implies that $\mathrm{Mod}_Q \Gamma = 0$ in the metric measure space $(S_\theta(\mathbb{R}^n), \widehat{D}_\theta, \mu')$. On the other hand, $\mathrm{Mod}_Q \Gamma > 0$ in the metric measure space $(\mathbb{R}^n, D, \mu)$ (see the proof of Theorem 3.2). Since D and $\widehat{D}_\theta$ are bilipschitz equivalent on C, and μ and μ' are also comparable on C, we have $\mathrm{Mod}_Q \Gamma > 0$ in the metric measure space $(S_\theta(\mathbb{R}^n), \widehat{D}_\theta, \mu')$, a contradiction. Hence $F(\xi_0) = \xi_0$.

Next we prove that F is bilipschitz with respect to the metric D. Since the map $F : (\partial G_A, d_{p,\epsilon}) \to (\partial G_A, d_{p,\epsilon})$ is quasisymmetric, Lemma 5.6 implies that

$$F : (\partial G_A \backslash \{\xi_0\}, (d_{p,\epsilon})_{\xi_0}) \to (\partial G_A \backslash \{\xi_0\}, (d_{p,\epsilon})_{\xi_0})$$

is also a quasisymmetric map. By Proposition 5.4, $id : (\partial G_A \backslash \{\xi_0\}, (d_{p,\epsilon})_{\xi_0}) \to (\partial G_A \backslash \{\xi_0\}, d_{\xi_0,p,\epsilon})$ is bilipschitz, where $d_{\xi_0,p,\epsilon}$ is a parabolic visual metric. It follows that

$$F : (\partial G_A \backslash \{\xi_0\}, d_{\xi_0,p,\epsilon}) \to (\partial G_A \backslash \{\xi_0\}, d_{\xi_0,p,\epsilon})$$

is quasisymmetric. By Proposition 5.2, any two parabolic visual metrics are quasisymmetrically equivalent. By the discussion following Proposition 5.1, it follows that $F : (\partial G_A \backslash \{\xi_0\}, D) \to (\partial G_A \backslash \{\xi_0\}, D)$ is quasisymmetric. Now the result follows from Theorem 4.1. $\square$

6 Consequences

In this section we will prove the corollaries from the introduction.

We note that because G_A has sectional curvature $-\alpha_r^2 \leq K \leq -\alpha_1^2$, G_A is a proper geodesic δ-hyperbolic space with δ depending only on α_1.

Proof of Corollary 1.3.. Suppose there is a quasiisometry $f : G_A \to G$ from G_A to a finitely generated group G, where G is equipped with a fixed word metric. Since

G_A is Gromov hyperbolic, it follows that G is Gromov hyperbolic and f induces a quasisymmetric map $\partial f : \partial G_A \to \partial G$. The left translation of G on itself induces an action of G on the Gromov boundary ∂G by quasisymmetric maps. By conjugating this action with ∂f we obtain an action of G on ∂G_A by quasisymmetric maps. By Theorem 1.1, this action has a global fixed point. It follows that the action of G on ∂G has a global fixed point. This can happen only when G is virtually infinite cyclic, in which case the Gromov boundary ∂G consists of only two points. This contradicts the fact that ∂G_A is a sphere of dimension $n \geq 2$ (since $r \geq 2$). □

The proofs of Corollaries 1.4 and 1.2 require some preparation.

Let X be a proper geodesic δ-hyperbolic space and $\xi_1, \xi_2, \xi_3 \in \partial X$ be three distinct points in the Gromov boundary. For any constant $C \geq 0$, a point $x \in X$ is called a *C-quasicenter* of the three points ξ_1, ξ_2, ξ_3 if for each $i = 1, 2, 3$, there is a geodesic σ_i joining ξ_i and ξ_{i+1} ($\xi_4 := \xi_1$) such that the distance from x to σ_i is at most C. For any $C \geq 0$, there is a constant C' that depends only on δ and C such that the distance between any two C-quasicenters of ξ_1, ξ_2, ξ_3 is at most C'.

The following three lemmas hold in all Hadamard manifolds with pinched negative sectional curvature.

Lemma 6.1. *Let $(x, t) \in G_A = \mathbb{R}^n \times \mathbb{R}$ be an arbitrary point, and σ a geodesic through (x, t) and tangent to the horosphere $\mathbb{R}^n \times \{t\}$. Let $p, q \in \mathbb{R}^n \approx \partial G_A \backslash \{\xi_0\}$ be the two endpoints of σ. Then (x, t) is a 12δ-quasicenter for p, q, ξ_0.*

Proof. As an ideal geodesic triangle in a δ-hyperbolic space, $\sigma \cup \gamma_p \cup \gamma_q$ is 4δ-thin. Hence there is some point $m \in \gamma_p \cup \gamma_q$ with $d((x, t), m) \leq 4\delta$. We may assume $m = (p, t') \in \gamma_p$ for some $t' \in \mathbb{R}$. We may further assume that (p, t') is the point on γ_p nearest to (x, t). Then the geodesic segment from (x, t) to (p, t') must be perpendicular to the geodesic γ_p. This implies $t' > t$. Since (x, t) is the highest point on σ and is more than 4δ below the horosphere through $(p, t' + 4\delta)$, we have $d((p, t' + 4\delta), \sigma) > 4\delta$. Now the thin triangle condition applied to the point $(p, t' + 4\delta)$ and the triangle $\sigma \cup \gamma_p \cup \gamma_q$ implies there is some $(q, t'') \in \gamma_q$ with $d((p, t' + 4\delta), (q, t'')) \leq 4\delta$. The triangle inequality together with $d((p, t'), (p, t' + 4\delta)) = 4\delta$ implies $d((x, t), (q, t'')) \leq 12\delta$. Hence (x, t) is a 12δ-quasicenter for p, q, ξ_0.

□

Let M be a simply connected Riemannian manifold with sectional curvature $-b^2 \leq K \leq -a^2$, where $b > a > 0$. For any $\xi \in \partial M$, any horosphere $\mathcal{H}$ centered at ξ, and any two points $x, y \in \mathcal{H}$, the distance $d_{\mathcal{H}}(x, y)$ between x and y in the horosphere is related to $d(x, y)$ by (see [HI]):

$$\frac{2}{a} \sinh\left(\frac{a}{2}\, d(x, y)\right) \leq d_{\mathcal{H}}(x, y) \leq \frac{2}{b} \sinh\left(\frac{b}{2}\, d(x, y)\right). \tag{6.1}$$

For any $s > 0$, let $\mathcal{H}_s$ be the horosphere centered at ξ that is closer to ξ than $\mathcal{H}$ and is at distance s from $\mathcal{H}$. Let $\phi_s : \mathcal{H} \to \mathcal{H}_s$ be the map which sends each $x \in \mathcal{H}$ to the unique intersection point of $x\xi$ with $\mathcal{H}_s$. Then for each tangent vector $v \in T_x\mathcal{H}$ of $\mathcal{H}$ at x we have (see [HI]): $e^{-bs}\|v\| \leq \|d\phi_s(v)\| \leq e^{-as}\|v\|$. It follows that for any rectifiable curve c in $\mathcal{H}$, the lengths of c and $\phi(c)$ are related by $e^{-bs}\ell(c) \leq \ell(\phi(c)) \leq e^{-as}\ell(c)$.

Lemma 6.2. *Let $p, q \in \mathbb{R}^n \approx \partial G_A \backslash \{\xi_0\}$ and suppose that $D_e(p,q) = e^{t_0}$. Then (p, t_0) is a C-quasicenter for p, q, ξ_0, where C depends only on α_1 and α_r.*

Proof. Let σ be the geodesic in G_A joining $p, q \in \partial G_A \backslash \{\xi_0\}$, and (x,t) the highest point on σ. We may assume $d((x,t), \gamma_p) \leq 4\delta$. Let $(p, t_1) \in \gamma_p$ be the point nearest to (x,t). Then $t_1 > t$ and the argument in the proof of Lemma 5.1 gives a point $(q, t_2) \in \gamma_q$ such that $d((p,t_1),(q,t_2)) \leq 8\delta$. It follows that $|t_1 - t_2| \leq 8\delta$. The triangle inequality then implies $d((p,t_1),(q,t_1)) \leq 16\delta$.

By the definition of D_e, we have $d_{\mathbb{R}^n \times \{t_0\}}(p,q) = 1$. Hence $d((p,t_0),(q,t_0)) \leq 1$. If $t_0 \leq t_1$, then the convexity of the distance function $f(t) := d(\gamma_p(t), \sigma) = d((p,t),\sigma)$ implies $d((p,t_0),\sigma) \leq d((p,t_1),\sigma) \leq 4\delta$. In this case, (p,t_0) is a $\max\{1, 4\delta\}$-quasicenter of p, q, ξ_0. Now we suppose $t_0 > t_1$. Join (p,t_1) and (q,t_1) by a shortest path c in the horosphere $\mathcal{H} := \mathbb{R}^n \times \{t_1\}$. By (6.1) we have $\ell(c) \leq \frac{2}{\alpha_r}\sinh(8\alpha_r\delta)$. The projection $\phi_{t_0-t_1}(c)$ is a path in the horosphere $\mathbb{R}^n \times \{t_0\}$ joining (p,t_0) and (q,t_0). Hence

$$1 = d_{\mathbb{R}^n \times \{t_0\}}(p,q) \leq \ell(\phi_{t_0-t_1}(c)) \leq e^{-(t_0-t_1)\alpha_1}\ell(c) \leq \frac{2}{\alpha_r}e^{-(t_0-t_1)\alpha_1}\sinh(8\alpha_r\delta).$$

It follows that $t_0 - t_1 \leq C_1$, where

$$C_1 = \frac{\ln[\frac{2}{\alpha_r}\sinh(8\alpha_r\delta)]}{\alpha_1}.$$

The triangle inequality then implies $d((p,t_0),\sigma) \leq C_1 + 4\delta$. Hence (p,t_0) is a C-quasicenter for p, q, ξ_0, where $C = \max\{1, C_1 + 4\delta\}$.

□

Lemma 6.3. *Let $p, q \in \mathbb{R}^n \approx \partial G_A \backslash \{\xi_0\}$ and suppose that $D_e(p,q) = e^{t_0}$.*
(1) If $t_1, t_2 < t_0$, then $|d((p,t_1),(q,t_2)) - (t_0 - t_1) - (t_0 - t_2)| \leq C$, where C depends only on α_1 and α_r;
(2) If $t_1 \geq t_0$ or $t_2 \geq t_0$, then $|t_1 - t_2| \leq d((p,t_1),(q,t_2)) \leq |t_1 - t_2| + 1$.

Proof. (1) Let σ be the geodesic in G_A joining p and q, and (x,t) the highest point on σ. By Lemmas 6.1 and 6.2, the three points (p,t_0), (q,t_0) and (x,t) are all c_1-quasicenters of ξ_0,p, q, where c_1 depends only on α_1 and α_r. Hence $d((p,t_0),(x,t)) \leq c_2$ and $d((q,t_0),(x,t)) \leq c_2$ for some $c_2 = c_2(c_1,\delta) = c_2(\alpha_1,\alpha_r)$. Since $t_1 < t_0$, the convexity of distance function implies that $d((p,t_1),m_1) \leq d((p,t_0),(x,t)) \leq c_2$ for some point $m_1 \in \sigma$ lying between (x,t) and p. Similarly, there is some point $m_2 \in \sigma$ between (x,t) and q with $d((q,t_2),m_2) \leq c_2$. By triangle inequality we have $|d((p,t_1),(q,t_2)) - d(m_1,m_2)| \leq 2c_2$. Since $d((p,t_0),(x,t)) \leq c_2$ and $d((p,t_1),m_1) \leq c_2$, the triangle inequality also implies $|d((p,t_0),(p,t_1)) - d(m_1,(x,t))| \leq 2c_2$. Similarly, $|d((q,t_0),(q,t_2)) - d(m_2,(x,t))| \leq 2c_2$. Since $d(m_1,m_2) = d(m_1,(x,t)) + d((x,t),m2)$ and $d((p,t_0),(p,t_1)) = t_0 - t_1$, $d((q,t_0),(q,t_2)) = t_0 - t_2$, the above estimates together yield $|d((p,t_1),(q,t_2)) - (t_0 - t_1) - (t_0 - t_2)| \leq 6c_2$.

(2) We may assume $t_1 \geq t_0$. Then the convexity of distance function and the definition of D_e imply

$$d((p,t_1),(q,t_1)) \leq d((p,t_0),(q,t_0)) \leq d_{\mathbb{R}^n \times \{t_0\}}((p,t_0),(q,t_0)) = 1.$$

Now (2) follows from the triangle inequality and (2.1).

□

Corollary 1.4 follows from Theorem 1.1 and the following lemma. Notice that, by Theorem 1.1, for any quasiisometry $f: G_A \to G_A$, the boundary map ∂f fixes ξ_0 and restricts to a homeomorphism of $\partial G_A \setminus \{\xi_0\}$, which we still denote by ∂f.

Lemma 6.4. *Let $f : G_A \to G_A$ be a quasiisometry. Then f is height-respecting if and only if $\partial f : (\partial G_A \setminus \{\xi_0\}, D) \to (\partial G_A \setminus \{\xi_0\}, D)$ is a bilipschitz map.*

Proof. Dymarz ([D, Lemma 7]) proved that the boundary map of a height-respecting quasiisometry is a bilipschitz map with respect to the quasimetric D_s. It follows that the boundary map is also bilipschitz with respect to the metric D. Hence we only prove the "if" part. So we assume ∂f is bilipschitz w.r.t. D. Notice that it is also bilipschitz w.r.t. D_e. Hence there is a constant $L \geq 1$ such that for all $p, q \in \partial G_A \setminus \{\xi_0\} = \mathbb{R}^n$,

$$D_e(p,q)/L \leq D_e(\partial f(p), \partial f(q)) \leq L D_e(p,q).$$

Let $(x,t) \in G_A = \mathbb{R}^n \times \mathbb{R}$. Pick any geodesic σ through (x,t) that is tangent to the horosphere $\mathbb{R}^n \times \{t\}$. Then the two endpoints p, q of σ are in $\partial G_A \setminus \{\xi_0\} = \mathbb{R}^n$. If t_0 is the real number such that $d_{\mathbb{R}^n \times \{t_0\}}((p,t_0),(q,t_0)) = 1$, then by the definition of D_e we have $D_e(p,q) = e^{t_0}$. By Lemmas 6.1 and 6.2 both (x,t) and (p,t_0) are c_1-quasicenters of the three points $p, q, \xi_0 \in \partial G_A$, where c_1 depends only on α_1 and α_r. Hence there is a constant c_2 depending only on c_1 and δ such that $d((x,t),(p,t_0)) \leq c_2$. By (2.1), we have $|t - t_0| \leq c_2$. Let t'_0 be the real number such that $d_{\mathbb{R}^n \times \{t'_0\}}((\partial f(p), t'_0), (\partial f(q), t'_0)) = 1$. Then $D_e(\partial f(p), \partial f(q)) = e^{t'_0}$. Since f is a quasiisometry between δ-hyperbolic spaces, $f(x,t)$ is a c_3-quasicenter of $\partial f(p), \partial f(q), \partial f(\xi_0) = \xi_0$, where c_3 depends only on δ, c_1 and the quasiisometry constants of f. As $(\partial f(p), t'_0)$ is a c_1-quasicenter of these three points, we have $d((\partial f(p), t'_0), f(x,t)) \leq c_4$, with c_4 depending only on c_1, c_3 and δ. Let t' be the height of $f(x,t)$. Then by (2.1) again, $|t' - t'_0| \leq c_4$.

The bilipschitz assumption of ∂f and the formulas $D_e(\partial f(p), \partial f(q)) = e^{t'_0}$ and $D_e(p,q) = e^{t_0}$ imply that $|t_0 - t'_0| \leq \ln L$. Combining this with $|t - t_0| \leq c_2$ and $|t' - t'_0| \leq c_4$, we obtain $|t - t'| \leq \ln L + c_2 + c_4$. Hence the heights of any point (x,t) and its image $f(x,t)$ differ by at most a constant that is independent of (x,t). The corollary follows. □

Proof of Corollary 1.2. Let $f : G_A \to G_A$ be an (L, A)-quasiisometry. By Theorem 1.1, the boundary map $\partial f : \partial G_A \to \partial G_A$ fixes the point ξ_0. Let $(x_1, t_1), (x_2, t_2) \in \mathbb{R}^n \times \mathbb{R} = G_A$. Suppose $D_e(x_1, x_2) = e^{t_0}$. We only consider the case $t_0 > t_1, t_2$, the other cases being similar. By Lemma 6.3 there is a constant $c_1 = c_1(\alpha_1, \alpha_r)$ such that

$$|d((x_1, t_1), (x_2, t_2)) - (t_0 - t_1) - (t_0 - t_2)| \leq c_1. \tag{6.2}$$

Let t'_i $(i = 1, 2)$ be the height of $f(x_i, t_i)$. By Corollary 1.4, there is a constant $c_2 \geq 0$ such that $|t_i - t'_i| \leq c_2$. Since $f(\gamma_{x_i})$ is an (L, A)-quasigeodesic joining ξ_0 and $\partial f(x_i)$, there is a constant c_3 depending only on L, A and δ such that the Hausdorff distance between $f(\gamma_{x_i})$ and $\gamma_{\partial f(x_i)}$ is at most c_3. Hence there is some t''_i such that $d((\partial f(x_i), t''_i), f(x_i, t_i)) \leq c_3$. It follows that $|t'_i - t''_i| \leq c_3$ and hence $|t_i - t''_i| \leq c_2 + c_3$.

Suppose $D_e(\partial f(x_1), \partial f(x_2)) = e^{t'_0}$. By Lemma 6.2 $(\partial f(x_1), t'_0)$ is a c_4-quasicenter for ξ_0, $\partial f(x_1)$, $\partial f(x_2)$, where $c_4 = c_4(\alpha_1, \alpha_r)$. Similarly, (x_1, t_0) is a c_4-quasicenter for ξ_0, x_1, x_2. On the other hand, since f is an (L, A) quasiisometry, $f(x_1, t_0)$ is a c_5-quasicenter of ξ_0, $\partial f(x_1)$ and $\partial f(x_2)$, where $c_5 = c_5(L, A, c_4, \delta)$. It follows that $d((\partial f(x_1), t'_0), f(x_1, t_0)) \leq c_6$ for some constant $c_6 = c_6(c_4, c_5, \delta)$. Let t''_0 be the height of $f(x_1, t_0)$. Then $|t'_0 - t''_0| \leq c_6$. By Corollary 1.4 we have $|t_0 - t''_0| \leq c_2$. Hence $|t_0 - t'_0| \leq c_6 + c_2$.

Next we consider two cases:

Case 1. Both $t''_1, t''_2 < t'_0$.
In this case, by Lemma 6.3 (1) again we have

$$|d((\partial f(x_1), t''_1), (\partial f(x_2), t''_2)) - (t'_0 - t''_1) - (t'_0 - t''_2)| \leq c_1.$$

Combining this with (6.2) and the estimates $|t_i - t''_i| \leq c_2 + c_3$, $|t_0 - t'_0| \leq c_6 + c_2$, and $d((\partial f(x_i), t''_i), f(x_i, t_i)) \leq c_3$, we obtain

$$|d((x_1, t_1), (x_2, t_2)) - d(f(x_1, t_1), f(x_2, t_2))| \leq 2c_1 + 4c_2 + 4c_3 + 2c_6.$$

Case 2. Either $t''_1 \geq t'_0$ or $t''_2 \geq t'_0$.
Without loss of generality, we may assume $t''_1 \geq t'_0$ and $t''_1 \geq t''_2$. Then Lemma 6.3 (2) implies

$$|d((\partial f(x_1), t''_1), (\partial f(x_2), t''_2)) - (t''_1 - t''_2)| \leq 1. \tag{6.3}$$

On the other hand, $t''_1 \geq t'_0$ and the assumption $t_0 > t_1$ together with $|t_0 - t'_0| \leq c_6 + c_2$ and $|t_i - t''_i| \leq c_2 + c_3$ imply that $|t_0 - t_1| \leq 2c_2 + c_3 + c_6$. Now it follows from (6.2) and the triangle inequality that

$$|d((x_1, t_1), (x_2, t_2)) - (t_1 - t_2)| \leq c_1 + 4c_2 + 2c_3 + 2c_6. \tag{6.4}$$

Now (6.3), (6.4), $|t_i - t''_i| \leq c_2 + c_3$, and $d((\partial f(x_i), t''_i), f(x_i, t_i)) \leq c_3$ imply

$$|d((x_1, t_1), (x_2, t_2)) - d(f(x_1, t_1), f(x_2, t_2))| \leq 1 + c_1 + 6c_2 + 6c_3 + 2c_6.$$

□

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