Singular Functions on Metric Measure Spaces*

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Abstract

On relatively compact domains in metric measure spaces we construct singular functions that play the role of Green functions of the p-Laplacian. We give a characterization of metric spaces that support a global version of such singular function, in terms of capacity estimates at infinity of such metric spaces. In addition, when the measure of the space is locally Q-regular, we study quasiconformal invariance property associated with the existence of global singular functions.

1 Introduction

A Green's function $g = g(\,\cdot\,, y)$ for the p-Laplace equation on a non-compact Riemannian manifold M is (if it exists) a certain positive solution of

$$-\mathrm{div}\big(|\nabla g|^{p-2}\nabla g\big) = \delta_y, \quad y \in M\,, \tag{1}$$

in the sense of distributions, that is,

$$\int_M \big\langle |\nabla g|^{p-2}\nabla g, \nabla\varphi\big\rangle\, dm = \varphi(y)$$

for every $\varphi \in C_0^\infty(M)$. It is known that a Green's function for the p-Laplacian exists on M if and only if the p-capacity of some compact set $K \subset M$ with respect to M is positive, in which case M is called p-hyperbolic. For complete manifolds M, p-hyperbolicity can be characterized in terms of the volume growth under very weak conditions on the manifold M; see [CHSC], [Ho2], and [HoK] for details. In recent years there has been a growing interest in analysis on general metric spaces equipped with a measure, for example, in studying quasiconformal maps, Sobolev spaces, differentiability of Lipschitz

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functions, and calculus of variations on such spaces. The notion of an upper gradient of a function has turned out to be extremely important in these studies. Furthermore, concepts like p-capacity and p-harmonic functions have been introduced and their basic properties have been explored. We refer to [C], [HaK], [He], [HeK1], [HKST], [KiSh], [Se], [Sh1], [Sh2], and [Sh3] for these studies. In this paper we propose a definition for a p-harmonic Green's function, called p-singular function here, on metric measure spaces X. Our approach uses minimal upper gradients and p-harmonic functions that are, by definition, p-energy minimizers among functions with the same boundary values in compact subsets. We are unable to use an equation like (1) in our definition simply because the Euler-Lagrange equation corresponding to the p-energy functional is not available on a general metric measure space. Fortunately, a Green's function g on a Riemannian manifold satisfies an equation for p-capacities of level sets of g; see [Ho1]. Following this observation we suggest here a definition that uses inequalities for p-capacities of level sets. First we prove the existence of a p-singular function on relatively compact domains in X. Then we study the existence of a global (positive) p-singular function and prove that such a function exists if and only if X is p-hyperbolic provided X satisfies certain natural assumptions. Finally, we apply our results to the existence questions of quasiconformal mappings between metric spaces of locally bounded geometry. Further applications of p-singular functions are given in [BBSh] and [HoShT]. In [HoShT] the authors, together with Tyson, use Q-singular functions in an Ahlfors Q-regular metric measure space setting to construct a conformal analogue for the Martin boundary. In [BBSh] a potential theoretic analog for the harmonic measure is constructed for the p-Laplacian by utilizing the p-singular functions; it is shown in [BBSh, Lemma 4.4] that a set $E \subset \partial\Omega$ has zero p-harmonic measure if it is of zero p-capacity. Thus the results developed in this paper explore the function-theoretic aspects of the objects studied in [BBSh] and provide means of measuring the largeness of the potential-theoretic boundary as in [HoShT].

2 Definitions and Notations

In this section we introduce the basic definitions and the standing assumptions on X. For reader's convenience these will be gathered up in Remark 2.4 at the end of this section.

We assume throughout the paper that X is a connected, locally compact, and non-compact metric measure space with an associated metric d and a non-trivial Borel regular measure μ supported on all of X. We furthermore assume that the measure is *locally doubling* - that is, there exists a constant $C \geq 1$ so that each point in X has a neighborhood U such that

$$\mu(B(x, 2r)) \leq C\,\mu(B(x, r))$$

for every open ball $B(x, r) \subset U$. A stronger condition than local doubling is the *local Q-regularity* where the measure μ is supposed to satisfy, for all balls $B(x,r)$ as above, a double inequality

$$C^{-1}r^Q \le \mu(B(x,r)) \le Cr^Q$$

with a fixed constant $Q>0$. All Riemannian manifolds satisfy the local Q-regularity condition with Q equals to the dimension. However, there are many situations where only the local doubling condition is satisfied. For example, the weights modifying the Lebesgue measure in $\mathbb{R}^n$ considered in [HKM] are in general not Q-regular for any $Q>0$.

A curve in X is the image of a continuous map from an interval to X. With a slight abuse in terminology, we also refer to the continuous map itself as a curve. A rectifiable curve is a curve whose Hausdorff 1-dimensional measure is finite. A non-negative Borel measurable function $\rho : X \to [0,\infty]$ is said to be an upper gradient of an extended real-valued function u on X if for every rectifiable curve γ in X,

$$|u(x)-u(y)| \le \int_\gamma \rho \, ds$$

where x and y denote the endpoints of γ.

Let $1<p<\infty$ be fixed. We assume that X supports a *local (weak)* $(1,q)$-*Poincaré inequality* for some $1 \le q < p$, that is, there exist constants $C>0$ and $\tau \ge 1$ so that each point in X has a neighborhood U such that, for all balls $B = B(x,r) \subset U$,

$$\fint_B |u-u_B| \, d\mu \le Cr \left(\fint_{\tau B} \rho^q \, d\mu \right)^{1/q}$$

whenever u is a measurable function on B and ρ is an upper gradient of u. Here

$$u_B = \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu$$

and $\tau B = B(x,\tau r)$. We note that, by Hölder's inequality, X supports then a local weak $(1,\tilde{q})$-Poincaré inequality for every $\tilde{q} \ge q$.

We also assume throughout that X is (locally) *linearly locally connected,* abbreviated by local LLC. By this we mean that there exists a constant $C \ge 1$ so that each point in X has a neighborhood U such that, for every ball $B(z,r) \subset U$ and for every pair of points $x_1, x_2 \in \bar{B}(z,2r)\setminus B(z,r)$, there exists a curve in $B(z,2Cr)\setminus B(z,r/C)$ joining the points x_1 and x_2; see [HeK1] for a (global) version of this notion without restrictions to the radius r. Note that by [HeK1, Section 3.12], a local Q-regular metric measure space supporting a local $(1,Q)$-Poincaré inequality is a local LLC space.

Remark 2.1. Since every open subset of X has positive measure and X is connected and satisfies the local Poincaré inequality, we note that the spheres $S(x,r) = \{x \in X : d(x,y) = r\}$ are non-empty for all $x \in X$ and for all sufficiently small r depending on x. This together with the local LLC property implies that, for each $x \in X$, there exists $r_x > 0$ such that every point $y \in \bar{B}(x, r_x)$ can be connected with x by a curve in $B(x, 2Cr_x)$. Finally, by a standard reasoning we obtain that X is path-wise connected. A further consequence of the local LLC property is that the subspace $X \setminus \{y\}$ is also path-wise connected for every $y \in X$.

The function space that plays the role of Sobolev spaces in the general setting of metric measure spaces in this paper is the *Newtonian space* $N^{1,p}(X)$ and $N_0^{1,p}(\Omega)$ is the corresponding Sobolev space of functions with zero boundary values on the domain Ω in X; see [Sh1], [KKiM], and [Sh2]. More specifically, $N^{1,p}(X)$ is a collection of equivalence classes of the set of all functions $f \in L^p(X)$ that have an upper gradient $\rho \in L^p(X)$ obtained by using the equivalence relation $u \sim v$ if

$$\|u - v\|_{1,p} := \|u - v\|_{L^p(X)} + \inf \|\rho\|_{L^p(X)},$$

the infimum being taken over all upper gradients ρ of $u - v$. The *p-capacity* of a set $A \subset X$ is the number

$$\mathrm{Cap}_p(A) := \inf_u \|u\|_{1,p}^p,$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$ such that $u\big|_A \geq 1$. The space $N_0^{1,p}(\Omega)$ is the collection of all elements of $N^{1,p}(X)$ whose representative functions vanish p-quasi-everywhere (p-q.e.) in $X \setminus \Omega$. In the paper [C] Cheeger gives another definition of a Sobolev type space, but for indices $p > 1$ Cheeger's construction yields the same space as $N^{1,p}(X)$; see [Sh1]. It is a deep theorem of Cheeger that if a metric measure space supports a doubling measure and a $(1,p)$-Poincaré inequality then the corresponding Sobolev-type space is reflexive; see [C]. The results of [C] are easily extendable to spaces supporting a locally doubling measure and a local $(1,p)$-Poincaré inequality. Using this space the papers [Sh2] and [KiSh] defined and explored properties of p-harmonic functions. Their definition of p-harmonic functions required such functions to be in the class $N^{1,p}(X)$. In this paper we consider a less restrictive definition of p-harmonic functions.

Definition 2.2. Let $\Omega \subset X$ be a domain. A function $u : X \to [-\infty, \infty]$ is said to be *p-harmonic* on Ω if $u \in N_{loc}^{1,p}(\Omega)$ and for all subsets U relatively compact in Ω and for all functions $\varphi \in N_0^{1,p}(U)$,

$$\int_U g_u^p \leq \int_U g_{u+\varphi}^p,$$

where g_u is the minimal (both pointwise a.e. and in the L_{loc}^p-class) weak upper gradient of u (see [C] for a proof of existence of such minimal functions). Here by "relatively compact in Ω" we mean that the closure of the subset is compact and lies in Ω.

Note by [Sh2] that if $u\big|_\Omega \in N^{1,p}(\Omega)$ and there is a function $\tilde{u} \in N^{1,p}(X)$ so that $\tilde{u} = u$ p-q.e. in $X\backslash\Omega$, then $u \in N^{1,p}(X)$ and hence is p-harmonic in the sense of [Sh2] and [KiSh]. In this case, the function $\tilde{u}$ is said to be the boundary data for the p-harmonic solution u in Ω. Given such boundary data, there is exactly one corresponding p-harmonic function $u \in N^{1,p}(X)$ with boundary data $\tilde{u}$ in the sense that if $u_1 \in N^{1,p}(X)$ is another such solution, then $u_1 = u$ p-q.e. on $\overline{\Omega}$.

As a consequence of the local LLC property, together with the local doubling property and the local Poincaré inequality, we note that a non-negative p-harmonic function on an annulus $B(y, Cr)\backslash B(y, r/C)$ satisfies a Harnack inequality on the sphere $S(y, r) = \{x \in X : d(x, y) = r\}$ for sufficiently small r; see for example [BMSh].

One of the most natural measurements of sets in the context of Sobolev type function spaces is the relative capacity.

Definition 2.3. The *relative p-capacity* of a set K with respect to an open set Ω, $K \subset \Omega$, is the number

$$\mathrm{Cap}_p(K; \Omega) := \inf_u \int_\Omega g_u^p,$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$ such that $u\big|_K \geq 1$ and $u\big|_{X\backslash\Omega} = 0$. If such functions do not exist, we set $\mathrm{Cap}_p(K; \Omega) = \infty$.

For more on capacities see [HeK2], [KiM], [KaSh], and the references therein.

Remark 2.4. To recapitulate, we assume throughout the paper that X is a connected, locally compact, non-compact metric measure space that is equipped with a non-trivial locally doubling measure, admits a local $(1, q)$-Poincaré inequality for some $1 \leq q < p$, and has a local LLC property. In particular, there exists a constant C so that, for each compact set $K \subset X$, there exists a constant $r_K = r(K)$ such that the local doubling property, the local $(1, q)$-Poincaré inequality, and the local LLC property hold with the constant C in every ball $B(x, r)$, with $x \in K$ and $r \in (0, r_K)$.

3 Existence of Singular Functions

One of the most important singular functions in analysis is the Green function, the fundamental solution of the Laplace operator. In the generalized setting of metric spaces we construct a singular function having most of the characteristics of the Green function. Since not all of the characteristics of the Green function hold in this general setting, we call these functions p-singular functions. The aim of this section is to give a viable definition of such singular functions and explore some conditions under which they exist.

Definition 3.1. Let $\Omega \subset X$ be a relatively compact domain and $y \in \Omega$. An extended real-valued function $g = g(\cdot, y)$ on Ω is said to be a *p-singular function with singularity at* y if it satisfies the following four criteria:

1. g is p-harmonic in $\Omega \setminus \{y\}$ and $g > 0$ on Ω;

2. $g\big|_{X\setminus\Omega} = 0$ p-q.e. and $g \in N^{1,p}(X\setminus B(y,r))$ for all $r > 0$;

3. y is a singularity; that is,

$$\lim_{x\to y} g(x) = \mathrm{Cap}_p(\{y\};\Omega)^{1/(1-p)},$$

   where we interpret this to mean $\lim_{x\to y} g(x) = \infty$ if $\mathrm{Cap}_p(\{y\};\Omega) = 0$;

4. whenever $0 \le a < b < \sup_{x\in\Omega} g(x)$,

$$\left(\frac{p-1}{p}\right)^{2(p-1)} \frac{1}{(b-a)^{p-1}} \le \mathrm{Cap}_p(\Omega^b;\Omega_a) \le \frac{p^2}{(b-a)^{p-1}}, \tag{2}$$

   where $\Omega^b = \{x \in \Omega \colon g(x) \ge b\}$ and $\Omega_a = \{x \in \Omega \colon g(x) > a\}$.

Since we have fixed the index p in this discussion, we shall simply call such functions singular functions, suppressing the reference to the index p. The double inequality (2) can be required to be an equality if we have a partial differential equation (the Euler-Lagrange equation) corresponding to the minimal energy condition; see [Ho1]. Hence if we use the derivatives constructed in [C] instead of the minimal weak upper gradients in the definition of p-harmonicity, then the corresponding construction of singular functions would satisfy the double inequality (2) with equality. However, to keep the arguments in this paper as geometric and simple as possible we use the minimal weak upper gradients.

In this section we show that every relatively compact domain $\Omega \subset X$ has a singular function with singularity at y, and we use such functions to construct singular functions on X. The proof of this result closely follows [Ho1].

Definition 3.2. If K is closed in a domain Ω, a function $u \in N^{1,p}(X)$ is called a *p-potential* of K with respect to Ω if

1. u is p-harmonic on $\Omega\setminus K$,

2. $u\big|_K = 1$, and

3. $u\big|_{X\setminus\Omega} = 0$.

By the following lemma p-potentials always exist on relatively compact domains if $\mathrm{Cap}_p(K;\Omega) < \infty$.

Lemma 3.3. *Let K be a closed subset of a relatively compact domain $\Omega \subset Y$ where Y is any metric measure space endowed with a non-trivial Borel regular measure. Then for every finite $p > 1$ there is a p-potential for K with respect to Ω provided $Cap_p(K;\Omega) < \infty$.*

Proof. Suppose that $\mathrm{Cap}_p(K;\Omega) < \infty$. A p-potential is a function $u \in N^{1,p}(X)$ with the properties that $u\big|_K = 1$, $u\big|_{Y\setminus\Omega} = 0$, and

$$\int_\Omega g_u^p = \mathrm{Cap}_p(K;\Omega).$$

We construct such a function as follows.

We can find a sequence of functions $u_i \in N_0^{1,p}(\Omega)$ so that

$$\int_\Omega g_{u_i}^p \to \mathrm{Cap}_p(K;\Omega)$$

and $0 \le u_i \le 1$, $u_i\big|_K = 1$, and $u_i\big|_{Y\setminus\Omega} = 0$. Note that $0 \le \int_\Omega u_i^p \le \mu(\Omega) < \infty$. Hence (u_i) is a bounded sequence in $N^{1,p}(Y)$, and hence by the reflexivity of $L^p(Y)$ we have a sequence of convex combinations of u_i and of g_{u_i}, respectively, that converge in $L^p(Y)$ and almost everywhere to functions $u \in L^p(Y)$ and $g \in L^p(Y)$ respectively, with

$$\int_\Omega g^p = \int_X g^p \le \mathrm{Cap}_p(K;\Omega). \tag{3}$$

By the argument in the proof of [Sh1, Lemma 3.6], g is a p-weak upper gradient of u, and as the convex combinations of u_i converges to u pointwise p-quasi-everywhere (see [KaSh, Lemma 3.1] or [Sh1, Lemma 4.11]), we have $u\big|_K = 1$ and $u\big|_{X\setminus\Omega} = 0$. Thus we have equality in equation (3) and hence u is a p-potential of K with respect to Ω because it is easy to see that u is p-harmonic in $\Omega\setminus K$.

Such a p-potential is unique in the following sense. If u_1 and u_2 are two p-potentials for a compact set K relative to Ω, then $g_{u_1-u_2} = 0$; see [C] for a proof of this fact. Thus if Y supports a local weak $(1,p)$-Poincaré inequality or if Y is an MEC_p space, then $u_1 = u_2$; see [Sh2]. □

The following theorem demonstrates that on relatively compact domains singular functions always exist.

Theorem 3.4. *If Ω is a relatively compact domain in X and $y \in \Omega$, then there exists a p-singular function on Ω with singularity at y. Moreover, if the measure on X is locally Q-regular and $p \le Q$, then every p-singular function g with singularity at y satisfies the condition $\lim_{x\to y} g(x) = \infty$.*

The proof of the above theorem requires the following lemma.

Lemma 3.5. *Let K be a compact subset of a relatively compact domain Ω, and let u be the p-potential of K with respect to Ω. For all a, b with $0 \leq a < b \leq 1$, we write $\Omega^b = \{x \in \Omega \colon u(x) \geq b\}$ and $\Omega_a = \{x \in \Omega \colon u(x) > a\}$. Then*

$$\frac{(p-1)^{2(p-1)}\, Cap_p(K;\Omega)}{p^{2(p-1)}(b-a)^{p-1}} \leq Cap_p(\Omega^b;\Omega_a) \leq \frac{p^2\, Cap_p(K;\Omega)}{(b-a)^{p-1}}. \tag{4}$$

Proof. For $a = 0$, the claim follows from [LM, Lemma 3.6 and Remark 3.7]; see also [BMSh, Lemma 5.4]. Thus we have

$$\frac{(p-1)^{p-1}\mathrm{Cap}_p(K;\Omega)}{(pb)^{p-1}} \leq \mathrm{Cap}_p(\Omega^b;\Omega) \leq \frac{p\mathrm{Cap}_p(K;\Omega)}{(b-a)^{p-1}}.$$

Then we note that $v := (u-a)/(1-a)$ is the p-potential of K with respect to Ω_a and $u \geq b$ if and only if $v \geq (b-a)/(1-a)$. Thus

$$\mathrm{Cap}_p(\Omega^b;\Omega_a) \leq \frac{p(1-a)^{p-1}\mathrm{Cap}_p(K;\Omega_a)}{(b-a)^{p-1}}.$$

Finally,

$$\mathrm{Cap}_p(K;\Omega_a) = \mathrm{Cap}_p(X\setminus\Omega_a; X\setminus K) \leq \frac{p\mathrm{Cap}_p(X\setminus\Omega; X\setminus K)}{(1-a)^{p-1}} = \frac{p\mathrm{Cap}_p(K;\Omega)}{(1-a)^{p-1}}$$

which proves the upper bound. The lower bound follows similarly. □

The following result is well-known in the Euclidean setting. Indeed, in $\mathbb{R}^n$, single points have positive p-capacity if and only if $p > n$.

Lemma 3.6. *If the measure on X is locally Q-regular, $1 < p \leq Q$, and $y \in \Omega$ where Ω is a domain in X, then*

$$\lim_{r\to 0} Cap_p(\overline{B(y,r)};\Omega) = 0. \tag{5}$$

Proof. Let $R > 0$ be sufficiently small so that the ball $B(y,R)$ is relatively compact in the domain Ω and $\mu(B(y,r)) \leq Cr^Q$ for all $r \leq R$. For each $i = 1, 2, \ldots$, write $B_i = B(y, 2^{-i}R)$. Then

$$\mathrm{Cap}_p(\overline{B_{i+1}};B_i) \leq \frac{C2^{-iQ}R^Q}{2^{-(i+1)p}R^p} \leq CR^{Q-p}.$$

By using [HoK, Lemma 2.1], we obtain

$$\mathrm{Cap}_p(\overline{B_{j+1}};\Omega) \leq \mathrm{Cap}_p(\overline{B_{j+1}};B_1) \leq \left(\sum_{i=1}^{j} \mathrm{Cap}_p(\overline{B_{i+1}};B_i)^{1/(1-p)}\right)^{1-p}$$

which tends to 0 as $j \to \infty$. □

As is seen in the proof above, only the upper estimate $\mu(B(y,r)) \le Cr^Q$ is needed in Lemma 3.6.

Proof of Theorem 3.4. Let $r_y > 0$ be such that the local doubling condition, $(1,q)$-Poincaré inequality, and the local LLC property hold for all balls in $B(y, r_y)$. Let $B_i = B(y, 2^{-i})$ for each positive integer i for which $B(y, 2^{-i})$ is relatively compact in Ω, and let u_i be the p-potential of $\overline{B_i}$ with respect to Ω. Here by $\overline{B}_i$ we mean the closed ball $\{x \in X : d(x,y) \le 2^{-i}\}$ rather than the topological closure of the open ball B_i. For $0 < r < \min\{r_K, d(y, \partial\Omega)\}$ set

$$\begin{aligned} m_i(r) &= \min\{u_i(x) \colon d(x,y) = r\}, \\ M_i(r) &= \max\{u_i(x) \colon d(x,y) = r\}. \end{aligned}$$

If $r \le 2^{-i}$, then $m_i(r) = M_i(r) = 1$. If $r > 2^{-i}$, then as $0 \le m_i(r) \le M_i(r) \le 1$, by the fact that u_i is p-harmonic in $\underline{B(y,r)} \setminus \overline{B_i}$ and by the maximum principle (see [Sh2]) we see that $u_i \ge m_i(r)$ on $\overline{B(y,r)}$. By the strong maximum principle (see [KiSh]) and by the fact that u_i is p-harmonic on $\Omega \setminus \overline{B(y,r)}$, we see that $u_i < M_i(r)$ on $\Omega \setminus \overline{B(y,r)}$. Hence as sets,

$$\{x \in \Omega \colon u_i(x) \ge M_i(r)\} \subset \overline{B(y,r)} \subset \{x \in \Omega \colon u_i(x) \ge m_i(r)\}.$$

Hence

$$\begin{aligned} \mathrm{Cap}_p(\{x \in \Omega \colon u_i(x) \ge m_i(r)\}; \Omega) &\ge \mathrm{Cap}_p(\overline{B(y,r)}; \Omega) \\ &\ge \mathrm{Cap}_p(\{x \in \Omega \colon u_i(x) \ge M_i(r)\}; \Omega). \end{aligned}$$

When the integer i is large enough so that $2^{-i} < \frac{r}{10} < \frac{r_y}{100}$, because X is locally LLC and by [KiSh] we have Harnack inequality for u_i on the sphere $S(y,r) := \{x \in \Omega \colon d(x,y) = r\}$; see [BMSh, Lemma 5.3]. Hence there is a constant $\lambda > 0$, independent of i and r so that

$$M_i(r) \le \lambda m_i(r).$$

By Lemma 3.5, with $b = m_i(r)$ and $a = 0$, we see that

$$\begin{aligned} M_i(r) \le \lambda m_i(r) &\le c \left(\frac{\mathrm{Cap}_p(\overline{B_i}; \Omega)}{\mathrm{Cap}_p(\{x \in \Omega \colon u_i(x) \ge m_i(r)\}; \Omega)} \right)^{1/(p-1)} \\ &\le c \left(\frac{\mathrm{Cap}_p(\overline{B_i}; \Omega)}{\mathrm{Cap}_p(\overline{B(y,r)}; \Omega)} \right)^{1/(p-1)}. \end{aligned}$$

Similarly taking $b = M_i(r)$ and $a = 0$ in Lemma 3.5, we see that

$$m_i(r) \ge c^{-1} \left(\frac{\mathrm{Cap}_p(\overline{B_i}; \Omega)}{\mathrm{Cap}_p(\overline{B(y,r)}; \Omega)} \right)^{1/(p-1)}.$$

Therefore

$$M_i(r) \approx m_i(r) \approx \left(\frac{\mathrm{Cap}_p(\overline{B_i}; \Omega)}{\mathrm{Cap}_p(\overline{B(y,r)}; \Omega)} \right)^{1/(p-1)} \approx u_i(x) \tag{6}$$

whenever $x \in S(y,r)$, with the comparison constant independent of i and r. Let

$$g_i = \frac{u_i}{\mathrm{Cap}_p(\overline{B_i}; \Omega)^{1/(p-1)}}.$$

Then for sufficiently small $r > 0$ and for all points x in the sphere $S(y,r)$,

$$0 \le g_i(x) \approx \mathrm{Cap}_p(\overline{B(y,r)}; \Omega)^{1/(1-p)}. \tag{7}$$

Hence by the maximum principle applied to the function g_i which is p-harmonic on $\Omega \setminus \overline{B(y,r)}$, we have

$$0 \le g_i \le c\mathrm{Cap}_p(\overline{B(y,r)}; \Omega)^{1/(1-p)}$$

on $\Omega \setminus \overline{B(y,r)}$. Thus (g_i) is a sequence of p-harmonic functions on $\Omega \setminus \overline{B(y,r)}$ that are uniformly bounded in that region. Hence by [Sh3, Proposition 4.1], there is a subsequence, also denoted by (g_i), converging uniformly to a function g which is p-harmonic on $\Omega \setminus \overline{B(y,r)}$. Letting $r \to 0$, and selecting a diagonal subsequence, we see that g is defined on $\Omega \setminus \{y\}$, $g_i \to g$ locally uniformly in $\Omega \setminus \{y\}$, and g is p-harmonic on $\Omega \setminus \{y\}$. As u_i, and hence g_i, vanishes on $X \setminus \Omega$, we see that $g\big|_{X \setminus \Omega} = 0$ p-q.e. Also by (7), we have

$$g \approx \mathrm{Cap}_p(\overline{B(y,r)}; \Omega)^{1/(1-p)} \tag{8}$$

on the sphere $S(y,r)$ for sufficiently small $r > 0$. In particular, if the measure on X is locally Q-regular and $p \le Q$, we have by Lemma 3.6,

$$\lim_{r \to 0} \mathrm{Cap}_p(\overline{B(y,r)}; \Omega)^{1/(1-p)} = \infty,$$

and therefore in this case $\lim_{x \to y} g(x) = \infty$. If $\mathrm{Cap}_p(\{y\}; \Omega) > 0$, then the function $g = \mathrm{Cap}_p(\{y\}; \Omega)^{1/(1-p)} u$ satisfies the required conditions.

It now only remains to prove condition 4 in the definition of a singular function.

By Lemma 3.5 again, if $0 \le a_i < b_i \le 1$, then

$$\mathrm{Cap}_p(\{x \in \Omega \colon u_i(x) \ge b_i\}; \{x \in \Omega \colon u_i(x) > a_i\}) \approx \frac{\mathrm{Cap}_p(\overline{B_i}; \Omega)}{(b_i - a_i)^{p-1}},$$

where the comparison constants are as in (4). Let $0 \le a < b < \sup_{x \in \Omega} g(x)$. Then $g_i \ge b$ if and only if $u_i \ge b\,\mathrm{Cap}_p(\overline{B_i}; \Omega)^{1/(p-1)}$, and $g_i > a$ if and only if $u_i > a\,\mathrm{Cap}_p(\overline{B_i}; \Omega)^{1/(p-1)}$. If $b\,\mathrm{Cap}_p(\overline{B_i}; \Omega)^{1/(p-1)} \le 1$, then taking $b_i = b\,\mathrm{Cap}_p(\overline{B_i}; \Omega)^{1/(p-1)}$ and $a_i = a\,\mathrm{Cap}_p(\overline{B_i}; \Omega)^{1/(p-1)}$, we have

$$\mathrm{Cap}_p(\{x \in \Omega \colon g_i(x) \ge b\}; \{x \in \Omega \colon g_i(x) > a\}) \approx \frac{1}{(b-a)^{p-1}}.$$

Note that the inequality $b\,\mathrm{Cap}_p(\overline{B_i};\Omega)^{1/(p-1)} \le 1$ holds true for sufficiently large i given b by the choice of upper bound on b and the uniform convergence of g_i. Now as $g_i \to g$ locally uniformly in $\Omega\setminus\{y\}$, and as p-capacity is a Choquet capacity (see [KiM]), we have the required condition 4 of a singular function. Hence g is a singular function for Ω with singularity at y.

Observe that as $g = 0$ on $X\setminus\Omega$ and $g > 0$ on Ω, the singular function g is necessarily non-constant. ☐

Remark 3.7. First note that a connected, locally compact metric space X is σ-compact; see e.g. [Sp]. Hence X is second countable. By using the local Poincaré inequality and the assumption that every open set in X has positive measure we see that X is locally connected. Hence by the (global) connectivity, second countability, and local compactness of X, there exists an exhaustion of X by relatively compact domains Ω_j. Furthermore, we may assume that each Ω_j is path-connected.

Let C be a compact subset of X, and (Ω_j) be an increasing sequence of relatively compact domains in X so that $\overline{\Omega_j} \subset \Omega_k$ if $k > j$, $C \subset \Omega_1$, and $X = \cup_j \Omega_j$. Then the numbers $\mathrm{Cap}_p(C;\Omega_j)$ form a decreasing sequence. We define the relative p-capacity of C with respect to X to be the number

$$\mathrm{Cap}_p(C;X) := \lim_{j\to\infty} \mathrm{Cap}_p(C;\Omega_j). \tag{9}$$

Lemma 3.8. *If C is a compact set in X and (Ω_j), (Ω'_j) are two increasing sequences of relatively compact domains in X such that $C \subset \Omega_1 \cap \Omega'_1$, and furthermore $X = \cup_j \Omega_j = \cup_j \Omega'_j$, then*

$$\lim_{j\to\infty} Cap_p(C;\Omega_j) = \lim_{j\to\infty} Cap_p(C;\Omega'_j),$$

and hence $Cap_p(C;X)$ is unambiguously defined. Moreover,

$$Cap_p(C;X) = \inf_u \int_X g_u^p,$$

where the infimum is taken over all compactly supported functions u, or, equivalently, over functions $u \in N^{1,p}(X)$, such that $u\big|_C = 1$.

The above lemma indicates that the definition (9) is consistent with the definition of relative p-capacity given at the end of Section 2.

Proof. Given Ω_j we have $k_j > 0$ so that $\Omega_j \subset \Omega'_k$ for every $k \ge k_j$. Similarly, given Ω'_j we have $k'_j > 0$ so that $\Omega'_j \subset \Omega_k$ whenever $k \ge k'_j$. Hence

$$\mathrm{Cap}_p(C;\Omega_k) \le \mathrm{Cap}_p(C;\Omega'_j)$$

whenever $k \ge k'_j$, and similarly

$$\mathrm{Cap}_p(C;\Omega'_k) \le \mathrm{Cap}_p(C;\Omega_j)$$

whenever $k \geq k_j$. The first equation in the lemma follows. By the fact that $\overline{\Omega}_j$ is compact in X, we see that $\mathrm{Cap}_p(C;\Omega_j) \geq \inf_u \int_X g_u^p$ where the infimum is taken over all functions u as in the statement of the lemma. Moreover, by the fact that if u is compactly supported in X then the support of u lies inside all but finitely many of the domains Ω_j, we have $\mathrm{Cap}_p(C;\Omega_j) \leq \inf_u \int_X g_u^p$. The second equation in the lemma now follows. Furthermore, since we know that compactly supported functions are dense in $N^{1,p}(X)$, we can also conclude that

$$\mathrm{Cap}_p(C;X) = \inf_u \int_X g_u^p,$$

where the infimum is now taken over functions $u \in N^{1,p}(X)$ so that u takes on the value 1 on C. $\square$

Remark 3.9. Note that we can enlarge the class of test functions used in the expression of $\mathrm{Cap}_p(C;X)$ at the end of the above proof. Let $L^{1,p}(X)$ be the closure of $N^{1,p}(X)$, or, equivalently, the closure of the set of all Lipschitz functions with compact support, in the Dirichlet seminorm $|u| := \|u\|_{1,p} - \|u\|_{L^p(X)}$. We can choose test functions from this class (called the Dirichlet space) in the expression of $\mathrm{Cap}_p(C;X)$ given at the end of the above proof. Hence the existence of a compact set $C \subset X$ with the property that $\mathrm{Cap}_p(C;X) > 0$ tells us that we cannot approximate non-zero constant functions in the Dirichlet seminorm by functions in $N^{1,p}(X)$.

Remark 3.10. By the results in [KaSh], if X is proper, locally quasiconvex, equipped with a locally doubling measure, and supports a local $(1,p)$-Poincaré inequality (any path-connected locally doubling metric measure space supporting a local Poincaré inequality is locally quasiconvex; see [HaK, Section 4]), then

$$\mathrm{Cap}_p(K;\Omega_j) = \mathrm{Mod}_p(\Gamma(K;\Omega_j)),$$

where $\mathrm{Mod}_p(\Gamma(K;\Omega_j))$ is the collection of all rectifiable curves that connect K to $X \backslash \Omega_j$. Hence in this setting,

$$\mathrm{Cap}_p(K;X) = \mathrm{Mod}_p(\Gamma(K;\{\infty\})),$$

where $\mathrm{Mod}_p(\Gamma(K;\{\infty\}))$ is the collection of all locally rectifiable curves that start from points in K and leave every compact set in X.

Next we look for the existence of global singular functions. If $\Omega_1 \subset \Omega_2$ are two relatively compact domains in X and $y \in \Omega_1$, by the above construction we have singular functions g_1 and g_2 on Ω_1 and Ω_2 respectively corresponding to the same subsequence of balls $B_i \subset \Omega_1$, with singularity at y. Consider such singular functions obtained via the above construction. Let $u_{1,i}$ and $u_{2,i}$ be the p-potentials used in the construction of g_1 and g_2 respectively. Note that $u_{j,i}$ is the p-potential of $\overline{B_i}$ with respect to Ω_j. By the comparison theorem for p-harmonic functions and by the fact that $\Omega_1 \subset \Omega_2$, we have $u_{2,i} \geq u_{1,i}$ on Ω_1 and

$$\mathrm{Cap}_p(\overline{B_i};\Omega_2) \leq \mathrm{Cap}_p(\overline{B_i};\Omega_1).$$

Hence $g_{1,i} \le g_{2,i}$ on Ω_1, and we can conclude that $g_1 \le g_2$. Thus if we have a nested sequence of relatively compact domains exhausting X, we can use the corresponding singular functions to obtain a singular function defined on the entire space X. This is the content of the next theorem; see also [Ho1, Theorem 3.27].

Definition 3.11. A non-constant extended real-valued function g on X is said to be a *p-singular function* on X with singularity at $y \in X$ if the following four criteria are met:

1. $\lim_{x\to y} g(x) = \mathrm{Cap}_p(\{y\};X)^{1/(1-p)}$, where we adopt a convention that $\mathrm{Cap}_p(\{y\};X)^{1/(1-p)} = \infty$ if $\mathrm{Cap}_p(\{y\};X) = 0$;

2. $g > 0$ on X and is p-harmonic on $X\setminus\{y\}$;

3. for sufficiently small $r > 0$, whenever $x \in S(y,r)$, we have
$$g(x) \approx \lim_{j\to\infty} \mathrm{Cap}_p(\overline{B(y,r)};\Omega_j)^{1/(1-p)}$$
with the comparison constant depending only on the singularity y;

4. there exists $b_0 > 0$ so that for all b with $b_0 \le b \le \mathrm{Cap}_p(\{y\};X)^{1/(1-p)}$ and for all a with $0 \le a < b < \infty$,
$$\left(\frac{p-1}{p}\right)^{2(p-1)} \frac{1}{(b-a)^{p-1}} \le \mathrm{Cap}_p(X^b;X_a) \le \frac{p^2}{(b-a)^{p-1}},$$
where $X^b = \{x \in X\colon g(x) \ge b\}$ and $X_a = \{x \in X\colon g(x) > a\}$.

Theorem 3.12. *Let (Ω_j) be an increasing sequence of relatively compact domains in X so that $\overline{\Omega_j} \subset \Omega_k$ whenever $k > j$ and $X = \cup_j \Omega_j$. Suppose there exist $y \in \Omega_1$ and a positive number $r < r_y$ so that $B(y,r) \subset \Omega_1$ and*
$$\mathit{Cap}_p(\overline{B(y,r)};X) = \lim_{j\to\infty} \mathit{Cap}_p(\overline{B(y,r)};\Omega_j) = C_0 > 0. \tag{10}$$
Then there is a singular function on X with singularity at y.

Proof. By the above discussion, we have a corresponding monotonically increasing sequence of singular functions g_j for the sequence of domains Ω_j with singularity at y. By the relation (8), we have
$$g_j \approx \mathrm{Cap}_p(\overline{B(y,r)};\Omega_j)^{1/(1-p)} \le \frac{C_y}{C_0}$$
on the sphere $S(y,r)$ for sufficiently small radii $r > 0$. Hence the sequence g_j is uniformly bounded on $S(y,r)$. Thus by the local Harnack inequality together with the fact that $X\setminus\{y\}$ is path-connected, we can see that $\{g_j\}_{j=i}^{\infty}$ is locally uniformly bounded on $\Omega_i\setminus\{y\}$. Therefore by [Sh3, Corollary 3.6], the function

$g = \lim_{j\to\infty} g_j$ is p-harmonic on $X\setminus\{y\}$, and moreover, as the sequence of functions g_j is monotonic increasing and $\lim_{x\to y} g_j(x) = \mathrm{Cap}_p(\{y\};\Omega_j)^{1/(1-p)}$, we see that $\lim_{x\to y} g(x) = \mathrm{Cap}_p(\{y\};X)^{1/(1-p)}$. The third condition in the definition of global singular function is easily verified by using the relation (8).

By condition 4 in the definition of singular functions on relatively compact domains, we see that for sufficiently large b (independent of Ω_j),

$$\mathrm{Cap}_p(\{x\in\Omega_j\colon g_j(x)\ge b\};\{x\in\Omega_j\colon g_j(x)>a\}) \approx \frac{1}{(b-a)^{p-1}}.$$

Note that

$$\{x\in X\colon g(x)>a\} = \bigcup_i \{x\in\Omega_i\colon g_i(x)>a\},$$

and

$$\{x\in X\colon g(x)\ge b\} = \bigcap_j \bigcup_i \{x\in\Omega_i\colon g_i(x)\ge b-1/j\},$$

where the two unions are of an increasing sequence of sets. Hence by the fact that g, g_i are continuous on the open sets $\Omega_i\setminus\{y\}$ and the fact that p-capacity is a Choquet capacity (see [KiM]), we obtain the fourth condition in the definition of global singular function. By Condition 4 and by the fact that the p-capacity of the set where $g>b$ relative to X is strictly positive, it is clear that g is non-constant. This completes the proof that g is a singular function on X with singularity at y.

□

Using the above theorem we obtain a characterization theorem similar to [Ho1, Theorem 3.27]. Following the terminology of [CHSC] and [Tr] we give the following definition.

Definition 3.13. We say that X is *p-hyperbolic* if there is a compact set C so that $\mathrm{Cap}_p(C;X)>0$.

We refer to [CHSC], [Ho2], and [HoK] for various conditions on manifolds and metric spaces that imply p-hyperbolicity. It is useful to classify metric measure spaces according to whether they are hyperbolic or not, since hyperbolicity is preserved by quasiconformal maps; see Theorem 4.5 below. Theorem 3.14 gives an equivalent criterion for verifying hyperbolicity in terms of singular functions, and hence is also useful in the theory of stochastic processes.

Theorem 3.14. *The space X is p-hyperbolic if and only if for every $y\in X$ there is a singular function with singularity at y.*

To prove the theorem, we need the following lemma.

Lemma 3.15. *Suppose there is a compact subset K of X so that $\mathrm{Cap}_p(K;X)$ is positive. Then every compact set $F\subset X$ of positive measure has positive p-capacity; $\mathrm{Cap}_p(F;X)$ is positive.*

Proof. Suppose that there is a compact set $F \subset X$ of positive measure so that $\operatorname{Cap}_p(F;X)$ is zero. Let (Ω_j) be a sequence of relatively compact domains in X so that $F \cup K \subset \Omega_1$, $\overline{\Omega}_j \subset \Omega_{j+1}$, and $X = \bigcup_j \Omega_j$. For each j there is a p-potential u_j of F with respect to Ω_j. Since $\operatorname{Cap}_p(F;X) = 0$, it is easy to see that $\int_X g_{u_j}^p = \operatorname{Cap}_p(F;\Omega_j) \to 0$ as $j \to \infty$. Note that by hypothesis $\mu(F) > 0$.

Consider the sequence $\{u_j\}_{j \geq j_0}$. By the comparison principle (see [Sh1]) this is an increasing sequence of functions that is equibounded and (by the results in [KiSh]) equicontinuous on $\Omega_{j_0} \backslash F$. Hence this sequence converges locally uniformly to a function w_{j_0}. Letting $j_0 \to \infty$, we see that the sequence (u_j) converges locally uniformly in $X \backslash F$ to a function w. Moreover, as the sequence (g_{u_j}) is bounded in $L^p(X)$, by Mazur's lemma a convex combination of functions in this sequence converges to a weak upper gradient g_w of w; see [KaSh, Lemma 3.1] or [Sh1, Lemma 4.11]. It is again easy to see that $\int_X g_w^p = 0$, and hence $g_w = 0$ almost everywhere in X. By the arguments in [Sh1, Section 6] (note that metric measure spaces supporting a local $(1,p)$-Poincaré inequality are MEC_p-spaces in the sense of [Sh1]), we see that w is a constant function. As $w\big|_F = 1$ and $\mu(F) > 0$, we have that $w = 1$ p-quasi-everywhere. Fix $0 < \epsilon < 1/2$. Now by the local uniform convergence of u_j and by the fact that K is a compact set, there is a positive integer j_0 so that for every $j > j_0$ we have $u_j \geq 1 - \epsilon$ on K. Consider the function $v_j := u_j/(1-\epsilon)$. Then $v_j\big|_{X \backslash \Omega_j} = 0$ and $v_j\big|_K \geq 1$. Hence

$$\operatorname{Cap}_p(K;\Omega_j) \leq \int_X g_{v_j}^p \leq \frac{1}{(1-\epsilon)^p} \int_X g_{u_j}^p \to 0.$$

This yields a contradiction, as the above inequality indicates that $\operatorname{Cap}_p(K;X)$ is zero. Hence the original supposition that $\operatorname{Cap}_p(F;X) = 0$ is false. This completes the proof of the lemma. □

Remark 3.16. Note that if we do not have the requirement of local $(1,p)$-Poincaré inequality, then the above lemma would fail. For example, let X_1 be a n-dimensional p-hyperbolic Riemannian manifold and X_2 be the n-dimensional infinite cylinder $S^{n-1} \times \mathbb{R}$. Choose two points $x_1 \in X_1$ and $x_2 \in X_2$, and consider the metric measure space X obtained by gluing X_1 to X_2 by identifying x_1 and x_2. Note that the p-capacity of x_1 in X_1 and the p-capacity of x_2 in X_2 are zero, and X does not support a $(1,p)$-Poincaré inequality at the identified point $x_1 = x_2$. It is easy to see that compact sets $K \subset X_1$ with non-empty interior have the property that $\operatorname{Cap}_p(K;X) > 0$, but for every compact set $F \subset X_2$ we have $\operatorname{Cap}_p(F;X) = 0$.

Proof of Theorem 3.14. Suppose that X is p-hyperbolic and $y \in X$. Then Lemma 3.15 implies that $\operatorname{Cap}_p(\overline{B(y,r)};X) > 0$ for every $r > 0$. Hence a singular function on X with a singularity at y exists by Theorem 3.12. Suppose then that we have a singular function g on X with a singularity at $y \in X$. By condition 3, for sufficiently small $r > 0$ and $x \in S(y,r)$, we have

$$\lim_{j \to \infty} \operatorname{Cap}_p(\overline{B(y,r)};\Omega_j) \approx g(x)^{1-p} > 0$$

since $g(x) < \infty$. Thus $\mathrm{Cap}_p(\overline{B(y,r)}; X) > 0$, and so X is p-hyperbolic. $\square$

4 Hyperbolicity and Quasiconformal Maps

In this section we show that under certain circumstances hyperbolicity is preserved by quasiconformal maps. We also explore some other harmonic properties of metric spaces that are preserved by quasiconformal maps.

Definition 4.1. A homeomorphism f between two metric spaces X and Y is said to be *quasiconformal* if

$$H(x) := \limsup_{r \to 0} \frac{\sup_{d_X(x,y) \le r} d_Y(f(x), f(y))}{\inf_{d_X(x,y) \ge r} d_Y(f(x), f(y))} \le H$$

for each $x \in X$.

The above definition of quasiconformality is from [HeK1]. Under certain geometric constraints on X and Y there are corresponding geometrically and analytically equivalent definitions of quasiconformality; see [HKST, Theorem 9.8]. The following geometric criterion is from [HKST].

Definition 4.2. A metric measure space X is said to be of *locally Q-bounded geometry,* with $Q > 1$, if X is a path-connected, locally compact metric space equipped with a locally Q-regular measure that admits a local $(1, Q)$-Poincaré inequality.

We recall from Section 2 that the measure μ is locally Q-regular if there exists a constant $C \ge 1$ so that each point in X has a neighborhood U such that

$$C^{-1} r^Q \le \mu(B(x,r)) \le C r^Q.$$

We also recall that a connected, locally compact metric space is separable. The following theorem from [HKST, Theorem 9.8] characterizes quasiconformal maps between metric spaces of locally Q-bounded geometry.

Theorem 4.3 ([HKST]). *Let $f : X \to Y$ be a homeomorphism between metric spaces of locally Q-bounded geometry. Then the following four conditions are equivalent:*

1. *f is quasiconformal;*

2. *f is locally quasisymmetric;*

3. *$f \in N^{1,Q}_{loc}(X; Y)$ and $Lip f(x)^Q \le K J_f(x)$ for a.e. $x \in X$;*

4. *for every collection Γ of paths in X the relation*

$$\mathrm{Mod}_Q \Gamma \approx \mathrm{Mod}_Q f\Gamma$$

holds.

Moreover, if any one of the above conditions holds for f, then f is absolutely continuous in measure (that is, Lusin's condition (N) is satisfied), and f^{-1} is also quasiconformal.

In the above theorem,

$$\mathrm{Lip} f(x) := \limsup_{r\to 0} \sup_{d_X(x,y)\le r} \frac{d_Y(f(x), f(y))}{r}$$

and

$$J_f(x) := \limsup_{r\to 0} \frac{\mu_Y(f(B(x,r)))}{\mu_X(B(x,r))}$$

is the Radon-Nikodym volume derivative of the pull-back measure of μ_Y with respect to the underlying measure μ_X. The Sobolev space of functions from X to Y is denoted $N^{1,Q}_{loc}(X;Y)$; see [HKST] for a definition.

As a consequence of the above theorem we have a change of variables formula useful in the lemmata found in this section.

Corollary 4.4 ([HKST, Theorem 9.10]). *If f is a quasiconformal map between metric spaces X and Y of locally Q-bounded geometry and $u \in N^{1,Q}_{loc}(Y)$, then $u\circ f \in N^{1,Q}_{loc}(X)$ and for every relatively compact open set $\Omega \subset X$*

$$\int_\Omega g^Q_{u\circ f}\, d\mu_X \le C \int_{f\Omega} g^Q_u\, d\mu_Y,$$

where $C > 1$ depends only on the quasiconformality constants of f and the data associated with the local Q-boundedness of spaces X and Y.

The following theorem demonstrates that Q-hyperbolicity is a quasiconformally invariant property.

Theorem 4.5. *Let X and Y be metric spaces of locally Q-bounded geometry and suppose that $f : X \to Y$ is a quasiconformal homeomorphism. Then X is Q-hyperbolic if and only if Y is Q-hyperbolic.*

Proof. Suppose that X is Q-hyperbolic. Let K be a compact subset of X and let (Ω_j) be an increasing sequence of relatively compact domains exhausting X so that $K \subset \Omega_1$ and $\mathrm{Cap}_Q(K;X) = \lim_{j\to\infty} \mathrm{Cap}_Q(K;\Omega_j) = C_0 > 0$. Then for all j, $\mathrm{Cap}_Q(K;\Omega_j) \ge C_0$. Let φ_j be a Lipschitz function with $\varphi_j\big|_{fK} = 1$, $\varphi_j\big|_{Y\setminus f\Omega_j} = 0$, and

$$\mathrm{Cap}_Q(fK; f\Omega_j) \ge \int_{f\Omega_j} g^Q_{\varphi_j}\, d\mu_Y - 1/j.$$

By Corollary 4.4,

$$\int_{\Omega_j} g^Q_{\varphi_j\circ f}\, d\mu_X \le C \int_{f\Omega_j} g^Q_{\varphi_j}\, d\mu_Y.$$

Since
$$\int_{\Omega_j} g^Q_{\varphi_j\circ f}\, d\mu_X \geq \mathrm{Cap}_Q(K;\Omega_j) \geq C_0,$$
we get
$$C_0 \leq C\int_{f\Omega_j} g^Q_{\varphi_j}\, d\mu_Y \leq C\left(\mathrm{Cap}_Q(fK; f\Omega_j) + 1/j\right).$$
Letting $j \to \infty$, we see that $\mathrm{Cap}_Q(fK;Y) \geq C_0/C > 0$. Hence Y is Q-hyperbolic. Repeating the argument for the quasiconformal map f^{-1} completes the proof. $\square$

The next lemma shows that quasiconformal maps between spaces of local Q-bounded geometry preserve the class of Q-quasiminimizers. See [KiSh] for more on quasiminimizers.

Lemma 4.6. *Let X and Y be metric spaces of locally Q-bounded geometry and let $f : X \to Y$ be a quasiconformal mapping. Then f preserves the class of Q-quasiminimizers on relatively compact domains.*

Proof. Let Ω_Y be a relatively compact domain in Y and $\Omega_X := f^{-1}\Omega_Y$. Then Ω_X is a relatively compact domain in X. Let $u \in N^{1,Q}(Y)$ be a Q-quasiminimizer on Ω_Y and let $v = u\circ f$. Then by [C] and by Corollary 4.4 we see that $v \in N^{1,Q}(X)$ and
$$\int_{\Omega_X} g^Q_v\, d\mu_X \leq C\int_{\Omega_Y} g^Q_u\, d\mu_Y.$$
Let $\varphi \in N^{1,Q}_0(\Omega_X)$ and $\psi = (v+\varphi)\circ f^{-1} - u$. Then by Corollary 4.4 again
$$\int_{\Omega_Y} g^Q_{u+\psi}\, d\mu_Y \leq C\int_{\Omega_X} g^Q_{v+\varphi}\, d\mu_X.$$
Combining these with the quasiminimizing property of u yields
$$\int_{\Omega_X} g^Q_v\, d\mu_X \leq C\int_{\Omega_Y} g^Q_u\, d\mu_Y \leq CK\int_{\Omega_Y} g^Q_{u+\psi}\, d\mu_Y \leq C^2K\int_{\Omega_X} g^Q_{v+\varphi}\, d\mu_X,$$
and hence v is a Q-quasiminimizer on Ω_X. $\square$

Note by [KiSh] that if X has a globally doubling measure and X supports a global $(1,p)$-Poincaré inequality for some $p < Q$, then every positive Q-quasiminimizer on X satisfies a global Harnack inequality and hence must be constant. Thus we obtain the following result.

Proposition 4.7. *Suppose that X and Y are of locally Q-bounded geometry so that the measure on X is globally doubling, X supports a global $(1,p)$-Poincaré inequality, with $1 \leq p < Q$, and Y admits a non-constant positive Q-harmonic function. Then there can be no quasiconformal mappings between X and Y.*

Proof. Suppose that there exists a quasiconformal map $f : X \to Y$. Let u be a non-constant positive Q-harmonic function on Y. By Lemma 4.6, the pull-back $u \circ f$ is a Q-quasiminimizer on every relatively compact domain $\Omega \subset X$ with a quasiminimizer constant independent of Ω. Hence $u \circ f$ is a positive non-constant Q-quasiminimizer on X. This contradicts with the fact that X admits no non-constant positive Q-quasiminimizer. Hence there can be no such quasiconformal map f between X and Y. ☐

Corollary 4.8. *Let X and Y be metric spaces of locally Q-bounded geometry so that the measure on X is globally doubling and supports a global $(1, p)$-Poincaré inequality with $1 \le p < Q$. Suppose, furthermore, that Y is Q-hyperbolic and admits a local $(1, q)$-Poincaré inequality with $1 \le q < Q$. Then for each $y \in Y$, X is not quasiconformally equivalent to $Y \setminus \{y\}$.*

Proof. Observe that as Y is of local Q-bounded geometry and a singleton has zero p-capacity for every $p \le Q$, the set $\Omega := Y \setminus \{y\}$ also is of local Q-bounded geometry. Suppose that $f : X \to \Omega$ is quasiconformal. Let g be a Q-singular function on Y with singularity at y. Then g is Q-harmonic in Ω and we reach a contradiction since the pull-back $g \circ f$ is a non-constant positive Q-quasiminimizer on X. ☐

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