Prime ends on metric spaces

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Abstract. In this paper we propose a notion of prime ends for domains in metric measure spaces, and study some connections between the boundary consisting of these prime ends and the Mazurkiewicz boundary.

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1. Introduction and preliminaries

The classical Dirichlet problem associated with a differential operator L is the problem of finding a function u on a domain Ω that satisfies the equation $Lu = 0$ on Ω and $u = f$ on $\partial\Omega$ for a given boundary function $f : \partial\Omega \to \mathbb{R}$. This problem has classically been well-studied for the linear operator $L = \Delta$ by Oskar Perron in [52], and more recently for its nonlinear counterpart $L = \Delta_p$. For the most general class of boundary functions f the problem has a (perhaps not unique) solution, called the Perron solution (see for example [28] and the notes therein). In the setting of metric measure spaces whose measures are doubling and support a p-Poincaré inequality, the method of Perron, has been extended in [11].

The Dirichlet problem, as posed above, is in many cases quite restrictive. For example, in the case of a slit disc in the plane, one should consider points on the slit not as single points in the boundary of the slit disc domain, but as two different points on an appropriate boundary, as seen by approaching the point from one side as opposed to the other side. Indeed, in the setting of a wide class of Euclidean and Riemannian domains, if the boundary of the domain is sufficiently regular (for example, rectifiable), then the Dirichlet problem for the operator $L = \Delta$ can be posed to take into account this notion of different ways of "approaching" the boundary points. This is the method of Martin boundary developed by, for example, Anderson–Schoen [8], Ancona [6], [7]. The minimal Martin kernel functions, which compose the Martin boundary of the domain, are analogs of Poisson kernels to more irregular setting and provide us with integral representations of the solution to the corresponding Dirichlet problem. In the setting of a slit disc one can see that there are two distinct minimal Martin kernels corresponding to each point on the slit (except for the tip of the slit).

Although the notion of Martin boundary does make sense even for nonlinear subelliptic operators such as $L = \Delta_p$ (see for example Holopainen–Shanmugalingam–Tyson [31], Lewis–Nyström [40]), because the operator is not linear we cannot hope to use the Martin boundary as a kernel to solve the corresponding Dirichlet problem. The goal of this paper is to instead use an alternative notion of boundary, called the prime end boundary, to pose a more general Dirichlet problem. The last two sections of this paper will focus on modifying the Perron method for the setting of prime end boundary.

The notion of prime end boundary was first proposed by Carathéodory [16], and used successfully by Ahlfors [3], Beurling [9], Näkki [49], Minda–Näkki [45], Ohtsuka [51] in some settings to study problems related to boundary regularity of conformal and quasiconformal mappings. Others who formulated versions of prime ends include Epstein [21], Mazurkiewicz [43], and Kaufmann [36]. More recently prime ends have been used by others including Ancona [5] and Rempe [53] in various settings to study problems related to potential theory and dynamical systems. These studies are set in Euclidean domains or domains in a topological manifold (as in [43]), and generally require that the domain be simply connected (in the planar case) or, at least, be locally connected at the boundary (quasiconformally collared domains). The literature on prime ends is quite substantial, and we cannot hope to provide an exhaustive list of references here; we recommend the interested reader to also consider papers cited in the above references.

In this note we construct a modified version of prime ends in the setting of general domains in metric setting, for the purpose of studying Dirichlet problems for p-harmonic functions ($L = \Delta_p$ in the Euclidean setting) on domains. The results of this paper are new even in the Euclidean setting, for example when the Euclidean domain is a nonsimply connected one. In the future authors intend to use the prime ends constructed in this paper to further study the Dirichlet problem potential theory of various metric measure spaces such as Riemannian manifolds, Heisenberg groups, and more general sub-Riemannian manifolds.

The rest of this section is devoted to the preliminary notions and definitions needed in the paper. In Section **[1.1: label! We have to use labels! /A]** we set up the definition of prime end boundary of domains in metric measure spaces and the associated topology. In Section **[1.2: label! We have to use labels! /A]** we study the structure of prime end boundary in relation to a more metric-driven boundary called the inner diameter boundary, and prove that if the domain is a John domain then the prime end boundary is equivalent to the inner diameter boundary and has single-point impressions. In some circumstances a domain might have some points in its topological boundary that may not be covered by prime ends; thus it is beneficial for us to also understand the more general notion of ends – which prime ends are in some sense the minimal ones (this is analogous to the fact that not all Martin kernels are minimal Martin kernels). In some circumstances as in Example 5.1 it is more beneficial to consider the wider class of ends than just prime ends; however, in this note we will focus solely on the prime end boundary. **[1.3: Put descriptions of sections here once reorganized.]**

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2. Preliminaries

Let (X, d, μ) be a metric measure space equipped with a metric d and a measure μ (and containing more than one point). We will assume that μ is a Borel measure such that $0 < \mu(B) < \infty$ for all balls B in X.

We also let $p \geq 1$ be fixed. We shall later impose additional assumptions on p, in particular, we shall require that $1 \leq p \in Q(x)$, where $Q(x)$ is the pointwise dimension set defined below.

Throughout the paper, Ω will be a proper bounded domain in X, i.e. a proper bounded nonempty connected open subset of X.

A wide class of metric measure spaces of current interest, including weighted and unweighted Euclidean spaces, Riemannian manifolds, Heisenberg groups, and other Carnot–Carathéodory spaces, all have locally doubling measures that support a Poincaré inequality locally. Since we are interested in unifying the potential theory on all these spaces, we will assume these properties for the metric spaces considered in this paper. Because the domain under consideration is assumed to be bounded, there is no loss of generality in assuming the doubling and Poincaré inequality properties as global properties, with only simple modifications needed to go from spaces with globally held properties to spaces with locally held properties.

A measure μ is said to be *doubling* if there is a constant $C_\mu > 0$ **[2.1: I changed to C_μ as the constant dep. on μ. /A]** such that for all balls $B = B(x, r) = \{y \in X : d(x, y) < r\}$,

$$\mu(2B) \leq C_\mu \mu(B),$$

where $\lambda B(x, r) = B(x, \lambda r)$. A consequence of the doubling property is the following polynomial-type lower mass bound on the decay of measures of balls. There are constants $C, Q > 0$ such that for all $x \in X$, $0 < r \leq R$ and $y \in B(x, R)$,

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq \frac{1}{C}\left(\frac{r}{R}\right)^Q. \tag{2.1}$$ lower-mass-bound

Indeed, $Q = \log_2 C_\mu$ will do, but there may be a better exponent. Note also that if (2.1) is satisfied then μ is doubling, so that μ is doubling if and only if there is an exponent Q such that (2.1) holds.

If μ is doubling, then X is complete if and only if it is proper (i.e. every closed bounded set is compact), see Björn–Björn [10], Proposition 3.1.

The following lemma is an easy consequence of the doubling property. It will often be used without further notice; see for example Heinonen [26], **[2.2: Provide specific ref. /A]**, or Lemma 3.6 in [10].

lem-comp-B'-B **Lemma 2.1.** *Assume that μ is doubling. Let $B = B(x, r)$ and $B' = B(x', r')$ be two balls such that $d(x, x') \le ar$ and $r/a \le r' \le ar$. Then $\mu(B') \simeq \mu(B)$ with the comparison constant depending only on $a > 1$ and the doubling constant C_μ.*

If X is also connected then there exist constants $C > 0$ and $q > 0$ such that for all $x \in X$, $0 < r \le R$ and $y \in B(x, R)$,

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \le C \left(\frac{r}{R}\right)^q. \tag{2.2}$$

upper-mass-bound

Note that we always have $0 < q \le Q$ and that any $0 < q' < q$ and $Q' > Q$ will do as well.

X is *Ahlfors Q_0-regular* if there is a constant C such that

$$\frac{1}{C} r^{Q_0} \le \mu(B(x,r)) \le C r^{Q_0}$$

for all balls $B(x, r) \subset X$ with $r < 2 \operatorname{diam} X$. In this case, the best choices for q and Q are to let $q = Q = Q_0$. We emphasize that in this paper we do *not* restrict ourselves to Ahlfors regular metric spaces.

Garofalo–Marola [22] introduced the *pointwise dimension* $q_0(x)$ as the supremum of all $q > 0$ such that

$$\frac{\mu(B(x,r))}{\mu(B(x,R))} \le C_q \left(\frac{r}{R}\right)^q.$$

for some $C_q > 0$ and all $0 < r \le R$. Since the analysis considered in this paper is local, we do not need the global nature associated with requiring the above inequality for all $R > 0$.

ptwise-dim **Definition 2.2.** Given $x \in X$ we consider the *pointwise dimension set* $Q(x)$ of all possible $q > 0$ for which there are constants $C_q > 0$ and $R_q > 0$ such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \le C_q \left(\frac{r}{R}\right)^q. \tag{2.3}$$

upper-mass-bound-q

for all $0 < r \le R \le R_q$ and all $y \in B(x, R)$.

Notice that $Q(x)$ now denotes a set of positive numbers, and that $Q(x)$ is a bounded interval, and indeed $Q(x) = (0, q_0)$ or $Q(x) = (0, q_0]$ for some positive number q_0. If the measure μ is Ahlfors Q_0-regular at x, then $q_0 = Q_0$ and $Q(x) = (0, Q_0]$.

The pointwise dimension $Q(x)$ will appear in some of our results in connection with the capacity and the modulus of curve families, see Section 6. Note that

$$q \le q_0 \le Q,$$

where q and Q are as in (2.1) and (2.2).

Example 2.3. Let $X = \mathbb{R}^n$ be equipped with the doubling measure $d\mu(x) = |x|^\alpha \, dx$ for some fixed $\alpha > -n$. Then $\mu(B(0, r))$ is comparable to $r^{n+\alpha}$, while for $x \ne 0$, $\mu(B(x, r))$ is comparable to r^n with comparison constants depending on $|x|$, and μ is globally doubling. It follows that (2.1) and (2.2) hold with $q = \min\{n, n + \alpha\}$ and $Q = \max\{n, n + \alpha\}$. Note that for α close to $-n$ we have q close to 0.

We follow Heinonen–Koskela [27] in introducing upper gradients as below ([27] calls upper gradients as very weak gradients).

[2.3: I changed path to curve throughout (not in pathconnected) as we used curve more often than path. /A]

Definition 2.4. A nonnegative Borel measurable function g on X is an *upper gradient* of an extended real-valued function $u : X \to [-\infty, \infty]$ if for all rectifiable curves $\gamma : [0, l_\gamma) \to X$, parameterized by arc length ds, we have

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \le \int_\gamma g\, ds, \tag{2.4}$$

eq:upperGrad

whenever both $f(\gamma(0))$ and $f(\gamma(l_\gamma))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. If g is a nonnegative measurable function on X and if (2.4) holds for p-a.e. rectifiable curve, then g is a *p-weak upper gradient* of f.

By saying that (2.4) holds for p-a.e. rectifiable curve, we mean that it fails only for a curve family with zero p-modulus, see (6.1) below. It is implicitly assumed that $\int_\gamma g\, ds$ is defined (with a value in $[0, \infty]$) for p-a.e. rectifiable curve.

Here and throughout the paper we require curves to be nonconstant, unless otherwise stated explicitly.

The p-weak upper gradients were introduced in Koskela–MacManus [39]. They also showed that if $g \in L^p(X)$ is a p-weak upper gradient of f, then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of f such that $g_j \to g$ in $L^p(X)$. If f has an upper gradient in $L^p(X)$, then it has a *minimal p-weak upper gradient* $g_f \in L^p(X)$ in the sense that for every p-weak upper gradient $g \in L^p(X)$ of f, $g_f \le g$ a.e., see Corollary 3.7 in Shanmugalingam [56].

Next we define a version of Sobolev spaces on the metric space X due to Shanmugalingam [55].

def:Np

Definition 2.5. Whenever $u \in L^p(X)$, let

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

We say that X supports a *p-Poincaré inequality*, i.e. there exist constants $C > 0$ and $\lambda \ge 1$ such that for all balls $B \subset X$, all integrable functions f on X and for all upper gradients g of f,

$$\fint_B |f - f_B| \, d\mu \le C(\operatorname{diam} B) \left(\fint_{\lambda B} g^p \, d\mu \right)^{1/p}, \tag{2.5}$$

PI-ineq

where $f_B := \fint_B f \, d\mu := \int_B f \, d\mu / \mu(B)$.

Such Poincaré inequalities are often called *weak* since we allow for $\lambda > 1$.

By the Hölder inequality, it is easy to see that if X supports a p-Poincaré inequality, then it supports a q-Poincaré inequality for every $q > p$. A deep theorem of Keith–Zhong [37] shows that if X is complete, $p > 1$ and μ is doubling, then it even supports a $\overline{p}$-Poincaré inequality for some $\overline{p} < p$. Such a q-Poincaré inequality for some $q < p$ was earlier a standard assumption for the theory of p-harmonic functions on metric spaces. In the definition of the Poincaré inequality we can equivalently assume that g is a p-weak upper gradient.

3. Carathéodory ends and prime ends

sect-Car

Let us take a quick look at how Carathéodory [16] defined ends and prime ends when he introduced the topic in 1913 (for simply connected planar domains).

Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain. A *cross cut* of Ω is a closed Jordan arc in Ω with endpoints on the boundary of Ω. A sequence $\{c_k\}_{k=1}^\infty$ of cross cuts is called a *chain* if for every k, (1) $\bar{c}_k \cap \bar{c}_{k+1} = \varnothing$, and (2) every cross cut c_k separates Ω into exactly two subdomains, one containing c_{k-1} another containing c_{k+1}, let D_k be the latter subdomain. The *impression* of the chain is $\bigcap_{k=1}^\infty \overline{D}_k$, which is a nonempty connected compact set.

Carathéodory then defines when a chain divides another chain, and says that two chains are equivalent if they divide each other. This leads to an equivalence relation for which the equivalence classes are called *ends*. The impression is independent of the representing [**3.1: representing/representative? (also elsewhere)**] chain.

The ends are then naturally partially ordered by division, and he says that a *prime end* is an end which is only divided by itself, or in other terms is minimal with respect to the partial ordering. The impression of a prime end is always a subset of $\partial\Omega$.

Later it was realised that if one imposes some extra condition on the chains, such as an extremal length condition, then the corresponding ends are automatically minimal, and they are therefore called prime ends, so that when this approach is used there are no ends (other than prime ends) and no need for weeding out bad ends. This approach leads to the same prime ends as in Carathéodory's approach. According to our investigations, the first use of extremal length in connection with prime ends is due to Schlesinger [54].

Prime ends are an important tool in various situations, and the theory works very well for simply (and finitely) connected planar domains. For infinitely connected domains, as well as in higher dimensions, the theory is not working quite so well, at least not for general domains. However, when restricting to certain domains it has proved useful also in higher dimensions.

We want to study prime ends in quite general situations, and see how far the theory can be developed. We therefore give two approaches. In the first we start by defining ends and then say that the prime ends are the ends which are minimal (with respect to the partial order). In the other approach we require the ends initially to satisfy a p-modulus condition, and to distinguish these ends from the earlier ones we call them Mod_p-ends. Here we have a choice of p (a real number larger than 1), leading us to different notions. The p-modulus condition is a generalization of extremal length, the latter being connected with the 2-modulus, and thus seems natural to consider.

In our generality it is possible that the Mod_p-ends are not minimal, and we therefore also introduce Mod_p-prime ends.

4. Ends and prime ends

From now on we assume that X is complete and supports a p-Poincaré inequality, and that μ is doubling.

We are now ready to give our definition of ends and prime ends.

Definition 4.1. A bounded connected set $E \subsetneq \Omega$ is an *acceptable* set of Ω if $\overline{E} \cap \partial\Omega$ is nonempty.

Since an acceptable set E is bounded, $\overline{E}$ is compact, and as E is connected, $\overline{E}$ is connected as well. Moreover, E is infinite, as otherwise we would have $\overline{E} = E \subset \Omega$.

Therefore, $\overline{E}$ is a continuum. Recall that a *continuum* is a connected compact set containing more than one point.

deff-chain **Definition 4.2.** A sequence $\{E_k\}_{k=1}^\infty$ of acceptable sets is a *chain* if

it-subset (a) $E_{k+1} \subset E_k$ for all $k = 1, 2, \ldots$;

pos-dist (b) $\operatorname{dist}(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k) > 0$ for all $k = 1, 2, \ldots$;

impr (c) The *impression* $\bigcap_{k=1}^\infty \overline{E}_k \subset \partial\Omega$.

rmk-interior **Remark 4.3.** As $\{\overline{E}_k\}_{k=1}^\infty$ is a decreasing sequence of continua, the impression is either a point or a continuum. Moreover, (a) and (b) above imply that $E_{k+1} \subset \operatorname{int} E_k$. In particular $\operatorname{int} E_k \neq \varnothing$. **[4.1: I deduce this without using connectedness. /A]**

Definition 4.4. We say that a chain $\{E_k\}_{k=1}^\infty$ *divides* the chain $\{F_k\}_{k=1}^\infty$ if for each k there exists l such that $E_l \subset F_k$. Two chains are *equivalent* if they divide each other, in which case we write $\{E_k\}_{k=1}^\infty \sim \{F_k\}_{k=1}^\infty$.

A class of equivalent chains is called an *end* and denoted $[E_k]$, where $\{E_k\}_{k=1}^\infty$ is any of the chains in the equivalence class. The *impression of* $[E_k]$, denoted $I[E_k]$, is defined as the impression of any representative chain.

The collection of all ends is called the *end boundary* and is denoted by $\partial_E\Omega$.

(In this definition we implicitly assumed that k and l are positive integers. We make similar implicit assumptions throughout the paper to enhance readability.)

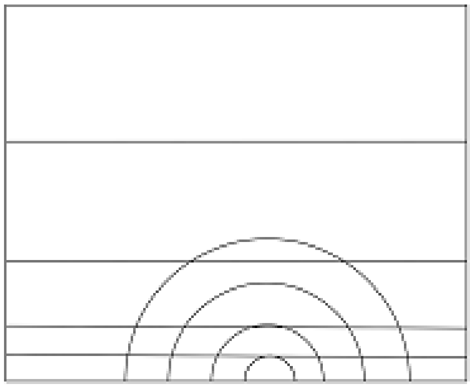
Note that if a chain $\{F_k\}_{k=1}^\infty$ divides $\{E_k\}_{k=1}^\infty$, then it divides every chain equivalent to $\{E_k\}_{k=1}^\infty$. Furthermore, if $\{F_k\}_{k=1}^\infty$ divides $\{E_k\}_{k=1}^\infty$, then every chain equivalent to $\{F_k\}_{k=1}^\infty$ also divides $\{E_k\}_{k=1}^\infty$. Therefore, the relations of dividing and equivalence extend in a natural way from chains to ends, the former becomes a partial order and the second becomes equality. Note also that the impression is independent of the choice of representing chain. Indeed, if $\{E_k\}_{k=1}^\infty$ divides $\{F_k\}_{k=1}^\infty$ then $I[E_k] \subset I[F_k]$ and the opposite inclusion holds similarly.

rmk-open **Remark 4.5.** Let $\{E_k\}_{k=1}^\infty$ be a chain. By Remark 4.3, $E_{k+1} \subset \operatorname{int} E_k$. Unfortunately $\operatorname{int} E_k$ is not necessarily connected, but if we let G_k be the component of $\operatorname{int} E_k$ containing E_{k+1}, then G_k is an open acceptable set. As $\partial G_k \subset \partial E_k$ we get that $\{G_k\}_{k=1}^\infty$ is a chain. Since $E_{k+1} \subset G_k$ for all k, we see that $\{G_k\}_{k=1}^\infty$ is divisible by $\{E_k\}_{k=1}^\infty$. On the other hand, $\{G_k\}_{k=1}^\infty$ clearly divides $\{E_k\}_{k=1}^\infty$, and thus they are equivalent and $[G_k] = [E_k]$.

As a consequence we could have required that acceptable sets are open without any consequences whatsoever for our theory. On the other hand, with our definition we have a bit more freedom when we construct examples.

[4.2: Is there any point *not* requiring that acceptable sets be open? /A]

Let us next show that there is a certain redundancy in the collection of ends.



needPrime1 **Example 4.6.** Let $\Omega = (0,10) \times (0,1)$ be the unit square in the plane and let $E_k = (0,1) \times (0,1/k)$ and $F_k = \Omega \cap B\bigl(\frac{1}{2}, 1/2k\bigr)$, $k = 1,2,\ldots$. (For simplicity we often see $\mathbb{R}$ as a subset of $\mathbb{R}^n$.) Then the chain $\{E_k\}_{k=1}^\infty$ is divisible by the chain $\{F_k\}_{k=1}^\infty$, but $\{F_k\}_{k=1}^\infty$ is not divisible by $\{E_k\}_{k=1}^\infty$. The above figure illustrates this example.

Such a redundancy might not cause a problem in some applications (see e.g. Miklyukov [44], where the analogs of acceptable sets are not even required to be connected), but since one of our aims is to use the notion of ends to construct a more general boundary of a domain such a redundancy creates a difficulty in using the collection of all ends as boundary. To overcome this type of redundancy, we consider the smallest ends in the following sense.

prime-end **Definition 4.7.** An end $[E_k]$ is a *prime end* if the only end dividing it is $[E_k]$ itself. The collection of all prime ends is called the *prime end boundary* and is denoted by $\partial_P\Omega$.

Prime ends are minimal with respect to the division partial order on the collection of ends.

5. Examples and comparison with Carathéodory's definition

We shall see later that in nice domains, every boundary point corresponds to at least one prime end. However, the following example illustrates that in some situations one may need to also consider ends which are not prime ends.

comb **Example 5.1.** Let Ω be the topologist's comb, i.e. the unit square $(0,1)^2 \subset \mathbb{R}^2$ with the segments $S_k = \bigl(\frac{1}{2}, 1\bigr) \times \{2^{-k}\}$ removed. Let $x_0 = \bigl(\frac{1}{2}, 0\bigr)$ and let $I = \bigl(\frac{1}{2}, 1\bigr] \times \{0\}$ be the set of inaccessible points, see Definition 7.5. Then the sets

$$E_k = \{(x,y) \in \Omega : \tfrac{1}{2} < x < 1 \text{ and } 0 < y < 2^{-k}\} \cup \bigl(B(x_0, 2^{-k}) \cap \Omega\bigr),$$

$k = 1,2,\ldots$, define an end with the impression $I \cup \{x_0\}$. However, this is not a prime end, as it is divided by the chain $\{B(x_0, 2^{-k}) \cap \Omega\}_{k=1}^\infty$, which defines a prime end with impression $\{x_0\}$. Note that there is no prime end with impression containing a point from I, cf. Corollary 7.12. We point out here that the prime ends of this domain are also Mod_p-prime ends, for $1 \le p \le 2$, in the sense defined in Definition 6.2, by Proposition 7.4.

It should be observed that with Carathéodory's prime ends the situation is different. In this case $\{x_0\}$ is not the impression of any prime end, instead $I \cup \{x_0\}$ is the impression of a Carathéodory prime end.

The following example is a major motivation for us.

ex-slit-disc **Example 5.2.** Let Ω be the *slit disc* $B(0,1) \setminus (-1,0] \subset \mathbb{R}^2$. Then for each $x \in [-1,0)$ there are two prime ends with the impression $\{x\}$ (one coming from the positive half-space and one from the negative half-space). For $x \in \partial\Omega \setminus [-1,0)$ there is exactly one prime end with the impression $\{x\}$, and these are all prime ends.

prop-car-our-end **Proposition 5.3.** *Let Ω be a bounded simply-connected domain in the plane. If $\{c_k\}_{k=1}^{\infty}$ is a Carathéodory prime end, then $[D_k]$ is an end, where D_k are defined as in Section 3.*

In fact the proof below shows that the corresponding result is true for all Carathéodory ends with impression in the boundary. (Of course, conversely if $\{c_k\}_{k=1}^{\infty}$ is a Carathéodory end with impression containing some point in Ω, then $[D_k]$ is not an end.)

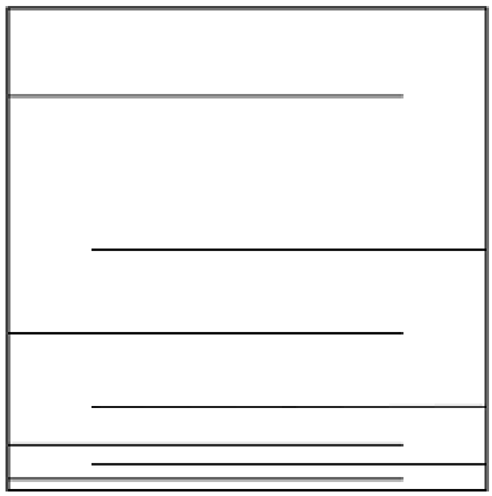
[5.1: In Section 3 I've remarked that all Caratéodory prime ends have impressions in the boundary, which is a well-known fact and I don't see a reason for us proving it. Using this made the proof below considerably shorter than Tomasz's proof in the extra notes. /A]

Proof. As the impression is a nonempty subset of $\partial\Omega$, see Section 3, each D_k is an acceptable set. It thus follows that $\{D_k\}_{k=1}^{\infty}$ is a chain, the condition (b) in Definition 4.2 being satisfied since the cross cuts are disjoint. Hence $[D_k]$ is an end. $\square$

Observe that not every Carathéodory prime end gives a *prime* end. This is due to the fact that we have more ends in some cases, see Example 5.1, which again depends on the fact that we only require that an acceptable set E is connected, not that its boundary $\Omega \cap \partial E$ is connected.

That we do not recover Carathéodory's prime ends in the simply connected planar case is a drawback with our theory, and in many situations our theory is inferior to Carathéodory's. On the other hand, it is well known that Carathéodory's theory to it is full extent is limited to simply and finitely connected planar domains. We will see in Section 7 that there is a close connection between our prime ends and accessibility of boundary points, a connection lost with Carathéodory's definition, as shown by Example 5.1 (x_0 is an accessible point but there is no Carathéodory prime end with impression equal to $\{x_0\}$). This connection is crucial for our results in Sections 9–11.

Example 5.4. Let us give one more example, the double equilateral comb shows (see picture below). From the point of view of Carathéodory's theory the limiting bottom segment is the impression of the prime end, while for us the prime end is associated only with the subinterval corresponding to the limit of the "common



parts" of the comb's teeth.

[5.2: The text below was too early, before we had defined our ends. As you can see I have now discussed the relation between our theory and Carathéodory's both in this section and in Section 3. The reason for just comparing with Carathéodory's theory is that it is very well known, while the theories of others like Näkki are much less known.

I'm leaving the text below here for now. We should clearly compare our definition with Näkki, Ohtsuka, Karmazin . . . , but perhaps in much less detail than what I've done with Carathéodory's theory. I'm also not sure where this should be put, possibly already in the introduction. Maybe we should leave this as point to be decided later rather than right now. /A]

Carathéodory [16] first developed a theory of prime ends in the setting of simply connected planar domains, see Section 3, and Näkki [49] developed a theory of prime ends in the higher-dimensional Euclidean setting using techniques of extremal length (related to our notion of Mod_p-ends in Section 6).

The boundaries of our acceptable sets correspond to Carathéodory's *cross-cuts*, and our acceptable sets correspond to the components D_n. Our definition differs from earlier definitions of prime ends. **[5.3: Has anyone defined prime ends using acceptable sets as we do? Probably Karmazin. /A]** There are several reasons for this. First, the topology of a metric space is more complicated than that of $\mathbb{R}^n$. (The reader should think of $\mathbb{R}^n$ with a number of holes removed as a particular example of a metric space under consideration.) If we require the boundary of an acceptable set to be connected, then it is not easy to construct ends using metric balls for example because boundaries of such balls need not be connected, and so it is not clear that there will be any end in the metric setting. Therefore, we have replaced cross-sets by acceptable sets. Analogs of cross-sets in our setting are the boundaries of the acceptable sets.

(1) Carathéodory and Näkki's cross-cuts are connected, while boundaries of acceptable sets need not be.
(2) Cross-cuts break the domain into exactly two components, whereas the boundaries of acceptable sets break the underlying domain into at least two components.

[5.4: I have referred to Kaufmann; his papers are in German there are no reviews, and Epstein's constructions do not seem to be so good, since

he a priori requires the chain to have "shrinking to zero" diameters, and so is too restrictive. These have been put in introduction, together with Ancona. /N]

6. Modulus ends and modulus prime ends

sect-Modp-ends

The notion of ends and prime ends discussed in the previous section does not take into account the potential theory associated with the domain. In this section we give a subclass of ends and prime ends associated with the potential theory, using the notion of p-modulus. Here, $1 \le p < \infty$.

Let Γ be a family of rectifiable curves in X. The *p-modulus* of the family Γ is

$$\operatorname{Mod}_p(\Gamma) := \inf_\rho \int_X \rho^p \, d\mu, \tag{6.1}$$

eq-deff-modulus

where the infimum is taken over all nonnegative Borel measurable functions ρ on X such that $\int_\gamma \rho \, ds \ge 1$ for every $\gamma \in \Gamma$. (As usual $\inf \varnothing := \infty$.) It is straightforward to verify that Mod_p is an outer measure on the collection of all rectifiable curves on X. If Γ_1 and Γ_2 are two families of rectifiable curves in X such that $\Gamma_1 \subset \Gamma_2$, then $\operatorname{Mod}_p(\Gamma_1) \le \operatorname{Mod}_p(\Gamma_2)$, this monotonicity will be useful in this paper. For more on p-modulus we refer the interested reader to Heinonen [26] and Väisälä [58].

For nonempty sets E, F and U in X, we let $\Gamma(E, F, U)$ denote the family of all rectifiable curves $\gamma : [0, l_\gamma] \to U \cup E \cup F$ such that $\gamma(0) \in E$ and $\gamma(l_\gamma) \in F$. As in [58], the modulus of the curve family $\Gamma(E, F, U)$ is the number

$$\operatorname{Mod}_p(E, F, U) := \operatorname{Mod}_p(\{\gamma \cap U : \gamma \in \Gamma(E, F, U)\}).$$

[6.1: Do we need this with $E, F \not\subset U$. In that case I don't see how to use $\gamma \cap U$ as it isn't a curve in general. I hope we don't need that. My hope is that we may define $\Gamma(E, F, U)$ to be the family of all rectifiable curves $\gamma : [0, l_\gamma] \to U$ (not $\to U \cup E \cup F$) such that $\gamma(0) \in E$ and $\gamma(l_\gamma) \in F$./A]

Definition 6.1. A chain $\{E_k\}_{k=1}^\infty$ is a Mod_p*-chain* if

$$\lim_{k \to \infty} \operatorname{Mod}_p(K, E_k, \Omega) = 0 \tag{6.2}$$

for every compact set $K \subset \Omega$.

Note that if $\{E_k\}_{k=1}^\infty$ is a Mod_p-chain and $\{F_k\}_{k=1}^\infty$ divides $\{E_k\}_{k=1}^\infty$, then $\{F_k\}_{k=1}^\infty$ is also a Mod_p-chain; this follows from the fact that whenever $K \subset \Omega$ is a compact set, then for each k there exists n_k such that $F_{n_k} \subset E_k$. Thus $\Gamma(K, F_{n_k}, \Omega) \subset \Gamma(K, E_k, \Omega)$ and $\operatorname{Mod}_p(K, F_{n_k}, \Omega) \le \operatorname{Mod}_p(K, E_k, \Omega)$.

deff-Modp-end

Definition 6.2. An end $[E_k]$ is a Mod_p*-end* if there is a Mod_p-chain representing it. A Mod_p-end $[E_k]$ is a Mod_p*-prime end* if the only Mod_p-end dividing it is $[E_k]$ itself.

As above it follows that any chain representing a Mod_p-end is a Mod_p-chain.

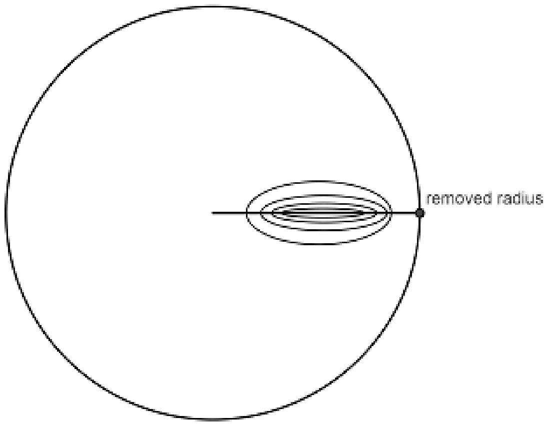
lem-en[illegible] it-prime-end-eq

Lemma 6.3. (a) *An end dividing a* Mod_p*-end is also a* Mod_p*-end.*
(b) *A* Mod_p*-end is a prime end if and only if it is a* Mod_p*-prime end.*

Proof. (a) The first part follows just as above from the monotonicity of Mod_p.

(b) Let E be a Mod_p-end. If E is a prime end, then there is no other end dividing it, let alone any other Mod_p-end dividing it. Thus E must be a Mod_p-prime end.

Conversely, if E is a Mod_p-prime end and F is an end dividing E, then F is a Mod_p-end, by (a). Hence $F = E$, and thus E is a prime end. □



needPrime

Example 6.4. Let Ω be the unit ball in $\mathbb{R}^n$, $n \geq 3$, with a radius removed. Then for every boundary point $x \in \partial\Omega$ there is a prime end P_x with it as impression, see Proposition 7.1 and Corollary 7.7.

Let I be a closed subsegment of the removed radius and let

$$E_k = \{x \in \Omega : \operatorname{dist}(x, I) < 1/k\}.$$

Then $[E_k]$ is an end with I as impression. This is not a prime end as it is divided by P_x for $x \in I$. If $p \leq n - 1$, then $\operatorname{Mod}_p(K, E_k, \Omega) \to 0$ as $k \to \infty$, and thus $[E_k]$ is a Mod_p-end but not a Mod_p-prime end.

Under some conditions **[6.2: I don't think one can use the word "circumstance" here. Also in some other places the word "might" was used where I think that it logically more correct to write "may". /A]** all Mod_p-ends are Mod_p-prime ends, and in this case one does not need to do the further subdivisions, see e.g. Section 11.

The notion of Mod_p-prime end is similar in flavour to the concept of p-parabolic prime ends discussed in Miklyukov [44] and Karmazin [35]. The name p-parabolicity has been used in the literature to denote spaces where there is not enough room out at infinity (in the sense that the collection of all curves that start from a fixed ball in the space and leave every compact subset of the space has p-modulus zero). See **[6.3: Is this list good? Let us leave this question to later. /A]** [29], [30], [18], [41], [23], [24] [42], [47], [33], and [19] for some applications of the notion of parabolic ends. A prime end is a Mod_p-prime end if there is insufficient room close to the impression of the prime end. In this sense one could think of a Mod_p-prime end as a p-parabolic end of the domain.

Recall that if $\{E_k\}_{k=1}^\infty$ is a chain, then $\{\overline{E}_k\}_{k=1}^\infty$ is a decreasing sequence of continua, and so the impression is either a point or a continuum. Corollary A.7 implies that condition (c) in Definition 4.2 follows if $\{E_k\}_{k=1}^\infty$ is a Mod_p-chain and $Q - 1 < p$ **[6.4: This doesn't make sense now. What did we want to say in this sentence? Is it still relevant? /A]**.

The condition $\lim_{k\to\infty} \operatorname{Mod}_p(K, E_k, \Omega) = 0$ depends heavily on p. For example, if $1 \leq p \notin Q(x)$, then the collection of all curves in X passing through x has positive p-modulus, and hence in general there are no chains with x in their impressions. (Here, $Q(x)$ denotes the pointwise dimension set from Definition 2.2.) However, it can happen that for some $x \in \partial\Omega$, and every $K \Subset \Omega$ we have $\operatorname{Mod}_p(K, \{x\}, \Omega) = 0$, even if $1 \leq p \notin Q(x)$. This is the case e.g. if $\Omega \subset \mathbb{R}^n$ (unweighted) has an outward polynomial cusp of degree m and $p \leq m + n - 1$, see [20, Example 2.2].

importante **Remark 6.5.** Many p-modulus estimates are not available in the nonconformal case. In $\mathbb{R}^n$, Näkki [49] uses the condition

$$0 < \mathrm{Mod}_n(\Omega \setminus E_k, E_{k+1}, \Omega) < \infty \tag{6.3}$$ eq-Modn

instead of condition (b) in Definition 4.2. For $p = n$ in the Euclidean setting, the conditions are equivalent. In Ahlfors Q-regular metric spaces, formula (3.9) in Theorem 3.6 of Heinonen–Koskela [27] shows the same equivalence (with n replaced by Q in (6.3)). See also the discussion on p. 16 and Remark 3.28 of [27]. In more general metric spaces there is no value of p for which the corresponding equivalence is true (see Example 2.7 of [1] and Example 6.6 below), and so we have explicitly required that chains $\{E_k\}_{k=1}^\infty$ satisfy $\mathrm{dist}(\Omega \cap \partial E_k, \Omega \cap \partial E_{k+1}) > 0$. This modification automatically implies that $\mathrm{Mod}_p(\Omega \setminus E_k, E_{k+1}, \Omega) < \infty$, since the function $\rho = \chi_\Omega/\tau$, with $\tau = \mathrm{dist}(\Omega \cap \partial E_k, \Omega \cap \partial E_{k+1})$, is admissible in the definition of $\mathrm{Mod}_p(\Omega \setminus E_k, E_{k+1}, \Omega)$. By Definition 4.2, the sets $\Omega \setminus E_k$ and E_{k+1} are disjoint and have nonempty interiors. It therefore follows from Lemma A.3 that $\mathrm{Mod}_p(\Omega \setminus E_k, E_{k+1}, \Omega) > 0$.

rem_p_n **Example 6.6. [6.5: (For Anders: I still need to check this example.)]**

Let $\Omega = B(0,2) \subset \mathbb{R}^2$, $E = [-1,0] \times \{0\}$ and $F = [0,1] \times \{0\}$. If $1 < p < 2$, then $\mathrm{dist}(E,F) = 0$ even though $\mathrm{Mod}_p(E,F,\Omega) < \infty$, as we shall next see.

Let Γ_0 be the family of curves in Ω passing through the origin. Since singletons have zero p-capacity in $\mathbb{R}^2$, we have $\mathrm{Mod}_p(\Gamma_0) = 0$. We shall therefore in this example only consider curves which do not pass through the origin. Let $\gamma : [a,b] \to \Omega$ be such a curve connecting E to F in Ω. Joining γ with its reflection in the real axis makes a closed curve $\tilde{\gamma}$ in Ω around the origin. The residue theorem now yields that

$$\int_{\tilde{\gamma}} \frac{\bar{z} dz}{|z|^2} = \int_{\tilde{\gamma}} \frac{dz}{z} = 2\pi i$$

and considering the imaginary part of $\bar{z}\, dz$, we obtain using symmetry and the Cauchy–Schwarz inequality that

$$\pi = \int_\gamma \frac{x\, dy - y\, dx}{x^2 + y^2} \le \int_a^b \frac{|\gamma'(t)|}{|\gamma(t)|}\, dt = \int_\gamma \frac{ds}{|\gamma(s)|},$$

where ds is the arc length parameterization of γ. It follows that the function $\rho(z) = 1/\pi|z|$ is admissible in the definition of $\mathrm{Mod}_p(E,F,\Omega)$ and hence

$$\mathrm{Mod}_p(E,F,\Omega) \le \int_\Omega \rho^p\, dx\, dy = 2\pi^{1-p} \int_0^2 r^{1-p}\, dr = \frac{2^{3-p}\pi^{1-p}}{2-p} < \infty.$$

7. Singleton impressions and accessibility

sect-access

It is clearly useful to have criteria for when ends are prime ends and Mod_p-prime ends.

The ends are naturally divided into two classes, those with singleton impressions and those with larger (continuum) impressions. The former are not surprisingly simpler to handle, and our many focus in the later sections will be on singleton-impression ends.

It turns out that ends with singleton impressions are always prime ends.

prop-end-single **Proposition 7.1.** *If an end has a singleton impression, then it is a prime end.*

[7.1: This was a corollary, but the essence of the proposition before was this, and I couldn't see any reason to leave the statement as it was. /A]

Note, however, that there are prime ends with nonsingleton impressions, see also below for more details.

Before proving this result let us give a characterization of singleton-impression ends.

prop-single-char

Proposition 7.2. *Let* $[E_k]$ *be an end. Then it has a singleton impression if and only if* $\operatorname{diam} E_k \to 0$ *as* $k \to \infty$.

Proof. Assume first that $\operatorname{diam} E_k \to 0$. As $\operatorname{diam} I[E_k] \le \operatorname{diam} \overline{E}_k = \operatorname{diam} E_k$ for all k it follows that $\operatorname{diam} I[E_k] = 0$, i.e. that $I[E_k]$ contains at most one point. Since $I[E_k]$ is nonempty it must be a singleton.

Conversely, assume that $\operatorname{diam} E_k > 4\delta > 0$ for all k, and let $x \in I[E_k]$. Then $A_k := \overline{E}_k \setminus B(x,\delta)$, $k = 1,2,\dots$, are nonempty compact sets. Hence there is $y \in \bigcap_{k=1}^\infty A_k$. Thus also $y \in I[E_k]$ and the impression is nonsingleton. □

The following observation will also be useful.

rem-connected-diam

Remark 7.3. If a connected set $F \subset \Omega$ intersects both A and $\Omega \setminus A$, then $F \cap (\Omega \cap \partial A)$ is nonempty.

A direct consequence is that if E_k, E_{k+1} and F are connected subsets of Ω with $E_{k+1} \subset E_k$, $E_{k+1} \cap F \ne \varnothing$ and $F \setminus E_k \ne \varnothing$, then F meets both $\Omega \cap \partial E_{k+1}$ and $\Omega \cap \partial E_k$, which implies in turn that $\operatorname{dist}(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k) \le \operatorname{diam} F$.

Proof of Proposition 7.1. Let $[F_k]$ be an end with a singleton impression. Assume that $[F_k]$ is not a prime end. Then there exists an end $[E_k]$ dividing $[F_k]$ such that $[F_k]$ does not divide $[E_k]$. It follows that there is l such that for each n there is $m_n \ge n$ with $F_{m_n} \setminus E_l \ne \varnothing$. By the nested property of the chain $\{F_k\}_{k=1}^\infty$ we get that $F_k \setminus E_l \ne \varnothing$ for all k. From this we infer that for all k there exists $y_k \in F_k \setminus E_l$.

As $[E_k]$ divides $[F_k]$, for every k there exists $j_k \ge l+1$ such that $E_{j_k} \subset F_k$. Let $x_k \in E_{j_k}$ be arbitrary. Then $x_k \in F_k \cap E_{l+1}$ and $y_k \in F_k \setminus E_l$. As F_k is connected, Remark 7.3 implies that

$$\operatorname{dist}(\Omega \cap \partial E_{l+1}, \Omega \cap \partial E_l) \le \operatorname{diam} F_k \to 0 \quad \text{as } k \to \infty,$$

by Proposition 7.2. Thus $\operatorname{dist}(\Omega \cap \partial E_{l+1}, \Omega \cap \partial E_l) = 0$, contradicting the fact that $\{E_k\}_{k=1}^\infty$ is a chain. □

For Mod_p-prime ends we have the following result.

prop-single-Modp

Proposition 7.4. *If* $[E_k]$ *is an end with singleton impression* $I[E_k] = x$ *and* $1 \le p \in Q(x)$, *then* $[E_k]$ *is a* Mod_p*-prime end.*

Proof. By Proposition 7.2, $\operatorname{diam} E_k \to 0$ as $k \to \infty$, and thus $[E_k]$ is a Mod_p-end by Lemma A.2. Moreover, Proposition 7.1 shows that $[E_k]$ is a prime end, and hence it is a Mod_p-prime end. □

deff-access-pt

Definition 7.5. We say that a point $x \in \partial\Omega$ is an *accessible* boundary point if there is a (possibly nonrectifiable) curve $\gamma : [0,1] \to X$ such that $\gamma(1) = x$ and $\gamma([0,1)) \subset \Omega$.

Moreover, if $[E_k]$ is an end and there is a curve γ as above such that for every k there is $0 < t_k < 1$, with $\gamma([t_k,1)) \subset E_k$, then $x \in \partial\Omega$ is *accessible through* $[E_k]$.

lem-curve-imp-prime-end

Lemma 7.6. *Let* $\gamma : [0,1] \to X$ *be a curve such that* $\gamma([0,1)) \subset \Omega$ *and* $\gamma(1) = x \in \partial\Omega$. *Let also* $\{r_k\}_{k=1}^\infty$ *be a strictly decreasing sequence converging to zero as* $k \to \infty$. *Then there exist a sequence* $\{\delta_k\}_{k=1}^\infty$ *of positive numbers smaller than* 1 *and a prime end* $[F_k]$ *such that* $I[F_k] = \{x\}$, $\gamma([\delta_k,1)) \subset F_k$ *and* F_k *is a connected component of* $\Omega \cap B(x,r_k)$ *for all* $k = 1,2,\dots$. *If* $1 \le p \in Q(x)$, *then this prime end is also a* Mod_p*-prime end.*

Proof. Note first that by the continuity of γ, for each $k = 1, 2, \dots$, there exists $0 < \delta_k < 1$ such that

$$\gamma([\delta_k, 1)) \subset \Omega \cap B(x, r_k).$$

Let F_k be the connected component of $\Omega \cap B(x, r_k)$ containing $\gamma(\delta_k)$. It follows directly that $\gamma([\delta_k, 1)) \subset F_k$ and hence $x \in \overline{F}_k$, showing that F_k is an acceptable set. Also, by construction, $F_{k+1} \subset F_k$ for all $k = 1, 2, \dots$.

Since $\Omega \cap \partial F_k \subset \partial B(x, r_k)$, it follows that for all $k = 1, 2, \dots$,

$$\operatorname{dist}(\Omega \cap \partial F_k, \Omega \cap \partial F_{k+1}) > 0.$$

Also, as $F_k \subset B(x, r_k)$, we have that $I[F_k] = \{x\}$.

Finally, Proposition 7.1 implies that $[F_k]$ is a prime end. Moreover, if $1 \le p \in Q(x)$, then by Proposition 7.4 it is also a Mod_p-prime end. □

access1 **Corollary 7.7.** *Let x be an accessible boundary point of Ω. Then there is a prime end $[F_k]$ with $I[F_k] = \{x\}$. If $p \le Q(x)$, then this prime end is a Mod_p-prime end.*

-prime-end-iff-diam-0 **Proposition 7.8.** *Let $[E_k]$ be an end and $x \in I[E_k]$ be accessible through $[E_k]$. Then the following are equivalent:*

it-prime (a) *$[E_k]$ is a prime end;*

it-x (b) $I[E_k] = \{x\}$;

it-diam (c) $\lim_{k\to\infty} \operatorname{diam} E_k = 0$.

If $1 \le p \in Q(x)$, then also the following statement is equivalent to the statements above:

it-Modp (d) *$[E_k]$ is a Mod_p-prime end.*

Proof. (c) ⇒ (b) This also follows from Proposition 7.2, as $x \in I[E_k]$ by assumption.

(b) ⇒ (a) This follows from Proposition 7.1.

(a) ⇒ (c) As x is accessible through $[E_k]$, there exists a curve $\gamma : [0, 1] \to X$ and an increasing sequence of positive numbers $t_k \to 1$, as $k \to \infty$, such that $\gamma(1) = x$ and for $k = 1, 2 \dots$, $\gamma([t_k, 1)) \subset E_k$. Lemma 7.6 with e.g. $r_k = 2^{-k}$ provides us with a prime end $[F_k]$ such that $I[F_k] = \{x\}$ and $\gamma([\delta_k, 1)) \subset F_k$ for some $0 < \delta_k < 1$, $k = 1, 2, \dots$.

We shall show that $[F_k]$ divides $[E_k]$. If not, then there exists k such that for every $l \ge k + 1$ there is a point $x_l \in F_l \setminus E_k$. Since $t_j \to 1$ as $j \to \infty$, for every $l \ge k+1$ we can find $j_l \ge l+1$ such that $t_{j_l} \ge \delta_l$ and hence $y_l := \gamma(t_{j_l}) \in E_{j_l} \subset E_{k+1}$. As $x_l \notin E_k$ and $y_l \in E_{k+1}$, Remark 7.3 yields

$$\operatorname{dist}(\Omega \cap \partial E_k, \Omega \cap \partial E_{k+1}) \le \operatorname{diam} F_l \to 0 \quad \text{as } l \to \infty,$$

which contradicts the definition of chains.

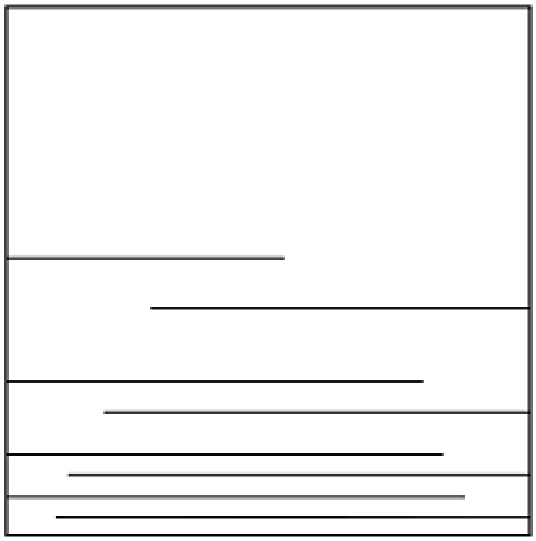
Hence, $[F_k]$ divides $[E_k]$, and as $[E_k]$ is a prime end, it follows that $[E_k] = [F_k]$, and in particular, $\operatorname{diam} E_k \to 0$ as $k \to \infty$.

Let us finally assume that $1 \le p \in Q(x)$.

(b) ⇒ (d) This follows from Proposition 7.4.

(d) ⇒ (a) This follows from Lemma 6.3 (b). □

The following example shows that the assumption of accessibility is essential in Proposition 7.8.



maze **Example 7.9.** Let $\Omega \subset \mathbb{R}^2$ be the domain obtained from the unit square $(0,1) \times (0,1)$ by removing the line segments $(0, 1-1/k) \times \{1/k\}$ for positive even integers k and removing the line segments $(1/k, 1) \times \{1/k\}$ for odd integers $k \geq 3$. The end $[E_k]$ given by $E_k = (0,1) \times (0, 1/(k+2))$ **[7.2: E_k changed. Ok? /A]** is a Mod_p-prime end for all $1 \leq p < \infty$ with the impression $I[E_k] = [0,1] \times \{0\}$ and $\lim_{k\to\infty} \operatorname{diam} E_k = 1$.

In fact, we have the following result.

prop1A-A **Proposition 7.10.** *If $[E_k]$ is an end and $I[E_k] = \{x\}$, then x is accessible through $[E_k]$.*

Proof. By Remark 4.5, we can assume that each E_k is open. As X is quasiconvex **[7.3: We should define and talk about this at some point, but let us decide where later. /A]**, Lemma 4.38 in Björn–Björn [10] then implies that E_k is pathconnected. Choose $x_k \in E_k \setminus E_{k+1}$ for $k = 1, 2, \ldots$. Since both x_k and x_{k+1} belong to E_k and E_k is pathconnected, there exists a curve $\gamma_k : [1-1/k, 1-1/(k+1)] \to E_k$ connecting x_k to x_{k+1}. Let γ be the union of all these curves. More precisely, let $\gamma : [0,1] \to X$ be given by $\gamma(t) = \gamma_k(t)$ if $t \in [1-1/k, 1-1/(k+1)]$, $k = 1, 2, \ldots$, and $\gamma(1) = x$. As $\operatorname{diam} E_k \to 0$, γ is continuous at 1, and hence x is accessible along γ through $[E_k]$. □

cor1A **Corollary 7.11.** *If $[E_k]$ is a prime end and $x \in I[E_k]$, then the following are equivalent:*

ii-x (a) $I[E_k] = \{x\}$;
ii-acc (b) *x is accessible through* $[E_k]$;
ii-diam (c) $\lim_{k\to\infty} \operatorname{diam} E_k = 0$.

Proof. (a) ⇒ (b) This follows directly from Proposition 7.10.
(b) ⇒ (c) This follows from Proposition 7.8.
(c) ⇒ (a) This follows from Proposition 7.2 as $x \in I[E_k]$. □

cor-access-equiv-end **Corollary 7.12.** *Let $x \in \partial\Omega$. Then the following are equivalent:*

i2-acc (a) *x is accessible;*
i2-end (b) *there is an end $[E_k]$ with $I[E_k] = \{x\}$;*
i2-prime-end (c) *there is a prime end $[E_k]$ with $I[E_k] = \{x\}$.*

If $1 \le p \in Q(x)$, then also the following statements are equivalent to the statements above:

i2-Modp (d) *there is a* Mod_p*-end* $[E_k]$ *with* $I[E_k] = \{x\}$;

i2-Modp-prime (e) *there is a* Mod_p*-prime end* $[E_k]$ *with* $I[E_k] = \{x\}$.

Proof. (a) ⇒ (c) This follows from Corollary 7.7.

(c) ⇒ (b) This is trivial.

(b) ⇒ (a) This follows from Proposition 7.10.

Let us finally assume that $1 \le p \in Q(x)$.

(b) ⇒ (e) This follows from Proposition 7.4

(e) ⇒ (d) ⇒ (b) These implications are trivial. □

8. The topology on ends and prime ends

sect-top

We would like to find homeomorphisms between $\partial_P\Omega$ and other boundaries. To do so we need to have a topology on $\partial_P\Omega$, or really on $\Omega \cup \partial_P\Omega$. Let us be a little more general and introduce a topology on $\Omega \cup \partial_E\Omega$. It then naturally induces a topology on $\Omega \cup \partial_P\Omega$ and also on the boundaries connected with Mod_p-prime ends introduced in Section 6.

seqtoprime **Definition 8.1.** We say that a *sequence of points* $\{x_n\}_{n=1}^\infty$ in Ω *converges to the end* $[E_k]$, and write $x_n \to [E_k]$, as $n \to \infty$, if for all k there exists n_k such that $x_n \in E_k$ whenever $n \ge n_k$.

[8.1: I have avoided using $\lim x_n$ as the limits here need not be unique. See also comment to the reader below. /A]

Observe that if $x_n \to [E_k]$ and $[E_k]$ divides $[F_k]$, then x_n also converges to $[F_k]$. Thus the limit of a sequence need not be unique, and we therefore avoid writing $\lim_{n\to\infty} x_n$. It is less obvious that this problem remains even if we restrict our attention to prime ends, see Example 8.6 below.

conv **Definition 8.2.** We say that a *sequence of ends* $\{[E_k^n]\}_{n=1}^\infty$ *converges to the end* $[E_k^\infty]$ if for every k, there is n_k such that for each $n \ge n_k$ there exists $l_{n,k}$ such that $E_{l_{n,k}}^n \subset E_k^\infty$.

Note that the integers n_k and $l_{n,k}$ in Definitions 8.1 and 8.2 depend on the representative chain of the corresponding ends. However, both notions of convergence are independent of the choice of representative chain.

Definition 8.3. Convergence of points and ends defines a topology on $\Omega \cup \partial_E\Omega$ by saying that a family $C \subset \Omega \cup \partial_E\Omega$ of points and ends is *closed* if whenever (a point or an end) $y \in \Omega \cup \partial_E\Omega$ is a limit of a sequence in C, then $y \in C$.

Here, a sequence $\{x_n\}_{n=1}^\infty$ of points in Ω converges to a point $y \in \Omega$ as given by the metric space.

It is not hard to verify that the open sets in the topology are given as $G_1 \cup G_2^E$, where G_1 and G_2 are open subsets of Ω and $G_2^E \subset \Omega \cup \partial_E\Omega$ is the union of G_2 and all the ends $[E_k]$ such that $E_k \subset G_2$ for some k.

Theorem 8.4. *The topology defined above is indeed a topology on* $\Omega \cup \partial_E\Omega$.

Proof. (1) The empty set and the collection of all ends are clearly closed.

(2) Let C_1 and C_2 be closed subsets of $\Omega \cup \partial_E\Omega$. Assume that $\{y_n\}_{n=1}^\infty$ is a sequence in $C_1 \cup C_2$ such that $y_n \to y_\infty$. There is either a subsequence, y_{n_k} in C_1, or else a subsequence y_{n_k} lies in C_2. Since a subsequence of a convergent sequence converges to the same limit, it follows that $y_\infty \in C_1$ or $y_\infty \in C_2$. Hence

$y_\infty \in C_1 \cup C_2$. By induction, for any positive integer N we have that $\bigcup_{n=1}^N C_n$ is closed whenever $C_1, \ldots, C_N$ are closed.

(3) Now let $\{C_i\}_{i\in\mathcal{I}}$ be a collection of closed subsets of $\Omega \cup \partial_E\Omega$. Consider a sequence $\{y_n\}_{n=1}^\infty \in \bigcap_{i\in\mathcal{I}} C_i$. If $y_n \to y_\infty$, as $n \to \infty$, then, since the C_i are closed, $y_\infty \in C_i$ for all $i \in \mathcal{I}$. Therefore, $y_\infty \in \bigcap_{i\in\mathcal{I}} C_i$ and the intersection is closed. □

If E and F are two distinct ends such that E divides F, then any neighbourhood of F contains E, and thus the topology does not satisfy the T1 separation condition.

If we however restrict ourselves to prime ends, i.e. to $\Omega \cup \partial_P\Omega$, then the T1 separation condition is satisfied.

Proposition 8.5. *The topology on $\Omega \cup \partial_P\Omega$ satisfies the T1 separation condition.*

Proof. If $x \in \Omega$, then $\{x\}$ is closed in our topology. Thus to verify the T1 separation condition we need to show that $\{P\}$ is closed for any prime end P. We thus need to consider the sequence $\{P_n\}_{n=1}^\infty$, with $P_n = P$ for all n. Assume that $P_n \to P_\infty$ as $n \to \infty$, where P_∞ is a prime end. As the sequence is constant it is not hard to see that P must divide P_∞. Since P_∞ is a prime end, we thus must have $P = P_\infty$. Hence $\{P\}$ is closed. □

The topology obtained on $\Omega \cup \partial_P\Omega$ does not need to satisfy the T2 separation condition, and can thus be nonmetrizable, as shown by the following example. In Section **[8.2: label! We have to use labels! /A]** we will study a condition under which this topology is metrizable.

ex-Jana-two-limits **Example 8.6.** Let $\Omega \subset \mathbb{R}^3$ be obtained from removing the following 2-dimensional sets from the cube $(-1,1)\times(0,2)\times(0,2)$:

$$\begin{aligned} &[-1,-1/(2k+1)] \times \{1/(2k+1)\} \times [0,2] \\ \text{and}\quad &[1/(2k+1),1] \times \{1/(2k+1)\} \times [0,2] \text{ for } k=1,2,..., \end{aligned}$$

and

$$[1/2k-1, 1-1/2k] \times \{1/2k\} \times [1/2k, 2] \text{ for } k = 2,4,6,...$$

and

$$[1/2k-1, 1-1/2k] \times \{1/2k\} \times [0, 2-(1/2k)] \text{ for } k = 1,3,5,...$$

There are two prime ends, with impressions $[-1,0]\times\{0\}\times\{1\}$ and $[0,1]\times\{0\}\times\{1\}$, but the sequence $\{(0,1/n,1)\}_{n=1}^\infty$ converges to both of them.

It follows that any neighbourhood of any of these two prime ends contains all but a finite number of points from this sequence. Hence these two prime ends do not have disjoint neighbourhoods, or in other terms the T2 separation condition fails.

Observe that Ω is simply connected.

[8.3: I've moved the following convergence discussion. I'm not sure we're we should have it. Do we really need it? Should we delete it? /A]

Definition 8.2 implies that there are sequences $\{x_i^n\}_{i=1}^\infty$ in Ω, $n = 1,2,\ldots$, and a sequence $\{x_i^\infty\}_{i=1}^\infty$ in Ω with the properties:

(1) $x_i^n \to [E_k^n]$, as $i \to \infty$;

(2) $x_i^\infty \to [E_k^\infty]$, as $i \to \infty$;

(3) $\limsup_{n\to\infty} \limsup_{i\to\infty} d(x_i^n, x_i^\infty) = 0$. **[8.4: Why? /A]**

However, even with the additional assumption that $\operatorname{diam}(E_k) \to 0$, this sequential criterion does not imply convergence of prime ends — consider e.g. the slit disc, see Example 5.2, and let x_i^n converge to a point on the slit from one side and x_i^∞ from the other side.

Instead we should use the Mazurkiewicz distance.

Definition 8.7. We define the *Mazurkiewicz distance* d_M on Ω by

$$d_M(x,y) = \inf \operatorname{diam} E,$$

where the infimum is over all connected sets $E \subset \Omega$ containing $x, y \in \Omega$.

Lemma 8.8. *The sequence of ends $\{[E_k^n]\}_{n=1}^\infty$ converges to the end $[E_k^\infty]$ with $\lim_{k\to\infty} \operatorname{diam}(E_k^\infty) = 0$ if and only if whenever $\{x_i^n\}_{i=1}^\infty$, $n = 1, 2, \ldots$, and $\{x_i^\infty\}_{i=1}^\infty$ are sequences in Ω such that*
(a) *$x_i^n \to [E_k^n]$, as $i \to \infty$;*
(b) *$x_i^\infty \to [E_k^\infty]$, as $i \to \infty$;*
we must have

$$\limsup_{n\to\infty} \limsup_{i\to\infty} d_M(x_i^n, x_i^\infty) = 0.$$

Proof. Assume that a sequence of ends $\{[E_k^n]\}_{n=1}^\infty$ converges to the end $[E_k^\infty]$. Let $\{x_i^n\}_{i=1}^\infty$ and $\{x_i^\infty\}_{i=1}^\infty$ converge to $[E_k^n]$ and $[E_k^\infty]$, respectively. By the definition of convergence of a sequence we have that for each k and n there exists $m_{k,n}$ such that for all $i \geq m_{k,n}$ we have $x_i^n \in E_k^n$. Similarly there is N_k such that for $j \geq N_k$ we have that $x_j^\infty \in E_k^\infty$. The definition of convergence of a sequence of ends to an end implies that there exists n_k such that for all $n > n_k$ we can find $l_{n,k}$ with the property that if $l > l_{n,k}$, then $E_l^n \subset E_k^\infty$. We may take $l_{n,k} \geq \max\{m_{k,n}, N_k\}$ and therefore there are $x_l^n \in E_l^n \subset E_k^\infty$ and $x_l^\infty \in E_k^\infty$ such that $d_M(x_l^n, x_l^\infty) \leq \operatorname{diam}(E_k^\infty)$ whenever $l \geq l_{n,k}$. Since $\operatorname{diam}(E_k^\infty) \to 0$ we have that

$$0 \leq \limsup_{n\to\infty} \limsup_{l\to\infty} d_M(x_l^n, x_l^\infty) \leq \operatorname{diam}(E_k^\infty) \to 0.$$

To prove the opposite implication we proceed by reductio ad absurdum. Let $\{x_i^\infty\}_{i=1}^\infty$ converge to an end $[E_k^\infty]$ and assume $\{[E_k^n]\}_{n=1}^\infty$ does not converge to the end $[E_k^\infty]$. Then there exists k_0 such that for all n there is $m_n \geq n$ with the property that $E_l^{m_n} \setminus E_{k_0}^\infty \neq \varnothing$ for all l. For each positive integer n we can construct a sequence $\{x_i^n\}_{i=1}^\infty$ by choosing $x_i^n \in E_i^n \setminus E_{k_0}^\infty$ if this set is not empty and $x_i^n \in E_i^n$ otherwise. Clearly $\lim_{i\to\infty} x_i^n = [E_k^n]$, and so by hypothesis we must have

$$\limsup_{n\to\infty} \limsup_{i\to\infty} d_M(x_i^n, x_i^\infty) = 0,$$

contradicting the assumption that $E_l^{m_n} \setminus E_{k_0}^\infty \neq \varnothing$, since then for large i we have that $d_M(x_i^\infty, x_i^l) > c > 0$ for a constant c depending on $\operatorname{diam}(E_{k_0}^\infty)$ **[8.5: Why? /A]**. Furthermore, to see that $\operatorname{diam}(E_k^\infty) \to 0$, we notice that otherwise $\limsup_{k\to\infty} \operatorname{diam}(E_k^\infty) > 0$. We can choose sequences $\{x_k^\infty\}_{k=1}^\infty$ and $\{y_k^\infty\}_{k=1}^\infty$ such that $x_k^\infty, y_k^\infty \in E_k^\infty$ and $\operatorname{dist}(x_k^\infty, y_k^\infty) \geq \frac{1}{2} \operatorname{diam}(E_k^\infty)$. For each positive integer n take any sequence $\{a_k^n\}_{k=1}^\infty$ with $a_k^n \in E_k^n$. With such a choice of sequences we have by the triangle inequality that

$$\limsup_{n\to\infty} \limsup_{k\to\infty} d_M(a_k^n, x_k^\infty) \neq 0 \quad \text{or} \quad \limsup_{n\to\infty} \limsup_{k\to\infty} d_M(a_k^n, y_k^\infty) \neq 0,$$

resulting in a contradiction. □

9. Prime ends and the Mazurkiewicz boundary

sect-Mazur

We now focus on describing the relations between the prime end boundary and two other boundaries, the topological boundary and the Mazurkiewicz boundary. Our investigations are motivated by the fact that having established homeomorphism or embedding between two kinds of boundaries one may discuss the correspondence between ends (or prime ends) and their impressions.

Let us mention that in Björn–Björn–Shanmugalingam [13] the Dirichlet problem for p-harmonic functions with respect to the Mazurkiewicz boundary is studied in domains which are finitely connected at the boundary (see the next section for the definition of finite connectivity on the boundary). By Theorem 10.8 this is equivalent to studying the Dirichlet problem with respect to the prime end boundary for such domains. We refer to [13] for further details on the Dirichlet problem, but this is another important motivation for this and the next section.

The discussion in the previous section indicates that path accessibility of a boundary point determines whether there is a prime end with a singleton impression. Motivated by this, we consider the following Mazurkiewicz metric associated with the connectedness properties of the domain at the boundary points. Recall also from the previous section that if $[E_k]$ is a singleton end, then it is always a prime end, and moreover, if $1 \le p \in Q(x)$, then it is also automatically a Mod_p-prime end.

innerDiam

Definition 9.1. We define the *Mazurkiewicz distance* d_M on Ω by

$$d_M(x,y) = \inf \operatorname{diam} E,$$

where the infimum is over all connected sets $E \subset \Omega$ containing $x, y \in \Omega$.

The completion of the metric space (Ω, d_M) is denoted $\overline{\Omega}^M$ and d_M extends in the usual way to $\overline{\Omega}^M$. Namely, for d_M-Cauchy sequences $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \in \Omega$ we define the equivalence relation

$$\{x_n\}_{n=1}^\infty \sim \{y_n\}_{n=1}^\infty \quad \text{if} \quad \lim_{n\to\infty} d_M(x_n, y_n) = 0.$$

The collection of all equivalence classes of d_M-Cauchy sequences can be formally considered to be $\overline{\Omega}^M$, but we will identify equivalence classes of d_M-Cauchy sequences that have a limit in Ω with that limit point. By considering equivalence classes of d_M-Cauchy sequences without limits in Ω we define the boundary of Ω with respect to d_M as $\partial_M\Omega = \overline{\Omega}^M \setminus \Omega$. Because X is quasiconvex, we know that Ω is locally compact; it follows that Ω is an open subset of $\overline{\Omega}^M$. We extend the original metric d_M on Ω to $\overline{\Omega}^M$ by letting

$$d_M(x^*, y^*) = \lim_{n\to\infty} d_M(x_n, y_n),$$

if $x^* = \{x_n\}_{n=1}^\infty \in \overline{\Omega}^M$ and $y^* = \{y_n\}_{n=1}^\infty \in \overline{\Omega}^M$. This is well defined and an extension of d_M.

preserveLength

Remark 9.2. Clearly, d_M is a metric on Ω. When $x, y \in \Omega$, we have $d_M(x,y) \ge d(x,y)$. Observe that if $B(x,r) \subset \Omega$, then by the L-quasiconvexity of X, we have for all $y \in B(x, r/L)$ that $d_M(x,y) \le Ld(x,y)$. Thus, d_M and d are locally biLipschitz equivalent in Ω and define the same topology inside Ω.

By Remark 9.2, every point in Ω can be identified with exactly one equivalence class of d_M-Cauchy sequences in Ω. This is, of course, not true on the boundary of Ω in general.

lem-M-bdry-to-bdry

Lemma 9.3. *There is a continuous map* $\Psi : \partial_M\Omega \to \partial\Omega$.

In fact, there is a continuous map $\Psi : \overline{\Omega}^M \to \overline{\Omega}$ *such that* $\Psi|_\Omega$ *is the identity map and* $\Psi|_{\partial_M\Omega} : \partial_M\Omega \to \partial\Omega$.

This mapping need not be surjective nor injective in general, as demonstrated by the topologist's comb considered in Example 5.1 and the slit disc considered in Example 5.2, respectively.

Proof. Let $\{x_n\}_{n=1}^\infty$ be a d_M-Cauchy sequence in Ω representing a point in $\overline{\Omega}^M$. Since $d(x_i, x_j) \le d_M(x_i, x_j)$, it follows that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in the given metric d **[9.1: We say given metric here as in our other papers. /A]** as well, and so by the completeness of X, we can set

$$\Psi\left(\{x_n\}_{n=1}^\infty\right) := \lim_{n\to\infty} x_n \in \overline{\Omega}.$$

The map Ψ is well defined, since every sequence representing the same point in $\overline{\Omega}^M$ converges to the same limit in the given metric d.

To prove the continuity of Ψ, consider $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \in \overline{\Omega}^M$ and let $x = \Psi\left(\{x_n\}_{n=1}^\infty\right)$ and $y = \Psi\left(\{y_n\}_{n=1}^\infty\right)$. Then by definition we have that

$$d_M(\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty) = \lim_{n\to\infty} d_M(x_n, y_n) \ge \lim_{n\to\infty} d(x_n, y_n) = d(x, y).$$

Therefore $d(\Psi(\{x_n\}_{n=1}^\infty), \Psi(\{y_n\}_{n=1}^\infty)) \le d_M(\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty)$, that is, Ψ is 1-Lipschitz continuous. □

Next, we show that under rather general assumptions, the prime end boundary and the Mazurkiewicz boundary coincide. Recall Remark 7.3. **[9.2: Should the last sentence be deleted? It doesn't make sense to me to have it here. /A]**

thm-homeo-primeends

Theorem 9.4. *Assume that every prime end in* Ω *has a singleton impression. Then there is a homeomorphism* $\Phi : \partial_P\Omega \to \partial_M\Omega$.

This is a special case of the following result.

thm-homeo-primeends-2

Theorem 9.5. *Let* $\partial_{SP}\Omega$ *be the set of all singleton ends* (*which are automatically prime ends by Proposition* 7.1).

Then there is a homeomorphism $\Phi : \Omega \cup \partial_{SP}\Omega \to \overline{\Omega}^M$ *such that* $\Phi|_\Omega$ *is the identity map and* $\Phi|_{\partial_{SP}\Omega} : \partial_{SP}\Omega \to \partial_M\Omega$.

Recall that by Proposition 7.2 an end $[E_k]$ has a singleton impression if and only if $\lim_{k\to\infty} \operatorname{diam} E_k = 0$. We will use this fact (implicitly) several times in the proof below.

Proof. Step 1. *Definition of* Φ. Let $[E_k] \in \partial_{SP}\Omega$. For each k choose $x_k \in E_k$. Then for $l \ge k$ we have that $x_k, x_l \in E_k$ and as E_k is connected, this implies that

$$d_M(x_k, x_l) \le \operatorname{diam} E_k \to 0, \quad \text{as } k \to \infty.$$

Thus, $\{x_k\}_{k=1}^\infty$ is a d_M-Cauchy sequence and corresponds to a point $y \in \overline{\Omega}^M$. If y belonged to Ω, then we would have $y \in \bigcap_{k=1}^\infty \overline{E}_k = I[E_k]$, which is a contradiction. Thus $y \in \partial_M\Omega$, and we define

$$\Phi([E_k]) = y.$$

For $x \in \Omega$ we, of course, define $\Phi(x) = x$.

Step 2. *Φ is well defined.* **[9.3: This step had two parts. However, the first was a special case of the second, hence I deleted it. Ok? /A]** To see this, we assume that $\{E_k\}_{k=1}^\infty \sim \{E'_k\}_{k=1}^\infty$ are equivalent chains and let $x_k \in E_k$ and $x'_k \in E'_k$, $k = 1, 2, \ldots$. Then for every k, there exists l_k such that $E_{l_k} \subset E'_k$. Hence for all $j \geq k$ and $l \geq l_k$, we have that $x_l \in E_l \subset E_{l_k} \subset E'_k$ and $x'_j \in E'_j \subset E'_k$. Thus

$$d_M(x_l, x'_j) \leq \operatorname{diam} E'_k \to 0, \quad \text{as } k \to \infty,$$

and it follows that $\{x_k\}_{k=1}^\infty$ and $\{x'_k\}_{k=1}^\infty$ are equivalent as d_M-Cauchy sequences. Hence Φ is well-defined.

Step 3. *Φ is surjective.* Let $\{x_n\}_{n=1}^\infty$ be a d_M-Cauchy sequence in Ω, corresponding to a point in $\partial_M \Omega$. We can assume that for all $j, k \geq n$,

$$d(x_j, x_k) \leq d_M(x_j, x_k) < 2^{-n-1}. \tag{9.1}$$

eq-surjective

It follows that $\{x_n\}_{n=1}^\infty$ is a d-Cauchy sequence and converges to some $x \in \partial\Omega$. Moreover,

$$d(x_k, x) \leq 2^{-k-1}. \tag{9.2}$$

eq-x_k-x

For each $k = 1, 2, \ldots$, let E_k be the connected component of $\Omega \cap B(x, 2^{-k})$ containing x_k. Then for all $j \geq k$, (9.1) implies that there exists a connected set $F \subset \Omega$ such that $x_j, x_k \in F$ and $\operatorname{diam} F < 2^{-k-1}$. Using (9.2), it follows that $F \subset \Omega \cap B(x, 2^{-k})$ and as F is connected and $x_k \in E_k$, we obtain that $F \subset E_k$ and $x_j \in E_k$ for all $j \geq k$. Letting $j \to \infty$ shows that $x \in \overline{E}_k$ for $k = 1, 2, \ldots$.

This also shows that $x_{k+1} \in E_k$ and as E_{k+1} is connected, we obtain that $E_{k+1} \subset E_k$ for all k.

Since $\Omega \cap \partial E_k \subset \partial B(x, 2^{-k})$, we obtain that

$$\operatorname{dist}(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k) \geq 2^{-k-1} > 0.$$

By construction we know that $\operatorname{diam} E_k \to 0$, and hence $\{E_k\}_{k=1}^\infty$ is a chain with impression $\{x\}$. By Proposition 7.1, $[E_k]$ is a prime end. Moreover, $\Phi([E_k]) = \{x_n\}_{n=1}^\infty$. Thus Φ is surjective. (That $\Phi|_\Omega$ is bijective is clear.)

Step 4. *Φ is injective.* Let $[E_k]$ and $[F_k]$ be two distinct singleton prime ends. So $\{F_k\}_{k=1}^\infty$ does not divide $\{E_k\}_{k=1}^\infty$. Hence, there exists k such that for each l we can find a point $y_l \in F_l \setminus E_k$. We need to show that $\{y_l\}_{l=1}^\infty$ is not equivalent to any sequence representing $\Phi([E_k])$. Let $x_n \in E_n$ for each n, and assume that $\{x_n\}_{n=1}^\infty \sim \{y_l\}_{l=1}^\infty$. Then for all sufficiently large $n, l \geq k+1$,

$$d_M(x_n, y_l) < \operatorname{dist}(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k),$$

and so there exist connected sets $K_{n,l} \subset \Omega$ such that $x_n, y_l \in K_{n,l}$ and

$$\operatorname{diam} K_{n,l} < \operatorname{dist}(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k).$$

As $x_n \in E_{k+1} \subset E_k$ and $y_l \in \Omega \setminus E_k \subset \Omega \setminus E_{k+1}$, the sets $K_{n,l}$ must meet both $\Omega \cap \partial E_{k+1}$ and $\Omega \cap \partial E_k$. Remark 7.3 yields that

$$\operatorname{dist}(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k) \leq \operatorname{diam} K_{n,l} < \operatorname{dist}(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k),$$

which is a contradiction. Thus $\{x_n\}_{n=1}^\infty$ and $\{y_l\}_{l=1}^\infty$ cannot be equivalent, and Φ is injective.

Step 5. *Φ is continuous.* **[9.4: Is it clear that continuity can be shown by sequential continuity? Similarly for Φ^{-1}. /A]** As $\Phi|_\Omega$ is continuous, it is enough to show that the image of every sequence with a limit in $\partial_{\mathrm{SP}}\Omega$ has the right limit. There are two such types of sequences we need to consider.

Assume first that the sequence of singleton prime ends $\{[E_k^n]\}_{n=1}^\infty$ converges to the singleton prime end $[E_k^\infty]$. This means that for each k there exists n_k such that whenever $n \ge n_k$ we can find $l_{n,k} \ge k$ so that $E^n_{l_{n,k}} \subset E^\infty_k$. Let $\Phi([E^n_k]) = \{x^n_k\}_{k=1}^\infty$ and $\Phi([E^\infty_k]) = \{x^\infty_k\}_{k=1}^\infty$.

If $\varepsilon > 0$, then there exists k such that $\operatorname{diam} E^\infty_k < \varepsilon$. Then for all $m \ge n \ge n_k$ we have $x^m_{l_{n,k}} \in E^n_{l_{n,k}} \subset E^\infty_k$ and $x^\infty_{l_{n,k}} \subset E^\infty_{l_{n,k}} \subset E^\infty_k$. Hence

$$d_M(x^m_{l_{n,k}}, x^\infty_{l_{n,k}}) \le \operatorname{diam} E^\infty_k < \varepsilon.$$

This shows that $\{\Phi([E^n_k])\}_{n=1}^\infty$ converges in d_M to $\Phi([E^\infty_k])$ as $n \to \infty$.

Assume next that $\Omega \ni y_n \to [E_k] \in \partial_{\mathrm{SP}}\Omega$, as $n \to \infty$. Thus for each k there is n_k such that $y_n \in E_k$ if $n \ge n_k$. As E_k is connected we see that

$$d_M(y_l, y_m) \le \operatorname{diam} E_k \quad \text{if } l, m \ge n_k.$$

Since $\operatorname{diam} E_k \to 0$, as $k \to \infty$, this shows that $\{y_n\}_{n=1}^\infty$ is a d_M-Cauchy sequence. Letting $x_k = y_{n_k}$ shows that $\Phi([E_k]) = \{x_k\}_{k=1}^\infty \sim \{y_n\}_{n=1}^\infty$, which is the limit of $\{y_n\}_{n=1}^\infty$ in $\overline{\Omega}^M$.

Thus we conclude that Φ is continuous.

Step 6. Φ^{-1} *is continuous.* As in Step 5 there are two types of sequence we need to consider.

Assume first that the sequence of singleton prime ends $\{[E^n_k]\}_{n=1}^\infty$ does not converge to the singleton prime end $[E^\infty_k]$. This means that there exists k and an increasing sequence $n_i \to \infty$ (depending on k) such that for all $i, l = 1, 2, \dots$, we have $E^{n_i}_l \not\subset E^\infty_k$. For $l \ge k+1$, choose $x^{n_i}_l \in E^{n_i}_l \setminus E^\infty_k$ and $x^\infty_l \in E^\infty_l \subset E^\infty_{k+1}$. Remark 7.3 implies that for all $i = 1, 2, \dots$ and $l \ge k+1$,

$$\delta := \operatorname{dist}(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k) \le d_M(x^{n_i}_l, x^\infty_l).$$

It follows that for all $i = 1, 2, \dots$,

$$d_M(\Phi([E^{n_i}_l]), \Phi([E^\infty_l])) \ge \delta > 0.$$

Thus, $\Phi([E^{n_i}_l])$ cannot converge to $\Phi([E^\infty_l])$ as $i \to \infty$, and hence the sequence $\{\Phi([E^n_k])\}_{n=1}^\infty$ does not converge to $\Phi([E^\infty_k])$ either.

Assume next that $\{y_n\}_{n=1}^\infty$ is a sequence of points in Ω which does not converge to the singleton prime end $[E^\infty_k]$. **[9.5: Details need to be added. /A]**

This shows that Φ^{-1} is continuous and so Φ is a homeomorphism. □

10. Finitely connected domains

sect-finconn

In general not all prime ends have singleton impressions. In this and the next section we explore conditions under which all prime ends have this property.

Here we present a condition which guarantees that all prime ends have singleton impressions, and moreover is equivalent to compactness of the prime end closure $\overline{\Omega}^P := \Omega \cup \partial_P\Omega$ in this case, see Theorem 10.10. (See the topologist comb in Example 5.1 for an example when $\overline{\Omega}^P$ is not compact. Observe that all the prime ends have singleton impression in this case example.)

Definition 10.1. We say that Ω is *finitely connected* at x_0 if for every $r > 0$ there is an open set G (open in X) such that $x_0 \in G \subset B(x_0, r)$ and $G \cap \Omega$ has only finitely many components.

The terminology above follows Näkki [48] who seems to have first used it in print (for $\mathbb{R}^n$). (Näkki [50] has informed us that he learned about the terminology from Väisälä, who however first seems to have used it in print in [58].)

Let us introduce some further notation. Fix $x_0 \in \partial\Omega$ (we do *not* assume that Ω is finitely connected yet). For each $r>0$ let $\{G_j(r)\}_{j=1}^{N(r)}$ be the components of $B(x_0,r)\cap\Omega$ which have x_0 in their boundary, i.e. $x_0 \in \overline{G_j(r)}$. Here $N(r)$ is either a nonnegative integer or ∞. Let further

$$H(r) = (B(x_0,r)\cap\Omega) \setminus \bigcup_{j=1}^{N(r)} G_j(r)$$

be the union of the remaining components (if any). (The sets $G_j(r)$ and $H(r)$ of course depends on x_0.)

[10.1: There was an example here with $N(r)=1$ for all r, but still not finitely connected, that I deleted. It fits more into mbdy. /A]

The following characterization of finite connectedness is useful.

prop1-fin

Proposition 10.2. *The set Ω is finitely connected at x_0 if and only if for all $r>0$, $N(r)<\infty$ and $x_0 \notin \overline{H(r)}$.*

See Björn–Björn–Shanmugalingam [14] for a proof.

lem-ex-G-j

Lemma 10.3. *Assume that Ω is finitely connected at a boundary point x_0. Let $[E_k]$ be an end with $x_0 \in I[E_k]$. Then there exists a decreasing sequence of positive numbers $r_k < 2^{-k}$ such that for each $k=1,2,\dots$, there is a connected component $G_{j_k}(r_k)$ of $B(x_0,r_k)$ satisfying $x_0 \in \overline{G_{j_k}(r_k)}$ and $G_{j_k}(r_k) \subset E_k$.*

Proof. As $\operatorname{dist}(\Omega\cap\partial E_k, \Omega\cap\partial E_{k+1})>0$, at least one of the two distances $\operatorname{dist}(x_0,\Omega\cap\partial E_k)>0$, $\operatorname{dist}(x_0,\Omega\cap\partial E_{k+1})$ must be positive. If $\operatorname{dist}(x_0,\Omega\cap\partial E_k)>0$, then there exists $0<r_k<2^{-k}$ such that $r_k<\operatorname{dist}(x_0,\Omega\cap\partial E_k)$. Consider the connected components $G_1(r_k),\dots,G_{N(r_k)}(r_k)$ of $B(x_0,r_k)\cap\Omega$ that have x_0 in their boundary. Let $H(r_k) = \Omega\cap B(x_0,r_k)\setminus\bigcup_{j=1}^{N(r_k)} G_j(r_k)$. As Ω is finitely connected at x_0, Lemma 10.2 shows that $x_0\notin\overline{H(r_k)}$, so at least one of $G_1(r_k),\dots,G_{N(r_k)}(r_k)$ has a nonempty intersection with E_k, say $G_1(r_k)$. Since $G_1(r_k)$ is connected and $r_k<\operatorname{dist}(x_0,\Omega\cap\partial E_k)$, we must have $G_1(r_k)\subset E_k$.

If instead $\operatorname{dist}(x_0,\Omega\cap\partial E_{k+1})>0$, then we find in the same way $G_1(r_{k+1})\subset E_{k+1}\subset E_k$ and let $r_k=r_{k+1}$.

By constructing the above sequence of positive numbers r_k inductively, we can also ensure that the sequence is a decreasing sequence. □

prop-ex-chain

Proposition 10.4. *Assume that Ω is finitely connected at x_0. Let $[E_k]$ be an end with $x_0\in I[E_k]$. Then there is a sequence of positive numbers r_k that decreases to 0, and a prime end $[F_k]$ which divides $[E_k]$, such that $F_k = G_{j_k}(r_k)$ for some $1\le j_k\le N(r_k)$, and $I[F_k]=\{x_0\}$.*

If moreover $[E_k]$ is a prime end, then $[E_k]=[F_k]$ and $I[E_k]=\{x_0\}$

[10.2: The proof below needs to be rewritten, who wrote it? It is said that the metric is on the tree, but it is defined for curves on the tree. Moreover, which curves are considered is not specified, I guess curves going down the tree. Also no motivation for why the completion is compact is given. /A]

Proof. Let the decreasing sequence of positive numbers r_k be as in Lemma 10.3, and T be the tree whose vertices are the components $G_j(r_k)$ provided by Lemma 10.3. Two vertices in this tree are connected by an edge if and only if the two vertices are $G_j(r_k)$ and $G_m(r_{k+1})$ for some $k\in\mathbb{N}$ and $1\le j\le N(r_k)$, $1\le m\le N(r_{k+1})$ and $G_m(r_{k+1})\subset G_j(r_k)$.

We introduce a metric on the tree T by $t(p,q) = 2^{-n}$, where n is the level where the curves p and q branch, i.e. they have a common ancestor $G_j(r_n)$ but belong to different branches corresponding to $G_m(r_{n+1})$ and $G_j(r_{n+1})$. This is a metric on T that turns T into a bounded metric space whose completion is compact.

For each positive integer k, let P_k be the collection of all vertices on geodesic curves in T which pass through the vertex corresponding to the component $G_j(r_k) \subset E_k$. By Lemma 10.3, each P_k is nonempty. Clearly, $P_k \supset P_{k+1}$ and each P_k is closed in the above metric. It follows that there exists a curve $p \in \bigcap_{k=1}^\infty P_k$. The vertices of this curve p correspond to a chain $\{G_{j_k}(r_k)\}_{k=1}^\infty$, and by Lemma 10.3 it divides $[E_k]$. Since $\operatorname{diam} G_{j_k}(r_k) \le 2r_k \le 2^{1-k}$, the obtained end is a prime end by Proposition 7.1.

If $[E_k]$ is a prime end, then we must have $[E_k] = [F_k]$ and thus $I[E_k] = \{x_0\}$. □

Definition 10.5. If Ω is finitely connected at $x_0 \in \partial\Omega$ and $N(r) = 1$ for all sufficiently small r in the definition of finite connectedness, then Ω is *locally connected* at x_0. If Ω is finitely (or locally) connected at every boundary point, then it is called finitely (or locally) connected *at the boundary.*

The following results are direct consequences of Proposition 10.4.

Corollary 10.6. *If Ω is finitely connected at $x_0 \in \partial\Omega$, then there exists a prime end $[F_k]$ with $I[F_k] = \{x_0\}$. Furthermore, if $[E_k]$ is a prime end such that $x_0 \in I[E_k]$ then $I[E_k] = \{x_0\}$.*

[10.3: I don't see how the existence is a direct consequence. However it may follow from the proof when it has been clarified. /A]

Corollary 10.7. *If Ω is locally connected at the boundary and $[E_k]$ is a prime end in Ω, then $I[E_k] = \{x\}$ for some $x \in \partial\Omega$ and there exist radii $r_k^x > 0$, such that*

$$B(x, r_k^x) \cap \Omega \subset E_k, \quad k = 1, 2, \dots.$$

thm-fin-con-homeo

Theorem 10.8. *If Ω is finitely connected at the boundary, then there is a homeomorphism $\Phi : \overline{\Omega}^P \to \overline{\Omega}^M$ such that $\Phi|_\Omega$ is the identity map. Furthermore, if*

$$1 \le p \in Q(\partial\Omega) := \{q : q \in Q(x) \text{ for all } x \in \partial\Omega\},$$

then $\partial_P\Omega$ is also the Mod_p*-prime end boundary.*

[10.4: I defined $Q(\partial\Omega)$ above. This should maybe be defined earlier. /A]

Proof. This first part follows immediately from Theorem 9.5 and Proposition 10.4, while the last part follows from Proposition 7.4. □

metric-lemma-5-12

Corollary 10.9. *If Ω is finitely connected at the boundary, then the prime end closure $\overline{\Omega}^P$ is metrizable with metric m_P defined as follows. If $y, z \in \overline{\Omega}^P$, then*

$$m_P(y,z) := d_M(\Phi(y), \Phi(z)).$$

The topology on $\overline{\Omega}^P$ given by this metric is equivalent to the topology given by the sequential convergence discussed in Section 8.

[10.5: The following is a more general result than before, taking into account the new discussion of $\partial_{\mathrm{SP}}\Omega$. /A]

thm-cl0mm-cpt-new
i3-fin

Theorem 10.10. *The following are equivalent:*

(a) *Ω is finitely connected at the boundary;*

i3-P (b) $\overline{\Omega}^P$ *is compact and all prime ends have singleton impressions*;
i3-SP (c) $\Omega \cup \partial_{\rm SP}\Omega$ *is compact*;
i3-m (d) $\overline{\Omega}^M$ *is compact.*

Proof. (a) $\Leftrightarrow$ (d) This is shown in Björn–Björn–Shanmugalingam [14].

(c) $\Leftrightarrow$ (d) This follows directly from Theorem 9.5.

(a) $\Rightarrow$ (b) By Proposition 10.4 all prime ends have singleton impressions. Hence $\overline{\Omega}^P = \Omega \cup \partial_{\rm SP}\Omega$, which is compact by the already shown implication (a) $\Rightarrow$ (c).

(b) $\Rightarrow$ (c) Since all prime ends have singleton impressions, we have that $\Omega \cup \partial_{\rm SP}\Omega = \overline{\Omega}^P$, which is compact by assumption. □

We say that Ω is *N-connected at a point* $x_0 \in \partial\Omega$ if Ω is finitely connected at x_0 and for sufficiently small $r > 0$ we have $N(r) = N$, see also Björn–Björn–Shanmugalingam [14].

lem-N-conn **Lemma 10.11.** *Assume that Ω is N-connected at $x_0 \in \partial\Omega$. Then there are exactly N distinct prime ends, $[E_k^1], \dots, [E_k^N]$ with impression $\{x_0\}$. Furthermore, there are no other prime ends with x_0 in their impressions.*

This follows from Theorem 9.5 together with a result in Björn–Björn–Shanmugalingam [14], but let us give a more direct proof.

Proof. Without loss of generality assume that $N(1) = N$. Then for positive integers k we consider $G_j(1/k)$, the connected components of $B(x_0, 1/k) \cap \Omega$ that have x_0 in their boundaries, for $j = 1, \dots, N$. We can label them in such a way that for $j = 1, \dots, N$ we have $G_j(1/m) \subset G_j(1/k)$ if $m \geq k$. It can be directly checked that the choice of $E_k^j = G_j(1/k)$ for $j = 1, \dots, N$ and for positive integers k gives us ends $[E_k^j]$ with impression $\{x_0\}$ for $j = 1, \dots, N$. If $j_1 \neq j_2$, then for every choice of positive integers k_1, k_2 we have $E_{k_1}^{j_1} \cap E_{k_2}^{j_2} = \varnothing$. Thus the ends $[E_k^j]$, $j = 1, \dots, N$, are pairwise distinct, neither dividing the other. By Proposition 7.1, these ends are prime ends.

[10.6: I added the following part of the proof which was entirely missing. /A]

It remains to show that there are no more prime ends with x_0 in the impression. Let $[E_k]$ be a prime end with $x_0 \in I[E_k]$. By Proposition 10.4, there is a sequence of positive numbers r_k that decreases to 0, and a singleton prime end $[F_k]$ which divides $[E_k]$, such that $F_k = G_{j_k}(r_k)$ for some $1 \leq j_k \leq N(r_k) = N$. As $G_{j_k}(r_k) = F_k \subset F_1 = G_{j_1}(r_1)$ we must have $j_k = j_1$. Hence $[F_k] = [E_k^{j_1}]$. Since $[E_k]$ is a prime end we must have $[E_k] = [E_k^{j_1}]$ showing that there are no more prime ends. □

cor-Upsilon **Corollary 10.12.** *If Ω is locally connected at the boundary, then there is a homeomorphism $\Upsilon : \overline{\Omega}^P \to \overline{\Omega}$ such that $\Upsilon|_\Omega$ is the identity map.*

Proof. Let $\Psi : \overline{\Omega}^M \to \overline{\Omega}$ be the continuous mapping defined in Lemma 9.3. By Lemma 10.11, Ψ is bijective. **[10.7: Show that Ψ^{-1} is continuous. /A]**

Letting $\Upsilon = \Psi \circ \Phi$, where Φ is from Theorem 10.8, and using Theorem 10.8 completes the proof. □

In Karmazin [34] another definition of prime ends is considered using curves in the domain Ω that accumulate towards some part of $\partial\Omega$; see also Suvorov [57]. Using such curves, they construct a chain of sets (not quite **[10.8: What does "not quite" mean? /A]** similar to our acceptable sets), and give a characterization (in terms of the Mazurkiewicz metric) of the curve for which the corresponding chain gives a prime end. We point out that their construction is for simply connected Euclidean domains, and hence are different from our construction of ends and prime ends. In their investigations ends play a crucial role in analysis of various compactifications, quasiconformal extension problem as well as in boundary behaviour of quasiregular mappings.

sect-John

11. John and uniform domains

In this section $\delta_\Omega(x)$ stands for the distance of the point $x \in \Omega$ to $X \setminus \Omega$ with respect to the given metric d.

Definition 11.1. A domain $\Omega \subset X$ is a *John domain* if there is a constant $C_\Omega > 0$, called a *John constant*, and a point $x_0 \in \Omega$, called a *John centre*, such that for every $x \in \Omega$ there exists a rectifiable curve $\gamma : [0, l(\gamma)] \to \Omega$ parameterized by arc length, such that $x = \gamma(0)$, $x_0 = \gamma(l(\gamma))$ and for every $t \in [0, l(\gamma)]$, we have

$$t \le C_\Omega \delta_\Omega(\gamma(t)). \tag{11.1}$$ eq-def-John

A domain $\Omega \subset X$ is a *uniform domain* if there is a constant $C > 0$, called a *uniform constant*, such that whenever $x, y \in \Omega$ there is a rectifiable curve $\gamma : [0, l(\gamma)] \to \Omega$, parameterized by arc length, connecting x to y and satisfying the following two conditions:

$$l(\gamma) \le Cd(x, y),$$

and for all points z in the trajectory of γ,

$$\min\{l(\gamma_{x,z}), l(\gamma_{y,z})\} \le C\delta_\Omega(z).$$

Here $\gamma_{x,z}$ denotes the subcurve of γ with end points x and z.

Observe that uniform domains are necessarily John domains.

In this section we will show that under some assumptions all Mod_p-ends are prime ends. Let us however first draw some consequences of the results in the previous section.

thm-John-Ncon

Theorem 11.2. *Let Ω be a John domain. Then there is a constant N depending only on the doubling constant C_μ and the John constant, such that Ω is at most N-connected at every boundary point.*

This follows from Lemma 4.3 in Aikawa–Shanmugalingam [4].

Corollary 11.3. *Let Ω be a John domain. Then the following are true:*

(a) *Every prime end has a singleton impression.*

(b) *There is a homeomorphism $\Phi : \overline{\Omega}^P \to \overline{\Omega}^M$ such that $\Phi|_\Omega$ is the identity map.*

(c) *If*
$$1 \le p \in Q(\partial\Omega) := \{q : q \in Q(x) \text{ for all } x \in \partial\Omega\},$$
then $\partial_P\Omega$ is also the Mod_p-prime end boundary.

(d) *The prime end closure $\overline{\Omega}^P$ is metrizable and compact.*

(e) *There is a positive integer N, depending only on the doubling constant C_μ and the John constant, such that for every $x \in \partial\Omega$ there are at most N prime ends of Ω that contain x in their impressions.*

This corollary follows directly from Theorem 11.2 together with the results from Section 10. Let us also point out the following special case of Corollary 10.12.

cor-John+loc-conn

Corollary 11.4. *Let Ω be a John domain, which is locally connected at the boundary. Then there is a homeomorphism $\Upsilon : \overline{\Omega}^P \to \overline{\Omega}$ such that $\Upsilon|_\Omega$ is the identity map.*

In particular, this holds for every uniform domain Ω.

[11.1: Why are uniform domains locally connected at the boundary? Provide reference? /A]

Note that there are plenty of John domains which are locally connected at the boundary, but which are not uniform.

Example 11.5. Let Ω be the inward cusp domain in $\mathbb{R}^2$. Clearly, Ω is a John domain, but not uniform. However, it satisfies the hypotheses of Corollary 11.4.

Example 11.6. Let Ω be the unit disc in $\mathbb{R}^2$ from which the closed discs $\overline{B(x_j, r_j)}$ with $x_j = (1 - 1/j, 0)$ and $r_j = 1/2j(j+1)$, $j = 1, 2, \ldots$, have been removed.

Then the distance between two consecutive balls is $1/j(j+1)(j+2)$, showing that any curve in Ω connecting the north and the south pole of some B_j has distance to $\mathbb{R}^2 \setminus \Omega$ comparable to $1/j(j+1)(j+2)$, while its length is comparable to $1/j(j+1)$.

We are now ready to formulate and prove one of our main results.

thm-b-John-cor **Theorem 11.7.** *If Ω is a John domain and $p > Q - 1$, then every Mod_p-end is a prime end with singleton impression.*

Note that the conclusion of the above theorem fails if $[E_k]$ is merely known to be an end. Furthermore, this theorem does not tell us that singleton Mod_p-ends exist, but as we will see later, if in addition $1 \le p \in Q(\partial\Omega)$, then singleton Mod_p-ends exist for every $x \in \partial\Omega$.

[11.2: I still need to go through everything from here on. /A]

For this we need the following lemma about chains of balls in John domains.

lem-John-chain **Lemma 11.8.** *Let Ω be a John domain with a John centre x_0 and a John constant C_Ω. Let $\rho_0 \le \delta_\Omega(x_0)/4\lambda$ and $A = C_\Omega \delta_\Omega(x_0)/\rho_0 \ge 4C_\Omega\lambda$. Then every $x \in \Omega$ can be connected to the ball $B_{0,0} := B(x_0, \rho_0)$ by a chain of balls $\{B_{i,j} : i = 0, 1, \ldots; j = 0, 1, \ldots, m_i\}$ satisfying the conditions (a)–(e) of Lemma A.4 with $M = 2A$.*

Proof. Let γ be a John curve connecting x to x_0 in Ω. Assume that γ is parameterized by its arc length and that $\gamma(0) = x$ and $\gamma(L) = x_0$, where $L = l(\gamma)$ is the length of γ. Choose $i_x \in \mathbb{N}$ such that $4\lambda C_\Omega \rho_{i_x} \le \delta_\Omega(x)/2$. Recall that $\rho_{i_x} = 2^{-i_x}\rho_0$.

The first ball $B_{0,0} = B(x_0, \rho_0)$ in the chain clearly satisfies $4\lambda B_{0,0} \subset \Omega$. Also, by (11.1),

$$d(x_0, x) \le L \le C_\Omega \delta_\Omega(x_0) = A\rho_0.$$

Assume that the ball $B_{i,j}$ has already been constructed and that it satisfies (a). Let

$$c = \inf\{t \in [0, L] : \gamma(t) \in B_{i,j}\}.$$

Assume first that $i < i_x$. If $c \ge 4\lambda C_\Omega \rho_i$, then let $B_{i,j+1} = B(x_{i,j+1}, \rho_i)$ with $x_{i,j+1} = \gamma(c)$ be the successor of $B_{i,j}$. Note that by construction and by (11.1),

$$4\lambda C_\Omega \rho_i \le c \le C_\Omega \delta_\Omega(x_{i,j+1}), \qquad (11.2)$$ eq-de-Om-j

i.e. $4\lambda B_{i,j+1} \subset \Omega$.

If $c < 4\lambda C_\Omega \rho_i$, then let $m_i = j$ and let $B_{i+1,0} = B(x_{i+1,0}, \rho_{i+1})$ with $x_{i+1,0} = \gamma(c)$ be the successor of $B_{i,j}$. Note that (11.2) implies

$$\delta_\Omega(x_{i+1,0}) \ge \delta_\Omega(x_{i,m_i}) - \rho_i \ge 4\lambda\rho_i - \rho_i \ge 4\lambda\rho_{i+1}$$

and hence $4\lambda B_{i+1,0} \subset \Omega$.

For $i = i_x$ and $c > 0$, let $B_{i_x,j+1} = B(x_{i_x,j+1}, \rho_{i_x})$ with $x_{i_x,j+1} = \gamma(c)$ be the successor of $B_{i_x,j}$. Note that if $c \ge 4\lambda C_\Omega \rho_{i_x}$, then (11.2) implies that $4\lambda B_{i_x,j+1} \subset \Omega$. On the other hand, if $0 < c < 4\lambda C_\Omega \rho_{i_x}$, then the same conclusion follows from the fact that

$$d(x_{i_x,j+1}, x) \le c < 4\lambda C_\Omega \rho_{i_x} \le \frac{1}{2}\delta_\Omega(x)$$

and hence

$$\delta_\Omega(x_{i_x,j+1}) \ge \delta_\Omega(x) - d(x_{i_x,j+1}, x) \ge \frac{1}{2}\delta_\Omega(x) \ge 4\lambda\rho_{i_x}.$$

If $i = i_x$ and $c = 0$ or if $i > i_x$, then let $B_{i+1,0} = B(x, \rho_{i+1})$ be the successor of $B_{i,j}$. Then clearly

$$4\lambda\rho_i \le 4\lambda\rho_{i_x} \le \frac{1}{2}\delta_\Omega(x)$$

and thus $4\lambda B_{i+1,0} \subset \Omega$.

The balls $\{B_{i,j} : i = 0, 1, \dots; j = 0, 1, \dots, m_i\}$ cover γ in the direction from x_0 to x and neighbouring balls always have nonempty intersection. Thus, (e) is satisfied. Also, (a) is satisfied by construction and the comments above.

As for the other properties, note first that if $i > i_x$, then there is only one ball with radius ρ_i and that ball is centred at x. This proves (d), so it remains to prove (b) and (c).

For $i = 0$ and all $j \le m_0$ we have that

$$0 \le d(x_{0,j}, x) \le c < L - j\rho_0 \le C_\Omega \delta_\Omega(x_0) - j\rho_0 = (A - j)\rho_0,$$

showing that $m_0 \le A$ and $d(x_{0,j}, x) \le A\rho_0$.

Similarly, for $0 < i \le i_x$ we have by construction that

$$0 \le d(x_{i,j}, x) \le c < 4\lambda C_\Omega \rho_{i-1} - j\rho_i = (8\lambda C_\Omega - j)\rho_i$$

and hence $j < 8\lambda C_\Omega \le 2A$. This also shows that $d(x_{i,j}, x) < 2A\rho_i$.

For $i > i_0$, (b)–(d) are obvious. □

cor-John-H<Mod

Corollary 11.9. *Let Ω be a John domain with a John centre x_0 and a John constant C_Ω. Let $\rho_0 \le \delta_\Omega(x_0)/4\lambda$ and $B = B(x_0, \rho_0)$. If $p > Q - 1$, then there exists a constant $C > 0$ depending only on C_Ω, B, p, the doubling constant and the constants in the p-Poincaré inequality, such that for all $E \subset \Omega \setminus B$,*

$$\mathcal{H}_1^\infty(E) \le C \operatorname{Mod}_p(E, B(x_0, r), \Omega).$$

Proof. This follows directly from Lemmas A.4 and 11.8. □

Proof of Theorem 11.7. As each E_k, $k = 1, 2, \dots$, is a connected set, Corollary 11.9 and Definition 4.2 then imply

$$\operatorname{diam}(E_k) \le \mathcal{H}_1^\infty(E_k) \le C \operatorname{Mod}_p(E_k, B, \Omega) \to 0 \quad \text{as } k \to \infty,$$

where the ball B is as in Corollary 11.9. Proposition 7.1 then shows that $[E_k]$ is a prime end. □

The conclusion of Theorem 11.7 holds for somewhat more general domains as well.

thm-singleton-gen

Theorem 11.10. *Let $p > Q - 1$. Assume that for all $0 < r < \operatorname{diam}(\Omega)$, there exists a closed set $F \subset \overline{\Omega}$ such that $\mathcal{H}_1^\infty(F) \le r$ and $\Omega \setminus F$ is a John domain, with a John constant depending on r. Then every Mod_p-end $[E_k]$ in Ω has a singleton impression and is a prime end and, if in addition $\max\{1, Q-1\} \le p \in Q(x)$, a Mod_p-prime end.*

Proof. Let $0 < r < \operatorname{diam}\Omega$ and F be the set associated with r as in the assumption of the theorem. Given an end (E_k) of Ω, let $E_k' = E_k \setminus F$ and $\Omega' = \Omega \setminus F$. Let x_0

be the John centre of Ω' and $B = B(x_0, \rho) \Subset \Omega'$. Every curve connecting $\overline{B}$ to E'_k in Ω' connects $\overline{B}$ to $E_k \supset E'_k$ in Ω and hence

$$\operatorname{Mod}_p(\overline{B}, E'_k, \Omega') \le \operatorname{Mod}_p(\overline{B}, E_k, \Omega).$$

As Ω' is a John domain, this together with Corollary 11.9 implies that

$$\mathcal{H}_1^\infty(E'_k) \le C \operatorname{Mod}_p(\overline{B}, E'_k, \Omega') \le C \operatorname{Mod}_p(\overline{B}, E_k, \Omega),$$

where C depends on r but not on E_k. Since E_k is connected, it follows that

$$\operatorname{diam} E_k \le \mathcal{H}_1^\infty(E_k) \le \mathcal{H}_1^\infty(F) + \mathcal{H}_1^\infty(E'_k) \le r + C \operatorname{Mod}_p(\overline{B}, E_k, \Omega).$$

Since $[E_k]$ is an end, $\lim_k \operatorname{Mod}_p(\overline{B}, E_k, \Omega) = 0$. Hence, we have

$$\limsup_{k\to\infty} \operatorname{diam} E_k \le r.$$

Letting $r \to 0$ shows that $\lim_k \operatorname{diam} E_k = 0$, and an application of Proposition 7.1 completes the proof. □

The following lemma is a consequence of Lemma 7.6 and Theorem 11.10.

Lemma 11.11. *Under the assumptions of Theorem 11.10, every prime end is a* Mod_p*-prime end and has a singleton impression.*

Our final result relates prime ends to the Mazurkiewicz boundary from Section 9. The conclusion about metrizability and compactness will be important in our forthcoming paper on Dirichlet problems with respect to prime end boundaries.

Theorem 11.12. *Let Ω be a John domain. If $\max\{1, Q-1\} \le p \in Q(\partial\Omega)$, then the* Mod_p*-end boundary and the prime end boundary coincide.*

Proof. **[11.3: Give proof or reference to above. /A]** □

Appendix A. Modulus and capacity estimates

appendix

In this section, we will provide several estimates for the modulus and capacity needed in our study of prime ends. While the proofs of these results are note directly pertinent to the discussion on prime ends developed in this paper, we include them here for completeness, since these results do not appear elsewhere in literature.

[A.1: I moved this lemma here. /A]

MP **Lemma A.1.** *For any choice of disjoint sets $E, F \subset \Omega$ we have*

$$\operatorname{Mod}_p(E, F, \Omega) = \operatorname{cap}_p(E, F, \Omega), \tag{A.1}$$

eq-capp=modp

where $\operatorname{cap}_p(E, F, \Omega)$ *is the* relative p-capacity *of the condenser* (E, F, Ω) *defined by*

$$\operatorname{cap}_p(E, F, \Omega) := \inf_u \int_\Omega g_u^p \, d\mu,$$

with the infimum taken over all $u \in N^{1,p}(\Omega)$ satisfying $0 \le u \le 1$ on Ω, $u = 1$ on E, and $u = 0$ on F.

If, moreover, X is quasiconvex then the infimum in the definition of cap_p can equivalently be taken over continuous $u \in N^{1,p}(\Omega)$ alone. **[A.2: I think it is necessary to have E and F closed for this to be true. /A]**

Proof. To see the validity of (A.1), note that clearly by (2.4) and the fact that for $u \in N^{1,p}(\Omega)$ there are upper gradients $g_j \to g_u$ in $L^p(\Omega)$, we have $\operatorname{cap}_p(E, F, \Omega) \geq \operatorname{Mod}_p(E, F, \Omega)$. On the other hand, if ρ is an admissible function used for computing $\operatorname{Mod}_p(E, F, \Omega)$, then we define a function f on X by

$$f(x) = \min \left\{ 1, \inf_{\gamma_{E,x}} \int_{\gamma_{E,x}} \rho \, ds \right\},$$

where we let $\rho = 0$ in $X \setminus \Omega$ and the infimum is taken over all rectifiable curves connecting E to x (including constant ones). Observe that $f = 0$ on E, $f = 1$ on F and ρ is an upper gradient of f, by Lemma 3.1 in Björn–Björn–Shanmugalingam [12] (or Lemma 5.25 in Björn–Björn [10]). By Järvenpää–Järvenpää–Rogovin–Rogovin–Shanmugalingam [32] the function f is measurable on X, and since $|f| \leq 1$ it follows that $f \in N^{1,p}(\Omega)$ and

$$\operatorname{cap}_p(E, F, \Omega) \leq \int_\Omega g_u^p \, d\mu \leq \int_\Omega \rho^p \, d\mu.$$

Hence, by taking infimum over all such ρ we get that

$$\operatorname{cap}_p(E, F, \Omega) \leq \operatorname{Mod}_p(E, F, \Omega). \qquad \square$$

lem-cap-0-mod-0 **Lemma A.2.** *Let $x \in \Omega$. If $1 \leq p \in Q(x)$, then for every compact $K \subset \Omega \setminus \{x\}$,*

$$\lim_{r \to 0} \operatorname{Mod}_p(B(x, r), K, \Omega) = 0.$$

[A.3: I think we need this for $x \in \partial\Omega$ in the proof of Proposition 7.4, in which case the proof at least needs to be rewritten. I don't know if we also need it for $x \in \Omega$. /A]

Proof. Let $\rho > 0$ be such that $B(x, 2\rho) \subset \Omega \setminus K$ and $\varepsilon > 0$ be arbitrary. As $1 \leq p \in Q(x)$, Theorems 3.2 and 3.3 in Garofalo–Marola [22] imply that $\operatorname{cap}_p(\{x\}, B) = 0$ for every ball B containing x.

Since cap_p is an outer capacity, by e.g. Theorem 6.16 in Björn–Björn [10], there exists $0 < r < \delta$ such that $\operatorname{cap}_p(B(x, r), B) < \varepsilon$. This means that there exists $u \in N_0^{1,p}(B)$ such that $u = 1$ on $B(x, r)$, $0 \leq u \leq 1$, and $\int_B g_u^p \, d\mu < \varepsilon$, where g_u is the minimal p-weak upper gradient of u.

Let $\eta(y) = (1 - \operatorname{dist}(y, B(x, \rho))/\rho)_+$. Then $v = u\eta \in N_0^{1,p}(B(x, 2\rho))$ and $v = 1$ on $B(x, r)$. It follows that for a.e. curve γ in Ω with endpoints $y \in B(x, r)$ and $z \in K$,

$$1 = |u(y) - u(z)| \leq \int_\gamma g_v \, ds.$$

Hence $\operatorname{Mod}_p(B(x, r), K, \Omega) \leq \int_X g_v^p \, d\mu$.

Since $g_v \leq \eta g_u + g_\eta$ the Poincaré inequality for $N_0^{1,p}$-functions then yields

$$\int_X g_v^p \, d\mu \leq 2^{p-1} \int_B g_u^p \, d\mu + \frac{2^{p-1}}{\delta^p} \int_B u^p \, d\mu \leq C(\delta, B) \int_B g_u^p \, d\mu \leq C(\delta, B)\varepsilon.$$

Since ε was arbitrary, this finishes the proof. $\square$

lem-mod>0 **Lemma A.3.** *Let $E, F \subset \Omega$ be disjoint and with nonempty interiors. Then*

$$\operatorname{Mod}_p(E, F, \Omega) = \operatorname{cap}_p(E, F, \Omega) > 0.$$

Proof. By (A.1), it is enough to show that

$$\int_\Omega g_u^p \, d\mu \geq c > 0$$

for every $u \in N^{1,p}(\Omega)$ such that $u = 1$ on E and $u = 0$ on F. Note that if there are no admissible functions u, then theorem trivially holds since then both quantities under study are infinite, and hence equal.

Let x and y be points in the interiors of E and F, respectively, and let $\gamma : [0, l_\gamma] \to \Omega$ be a rectifiable curve connecting x to y. Let $0 < r < \operatorname{dist}(\gamma, X \setminus \Omega)$ be such that both $B(x,r) \subset E$ and $B(y,r) \subset F$. Cover γ by balls $B_j = B(x_j, r)$, $j = 0, 1, \dots, n$, such that $B_0 = B(x,r)$, $B_n = B(y,r)$ and $B_j \cap B_{j+1}$ is nonempty, $j = 0, 1, \dots, n-1$.

Then $B_{j+1} \subset 2B_j \subset 3B_{j+1}$, $j = 0, 1, \dots, n-1$ and hence

$$|u_{B_j} - u_{B_{j+1}}| \leq |u_{B_j} - u_{2B_j}| + |u_{B_{j+1}} - u_{2B_j}| \leq C \fint_{2B_j} |u - u_{2B_j}|.$$

The p-Poincaré inequality and Lemma 2.1 then yield

$$\begin{aligned} 1 = |u_{B_0} - u_{B_n}| &\leq \sum_{j=0}^{n-1} |u_{B_j} - u_{B_{j+1}}| \leq C \sum_{j=0}^{n-1} \fint_{2B_j} |u - u_{2B_j}| \\ &\leq C \sum_{j=0}^{n-1} r \biggl(\fint_{2\lambda B_i} g_u^p \biggr)^{1/p} \leq \frac{Crn}{\mu(B_0)} \biggl(\int_\Omega g_u^p \biggr)^{1/p}, \end{aligned}$$

where C is independent of u. Taking infimum over all admissible functions u yields the desired result. $\square$

Next, we shall relate the modulus to the Hausdorff content.

Recall that the *s-dimensional Hausdorff content* $\mathcal{H}^s_\infty(E)$ of a set $E \subset X$ is the number

$$\mathcal{H}^s_\infty(E) := \inf \biggl\{ \sum_{j=1}^\infty r_i^s : E \subset \bigcup_{j=1}^\infty B(x_i, r_i) \biggr\}.$$

lem-chain-imp-length-est **Lemma A.4.** *Let $E \subset \Omega$ and $B(x_0, r) \subset \Omega \setminus E$. Assume that there exists a constant $M > 0$ such that for every $x \in E$ there exists $0 < \rho_0 \leq r$ such that x can be connected to the ball $B_{0,0} = B(x_0, \rho_0)$ by a chain of balls $\{B_{i,j} : i = 0, 1, \dots; j = 0, 1, \dots, m_i\}$ with the following properties:*

first (a) *For all balls B in the chain, we have $3\lambda B \subset \Omega$.*

second (b) *For all i and j, the ball $B_{i,j}$ has radius $\rho_i = 2^{-i} \rho_0$ and centre $x_{i,j}$ such that $d(x_{i,j}, x) \leq M \rho_i$.*

third (c) *For all i, we have $m_i \leq M$.*

fourth (d) *For large i, we have $m_i = 0$ and the balls $B_{i,0}$ are centred at x.*

last (e) *The balls $B_{i,j}$ are ordered lexicographically, i.e. $B_{i,j}$ comes before $B_{i',j'}$ if and only if $i < i'$ or $i = i'$ and $j < j'$. If $B_{i,j}$ and $B_{i',j'}$ are two neighbours with respect to this ordering, then $B_{i,j} \cap B_{i',j'}$ is nonempty.*

Let $s > 0$ and $p > Q - s$. Then there exists a constant C depending only on M, p, s, Q, r, the doubling constant C_d and on the constants in the Poincaré inequality such that

$$\mathcal{H}^s_\infty(E) \leq C \operatorname{cap}_p(E, B(x_0, r), \Omega) = C \operatorname{Mod}_p(E, B(x_0, r), \Omega).$$

Proof. By (A.1) and the comment after it, we can test $\operatorname{cap}_p(E, B(x_0, r), \Omega)$ by continuous functions. Let therefore $u \in N^{1,p}(\Omega)$ be continuous and such that $u = 0$

on $B(x_0,r)$ and $u=1$ on E. Consider $x\in E$ and let $\mathcal{C}_x=\{B_{i,j}: i=0,1,\dots; j=0,1,\dots,m_i\}$ be the corresponding chain. For each ball B in the chain let B^* be its immediate successor. Since $B\cap B^*$ is nonempty, we have $B^*\subset 3B$. Note also that properties (b) and (c) above imply that for all $i=0,1,\dots$ and $j=0,1,\dots,m_i$, we have

$$B_{i,j}\subset (M+1)B(x,\rho_i)\subset (M+1)B(x,\rho_0)\subset (2M+1)B_{0,0}. \tag{A.2}$$

eq-Bij-subset-Boo

Since x is a Lebesgue point of u, a telescopic argument together with assumption (d) implies

$$\begin{aligned} 1=|u(x)-u_{B_{0,0}}| &= \lim_{i\to\infty}|u_{B_{i,0}}-u_{B_{0,0}}| \\ &\le \sum_{B\in\mathcal{C}_x}|u_B-u_{B^*}| \le \sum_{B\in\mathcal{C}_x}(|u_B-u_{3B}|+|u_{B^*}-u_{3B}|). \end{aligned} \tag{A.3}$$

eq-telescopic

Lemma 2.1 and the p-Poincaré inequality yield

$$|u_{B^*}-u_{3B}|\le C\fint_{3B}|u-u_{3B}|\,d\mu\le Cr(B)\biggl(\fint_{3\lambda B}g_u^p\,d\mu\biggr)^{1/p},$$

where $r(B)$ is the radius of the ball B. The difference $|u_B-u_{3B}|$ is estimated similarly and inserting both estimates into (A.3) together with (2.1) and (A.2) implies

$$\begin{aligned} 1&\le C\sum_{B\in\mathcal{C}_x}\frac{r(B)}{\mu(3\lambda B)^{1/p}}\biggl(\int_{3\lambda B}g_u^p\,d\mu\biggr)^{1/p} \\ &\le \frac{C\rho_0^{\frac{Q}{p}}}{\mu(B_{0,0})^{1/p}}\sum_{B\in\mathcal{C}_x}r(B)^{1-Q/p}\biggl(\int_{3\lambda B}g_u^p\,d\mu\biggr)^{1/p}, \end{aligned}$$

where C depends only on M, p, Q, the doubling constant C_d and on the constants in the Poincaré inequality.

Since $p>Q-s$, we have $p-Q+s>0$ and hence

$$1=C\sum_{i=1}^\infty 2^{-i(p-Q+s)/p}\ge\frac{C}{M}\sum_{B\in\mathcal{C}_x}\biggl(\frac{r(B)}{\rho_0}\biggr)^{(p-Q+s)/p},$$

where C depends only on p and Q. Comparing the last two estimates we see that there exists a ball $B_x\in\mathcal{C}_x$ such that

$$\begin{aligned} \biggl(\frac{r(B_x)}{r}\biggr)^{(p-Q+s)/p} &\le \biggl(\frac{r(B_x)}{\rho_0}\biggr)^{(p-Q+s)/p} \\ &\le \frac{Cr^{\frac{Q}{p}}}{\mu(B_{0,0})^{1/p}}r(B_x)^{1-Q/p}\biggl(\int_{3\lambda B_x}g_u^p\,d\mu\biggr)^{1/p}, \end{aligned}$$

where C depends on the same constants as before, but not on u or x.

Repeating this argument for every $x\in E$, we obtain balls B_x, $x\in E$, such that

$$r(B_x)^s\le\frac{Cr^{p+s}}{\mu(B_{0,0})}\int_{3\lambda B_x}g_u^p\,d\mu.$$

The balls $\{3\lambda B_x\}_{x\in E}$ cover E and hence the 5-covering lemma allows us to choose pairwise disjoint balls $3\lambda B_{x_i}$, $i=1,2\dots$, so that $E\subset\bigcup_{i=1}^\infty 15\lambda B_{x_i}$. Thus we get

$$\begin{aligned} \mathcal{H}^s_\infty(F)&\le\sum_{i=1}^\infty r(15\lambda B_{x_i})^s=15^s\lambda^s\sum_{i=1}^\infty r(B_{x_i})^s \\ &\le\frac{Cr^{p+s}}{\mu(B_{0,0})}\sum_{i=1}^\infty\int_{3\lambda B_x}g_u^p\,d\mu\le\frac{Cr^{p+s}}{\mu(B_{0,0})}\int_\Omega g_u^p. \end{aligned}$$

Taking infimum over all admissible functions u finishes the proof. □

lem-ex-chain

Lemma A.5. *Let $F \Subset \Omega$ and $B = B(x_0, r) \Subset \Omega \setminus F$. Then there exists $0 < \rho_0 < r$ such that every $x \in F$ can be connected to the ball $B_{0,0} = B(x_0, \rho_0)$ by a chain $\mathcal{C}_x$ satisfying the conditions (a)–(e) of Lemma A.4.*

Proof. Since Ω is connected, there exists $0 < \varepsilon < r$ such that both $\overline{B}$ and $\overline{F}$ belong to the same connected component of

$$\Omega_\varepsilon := \{x \in \Omega \cap B(x_0, 1/\varepsilon) : \operatorname{dist}(x, X \setminus \Omega) > \varepsilon\}.$$

Choose $0 < \rho_0 \le \varepsilon/4\lambda$ and let $B_i = B(x_i, \rho_0/2)$, $i = 1, \dots, N$, be a maximal pairwise disjoint collection of balls with centres in Ω_ε. By the doubling property, there are only finitely many such balls and their number N depends only on ε, ρ_0 and the doubling constant. The balls $2B_i$, $i = 1, 2, \dots, N$, cover Ω_ε and $4\lambda B_i \subset \Omega$ for all $i = 1, 2, \dots, N$.

Let $x \in F$ be arbitrary. By connectedness, there exists a curve in Ω_ε from x_0 to x. We can therefore from $2B_i$, $i = 1, 2, \dots, N$, choose a chain of balls covering γ. Number these balls in the direction from x_0 to x and call them $B_{0,j}$, $j = 1, 2, \dots, m_0$. Clearly, $m_0 \le N$ and neighbouring balls in the chain have nonempty intersection. Complete the chain by the balls $B_{i,0} = B(x, \rho_i)$, where $\rho_i = 2^{-i}\rho_0$, $i = 1, 2, \dots$.

It remains to verify that the conditions (a)–(e) of Lemma A.4 are satisfied. The only property that needs some justification is that $d(x_{i,j}, x) \le M\rho_i$ with $M = \max\{N, 2/\varepsilon\rho_0\}$. For $i \ge 1$, this is trivial and for $i = 0$ we have $d(x_{0,j}, x) \le \operatorname{diam} \Omega_\varepsilon \le 2/\varepsilon$. The other properties follow by construction. □

Remark A.6. The proof of Lemma A.5 shows that $M = \max\{N, 2/\varepsilon\rho_0\}$. It follows that M (and hence also C in Lemma A.4) depends on $\operatorname{dist}(F, X \setminus \Omega)$. The estimate in Lemma A.4 therefore does not apply if we only know that $F \subset \overline{\Omega}$. Indeed, in the topologist's comb from Example 5.1, the interval $I \subset \Omega$ is not accessible by any curve and hence $\operatorname{Mod}_p(I, K, \Omega) = 0$ for all $K \Subset \Omega$. See, however, Lemma 11.8 below.

cor-modp-Q-1

Corollary A.7. *Let $F \subset \Omega$ be a continuum and $B = B(x_0, r) \Subset \Omega \setminus F$. If $p > Q-1$, then $\operatorname{Mod}_p(F, B, \Omega) > 0$.*

Proof. Since F is a continuum, we have $0 < \operatorname{diam} F \le \mathcal{H}^1_\infty(F)$ and the result follows directly from Lemmas A.4 and A.5. □

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