PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

A PROBLEM OF BAERNSTEIN ON THE EQUALITY OF THE p-HARMONIC MEASURE OF A SET AND ITS CLOSURE

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Abstract. A. Baernstein II, [*Comparison of p-harmonic measures of subsets of the unit circle*, St. Petersburg Math. J. **9** (1998), 543–551, p. 548], posed the following question: If G is a union of m open arcs on the boundary of the unit disc $\mathbf{D}$, is then $\omega_{a,p}(G) = \omega_{a,p}(\overline{G})$, where $\omega_{a,p}$ denotes the p-harmonic measure? (Strictly speaking he stated this question for the case $m = 2$.) For $p = 2$ the positive answer to this question is well known. Recall that for $p \neq 2$ the p-harmonic measure, being a nonlinear analogue of the harmonic measure, is not a measure in the usual sense.

The purpose of this note is to answer a more general version of Baernstein's question in the affirmative when $1 < p < 2$. In the proof, using a deep trace result of Jonsson and Wallin, it is first shown that the characteristic function χ_G is the restriction to $\partial\mathbf{D}$ of a Sobolev function from $W^{1,p}(\mathbf{C})$.

For $p \geq 2$ it is no longer true that χ_G belongs to the trace class. Nevertheless, we are able to show equality for the case $m = 1$ of one arc for all $1 < p < \infty$, using a very elementary argument. A similar argument is used to obtain a result for starshaped domains.

Finally we show that in a certain sense the equality holds for almost all relatively open sets.

1. Introduction

Baernstein [1], p. 548, posed the following question: If $G \subset \mathbf{D}$ is a union of m open arcs, is then $\omega_{a,p}(G) = \omega_{a,p}(\overline{G})$? Here $\omega_{a,p}$ denotes the p-harmonic measure. (Strictly speaking he stated this question for the case $m = 2$.) For the linear case $p = 2$ the positive answer to this question is well known. Recall that for $p \neq 2$ the p-harmonic measure, being a nonlinear analogue of the harmonic measure, is not a measure in the usual sense.

We answer Baernstein's question in the following proposition.

2000 *Mathematics Subject Classification.* Primary 31C45; Secondary 30C85, 31A25, 31B20, 31C15, 46E35.

Key words and phrases. Ahlfors regular, Dirichlet problem, d-set, Lipschitz domain, Minkowski dimension, p-harmonic function, p-harmonic measure, Sobolev function, starshaped, trace, unit disc.

We wish to thank Juha Heinonen for drawing our attention to the question of Baernstein.

The first two authors are supported by the Swedish Research Council and Gustaf Sigurd Magnuson's fund of the Royal Swedish Academy of Sciences. The second author did this research while she was at Lund University.

The third author is partly supported by NSF grant DMS 0243355.

Proposition 1.1. *Let $1 < p < \infty$ and let $\Omega = \mathbf{D}$ be the unit disc in the complex plane. Let $G \subset \partial\mathbf{D}$ be the union of m open arcs. If $m = 1$ or $p \le 2$, then*

$$\omega_{a,p}(G) = \omega_{a,p}(\overline{G}) \quad \textit{for all } a \in \mathbf{D}.$$

As far as we know, Baernstein's question remains open if $m > 1$ and $p > 2$.

In the proof for $1 < p < 2$, it is first shown that the characteristic function χ_G is the restriction to $\partial\mathbf{D}$ of a p-quasicontinuous Sobolev function from $W^{1,p}(\mathbf{C})$, using a deep trace result of Jonsson and Wallin (some earlier results would also suffice for the unit disc). This together with a recent result of the authors of this note [3], Corollary 5.2, demonstrates the equality in question.

For $p \ge 2$ it is no longer true that χ_G belongs to the trace class, and the above outlined proof breaks down. Nevertheless, we are able to show equality for the case $m = 1$ (a single arc) for all $1 < p < \infty$, using a very elementary argument. In Proposition 3.1 we obtain a similar result for starshaped domains Ω and a starshaped set on its boundary.

In the literature there are, of course, invariance results for the p-harmonic measure. Kurki [11], Theorem 1.1 (or Björn–Björn–Shanmugalingam [3], Corollary 7.5), states that if K is a compact set and $C_p(E) = 0$, then $\overline{\omega}_{a,p}(K) = \overline{\omega}_{a,p}(K \cup E)$. As far as we are aware, Kurki's result (together with its dual version) contains all other invariance results in the literature for $p \ne 2$. Unfortunately, as Baernstein points out, Kurki's result cannot be applied to our problem.

For bounded domains $\Omega \subset \mathbf{R}^n$ and $E \subset \partial\Omega$, we use the full power of Jonsson–Wallin's result to give conditions on Ω and E such that $\omega_{a,p}(\operatorname{int}_{\partial\Omega} E) = \omega_{a,p}(E) = \omega_{a,p}(\overline{E})$. For the full statement of the main result we refer to Section 4, but here we give a corollary.

Proposition 1.2. *Let $1 < p < n$ and let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain. Moreover, let $E \subset \partial\Omega$ be such that $\dim_M \partial_{\partial\Omega} E < n - p$. Then*

$$\omega_{a,p}(\operatorname{int}_{\partial\Omega} E) = \omega_{a,p}(E) = \omega_{a,p}(\overline{E}) \quad \textit{for all } a \in \Omega. \tag{1.1}$$

Here $\operatorname{int}_{\partial\Omega} E$ and $\partial_{\partial\Omega} E$ are the interior and the boundary, respectively, of E in the induced subset topology on $\partial\Omega$. The upper Minkowski dimension is denoted by $\dim_M$, see Federer [7], Section 3.2.37, or Mattila [13], Sections 5.3 and 5.5 for the definition.

If $n = 2$ and (1.1) holds for some $a \in \Omega$, then (1.1) holds for all $a \in \Omega$, by Manfredi [12], Theorem 2. Whether this implication holds for $n \ge 3$ is an open problem.

Finally we show in Proposition 3.2 that in a certain sense equality holds for almost all relatively open sets.

It should be pointed out that in Carnot–Carathéodory spaces a result similar to Theorem 4.1 can be obtained by replacing the trace result of Jonsson–Wallin by the trace result of Danielli–Garofalo–Nhieu [5], [6]. For general metric spaces see also Björn–Björn–Shanmugalingam [4].

2. Notation and definitions

We let $W^{1,p}(\Omega)$ be the standard *Sobolev space* of all functions in $L^p(\Omega)$ whose distributional gradients also belong to $L^p(\Omega)$. We also define the *Sobolev capacity*

$$C_p(E) = \inf \int_{\mathbf{R}^n} (|f|^p + |\nabla f|^p)\, dx,$$

where the infimum is taken over all $f \in W^{1,p}(\mathbf{R}^n)$ such that $f = 1$ in an open set containing E.

A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is *p-harmonic* if it is continuous and a weak solution of the equation

$$\Delta_p u := \operatorname{div}(|Du|^{p-2} Du) = 0 \quad \text{in } \Omega;$$

it is *p-superharmonic* if it is lower semicontinuous and

$$\Delta_p u \le 0 \quad \text{in } \Omega$$

in the distributional sense.

We define the *upper Perron solution* of f by letting

$$\overline{P} f(x) = \inf u(x), \quad x \in \Omega,$$

where the infimum is taken over all p-superharmonic functions u on Ω bounded below and such that

$$\liminf_{\Omega \ni y \to x} u(y) \ge f(x) \quad \text{for all } x \in \partial\Omega.$$

The *lower Perron solution* can be defined in a dual fashion, or simply by letting

$$\underline{P} f(x) = -\overline{P}(-f)(x).$$

If $\overline{P} f = \underline{P} f$, then we denote the common function by Pf, and f is said to be *resolutive*. It is always true that $\underline{P} f \le \overline{P} f$. Apart from the case when $\overline{P} f \equiv \pm\infty$, $\overline{P} f$ is a p-harmonic function.

It should be observed that if $f \in C(\partial\Omega)$, then f is resolutive, see, e.g., Heinonen–Kilpeläinen–Martio [9], Theorem 9.25, or Björn–Björn–Shanmugalingam [3], Theorem 6.1. In [3], Corollary 5.2, it was shown that if f is a p-quasicontinuous representative of an equivalence class in $W^{1,p}(\mathbf{R}^n)$, then f is resolutive; this will be essential to us. A function f is *p-quasicontinuous* if for every $\varepsilon > 0$ there is a set E with $C_p(E) < \varepsilon$ such that $f|_{\mathbf{R}^n \setminus E}$ is continuous.

The *upper and lower p-harmonic measures* of $E \subset \partial\Omega$ evaluated at $a \in \Omega$ are defined by

$$\overline{\omega}_{a,p}(E;\Omega) = \overline{\omega}_{a,p}(E) = \overline{P}\chi_E(a) \quad \text{and} \quad \underline{\omega}_{a,p}(E;\Omega) = \underline{\omega}_{a,p}(E) = \underline{P}\chi_E(a).$$

When $\underline{\omega}_{a,p}(E) = \overline{\omega}_{a,p}(E)$ we denote the common value by $\omega_{a,p}(E;\Omega) = \omega_{a,p}(E)$.

It should be observed that due to the nonlinearity of Δ_p the p-harmonic measure is not a measure when $p \ne 2$. For more on p-harmonic measures and their history, see [9], Chapter 11. (Let us also refer the interested reader to the results in Section 8 in Björn–Björn–Shanmugalingam [2] and Corollary 7.5 in [3].)

Throughout this paper, C denotes a positive constant whose exact value is unimportant, can change even within the same line, and depends only on fixed parameters. The ball with centre x and radius r is denoted by $B(x,r)$.

3. Elementary results

We start with some elementary proofs.

Proof of Proposition 1.1 *for the case* $m = 1$. We use complex notation to simplify some of the expressions. Let $G_\theta = \{z \in \partial\mathbf{D} : |\arg z| < \theta\}$. By rotating $\mathbf{D}$ if necessary, we can assume that $G = G_\delta$ for some $\delta \in (0,\pi)$.

Let $u(z) = \omega_{z,p}(G_\delta)$ and $\bar{u}(z) = \omega_{z,p}(\overline{G}_\delta)$. Note that the upper and lower p-harmonic measures coincide in this case, by Proposition 9.31 in Heinonen–Kilpeläinen–Martio [9] (or Corollary 7.4 in [3]).

For $0 < x < 1$, let $r(x) = |e^{i\delta} - x|$, $\delta(x) = \arg(e^{i\delta} - x)$, and for $z \in B(x, r(x))$, let

$$\begin{aligned} v_x(z) &= \omega_{z,p}(\{y \in \partial B(x,r(x)) : |\arg(y-x)| < \delta(x)\}; B(x,r(x))) \\ &= \omega_{(z-x)/r(x),p}(G_{\delta(x)}). \end{aligned}$$

By the comparison principle in the domain $\mathbf{D} \cap B(x, r(x))$, we see that $v_x \le u$ in $\mathbf{D} \cap B(x, r(x))$. On the other hand, by the comparison principle again, $u(z) \le \bar{u}(z) \le \omega_{z,p}(G_{\delta(x)}) = v_x(x + r(x)z)$ for all $z \in \mathbf{D}$ and $0 < x < 1$. Thus, using the continuity of the p-harmonic function $\bar{u}$ and by the fact that every $z \in \mathbf{D}$ belongs to $\mathbf{D} \cap B(x, r(x))$ for sufficiently small $x > 0$,

$$u(z) \le \bar{u}(z) = \lim_{x \to 0+} \bar{u}((z-x)/r(x)) \le \lim_{x \to 0+} v_x(z) \le u(z) \quad \text{for } z \in \mathbf{D}.$$

$\square$

The proof above immediately generalizes to spherical caps on balls in $\mathbf{R}^n$.

In the next proposition we apply the method of the above proof to starshaped domains.

Proposition 3.1. *Let $1 < p < \infty$ and let $\Omega \subset \mathbf{R}^n$ be a bounded domain which is starshaped with respect to $0 \in \partial\Omega$. Let $E \subset \partial\Omega$ be starshaped with respect to 0 and such that*

$$\delta\overline{E} \subset E \quad \textit{whenever } 0 < \delta < 1.$$

Then

$$\overline{\omega}_{a,p}(E) = \overline{\omega}_{a,p}(\overline{E}) \textit{ and } \underline{\omega}_{a,p}(E) = \underline{\omega}_{a,p}(\overline{E}) \quad \textit{for all } a \in \Omega.$$

If Ω is a p-regular domain, or $\delta\overline{E} \subset \operatorname{int}_{\partial\Omega} E$ when $0 < \delta < 1$, then

$$\omega_{a,p}(E) = \omega_{a,p}(\overline{E}) \quad \textit{for all } a \in \Omega.$$

For the definition of p-regularity see [2], [3] or [9].

The starshapedness of E does not in general guarantee that $\delta\overline{E} \subset E$ for $0 < \delta < 1$, even if we insist on the starcenter 0 lying in $\operatorname{int}_{\partial\Omega} E$, as exhibited by, e.g., letting $E = \mathbf{D} \setminus \left[\frac{1}{2}, 1\right) \subset \mathbf{C}$ or $E = \mathbf{D} \cup \{z : |z| < 2 \text{ and } 0 < \arg z < \pi/4\}$. Here we take the ambient space to be $\mathbf{R}^3$ and Ω to be the cube $(-2,2)^2 \times (0,4)$.

Note that if $\delta\overline{E} \subset \operatorname{int}_{\partial\Omega} E$ when $0 < \delta < 1$, then $\overline{E} = \overline{\operatorname{int}_{\partial\Omega} E}$, and thus also $\omega_{a,p}(\operatorname{int}_{\partial\Omega} E) = \omega_{a,p}(\overline{E})$.

This proposition can be applied to the situation when $\Omega \subset \mathbf{R}^2$ is a rectangle, and $E \subset \partial\Omega$ is connected, and either lies on one or two of the rectangle's sides or has a complement which lies on one or two sides. In the latter case one needs to use the duality condition $\overline{\omega}_{a,p}(A) = 1 - \underline{\omega}_{a,p}(\partial\Omega \setminus A)$.

Proof. Let $0 < \delta < 1$. Then by a rescaling and the comparison principle, we have

$$\overline{\omega}_{a,p}(\overline{E}) = \overline{\omega}_{\delta a,p}(\delta\overline{E}; \delta\Omega) \le \overline{\omega}_{\delta a,p}(E) \quad \text{for } a \in \Omega.$$

Letting $\delta \to 1-$ and using the continuity of the p-harmonic function $a \mapsto \overline{\omega}_{a,p}(E)$ (note that $\lim_{\delta \to 1-} \delta a = a$), we obtain

$$\overline{\omega}_{a,p}(\overline{E}) \le \overline{\omega}_{a,p}(E) \le \overline{\omega}_{a,p}(\overline{E}) \quad \text{whenever } a \in \Omega.$$

The proof for $\underline{\omega}$ is similar.

If Ω is a p-regular domain, then Proposition 9.31 in Heinonen–Kilpeläinen–Martio [9] (or Corollary 7.4 in [3]) shows that $\overline{\omega}_{a,p}(\overline{E}) = \underline{\omega}_{a,p}(\overline{E})$ from which the conclusion follows.

If $\delta\overline{E} \subset \operatorname{int}_{\partial\Omega} E$ when $0 < \delta < 1$, we proceed as follows. For $0 < \delta < 1$, we can find $f_\delta \in C(\partial\Omega)$ such that $f_\delta = 1$ on $\delta\overline{E}$ and $f_\delta = 0$ on $\partial\Omega \setminus E$. Thus, by the resolutivity of continuous functions, we obtain

$$\overline{\omega}_{a,p}(\overline{E}) = \overline{\omega}_{\delta a,p}(\delta\overline{E}; \delta\Omega) \le \overline{\omega}_{\delta a,p}(\delta\overline{E}; \Omega) \le Pf_\delta(\delta a) \le \underline{\omega}_{\delta a,p}(E) \quad \text{for } a \in \Omega.$$

Letting $\delta \to 1-$ and using the continuity of the p-harmonic function $a \mapsto \underline{\omega}_{a,p}(E)$ we obtain

$$\overline{\omega}_{a,p}(\overline{E}) \le \underline{\omega}_{a,p}(E) \le \overline{\omega}_{a,p}(\overline{E}) \quad \text{when } a \in \Omega,$$

which combined with the first part completes the proof. □

Proposition 3.2. *Let $1 < p < \infty$ and let $\Omega \subset \mathbf{R}^n$ be a bounded domain. Let $\{E_r\}_{r\in(0,1)}$ be a family of sets such that $\overline{E}_s \subset \operatorname{int}_{\partial\Omega} E_r \subset \partial\Omega$ if $0 < s < r < 1$. Then for almost every r (with respect to the Lebesgue measure on the interval $(0,1)$), we have*

$$\omega_{a,p}(\operatorname{int}_{\partial\Omega} E_r) = \omega_{a,p}(E_r) = \omega_{a,p}(\overline{E}_r) \quad \textit{for all } a \in \Omega.$$

This result holds also in the weighted Euclidean setting considered in Heinonen–Kilpeläinen–Martio [9] (since we do not use the invariance of the p-harmonic measure under affine transformations in the proof below) or even more generally in the metric space setting considered in Björn–Björn–Shanmugalingam [2] and [3].

Proof. The function $r \mapsto \overline{\omega}_{a,p}(\overline{E}_r)$ is a real-valued nondecreasing function of r and hence differentiable almost everywhere in the interval $(0,1)$. We will only need that it is continuous almost everywhere. Similarly $r \mapsto \underline{\omega}_{a,p}(\overline{E}_r)$ is continuous almost everywhere. Let A be a countable dense subset of Ω and $B \subset (0,1)$ be the set of points r such that for every $a \in A$, the functions $\rho \mapsto \overline{\omega}_{a,p}(\overline{E}_\rho)$ and $\rho \mapsto \underline{\omega}_{a,p}(\overline{E}_\rho)$ are continuous at $\rho = r$. Then almost every $r \in (0,1)$ belongs to B.

If $0 < s < r < 1$, then we can find $f \in C(\partial\Omega)$ such that $f = 1$ on $\overline{E}_s$ and $f = 0$ on $\partial\Omega \setminus \operatorname{int}_{\partial\Omega} E_r$. It follows that

$$\overline{\omega}_{a,p}(\overline{E}_s) \le Pf \le \underline{\omega}_{a,p}(\operatorname{int}_{\partial\Omega} E_r) \quad \text{for all } a \in \Omega.$$

Therefore for all $a \in A$ and $r \in B$,

$$\overline{\omega}_{a,p}(\overline{E}_r) = \lim_{s\to r-} \overline{\omega}_{a,p}(\overline{E}_s) \le \underline{\omega}_{a,p}(\operatorname{int}_{\partial\Omega} E_r).$$

Combining this inequality with the obvious inequalities

$$\underline{\omega}_{a,p}(\operatorname{int}_{\partial\Omega} E_r) \le \underline{\omega}_{a,p}(E_r) \le \underline{\omega}_{a,p}(\overline{E}_r) \le \overline{\omega}_{a,p}(\overline{E}_r)$$

and

$$\underline{\omega}_{a,p}(\operatorname{int}_{\partial\Omega} E_r) \le \overline{\omega}_{a,p}(\operatorname{int}_{\partial\Omega} E_r) \le \overline{\omega}_{a,p}(E_r) \le \overline{\omega}_{a,p}(\overline{E}_r),$$

we see that

$$\omega_{a,p}(\operatorname{int}_{\partial\Omega} E_r) = \omega_{a,p}(E_r) = \omega_{a,p}(\overline{E}_r) \quad \text{for } a \in A \text{ and } r \in B.$$

Thus for $r \in B$ the continuous functions $a \mapsto \omega_{a,p}(\operatorname{int}_{\partial\Omega} E_r)$, $a \mapsto \omega_{a,p}(E_r)$ and $a \mapsto \omega_{a,p}(\overline{E}_r)$ are equal on the dense subset A of Ω, and thus by continuity they are equal on all of Ω. □

4. The main theorem

The primary aim of this note is to study some conditions on the domain Ω and set $E \subset \partial\Omega$ which guarantee that the p-harmonic measure of E relative to Ω satisfies the condition

$$\omega_{a,p}(\operatorname{int}_{\partial\Omega} E) = \omega_{a,p}(E) = \omega_{a,p}(\overline{E}).$$

The main condition involves a Riesz integral on $\partial\Omega$ which satisfies the integrability hypothesis of the trace theorem of Jonsson–Wallin [10] (see also Hajłasz–Martio [8]).

We denote the d-dimensional Hausdorff measure by Λ_d.

Theorem 4.1. *Let $1 < p < n$, $\Omega \subset \mathbf{R}^n$ be a bounded domain, $E \subset \partial\Omega$, and assume that $C_p(\partial_{\partial\Omega} E) = 0$ (which in particular holds if $\Lambda_{n-p}(\partial_{\partial\Omega} E) < \infty$). Moreover, let $U_1, \dots, U_m \subset \partial\Omega$ be relatively open sets such that $\partial\Omega = \bigcup_{j=1}^m U_j$, and for $1 \le j \le m$ let $0 \le d_{1,j} \le d_{2,j} \le n$ be such that there is a finite Borel measure μ_j with $\mu_j(\mathbf{R}^n \setminus U_j) = 0$ and*

$$Cr^{d_{2,j}} \le \mu_j(B(x,r)) \le Cr^{d_{1,j}} \quad \text{for } x \in U_j \text{ and } 0 < r \le 1. \tag{4.1}$$

For each j, $1 \le j \le m$, assume that one of the following conditions hold:

(a) $C_p(U_j \cap E) = 0$ *(which in particular holds if $\Lambda_{n-p}(U_j \cap E) < \infty$);*
(b) $C_p(U_j \setminus E) = 0$ *(which in particular holds if $\Lambda_{n-p}(U_j \setminus E) < \infty$);*
(c) $d_{2,j} \le n - p$;
(d) $d_{2,j} < n$ *and*

$$\int_{U_j \cap E} \int_{U_j \setminus E} |x-y|^{n-p-2d_{2,j}}\, d\mu_j(y)\, d\mu_j(x) < \infty; \tag{4.2}$$

(e) $d_{2,j} < n$ *and*

$$\int_{U_j \cap \overline{E}} \int_{U_j \setminus \overline{E}} |x-y|^{n-p-2d_{2,j}}\, d\mu_j(y)\, d\mu_j(x) < \infty;$$

(f) $d_{2,j} < n$ *and*

$$\int_{U_j \cap \operatorname{int}_{\partial\Omega} E} \int_{U_j \setminus \operatorname{int}_{\partial\Omega} E} |x-y|^{n-p-2d_{2,j}}\, d\mu_j(y)\, d\mu_j(x) < \infty;$$

(g) $d_{2,j} < n$, $\dim_M U_j \cap \partial_{\partial\Omega} E < n - p - 2(d_{2,j} - d_{1,j})$ *and*

$$\sup_{x \in U_j \setminus \overline{E}} \frac{\operatorname{dist}(x, U_j \cap \partial_{\partial\Omega} E)}{\operatorname{dist}(x, U_j \cap E)} < \infty; \tag{4.3}$$

(h) $\dim_M U_j \cap \partial_{\partial\Omega} E < n - p$ *and U_j is a Lipschitz graph, i.e. there is a convex set $K_j \subset \mathbf{R}^{n-1}$, a Lipschitz function $\phi_j : \mathbf{R}^{n-1} \to \mathbf{R}$ and an affine bijection $\Gamma_j : \mathbf{R}^{n-1} \times \mathbf{R} \to \mathbf{R}^n$ such that $U_j = \{\Gamma_j(x', \phi_j(x')) : x' \in K_j\}$.*

Then χ_E belongs to the trace space of $W^{1,p}(\mathbf{R}^n)$ to $\partial\Omega$ and

$$\omega_{a,p}(\operatorname{int}_{\partial\Omega} E) = \omega_{a,p}(E) = \omega_{a,p}(\overline{E}) \quad \text{for all } a \in \Omega.$$

Let us start by obtaining Propositions 1.1 and 1.2 as corollaries.

Proof of Proposition 1.2. The boundary is locally as in (h) in Theorem 4.1. That a finite number of sets U_j are sufficient follows by compactness. We can therefore apply Theorem 4.1 to complete the proof. □

Proof of Proposition 1.1. This follows directly from Proposition 1.2 for $1 < p < 2$. For $p = 2$ it follows directly from linearity together with the well-known fact that the harmonic measure of a finite set is zero. For $p > 2$ (and $m = 1$) we gave a proof in Section 3. $\square$

The method of this proof of Proposition 1.1 for $1 < p < 2$ cannot be extended to cover the case $p > 2$. In fact even in the case when $E \subset \partial \mathbf{D}$ is an open arc, χ_E does not belong to the trace space of $W^{1,p}(\mathbf{R}^2)$ when $2 \le p < \infty$.

Let us next make some comments on Theorem 4.1.

The double inequality (4.1) is essential only for the cases (c)–(g). Moreover, the upper bound in (4.1) is used only in (g). For the conditions (a), (b) and (h) the measure μ_j is not needed at all. However (4.1) is trivially satisfied if we let $d_{1,j} = 0$ and $d_{2,j} = n$, so assuming (4.1) in all cases is no extra assumption. To see this we give the following construction for μ_j: Assume, without loss of generality, that $U_j \subset I := [0,1)^n$. We let $\mu_j(U_j) = 1$. Next divide I into 2^n cubes similar to I but with half the side length. Say that a of these cubes intersect U_j, then we let each of these a cubes have measure $1/a$. In the next step we consider one of these cubes and subdivide it into cubes with half its side length. Say that b of these cubes intersect U_j, then we let each of these b cubes have the measure $1/b$ times the measure of the parent cube, i.e. the measure is $1/ab$. We proceed in this manner to obtain a measure μ_j which satisfies (4.1) with $d_{1,j} = 0$ and $d_{2,j} = n$.

In view of (c), conditions (d)–(h) are only interesting if $d_{2,j} > n - p$. If (g) is fulfilled with $d_{2,j} > n-p$, then $0 < n-p-2(d_{2,j}-d_{1,j})$, i.e. $d_{1,j} > d_{2,j} - \frac{1}{2}(n-p) > \frac{1}{2}(n-p) > 0$. Since Ω is a bounded domain, $d_{2,j} \ge n-1$ for at least one j.

If $d_{1,j} = d_{2,j}$, then μ_j is equivalent to the Hausdorff measure $\Lambda_{d_{1,j}}|_{U_j}$, see Theorem 1 on p. 32 of Jonsson–Wallin [10]. In this case U_j is *Ahlfors* $d_{1,j}$*-regular* (or a $d_{1,j}$*-set*).

Note that (4.3) is equivalent to

$$\sup_{x \in U_j \cap E} \frac{\operatorname{dist}(x, U_j \cap \partial_{\partial\Omega} E)}{\operatorname{dist}(x, U_j \setminus \overline{E})} < \infty. \tag{4.4}$$

Moreover (c) $\Rightarrow$ (a) and (h) $\Rightarrow$ (g) $\Rightarrow$ (d), see the proof below.

Let us also point out that (d) and (e) are not equivalent. Let, e.g., $1 < p < 2$, $\Omega = B(0,2) \setminus [0,1] \subset \mathbf{R}^3$, $U = U_1 = \partial\Omega$ and $\mu|_{\partial B(0,2)} = \Lambda_2|_{\partial B(0,2)}$ and $\mu|_{[0,1]}$ be the restriction of the one-dimensional Lebesgue measure. Note that $d_{1,1} = 1$ and $d_{2,1} = 2$. Further, let $C \subset [0,1]$ be a Cantor set constructed starting with $[0,1]$ so that in the kth step 2^k open intervals of length α_k are removed and the length of each of the 2^{k+1} remaining closed intervals equal $l_k = \alpha_k^{1/2}$. Let $E = [0,1] \setminus C$ and observe that $C_p(\partial_{\partial\Omega} E) \le C_p([0,1]) = 0$. Since $\overline{E} = [0,1]$, (e) holds. Let us turn to (4.2) and start by interchanging the order of integration and letting $y \in C$. Then there exist pairwise disjoint open intervals $I_k \subset E$ of length α_k with $\operatorname{dist}(y, I_k) \le l_k$. It follows that

$$\int_{U \cap E} |x-y|^{n-p-2d_{2,1}}\, d\mu(x) \ge \sum_{k=0}^{\infty} \alpha_k (2l_k)^{-1-p} = 2^{-1-p} \sum_{k=0}^{\infty} l_k^{1-p} = \infty.$$

Since $\mu(C) = 1 - \sum_{k=0}^{\infty} 2^k \alpha_k = 1 - \sum_{k=0}^{\infty} 2^k l_k^2 \ge 1 - \sum_{k=0}^{\infty} 2^k 2^{-2k-2} = \frac{1}{2}$, (d) does not hold.

Proof of Theorem 4.1. Let us first observe that if $\Lambda_{n-p}(A) < \infty$, then $C_p(A) = 0$ by Theorem 2.27 in Heinonen–Kilpeläinen–Martio [9].

By a standard partition of unity argument, we can obtain functions $\eta_1, \dots, \eta_m \in \mathrm{Lip}_c(\mathbf{R}^n)$, the space of compactly supported Lipschitz functions, such that

$$\partial\Omega \cap \operatorname{supp} \eta_j \subset U_j \quad \text{and} \quad \sum_{j=1}^m \eta_j(y) = 1 \text{ for } y \in \partial\Omega.$$

Next, for each j, $1 \le j \le m$, we want to find a p-quasicontinuous function $f_j \in W^{1,p}(\mathbf{R}^n)$ which equals χ_E on U_j. Let us postpone this part of the proof for the moment. We proceed by setting

$$f = \sum_{j=1}^m f_j \eta_j.$$

It is straightforward to see that $f = \chi_E$ on $\partial\Omega$ and that f is p-quasicontinuous. Moreover since $\eta_j \in \mathrm{Lip}_c(\mathbf{R}^n)$ it follows that $f_j\eta_j \in W^{1,p}(\mathbf{R}^n)$ and hence that $f \in W^{1,p}(\mathbf{R}^n)$.

Since $C_p(\partial_{\partial\Omega} E) = 0$, the functions f, $f - \chi_{E\cap\partial_{\partial\Omega}E}$ and $f + \chi_{\partial_{\partial\Omega}E\setminus E}$ are p-quasicontinuous representatives of the same equivalence class in $W^{1,p}(\mathbf{R}^n)$. Hence, by Corollary 5.2 in Björn–Björn–Shanmugalingam [3], we have for $a \in \Omega$,

$$\begin{aligned}
\omega_{a,p}(E) &= Pf(a) &&= Hf(a),\\
\omega_{a,p}(\operatorname{int}_{\partial\Omega} E) &= P(f - \chi_{E\cap\partial_{\partial\Omega}E})(a) &&= H(f - \chi_{E\cap\partial_{\partial\Omega}E})(a),\\
\omega_{a,p}(\overline{E}) &= P(f + \chi_{\partial_{\partial\Omega}E\setminus E})(a) &&= H(f + \chi_{\partial_{\partial\Omega}E\setminus E})(a),
\end{aligned}$$

where Hg is the p-harmonic extension of $g \in W^{1,p}(\mathbf{R}^n)$ from $\partial\Omega$ to Ω, see [3]. The operator H is independent of which representative we choose from an equivalence class in $W^{1,p}(\mathbf{R}^n)$, and thus the terms on the right-hand side of the above three identities coincide, which in turn yields the equality of the left-hand side terms.

It remains to find a p-quasicontinuous function $f_j \in W^{1,p}(\mathbf{R}^n)$ which equals χ_E on U_j for each j, $1 \le j \le m$. Here the proof depends on which of the eight conditions holds for j:

(a) We let $f_j = \chi_{U_j\cap E}$ a function which is zero p-quasieverywhere.

(b) We let $f_j = \eta - \chi_{U_j\setminus E}$, where $\eta \in \mathrm{Lip}_c(\mathbf{R}^n)$ equals one in a neighbourhood of $\overline{\Omega}$.

(c) Let $0 < \varepsilon < \frac{1}{2}$ and cover U_j with balls $B(x_k, \varepsilon)$ such that $x_k \in U_j$. If $k \le l$ and $B(x_k,\varepsilon) \cap B(x_l,\varepsilon) \ne \emptyset$, then $B(x_l,\varepsilon) \subset B(x_k, 3\varepsilon)$. By taking a subsequence (also denoted $\{x_k\}_k$ by abuse of notation) we therefore have disjoint balls $B(x_k,\varepsilon)$ such that $U_j \subset \bigcup_k B(x_k, 3\varepsilon)$. It follows, denoting the $d_{2,j}$-dimensional Hausdorff content of U_j at scale ε by $\Lambda^{\varepsilon}_{d_{2,j}}(U_j)$, that

$$\Lambda^{3\varepsilon}_{d_{2,j}}(U_j) \le \sum_k (3\varepsilon)^{d_{2,j}} = C \sum_k \varepsilon^{d_{2,j}} \le C \sum_k \mu_j(B(x_k,\varepsilon)) \le C\mu_j(U_j).$$

Letting $\varepsilon \to 0$, this shows that $\Lambda_{d_{2,j}}(U_j) \le C\mu_j(U_j) < \infty$. Therefore $\Lambda_{n-p}(U_j)$ is finite, and thus (a) and (b) are immediately true. (In fact, we have shown that the upper Minkowski content at scale ε is $\le C\mu_j(U_j)$ and thus that $\dim_M U_j \le n-p$.)

(d) We first observe that

$$Cr^{d_{2,j}} \le \mu_j(B(x,r)) \quad \text{for } x \in \overline{U}_j \text{ and } 0 < r \le 1, \tag{4.5}$$

and that (4.2) holds also if U_j is replaced by $\overline{U}_j$, since $\mu_j(\mathbf{R}^n \setminus U_j) = 0$.

A function $\mathcal{E}\chi_E \in C^\infty(\mathbf{R}^n \setminus \overline{U}_j)$ is constructed on p. 157 of Jonsson–Wallin [10]. By the trace theorem on p. 182 in [10], $\mathcal{E}\chi_E \in W^{1,p}(\mathbf{R}^n)$. (We only use the extension part of the trace result, and hence the comment following Theorem 3 on p. 197 in [10] applies here, enabling us to only require the one-sided estimate (4.5).) We let

$$f_j = \begin{cases} \mathcal{E}\chi_E & \text{on } \mathbf{R}^n \setminus \overline{U}_j, \\ \chi_E & \text{on } \overline{U}_j. \end{cases}$$

From the contruction on p. 157 in [10], it is easy to see that f_j is continuous on $\mathbf{R}^n \setminus \partial_{\partial\Omega} E$, and since $C_p(\partial_{\partial\Omega} E) = 0$, it follows that f_j is p-quasicontinuous.

(e) We let

$$f_j = \begin{cases} \mathcal{E}\chi_{\overline{E}} & \text{on } \mathbf{R}^n \setminus \overline{U}_j, \\ \chi_{\overline{E}} & \text{on } \overline{U}_j, \end{cases}$$

and proceed as in (d).

(f) We let

$$f_j = \begin{cases} \mathcal{E}\chi_{\operatorname{int}_{\partial\Omega} E} & \text{on } \mathbf{R}^n \setminus \overline{U}_j, \\ \chi_{\operatorname{int}_{\partial\Omega} E} & \text{on } \overline{U}_j, \end{cases}$$

and proceed as in (d).

(g) If $d_{2,j} \le n-p$, then (c) holds. We therefore assume that $d_{2,j} > n-p$. Our aim is now to show that condition (d) is satisfied as well.

Let us start with some estimates which follow from the condition

$$\dim_M U_j \cap \partial_{\partial\Omega} E < d_{3,j} := n - p - 2(d_{2,j} - d_{1,j}).$$

Let

$$N_j(\rho) = \min\left\{ k : U_j \cap \partial_{\partial\Omega} E \subset \bigcup_{l=1}^{k} B(x_l, \rho) \text{ for some } x_l \in \mathbf{R}^n, l = 1, \dots, k \right\}.$$

Also, let $(x_{l,\rho})_{l=1}^{N_j(\rho)}$ be some finite collection of x_l for which the minimum is attained. Since $\dim_M U_j \cap \partial_{\partial\Omega} E < d_{3,j}$ there is $\varepsilon > 0$ such that

$$N_j(\rho) \le \frac{C}{\rho^{d_{3,j}-\varepsilon}}.$$

As

$$\{x \in U_j : \operatorname{dist}(x, U_j \cap \partial_{\partial\Omega} E) < \rho\} \subset \bigcup_{l=1}^{N_j(\rho)} B(x_{l,\rho}, 2\rho),$$

and $\mu_j(B(x_{l,\rho}, 2\rho)) \le C\rho^{d_{1,j}}$ (this estimate holds also if $x_{l,\rho} \notin U_j$ or $\rho > 1$, though the constant C may not be the same as in the statement of the theorem), it follows that

$$\mu_j(\{x \in U_j : \operatorname{dist}(x, U_j \cap \partial_{\partial\Omega} E) < \rho\}) \le C N_j(\rho) \rho^{d_{1,j}} \le C\rho^{d_{1,j}-d_{3,j}+\varepsilon}. \tag{4.6}$$

Hence, using (4.4), we obtain the estimate

$$\mu_j(\{x \in U_j \cap E : \operatorname{dist}(x, U_j \setminus \overline{E}) < \rho\}) \le C\rho^{d_{1,j}-d_{3,j}+\varepsilon}. \tag{4.7}$$

We want to prove that condition (d) holds, i.e. that

$$I := \int_{U_j \cap E} \int_{U_j \setminus E} |x-y|^{n-p-2d_{2,j}} \, d\mu_j(y) \, d\mu_j(x) < \infty.$$

Let us first estimate the inner integral, fixing $x \in U_j \cap \operatorname{int}_{\partial\Omega} E$. (Note that $\mu_j(\partial_{\partial\Omega} E) = 0$ by (4.6).) Let $\delta = \delta(x) = \operatorname{dist}(x, U_j \setminus E)$. Using $n-p-2d_{2,j}+d_{1,j} \le n-p-d_{2,j} < 0$, we then find that

$$\begin{aligned}
\int_{U_j \setminus E} |x-y|^{n-p-2d_{2,j}}\, d\mu_j(y) &\le \int_{U_j \setminus B(x,\delta)} |x-y|^{n-p-2d_{2,j}}\, d\mu_j(y) \\
&\le C \sum_{j=0}^{\infty} (2^j\delta)^{n-p-2d_{2,j}} \mu_j(B(x, 2^{j+1}\delta)) \\
&\le C \sum_{j=0}^{\infty} (2^j\delta)^{n-p-2d_{2,j}} (2^{j+1}\delta)^{d_{1,j}} \\
&\le C\delta^{n-p-2d_{2,j}+d_{1,j}} = C\delta^{d_{3,j}-d_{1,j}}.
\end{aligned}$$

Inserting the above estimate into I and using $\mu_j(\partial_{\partial\Omega} E) = 0$ and (4.7) yields

$$\begin{aligned}
I &\le C \int_{U_j \cap \operatorname{int}_{\partial\Omega} E} \delta(x)^{d_{3,j}-d_{1,j}}\, d\mu_j(x) \\
&\le C \int_0^{\operatorname{diam} \partial\Omega} t^{d_{3,j}-d_{1,j}} \mu_j\left(\{x \in U_j \cap E : \delta(x) < t\}\right) \frac{dt}{t} \\
&\le C \int_0^{\operatorname{diam} \partial\Omega} t^{d_{3,j}-d_{1,j}} t^{d_{1,j}-d_{3,j}+\varepsilon} \frac{dt}{t} < \infty.
\end{aligned}$$

(h) We show that under this condition the set U_j satisfies condition (g) as well. Let M be such that $|\Gamma_j(x',\phi_j(x')) - \Gamma_j(y',\phi_j(y'))| \le M|x'-y'|$ for $x', y' \in K_j$. We want to show that

$$\sup_{x \in U_j \setminus \overline{E}} \frac{\operatorname{dist}(x, U_j \cap \partial_{\partial\Omega} E)}{\operatorname{dist}(x, U_j \cap E)} < \infty.$$

Let $x \in U_j \setminus \overline{E}$ and $y \in U_j \cap E$ be such that $|x-y| \le 2\operatorname{dist}(x, U_j \cap E)$. We further let $x', y' \in K_j$ be such that $x = \Gamma_j(x', \phi_j(x'))$ and $y = \Gamma_j(y', \phi_j(y'))$. Then there exists a minimal $\theta \in (0,1]$ such that if $z' = (1-\theta)x' + \theta y'$, then $z := \Gamma_j(z', \phi_j(z')) \in \overline{E}$. It follows from the minimality that $z \in \partial_{\partial\Omega} E$. We find by the Lipschitz property of φ_j that

$$|x-z| \le M|x'-z'| = M\theta|x'-y'| \le C|x-y|,$$

(where C depends only M and the stretching of Γ_j) and thus

$$\frac{\operatorname{dist}(x, U_j \cap \partial_{\partial\Omega} E)}{\operatorname{dist}(x, U_j \cap E)} \le 2\frac{|x-z|}{|x-y|} \le C.$$

This shows that condition (g) is fulfilled with the measure $\mu_j = \Lambda_{n-1}|_{U_j}$ and $d_{1,j} = d_{2,j} = n-1$, see, e.g., Example 3 on p. 30 in Jonsson–Wallin [10] to see that μ_j satisfies our conditions. □

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