The Perron method for p-harmonic functions in metric spaces

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Abstract

We use the Perron method to construct and study solutions of the Dirichlet problem for p-harmonic functions in proper metric measure spaces endowed with a doubling Borel measure supporting a weak $(1, q)$-Poincaré inequality (for some $1 \leq q < p$). The upper and lower Perron solutions are constructed for functions defined on the boundary of a bounded domain and it is shown that these solutions are p-harmonic in the domain. It is also shown that Newtonian (Sobolev) functions and continuous functions are resolutive, i.e. that their upper and lower Perron solutions coincide, and that their Perron solutions are invariant under perturbations of the function on a set of capacity zero. We further study the problem of resolutivity and invariance under perturbations for semicontinuous functions. We also characterize removable sets for bounded p-(super)harmonic functions.

Key words: A-harmonic, capacity, Dirichlet problem, doubling measure, energy minimizer, Newtonian space, nonlinear, Perron solution, p-harmonic, p-subharmonic, p-superharmonic, Poincaré inequality, potential, resolutive, Sobolev function, removable.
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1 Introduction

The potential theoretic construction of a solution to the Dirichlet problem for the Laplace operator using the Perron method was introduced by Oskar Perron [33] in 1923. It has been studied extensively in Euclidean domains; see for example Brelot [6,7], Kilpeläinen [17], Lindqvist–Martio [28], Lukeš–Malý–Zajíček [29], Heinonen–Kilpeläinen–Martio [11], Bauer [2], and the references therein. The advantage of the Perron method lies in the fact that it allows us to construct a reasonable solution to the Dirichlet problem for boundary data which are not necessarily continuous. The study of Perron solutions has been extended to degenerate elliptic operators in Euclidean domains in [17,28,11], and Granlund–Lindqvist–Martio [10]. Recent development in the study of Perron solutions has been in the direction of applying the method to subelliptic operators, see Markina–Vodop'yanov [31,32].

The purpose of this paper is twofold: We present some results that are new, as far as we know, in the nonlinear case ($p \neq 2$) for $\mathbf{R}^n$, $n \geq 2$, (see below). Secondly, and perhaps more importantly, we extend the Perron method to the setting of proper metric spaces endowed with a doubling measure supporting a weak Poincaré type inequality, thus unifying the theory developed in the Euclidean setting and the theory developed in Markina–Vodop'yanov [31,32] for Hörmander vector fields (leading to the study of Carnot groups), as well as extending the theory to the more general setting of Riemannian and certain sub-Riemannian manifolds such as the Carnot–Carathéodory spaces and the spaces of Bourdon–Pajot [5] and Laakso [26]. It must be noted that even in the setting of Carnot groups, our results apply to a wider class of problems than those studied in [31,32], since the minimization problem considered in this paper can be based both on the horizontal gradients of functions as in [31,32], and on the length metric given by the Carnot–Carathéodory construction.

Lukeš–Malý–Zajíček [29] developed an axiomatic potential theory in which two of the axioms assumed are the axioms of sheaf and base (see p. 328 of [29]). In the general metric measure spaces considered here, it is an open question if these axioms hold. Indeed, energy minimizers may not satisfy the following sheaf property: if u is a minimizer in the balls B_1 and B_2, then u is a minimizer in $B_1 \cup B_2$, cf. the discussion in Section V.1 in Ladyzhenskaya–Uraltseva [27] (at the same time, for solutions of an Euler–Lagrange equation the sheaf axiom of [29] holds). Moreover, the base axiom stating that the topology of the metric space admits a base consisting solely of regular domains might fail in our setting. It holds in Euclidean spaces and on Riemannian manifolds (because balls are regular) and is strongly used in the study of Perron solutions in Kilpeläinen [17], Lindqvist–Martio [28] and Heinonen–Kilpeläinen–Martio [11]. However, it is not known whether the base axiom holds on the Heisenberg group and more general Carnot groups or not. Thus, our proofs of some classical results for Perron solutions (e.g. in Section 4) are necessarily different from the proofs given in the above references.

Kurki [25] used the obstacle problem to prove that if K is a compact set and E a set of zero capacity, then the p-harmonic measure $\omega(K \cup E) = \omega(K)$ (see Section 7). His work inspired our work and many results herein are generalizations of his result.

Some of the results presented in this paper are, as far as we know, new even in Euclidean spaces, although some of these may be folklore in the mathematical community. In particular, it is shown that quasicontinuous representatives of the classical Sobolev functions are resolutive (Corollary 5.2). Our results on resolutivity properties of semicontinuous functions (Section 7), on uniqueness of p-harmonic extensions of continuous boundary data (Corollary 6.2) and on invariance under perturbations on a set of zero capacity (Theorems 5.1, 6.1 and Proposition 7.3) also seem new for degenerate elliptic operators in the Euclidean setting.

The paper is organized as follows. In the next section, some basic definitions relating to Sobolev-type spaces on metric spaces are reviewed, and in Section 3 the upper and lower Perron solutions are constructed for general boundary data on bounded domains in the metric measure space. In Section 4, it is shown that Perron solutions are p-harmonic. In subsequent sections, the question of resolutivity is studied, i.e. whether the upper and lower Perron solution coincide. It is shown that continuous as well as certain Sobolev-type functions are resolutive, see Sections 5 and 6. Resolutivity of semicontinuous functions is studied in Section 7, in particular it is shown that bounded semicontinuous functions are resolutive on regular domains. Section 8 deals with the removability of small sets for bounded p-(super)harmonic functions. Resolutivity of L^1-functions in the linear case ($p = 2$) is studied in Section 9, and in the last section, some open problems related to the Perron solutions are stated.

2 Notation and preliminaries

We assume throughout the paper that $X = (X, d, \mu)$ is a proper (i.e., closed bounded sets are compact) pathconnected metric space endowed with a metric d and a positive Borel regular measure μ which is finite on bounded sets, positive on nonempty open sets, and is *doubling*, i.e., there exists a constant $C > 0$ such that for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in X,

$$\mu(2B) \le C\mu(B),$$

where $\lambda B = B(x_0, \lambda r)$. Moreover, we fix p with $1 < p < \infty$ and suppose that X supports a weak $(1, q)$-Poincaré inequality for some $q \in [1, p)$ (see Definition 2.1 below).

A Borel function g on X is an *upper gradient* of an extended real-valued function f on X if for all rectifiable paths $\gamma : [0, l_\gamma] \to X$ parameterized by the arc length ds,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \le \int_\gamma g\, ds$$

whenever both $f(\gamma(0))$ and $f(\gamma(l_\gamma))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. If the above condition fails only for a curve family with zero p-modulus (see Definition 2.1 in Shanmugalingam [34]), then g is a *p-weak upper gradient* of u. It is known that the L^p-closed convex hull of the set of all upper gradients of u that are in $L^p(X)$ is precisely the set of all p-weak upper gradients of u in $L^p(X)$; see Lemma 2.4 in Koskela–MacManus [24].

Definition 2.1 *We say that X supports a* weak $(1, q)$-Poincaré inequality *if there exist constants $C > 0$ and $\lambda \ge 1$ such that for all balls $B \subset X$, all measurable functions f on X and all upper gradients g of f,*

$$\fint_B |f - f_B|\, d\mu \le Cr \left(\fint_{\lambda B} g^q\, d\mu \right)^{1/q},$$

where r is the radius of B and

$$f_B := \fint_B f\, d\mu := \frac{1}{\mu(B)} \int_B f\, d\mu.$$

By Hölder's inequality it is easy to see that if X supports a weak $(1, q)$-Poincaré inequality, then it supports a weak $(1, s)$-Poincaré inequality for every $s > q$. In the above definition of Poincaré inequality we can equivalently assume that g is a p-weak upper gradient – see the comments above.

Following Shanmugalingam [34], we define a version of Sobolev spaces on the metric space X.

Definition 2.2 *Whenever $u \in L^p(X)$, let*

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p\, d\mu \right)^{1/p} + \inf_g \left(\int_X g^p\, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients of u. The Newtonian space *on X is the quotient space*

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

The space $N^{1,p}(X)$ is a Banach space and a lattice, see Shanmugalingam [34]. Note that all representatives of an equivalence class coincide p-q.e. (see below for the definition of p-q.e.) and are p-quasicontinuous, see [34]. This means that in the Euclidean setting, $N^{1,p}(X)$ is the refined Sobolev space as defined on p. 96 of Heinonen–Kilpeläinen–Martio [11].

Cheeger [8] gives an alternative definition of Sobolev spaces which leads to the same space, see Theorem 4.10 in [34]. Cheeger's definition yields a notion of partial derivatives in the following theorem (Theorem 4.38 in [8]).

Theorem 2.3 *Let X be a metric measure space equipped with a positive doubling Borel regular measure μ. Assume that X admits a weak $(1,p)$-Poincaré inequality for some $1 < p < \infty$.*

Then there exists a countable collection (U_α, X^α) of measurable sets U_α and Lipschitz "coordinate" functions $X^\alpha = (X_1^\alpha, \dots, X_{k(\alpha)}^\alpha) : X \to \mathbf{R}^{k(\alpha)}$ such that $\mu\bigl(X \setminus \bigcup_\alpha U_\alpha\bigr) = 0$ and for all α, the following hold.

The functions $X_1^\alpha, \dots, X_{k(\alpha)}^\alpha$ are linearly independent on U_α and $1 \le k(\alpha) \le N$, where N is a constant depending only on the doubling constant of μ and the constants from the Poincaré inequality. If $f : X \to \mathbf{R}$ is Lipschitz, then there exist unique measurable bounded vector-valued functions $d^\alpha f : U_\alpha \to \mathbf{R}^{k(\alpha)}$ such that for μ-a.e. $x_0 \in U_\alpha$, the following Taylor theorem holds:

$$\lim_{r \to 0+} \sup_{x \in B(x_0,r)} \frac{|f(x) - f(x_0) - \langle d^\alpha f(x_0), X^\alpha(x) - X^\alpha(x_0)\rangle|}{r} = 0.$$

The functions $d^\alpha f(x_0)$ clearly depend on the "basis" X^α. Following the discussion on p. 460 in Cheeger [8], we introduce a norm $|\cdot|_{1,x_0}$ of $d^\alpha f(x_0)$ such that

$$|d^\alpha f(x_0)|_{1,x_0} = g_f(x_0) := \inf_g \limsup_{r \to 0+} \fint_{B(x,r)} g \, d\mu,$$

where g_f is the *minimal p-weak upper gradient* of f (see Corollary 3.7 in Shanmugalingam [35] and Lemma 2.3 in Björn [4]) and the infimum is taken over all upper gradients g of f. Then one can find an inner product norm $|\cdot|_{x_0}$, which is C-quasiisometric to $|\cdot|_{1,x_0}$, where the constant C only depends on $k(\alpha)$, see p. 460 in [8].

We can assume that the sets U_α are pairwise disjoint and extend $d^\alpha f$ by zero outside U_α. Regard $d^\alpha f(x)$ as vectors in $\mathbf{R}^N$ and let $Df = \sum_\alpha d^\alpha f$. The differential mapping $D : f \mapsto Df$ is linear and it follows from the discussion above that there is a constant $C > 0$, depending only on N, such that for all

Lipschitz functions f and μ-a.e. $x \in X$,

$$\frac{1}{C}|Df(x)| \le g_f(x) \le C|Df(x)|. \tag{2.1}$$

Here and throughout this paper, by $|Df(x)|$ we mean $|d^\alpha f(x)|_x$ whenever $x \in U_\alpha$. Note that Df is bounded with respect to this inner product norm, since g_f is bounded for Lipschitz functions f.

By Proposition 2.2 in [8], $Df = 0$ μ-a.e. on every set where f is constant. By Theorem 4.47 in [8] or Theorem 4.1 in [34], the Newtonian space $N^{1,p}(X)$ is equal to the closure in the $N^{1,p}(X)$-norm of the collection of Lipschitz functions on X with finite $N^{1,p}(X)$-norm. By Theorem 10 in Franchi–Hajłasz–Koskela [9], there exists a unique "gradient" Du satisfying (2.1) for every $u \in N^{1,p}(X)$. Moreover, if $\{u_j\}_{j=1}^\infty$ is a sequence in $N^{1,p}(X)$, then $u_j \to u$ in $N^{1,p}(X)$ if and only if $u_j \to u$ in $L^p(X)$ and $Du_j \to Du$ in $L^p(X; \mathbf{R}^N)$ as $j \to \infty$.

Definition 2.4 *The* p-capacity *of a Borel set* $E \subset X$ *is the number*

$$C_p(E) = \inf \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ *such that* $u = 1$ *on* E.

Remark 2.5 *For equivalent definitions of the p-capacity we refer to Kilpeläinen–Kinnunen–Martio [18] and Kinnunen–Martio [20], where it is also proven that the p-capacity is a Choquet capacity. By Theorem 1.1 in Kallunki–Shanmugalingam [16] and Proposition 4.4 in Heinonen–Koskela [12], for compact sets E it is sufficient to consider only compactly supported Lipschitz functions u in the definition of p-capacity.*

We say that a property regarding points in X holds *p-quasieverywhere* (p-q.e.) if the set of points for which the property does not hold has p-capacity zero. The p-capacity is the correct gauge for distinguishing between two Newtonian functions. In particular, Corollary 3.3 in Shanmugalingam [34] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ μ-a.e., then $u = v$ p-q.e. and $\|u - v\|_{N^{1,p}(X)} = 0$. Moreover, if we redefine a function $u \in N^{1,p}(X)$ on a set of p-capacity zero, then the new function remains a representative of the same equivalence class in $N^{1,p}(X)$.

To be able to compare the boundary values of Newtonian functions we need a Newtonian space with zero boundary values. Let $\Omega \subset X$ be an open set and let

$$N_0^{1,p}(\Omega) = \{u \in N^{1,p}(X) : u = 0 \ p\text{-q.e. on } X \setminus \Omega\}.$$

Corollary 3.9 in Shanmugalingam [34] implies that $N_0^{1,p}(\Omega)$ equipped with the $N^{1,p}(X)$-norm is a closed subspace of $N^{1,p}(X)$. Note also that if $C_p(X \setminus \Omega) = 0$, then $N^{1,p}_0(\Omega) = N^{1,p}(X)$. By Theorem 4.8 in Shanmugalingam [35], the space $\mathrm{Lip}_c(\Omega)$ of Lipschitz functions with compact support in Ω is dense in $N^{1,p}_0(\Omega)$.

In the rest of this paper, unless otherwise stated, $\Omega \subset X$ will always denote a bounded domain (i.e. a nonempty open pathconnected set) in X such that $C_p(X \setminus \Omega) > 0$.

By a continuous function we always mean a real-valued continuous function, whereas a semicontinuous function is allowed to be extended real-valued, i.e. to take values in the extended real line $\overline{\mathbf{R}} := [-\infty, \infty]$.

3 Perron solutions and p-(super)harmonic functions

Definition 3.1 *Let Ω be an arbitrary domain in X. We say that $f \in N^{1,p}_{\mathrm{loc}}(\Omega)$ if for every $\eta \in \mathrm{Lip}_c(\Omega)$ we have $\eta f \in N^{1,p}(X)$. Furthermore, we say that $f_j \to f$ in $N^{1,p}_{\mathrm{loc}}(\Omega)$, as $j \to \infty$, if for every $\eta \in \mathrm{Lip}_c(\Omega)$, $\eta f_j \to \eta f$ in $N^{1,p}(X)$ as $j \to \infty$.*

Definition 3.2 *Let Ω be an arbitrary domain in X. A function u is* p-harmonic *in Ω if it is continuous, belongs to $N^{1,p}_{\mathrm{loc}}(\Omega)$, and satisfies*

$$\int_{\operatorname{supp}\varphi} g_u^p \, d\mu \le \int_{\operatorname{supp}\varphi} g_{u+\varphi}^p \, d\mu \quad \textit{for all } \varphi \in \mathrm{Lip}_c(\Omega). \tag{3.1}$$

A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is Cheeger p-harmonic *in Ω if it is continuous and*

$$\int_{\operatorname{supp}\varphi} |Du|^p \, d\mu \le \int_{\operatorname{supp}\varphi} |D(u+\varphi)|^p \, d\mu \quad \textit{for all } \varphi \in \mathrm{Lip}_c(\Omega). \tag{3.2}$$

In the above definition, inequality (3.2) can be replaced by the following equation to yield an equivalent definition of Cheeger p-harmonicity:

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, d\mu = 0 \quad \text{for all } \varphi \in \mathrm{Lip}_c(\Omega), \tag{3.3}$$

where the inner product is coming from the inner product norm, $|\cdot|_x$, see the comments after Theorem 2.3.

It should be noted that in some of the literature Cheeger p-harmonic functions are also called p-harmonic functions. However, in the Euclidean setting, with the Euclidean gradient playing the role of Cheeger derivative, the two definitions coincide since $|\nabla u| = |Du| = g_u$ in this case. All the results given in this paper for p-harmonic functions hold also for Cheeger p-harmonic functions, with easy modifications of the proofs, essentially just replacing g_u by $|Du|$.

The results and proofs given in this paper also hold for $\mathcal{A}$-harmonic functions as defined on p. 57 of Heinonen–Kilpeläinen–Martio [11], assuming that $\mathcal{A}$ satisfies the degenerate ellipticity conditions (3.3)–(3.7) on p. 56 of [11]. In Section 9 we give some results that we have only been able to obtain for Cheeger two-harmonic functions. They hold for $\mathcal{A}$-harmonic functions as well if $\mathcal{A}$ is linear in the second variable and $p = 2$.

Note that a p-harmonic function on Ω is a p-quasiminimizer in every subdomain $\Omega' \Subset \Omega$ in the sense of Kinnunen–Shanmugalingam [22]. Hence, the results of [22] apply to p-harmonic functions. Let us mention some of them. By Proposition 3.8 and Corollary 5.5 in [22], a function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ satisfying (3.1) can be modified on a set of p-capacity zero so that it becomes locally Hölder continuous in Ω. By a p-harmonic function we always mean this continuous representative of u. By Corollary 6.4 in [22], p-harmonic functions satisfy the strong maximum principle: If u attains its minimum or maximum in Ω, then it is constant. Nonnegative p-harmonic functions satisfy the Harnack inequality $\sup_K u \le C_K \inf_K u$ for all compact $K \subset \Omega$, by Corollary 7.7 in [22] together with a covering argument.

The sum of two p-harmonic functions is, in general, not a p-harmonic function (the sum of two Cheeger two-harmonic functions is however always a Cheeger two-harmonic function); nevertheless, if u is p-harmonic and $\alpha, \beta \in \mathbf{R}$, then $\alpha u + \beta$ is also p-harmonic.

Definition 3.3 *By the* p-harmonic extension *of $f \in N^{1,p}(X)$ to a bounded domain V with $C_p(X \setminus V) > 0$ we mean the function $H_V f \in N^{1,p}(X)$ which is p-harmonic in V and satisfies $H_V f = f$ in $X \setminus V$.*

When $V = \Omega$ we usually suppress the index and merely write Hf.

By saying that Hf is p-harmonic in Ω we mean that there is a representative in the equivalence class which is p-harmonic, and hence continuous, in Ω. When we refer to Hf in Ω we always refer to this continuous representative. In some proofs it is advantageous to have Hf defined in all of X, in these cases when referring to Hf we always refer to the representative that is continuous in Ω and equals f outside of Ω.

The existence and uniqueness of p-harmonic functions with prescribed Newtonian boundary data is proved in Theorem 5.6 in Shanmugalingam [35], see also Cheeger [8] and Heinonen–Kilpeläinen–Martio [11]. Note that for the coercivity of the p-energy functional $u \mapsto \int_\Omega g_u^p \, d\mu$ it is necessary and sufficient that $C_p(X \setminus \Omega)$ be positive, see Björn [4], i.e. under our standing assumption $C_p(X \setminus \Omega) > 0$, the function $H_V f$ in Definition 3.3 exists uniquely.

The comparison principle for p-harmonic extensions of functions from $N^{1,p}(X)$ says that $Hf_1 \le Hf_2$ in Ω whenever $f_1 \le f_2$ p-q.e. on $\partial\Omega$, see Theorem 6.4 in

Shanmugalingam [35]. Note that for the validity of the comparison principle in our setting it is essential that Ω is bounded, $C_p(X \setminus \Omega) > 0$ and X is proper. The comparison principle may fail if any of these assumptions is omitted.

Following Björn–Björn–Shanmugalingam [3] we consider the definition below (the operator called H here was called H_p in [3]).

Definition 3.4 *Given $\varphi \in C(\partial\Omega)$, the space of real-valued continuous functions on $\partial\Omega$, define $H\varphi : \Omega \to \mathbf{R}$ by*

$$H\varphi(x) = \sup_{\mathrm{Lip}(\partial\Omega) \ni \psi \le \varphi} H\psi(x), \quad x \in \Omega.$$

Here we abuse notation since if $\varphi \in N^{1,p}(X)$, then $H\varphi$ has already been defined by Definition 3.3. However, since continuous functions can be uniformly approximated by Lipschitz functions, the comparison principle shows that the two definitions of $H\varphi$ coincide in this case.

The function $H\varphi$ is called the *p-harmonic extension* of φ to Ω.

The comparison principle extends immediately to p-harmonic extensions of functions in $C(\partial\Omega)$ in the sense that if $\varphi, \psi \in C(\partial\Omega)$ such that $\varphi \ge \psi$, then $H\varphi \ge H\psi$ in Ω. Recall also the following lemma, Lemma 3.7 in [3].

Lemma 3.5 *Let $\varphi \in C(\partial\Omega)$. Then $H\varphi$ is a p-harmonic function in Ω and*

$$H\varphi(x) = \inf_{\mathrm{Lip}(\partial\Omega) \ni \psi \ge \varphi} H\psi(x) = \lim_{j\to\infty} H\varphi_j(x), \quad x \in \Omega,$$

for every sequence $\{\varphi_j\}_{j=1}^\infty$ of functions in Lip$(\partial\Omega)$ *converging uniformly to φ.*

Definition 3.6 *A point $x \in \partial\Omega$ is* p-regular *if*

$$\lim_{\Omega \ni y \to x} H\varphi(y) = \varphi(x) \quad \textit{for all } \varphi \in C(\partial\Omega).$$

If $x \in \partial\Omega$ is not p-regular, then we call it p-irregular. *If every $x \in \partial\Omega$ is p-regular, then the domain Ω is* p-regular.

We will demonstrate that $H\varphi$ can be replaced by the Perron solution $P\varphi$, see Theorem 6.1. The following result was proved in Björn–Björn–Shanmugalingam [3], Theorem 3.9.

Theorem 3.7 (The Kellogg property) *The set of all p-irregular points on $\partial\Omega$ has p-capacity zero.*

Let us also recall the following convergence theorems from Shanmugalingam [36], Theorem 1.1 and Proposition 4.1.

Proposition 3.8 *Assume that $\{f_j\}_{j=1}^\infty$ is a monotone sequence and that $f_j \to f$ in $N^{1,p}(X)$. Then a subsequence $Hf_j \to Hf$ both locally uniformly in Ω and weakly in $N^{1,p}(X)$.*

Proposition 3.9 *Let $\{u_j\}_{j=1}^\infty$ be a sequence of nonnegative p-harmonic functions on Ω. If there is some $x \in \Omega$ and a constant C such that $u_j(x) \le C$ for all j, then some subsequence of $\{u_j\}_{j=1}^\infty$ converges locally uniformly to a p-harmonic function on Ω.*

We follow Kinnunen–Martio [21], Section 7, in considering the following definition.

Definition 3.10 *A function $u : \Omega \to (-\infty, \infty]$ is* p-superharmonic *in Ω if*

(a) *u is lower semicontinuous;*
(b) *u is not identically ∞ in Ω;*
(c) *for every domain $\Omega' \Subset \Omega$ and all functions $v \in C(\overline{\Omega}') \cap N^{1,p}(\Omega')$, we have $H_{\Omega'} v \le u$ in Ω' whenever $v \le u$ on $\partial\Omega'$.*

A function $u : \Omega \to [-\infty, \infty)$ is p-subharmonic *if $-u$ is p-superharmonic.*

(Condition (c) can equivalently be required to hold for all $v \in \mathrm{Lip}_c(X)$.)

If u and v are p-superharmonic, $\alpha \ge 0$ and $\beta \in \mathbf{R}$, then $\alpha u + \beta$ and $\min\{u, v\}$ are p-superharmonic, but in general $u + v$ is not p-superharmonic. Moreover a p-superharmonic function u is *lower semicontinuously regularized*, i.e. $u(x) = \operatorname{ess\,lim\,inf}_{y \to x} u(y)$, see Kinnunen–Martio [21], Theorem 7.14.

Our principal source for the theory of p-superharmonic functions is [21]. For readers interested only in the (weighted) Euclidean case, all the necessary results from [21] can be found in Heinonen–Kilpeläinen–Martio [11], Chapters 3 and 7.

Following [11], Chapter 9, we define Perron solutions as follows.

Definition 3.11 *Given a function $f : \partial\Omega \to \overline{\mathbf{R}}$, let $\mathcal{U}_f$ be the set of all p-superharmonic functions u on Ω bounded below such that*

$$\liminf_{\Omega \ni y \to x} u(y) \ge f(x) \quad \textit{for all } x \in \partial\Omega.$$

Define the upper Perron solution *of f by*

$$\overline{P} f(x) = \inf_{u \in \mathcal{U}_f} u(x), \quad x \in \Omega.$$

Similarly, let $\mathcal{L}_f$ be the set of all p-subharmonic functions u on Ω bounded *above such that*

$$\limsup_{\Omega \ni y \to x} u(y) \le f(x) \quad \textit{for all } x \in \partial\Omega,$$

and define the lower Perron solution *of f by*

$$\underline{P}f(x) = \sup_{u \in \mathcal{L}_f} u(x), \quad x \in \Omega.$$

If $\overline{P}f = \underline{P}f$, then we let $Pf := \overline{P}f$, and f is said to be resolutive.

The following comparison principle shows that $\underline{P}f \le \overline{P}f$ for all functions f.

Theorem 3.12 (Kinnunen–Martio [21], Theorem 7.2) *Assume that u is p-superharmonic and that v is p-subharmonic in Ω. If*

$$\limsup_{\Omega \ni y \to x} v(y) \le \liminf_{\Omega \ni y \to x} u(y) \quad \textit{for all } x \in \partial\Omega,$$

and if both sides are not simultaneously ∞ or $-\infty$, then $v \le u$ in Ω.

4 Harmonicity of Perron solutions

Theorem 4.1 *For every function $f : \partial\Omega \to \overline{\mathbf{R}}$, the upper Perron solution $\overline{P}f$ is p-harmonic in Ω or is identically $\pm\infty$.*

In the Euclidean setting, the proofs of the above proposition used the base axiom (see the introduction); we have no such property in the general setting of metric measure spaces. Therefore the proof given below differs from the classical proof.

In order to prove this theorem, we need a *Poisson modification* for nonregular domains. See Heinonen–Kilpeläinen–Martio [11], Lemma 7.14, for an analogue of Lemma 4.2 for regular domains.

Lemma 4.2 *Let $\Omega' \Subset \Omega$ be a subdomain and u be a p-superharmonic function in Ω locally bounded from above. Let*

$$u'(x) = \begin{cases} u(x), & \textit{if } x \in \Omega \setminus \overline{\Omega}', \\ H_{\Omega'}u(x), & \textit{if } x \in \Omega', \\ \min\{u(x), \liminf_{\Omega' \ni y \to x} H_{\Omega'}u(y)\}, & \textit{if } x \in \partial\Omega'. \end{cases}$$

Then u' is p-superharmonic in Ω and $u' \le u$ in Ω.

By Corollary 7.8 of Kinnunen–Martio [21] we see that as u is locally bounded above in Ω, $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$. By $H_{\Omega'}u$ we mean the p-harmonic extension of

$u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ to Ω', which is defined as the p-harmonic extension of any function $\tilde{u} \in N^{1,p}(X)$ such that $\tilde{u} = u$ in a neighbourhood of $\overline{\Omega}'$. Note that in Ω', the p-harmonic extension is independent of the choice of $\tilde{u}$.

PROOF. Let

$$u''(x) = \begin{cases} u(x), & \text{if } x \in \Omega \setminus \Omega', \\ H_{\Omega'}u(x), & \text{if } x \in \Omega'. \end{cases}$$

Note that u' is the lower semicontinuous regularization of u''.

Since u is p-superharmonic, it follows from Lemma 3.4 in [21] that $H_{\Omega'}u \le u$ μ-a.e. in Ω'. Since u is lower semicontinuously regularized and $H_{\Omega'}u$ is continuous, it follows that $H_{\Omega'}u \le u$ in Ω'. Thus $u' \le u'' \le u$ in Ω and the last part is proved.

It remains to prove that u' is p-superharmonic in Ω. Parts (a) and (b) of Definition 3.10 are clear. Let $V \Subset \Omega$ be a domain and $v \in C(\overline{V}) \cap N^{1,p}(V)$ such that $v \le u'$ on ∂V. We need to show that $H_V v \le u'$ in V. Since $v \le u$ on ∂V, the comparison principle yields that $H_V v \le H_V u \le u$ in V (where the latter inequality is proved in the same way as the inequality $H_{\Omega'}u \le u$ above). This yields $H_V v \le u = u''$ in $V \setminus \Omega'$. In particular, $H_V v \le u''$ on $\partial V' \setminus \Omega'$, where $V' = V \cap \Omega'$. On $\partial V' \cap \Omega' \subset \partial V$ we have $H_V v = v \le u' = u''$. The comparison principle shows that

$$H_V v = H_{V'}(H_V v) \le H_{V'}u'' = u'' \quad \text{in } V'.$$

Thus $H_V v \le u''$ in V. Since $H_V v$ is continuous in V and $u' = u''$ in $\Omega' \cup (\Omega \setminus \overline{\Omega'})$, it follows that $H_V v \le u'$ in V. $\square$

Proof of Theorem 4.1 If $\mathcal{U}_f$ is empty, then $\overline{P}f = +\infty$. Assume therefore that $\mathcal{U}_f \ne \emptyset$. Let $\Omega'' \Subset \Omega' \Subset \Omega$ be subdomains.

Let $v \in \mathcal{U}_f$ be arbitrary and $v_m = \min\{v, m\}$, $m \in \mathbf{Z}_+$. Then v_m is p-superharmonic and $v_m \in N^{1,p}_{\mathrm{loc}}(\Omega)$, by Corollary 7.8 of Kinnunen–Martio [21]. Lemma 4.2 applied to v_m provides us with a new p-superharmonic function v'_m such that $v'_m = H_{\Omega'}v_m$ in Ω' and $v'_m \le v_m$ in Ω. Let $v' = \lim_{m\to\infty} v'_m$, which is p-superharmonic in Ω by Lemma 7.1 in [21] (note that $\{v'_m\}_{m=1}^\infty$ is an increasing sequence of functions, and that v' is not identically ∞ since $v' \le v$). The functions v'_m are p-harmonic in Ω', and hence by Proposition 3.9 so is v'. (Here we have used the fact that v'_1 is lower semicontinuous and hence bounded from below in Ω'.) Therefore, v' is continuous in Ω'. As $v' = v$ in $\Omega \setminus \overline{\Omega}'$, we have $\overline{P}f = \inf_{v\in\mathcal{U}_f} v'$. It follows that the upper Perron solution $\overline{P}f$ is upper semicontinuous in Ω'.

Since the measure on X is doubling, X is a separable metric space. Let $Z = \{z_1, z_2, \dots\}$ be a countable dense subset of Ω and for each $j = 1, 2, \dots$, find p-superharmonic functions $u_{j,k} \in \mathcal{U}_f$ so that $\lim_{k\to\infty} u_{j,k}(z_j) = \overline{P}f(z_j)$. As the minimum of two p-superharmonic functions is also p-superharmonic, we can by a diagonalization argument find a pointwise decreasing sequence $\{u_j\}_{j=1}^\infty$ in $\mathcal{U}_f$ so that $\overline{P}f(z) = \lim_{j\to\infty} u_j(z)$ for all $z \in Z$.

Let u'_j be the p-superharmonic functions obtained from u_j as above and $u = \lim_{j\to\infty} u'_j$. Then $\overline{P}f \le u$ in Ω. At the same time, we have for all $z \in Z$,

$$\overline{P}f(z) = \lim_{j\to\infty} u_j(z) \ge \lim_{j\to\infty} u'_j(z) = u(z),$$

i.e. $u = \overline{P}f$ on Z. The function u'_1 is p-harmonic on Ω', and thus bounded on Ω''. Applying Proposition 3.9 (to $\{(\sup_{\Omega''} u'_1) - u'_j\}_{j=1}^\infty$) shows that u is p-harmonic or is identically $-\infty$ in Ω''. If $u \equiv -\infty$, then also $\overline{P}f \equiv -\infty$. Otherwise, u is continuous in Ω'' and using the upper semicontinuity of $\overline{P}f$, we find that

$$u(x) \ge \overline{P}f(x) \ge \limsup_{Z \ni z \to x} \overline{P}f(z) = \limsup_{Z \ni z \to x} u(z) = u(x) \quad \text{for } x \in \Omega'',$$

i.e., $\overline{P}f = u$ is p-harmonic in Ω''.

Now let $\varphi \in \mathrm{Lip}_c(\Omega)$, then there are domains Ω' and Ω'' such that $\operatorname{supp} \varphi \subset \Omega'' \Subset \Omega' \Subset \Omega$. Since $\overline{P}f$ is p-harmonic in Ω'', it is continuous there and

$$\int_{\operatorname{supp} \varphi} g^p_{\overline{P}f} \, d\mu \le \int_{\operatorname{supp} \varphi} g^p_{\overline{P}f + \varphi} \, d\mu.$$

As $\varphi \in \mathrm{Lip}_c(\Omega)$ was arbitrary, it follows that $\overline{P}f$ is continuous in Ω, $\overline{P}f \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and $\overline{P}f$ is p-harmonic in Ω. □

Let us remark that the last paragraph of the proof above was needed since it is not known if p-harmonicity satisfies the sheaf property.

5 Resolutivity of Newtonian functions

Theorem 5.1 *If $f \in N^{1,p}(X)$, then f is resolutive and $Pf = Hf$.*

Note that every function $f \in N^{1,p}(X)$ is well-defined outside a set of zero p-capacity and is p-quasicontinuous. By stating that $f \in N^{1,p}(X)$ we mean that f is a function defined everywhere on X and is a representative of an equivalence class in $N^{1,p}(X)$. The classical (possibly weighted) Sobolev functions on Euclidean domains are well-defined only up to sets of measure zero and our construction of Sobolev-type spaces isolates the p-quasicontinuous representatives in each equivalence class. Hence, we have the following corollary of Theorem 5.1.

Corollary 5.2 *If f is a p-quasicontinuous function in the Sobolev space $W^{1,p}(\mathbf{R}^n)$, then f is resolutive and $Pf = Hf$.*

It should be observed that this result is not true if f is allowed to be an arbitrary representative of a Sobolev function. If the domain Ω is reasonably regular, then $\partial\Omega$ has zero Lebesgue measure, and hence any function on $\partial\Omega$ occurs as a restriction of a representative of a Sobolev function. Since Pf only depends on the restriction of f to $\partial\Omega$, it is natural to consider the trace class on $\partial\Omega$, which is obtained by taking restrictions of the p-quasicontinuous representatives in the Sobolev space.

Since Hf is independent of which (p-quasicontinuous) representative we choose from a given equivalence class, it also follows that Pf is independent of the choice of representative. Thus, the above theorem shows that all representatives in the equivalence class of a function in $N^{1,p}(X)$ agree with each other well enough to yield the same Perron solution in Ω.

In order to prove Theorem 5.1 we will need the following results.

Lemma 5.3 *Let $\{U_k\}_{k=1}^{\infty}$ be a decreasing sequence of open sets such that $C_p(U_k) < 1/2^{kp}$. Then there exists a decreasing sequence of nonnegative functions $\{\psi_j\}_{j=1}^{\infty}$ such that $\|\psi_j\|_{N^{1,p}(X)} < 1/2^j$ and $\psi_j \geq k - j$ in U_k whenever $k > j$. In particular, $\psi_j = +\infty$ on $\bigcap_{k=1}^{\infty} U_k$.*

PROOF. Since $C_p(U_k) < 1/2^{kp}$ there is a nonnegative function $f_k \in N^{1,p}(X)$ so that $f_k = 1$ in U_k and $\|f_k\|_{N^{1,p}(X)} < 1/2^k$. Let $\psi_j = \sum_{k=j+1}^{\infty} f_k$. Then $\|\psi_j\|_{N^{1,p}(X)} < 1/2^j$ and $\psi_j \geq k - j$ in U_k, $k > j$. $\square$

Next we follow Kinnunen–Martio [21], Section 3, for the definition of the obstacle problem.

Definition 5.4 *Let $\psi \in N^{1,p}(\Omega)$ and*

$$\mathcal{K}_\psi(\Omega) = \{v \in N^{1,p}(\Omega) : v - \psi \in N_0^{1,p}(\Omega) \text{ and } v \geq \psi \ \mu\text{-a.e. in } \Omega\}.$$

A function $u \in \mathcal{K}_\psi(\Omega)$ is a solution of the obstacle problem in Ω with obstacle and boundary values ψ *if*

$$\int_\Omega g_u^p \, d\mu \leq \int_\Omega g_v^p \, d\mu \quad \text{for all } v \in \mathcal{K}_\psi(\Omega).$$

By Theorem 3.2 in Kinnunen–Martio [21], there is a unique solution up to sets of p-capacity zero. By defining $u^*(x) = \operatorname{ess\,lim\,inf}_{y\to x} u(y)$, we obtain a unique lower semicontinuously regularized solution in the same equivalence class as u, see Theorem 5.1 in [21]. Moreover, if ψ is itself p-harmonic in Ω, then the unique solution to the obstacle problem with obstacle and boundary values ψ is ψ itself.

A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *p-superminimizer* in Ω, if it is a solution of the obstacle problem with itself as obstacle and boundary values in every subdomain $\Omega' \Subset \Omega$. The unique lower semicontinuously regularized representative of u is not only a p-superminimizer but also p-superharmonic. It is easily seen that every solution to an obstacle problem on Ω is a p-superminimizer. If u and $-u$ are p-superminimizers in Ω then u is a *p-energy minimizer* in Ω and there is a p-harmonic representative in the same equivalence class in $N^{1,p}_{\mathrm{loc}}(\Omega)$ as u, see Section 3 in [21].

The following result is from Kinnunen–Shanmugalingam [23]; for completeness we include the proof here. In the weighted Euclidean setting it appears as Theorem 3.79 in Heinonen–Kilpeläinen–Martio [11].

Proposition 5.5 *Let $\{\psi_j\}_{j=1}^\infty$ be a p-q.e. decreasing sequence of nonnegative functions in $N^{1,p}(X)$ so that $\psi_j \to \psi$ in $N^{1,p}(X)$. Let u_j be a solution to the obstacle problem in Ω with obstacle and boundary values ψ_j. Then there exists a function $u \in N^{1,p}(X)$ so that $\{u_j\}_{j=1}^\infty$ decreases p-q.e. in Ω to u and u is a solution to the obstacle problem in Ω with obstacle and boundary values ψ.*

The above proposition is a complementary result to Theorem 6.1 of Kinnunen–Martio [21]. To prove this proposition we need the following lemma.

Lemma 5.6 *Let $\psi \in N^{1,p}(X)$ be a nonnegative function and let v be a solution to the obstacle problem in Ω with obstacle and boundary values ψ. If u is a p-superminimizer in Ω with $\min\{u,v\} \in \mathcal{K}_\psi(\Omega)$, then $u \ge v$ p-q.e. in Ω. Moreover, denoting their lower semicontinuous regularizations by $\tilde u$ and $\tilde v$, we have $\tilde u \ge \tilde v$ everywhere in Ω.*

PROOF. The proof is similar to the proof of Lemma 3.5 in Kinnunen–Martio [21]. The function v is a p-superminimizer in Ω and by Lemma 3.3 in [21], so is $\min\{u,v\}$. As $v - \min\{u,v\} \in N^{1,p}_0(\Omega)$ is nonnegative, and $\mathrm{Lip}_c(\Omega)$ is dense in $N^{1,p}_0(\Omega)$, for every $\varepsilon > 0$ there is a nonnegative function $\varphi \in \mathrm{Lip}_c(\Omega)$ so that $\|v - \min\{u,v\} - \varphi\|_{N^{1,p}} < \varepsilon$. Let $\Omega' \Subset \Omega$ be a domain containing the support of φ. Then $\min\{u,v\} + \varphi \in \mathcal{K}_{\min\{u,v\}}(\Omega')$ and hence

$$\left(\int_\Omega g^p_{\min\{u,v\}}\,d\mu\right)^{1/p} \le \left(\int_\Omega g^p_{\min\{u,v\}+\varphi}\,d\mu\right)^{1/p} \le \left(\int_\Omega g^p_v\,d\mu\right)^{1/p} + \varepsilon.$$

Letting $\varepsilon \to 0$ and the fact that $\min\{u, v\} \in \mathcal{K}_\psi(\Omega)$ show that $\min\{u, v\}$ is a solution to the obstacle problem in Ω with the obstacle and boundary values ψ. By uniqueness, $v = \min\{u, v\}$ p-q.e. in Ω, which completes the first part of the proof. As for the second part, we have

$$\tilde{v}(x) = \operatorname{ess} \liminf_{y \to x} \tilde{v}(y) \le \operatorname{ess} \liminf_{y \to x} \tilde{u}(y) = \tilde{u}(x), \quad x \in \Omega. \qquad \square$$

Now we are ready to prove Proposition 5.5.

Proof of Proposition 5.5 Without loss of generality we may assume that u_j, $j = 1, 2, \dots$, are lower semicontinuously regularized.

As $u_j \ge \psi_j \ge \psi_{j+1}$ μ-a.e. in Ω we get that $\min\{u_j, u_{j+1}\} \ge \psi_{j+1}$ μ-a.e. in Ω. Furthermore,

$$\min\{u_j, u_{j+1}\} - \psi_{j+1} = \min\{\psi_j, \psi_{j+1}\} - \psi_{j+1} = 0 \quad p\text{-q.e. on } X \setminus \Omega,$$

i.e. $\min\{u_j, u_{j+1}\} \in \mathcal{K}_{\psi_{j+1}}(\Omega)$. Since u_j is a p-superminimizer it follows from Lemma 5.6 that $u_j \ge u_{j+1}$ in Ω. Hence $\{u_j\}_{j=1}^\infty$ is a decreasing sequence.

Let $u = \lim_{j\to\infty} u_j$. As $u_j \in L^p(\Omega)$ and $u_j \ge u \ge 0$, we see that $u \in L^p(\Omega)$. Also, as $\psi_j \in \mathcal{K}_{\psi_j}(\Omega)$, we have $\int_\Omega g_{u_j}^p \, d\mu \le \int_\Omega g_{\psi_j}^p \, d\mu$. Hence by Lemma 3.1 of Kallunki–Shanmugalingam [16] we see that $u \in N^{1,p}(X)$ with

$$\int_\Omega g_u^p \, d\mu \le \liminf_{j\to\infty} \int_\Omega g_{u_j}^p \, d\mu \le \liminf_{j\to\infty} \int_\Omega g_{\psi_j}^p \, d\mu = \int_\Omega g_\psi^p \, d\mu.$$

Since $u_j \ge \psi_j \ge \psi$ μ-a.e. in Ω, we have $u \ge \psi$ μ-a.e. in Ω, and hence $u \in \mathcal{K}_\psi(\Omega)$ because $u_j - \psi_j \in N_0^{1,p}(\Omega)$ and $\{\psi_j\}_{j=1}^\infty$ decreases to ψ p-q.e. in $X \setminus \Omega$. Let v be the lower semicontinuously regularized solution to the obstacle problem in Ω with obstacle and boundary values ψ. Then

$$\int_\Omega g_v^p \, d\mu \le \int_\Omega g_u^p \, d\mu.$$

Also by Lemma 5.6 again, $u_j \ge v$, and hence $u \ge v$. Let $\varphi_j = \max\{v, \psi_j\}$. Then we see that $\varphi_j \in \mathcal{K}_{\psi_j}(\Omega)$, and hence $\int_\Omega g_{u_j}^p \, d\mu \le \int_\Omega g_{\varphi_j}^p \, d\mu$. As $\{\psi_j\}_{j=1}^\infty$ decreases to ψ p-q.e. in Ω and $v \ge \psi$, we see that $\{\varphi_j\}_{j=1}^\infty$ decreases to v p-q.e. in Ω. Moreover, as $\psi_j \to \psi$ in $N^{1,p}(X)$, we see that $\varphi_j \to v$ in $L^p(\Omega)$. Furthermore, putting $E_j = \{x \in \Omega : \psi_j(x) > v(x)\}$, we have

$$\begin{aligned}
\left(\int_\Omega g_{\varphi_j - v}^p \, d\mu \right)^{1/p} &= \left(\int_{E_j} g_{\psi_j - \psi + \psi - v}^p \, d\mu \right)^{1/p} \\
&\le \left(\int_{E_j} g_{\psi_j - \psi}^p \, d\mu \right)^{1/p} + \left(\int_{E_j} g_{\psi - v}^p \, d\mu \right)^{1/p}.
\end{aligned}$$

The first integral on the right-hand side tends to zero since $\psi_j \to \psi$ in $N^{1,p}(X)$. As $g_{\psi-v} = 0$ μ-a.e. on the set where $\psi = v$, we have

$$\int_{E_j} g_{\psi-v}^p \, d\mu = \int_{\{x\in\Omega:\psi_j(x)>v(x)>\psi(x)\}} g_{\psi-v}^p \, d\mu,$$

which tends to zero by the absolute continuity of integrals, because the measure of the set $\{x \in \Omega : \psi_j(x) > v(x) > \psi(x)\}$ tends to zero as $j \to \infty$. Hence $\varphi_j \to v$ in $N^{1,p}(\Omega)$, and therefore,

$$\int_\Omega g_u^p \, d\mu \le \liminf_{j\to\infty} \int_\Omega g_{u_j}^p \, d\mu \le \liminf_{j\to\infty} \int_\Omega g_{\varphi_j}^p \, d\mu = \int_\Omega g_v^p \, d\mu.$$

Thus we see that $\int_\Omega g_u^p \, d\mu = \int_\Omega g_v^p \, d\mu$. By the uniqueness of solutions to the obstacle problems we have $u = v$ p-q.e., i.e., u is a solution to the obstacle problem in Ω with obstacle and boundary values ψ. $\square$

Proof of Theorem 5.1 Assume first that $f \ge 0$ and $f \in N^{1,p}(X)$.

Using the fact that Hf is p-quasicontinuous, we can find a decreasing sequence $\{U_k\}_{k=1}^\infty$ of open sets such that $C_p(U_k) \le 1/2^{kp}$ and $Hf|_{X\setminus U_k}$ is continuous. Consider the decreasing sequence of nonnegative functions $\{\psi_j\}_{j=1}^\infty$ given by Lemma 5.3.

Let $f_j = Hf + \psi_j$ and let φ_j be the lower semicontinuously regularized solution of the obstacle problem with obstacle and boundary values f_j.

If $m \in \mathbf{Z}_+$ and $\varepsilon > 0$ is arbitrary, by Lemma 5.3,

$$f_j \ge \psi_j \ge m \quad \text{on } U_{m+j} \cap \Omega. \tag{5.1}$$

Let $x \in \partial\Omega$. If $x \in U_{m+j}$, then setting $V_x = U_{m+j}$, by inequality (5.1) we see that in the neighbourhood V_x of x we have $f_j \ge m \ge \min\{f(x) - \varepsilon, m\}$. If $x \notin U_{m+j}$, then by the continuity of $Hf|_{X\setminus U_{m+j}}$ there is a neighbourhood V_x of x such that

$$f_j(y) \ge Hf(y) \ge Hf(x) - \varepsilon = f(x) - \varepsilon, \quad \text{if } y \in (V_x \cap \Omega) \setminus U_{m+j}. \tag{5.2}$$

Combining (5.1) and (5.2) we see that

$$\min\{f_j(y), m\} \ge \min\{f(x) - \varepsilon, m\} \quad \text{for } y \in V_x \cap \Omega.$$

Since $\varphi_j \ge f_j$ μ-a.e and φ_j is lower semicontinuously regularized, it follows that $\varphi_j(y) \ge \min\{f(x) - \varepsilon, m\}$ for $y \in V_x \cap \Omega$. Hence

$$\liminf_{\Omega\ni y\to x} \varphi_j(y) \ge \min\{f(x) - \varepsilon, m\}.$$

Letting $\varepsilon \to 0$ and $m \to \infty$, we see that

$$\liminf_{\Omega \ni y \to x} \varphi_j(y) \geq f(x) \quad \text{for all } x \in \partial\Omega.$$

Since φ_j is p-superharmonic, it follows that $\varphi_j \in \mathcal{U}_f$, and hence that $\varphi_j \geq \overline{P}f$.

Since Hf clearly is a solution of the obstacle problem with obstacle and boundary values Hf, we see by Proposition 5.5 that $\{\varphi_j\}_{j=1}^{\infty}$ decreases p-q.e. to Hf. Hence $\overline{P}f \leq Hf$ p-q.e. in Ω. Since $\overline{P}f$ and Hf are continuous we find that $\overline{P}f \leq Hf$ in Ω.

Finally, let $f \in N^{1,p}(X)$ be arbitrary. Then, by Proposition 3.8,

$$\overline{P}f \leq \lim_{m \to -\infty} \overline{P} \max\{f, m\} \leq \lim_{m \to -\infty} H \max\{f, m\} = Hf.$$

It then follows that $\underline{P}f = -\overline{P}(-f) \geq -H(-f) = Hf \geq \overline{P}f \geq \underline{P}f$, and hence that $Hf = \underline{P}f = \overline{P}f$. $\square$

We end this section with a uniqueness result.

Corollary 5.7 *Let $f \in N^{1,p}(X)$ be bounded. Assume that u is a bounded p-harmonic function in Ω and that there is a set $E \subset \partial\Omega$ with $C_p(E) = 0$ such that*

$$\lim_{\Omega \ni y \to x} u(y) = f(x) \quad \textit{for all } x \in \partial\Omega \setminus E.$$

Then $u = Pf$.

Note that if the word *bounded* is omitted, the result becomes false; consider for example, the Poison kernel in the unit disc $B((0,0),1) \subset \mathbf{R}^2$ with a pole at $(1,0)$ which is zero on $\partial B((0,0),1) \setminus \{(1,0)\}$.

PROOF. By adding a sufficiently large constant to both f and u, and then rescaling them simultaneously we may assume without loss of generality that $0 \leq u \leq 1$ and $0 \leq f \leq 1$. Hence $u \in \mathcal{U}_{f-\chi_E}$ and $u \in \mathcal{L}_{f+\chi_E}$. Therefore, by Theorem 5.1, we see that $u \geq \overline{P}(f - \chi_E) = Pf = \underline{P}(f + \chi_E) \geq u$. $\square$

6 Continuous functions

Theorem 6.1 *Let $f \in C(\partial\Omega)$ and g be a function which is zero p-q.e. Then $f + g$ is resolutive, and*

$$P(f+g) = Hf = Pf.$$

Recall that Hf was defined by Definition 3.4 for arbitrary $f \in C(\partial\Omega)$.

PROOF. For each $j = 1, 2, \dots$, there is a Lipschitz function $f_j \in \mathrm{Lip}_c(X) \subset N^{1,p}(X)$ such that $f - 1/j \le f_j \le f + 1/j$ on $\partial\Omega$. Using the comparison principle we see that

$$Hf - 1/j \le Hf_j \le Hf + 1/j.$$

Hence $Hf_j \to Hf$ uniformly, as $j \to \infty$. Similarly, it follows directly from Definition 3.11 that $\overline{P}f - 1/j \le \overline{P}f_j \le \overline{P}f + 1/j$, i.e. $\overline{P}f_j \to \overline{P}f$ uniformly, as $j \to \infty$. The uniform convergence of $\underline{P}f_j$, $\overline{P}(f_j + g)$ and $\underline{P}(f_j + g)$ is proved in the same way. As $f_j \in N^{1,p}(X)$, by Theorem 5.1 we have $P(f_j + g) = H(f_j + g) = Hf_j = Pf_j$. Letting $j \to \infty$ completes the proof. $\square$

A direct consequence is the following uniqueness result, which generalizes Proposition 3.13 in Björn–Björn–Shanmugalingam [3].

Corollary 6.2 *Let $f \in C(\partial\Omega)$. Assume that u is a bounded p-harmonic function in Ω and that there is a set $E \subset \partial\Omega$ with $C_p(E) = 0$ such that*

$$\lim_{\Omega \ni y \to x} u(y) = f(x) \quad \text{for all } x \in \partial\Omega \setminus E.$$

Then $u = Pf$.

As in Corollary 5.7 the word *bounded* is essential. The proof of Corollary 6.2 is the same as the proof of Corollary 5.7, with Theorem 6.1 playing the role of Theorem 5.1.

7 Semicontinuous functions

In this section we formulate a number of propositions for upper semicontinuous functions. There are immediate analogues for lower semicontinuous functions. Recall that f is upper semicontinuous at x if $f(x) \ge \limsup_{y \to x} f(y)$.

Proposition 7.1 *Let $x \in \partial\Omega$ be a p-regular boundary point and let f be a function on $\partial\Omega$ that is bounded from above. If f is upper semicontinuous at x, then*

$$\limsup_{\Omega \ni y \to x} \underline{P}f(y) \le \limsup_{\Omega \ni y \to x} \overline{P}f(y) \le f(x).$$

PROOF. Let $\varepsilon > 0$. Then there is a neighbourhood $U \subset \partial\Omega$ of x such that $f(y) < f(x) + \varepsilon$ for $y \in U$. We can then find a Lipschitz function g on $\partial\Omega$ such that $g(x) = f(x) + \varepsilon$ and $f \le g$ on $\partial\Omega$. Hence, using the regularity of x and Theorem 6.1 we find that

$$\limsup_{\Omega \ni y \to x} \overline{P} f(y) \le \limsup_{\Omega \ni y \to x} \overline{P} g(y) = \limsup_{\Omega \ni y \to x} Hg(y) = g(x) = f(x) + \varepsilon.$$

Letting $\varepsilon \to 0$ shows that $\limsup_{\Omega \ni y \to x} \underline{P} f(y) \le \limsup_{\Omega \ni y \to x} \overline{P} f(y) \le f(x)$. $\square$

Corollary 7.2 *Let f be a bounded function on $\partial\Omega$. Assume that $x \in \partial\Omega$ is a p-regular boundary point and that f is continuous at x. Then*

$$\lim_{\Omega \ni y \to x} \underline{P} f(y) = \lim_{\Omega \ni y \to x} \overline{P} f(y) = f(x).$$

PROOF. By applying the previous proposition to f and $-f$, we find that

$$f(x) \le \liminf_{\Omega \ni y \to x} \overline{P} f(y) \le \limsup_{\Omega \ni y \to x} \overline{P} f(y) \le f(x).$$

Thus $f(x) = \lim_{\Omega \ni y \to x} \overline{P} f(y)$. Similarly $f(x) = \lim_{\Omega \ni y \to x} \underline{P} f(y)$. $\square$

Proposition 7.3 *Let f be an upper semicontinuous function on $\partial\Omega$ bounded from above and g be a nonnegative function which is zero p-q.e. Then*

$$\overline{P}(f+g) = \overline{P} f = \inf_{\mathrm{Lip}(\partial\Omega) \ni \varphi \ge f} P\varphi = \inf_{\mathrm{Lip}(\partial\Omega) \ni \varphi \ge f} H\varphi. \tag{7.1}$$

If, in addition, $f(x) + g(x) \ge \sup_{\partial\Omega} f$ for all p-irregular $x \in \partial\Omega$, then $f + g$ is resolutive and $P(f+g) = \overline{P} f$.

It follows that if $K \subset \partial\Omega$ is compact and the p-harmonic measure $\omega_{x,p}$ is given by Definition 8.1 in Björn–Björn–Shanmugalingam [3], then $\omega_{x,p}(K) = \overline{P}\chi_K(x)$. Similarly if $G \subset \partial\Omega$ is relatively open, then $\omega_{x,p}(G) = \underline{P}\chi_G(x) = \sup_K \overline{P}\chi_K(x)$, where the supremum is taken over all compact subsets K of G.

In Heinonen–Kilpeläinen–Martio [11], Chapter 11, and Kurki [25], the p-harmonic measure of a set $E \subset \partial\Omega$, was defined as $\omega(E) = \overline{P}\chi_E$ (they were actually considering $\mathcal{A}$-harmonic measure in the weighted Euclidean setting). The main result in [25], Theorem 1.1, says that if $K \subset \partial\Omega$ is compact and $E \subset \partial\Omega$ has zero p-capacity, then $\omega(K) = \omega(K \cup E)$. Proposition 7.3 is therefore a generalization of this result. In fact, Kurki used the obstacle problem to show his result. His proof more or less directly generalizes to prove Theorem 6.1 and Proposition 7.3, once the necessary lemmas have been generalized to the metric case. In order to also establish Theorem 5.1 we have had to use the obstacle problem in a slightly more complicated manner.

Following Heinonen–Kilpeläinen–Martio [11], we say that a family $\mathcal{F}$ of functions is *downward directed* if for every pair of functions $f, g \in \mathcal{F}$ there exists a function $h \in \mathcal{F}$ such that $h \le \min\{f, g\}$.

Proof of Proposition 7.3 If $\mathcal{F} = \{\varphi \in \mathrm{Lip}(\partial\Omega) : \varphi \ge f\}$, then $\mathcal{F}$ is downward directed and $f = \inf_{\varphi\in\mathcal{F}} \varphi$, as f is upper semicontinuous. Hence by Theorem 9.3 of Heinonen–Kilpeläinen–Martio [11] (the proof of which is the same in our generality), we have $\overline{P}f = \inf_{\varphi\in\mathcal{F}} P\varphi$. Using Theorem 6.1, we find that

$$\overline{P}f \le \overline{P}(f+g) \le \inf_{\mathrm{Lip}(\partial\Omega)\ni\varphi\ge f} \overline{P}(\varphi+g) = \inf_{\mathrm{Lip}(\partial\Omega)\ni\varphi\ge f} P\varphi = \overline{P}f.$$

The last equality in (7.1) follows directly from Theorem 6.1.

Assume next that $f(x) + g(x) \ge \sup_{\partial\Omega} f$ for all p-irregular $x \in \partial\Omega$. Then

$$\limsup_{\Omega\ni y\to x} \overline{P}f(y) \le \sup_{\partial\Omega} f \le f(x) + g(x) \quad \text{for } p\text{-irregular } x \in \partial\Omega.$$

Moreover, by Proposition 7.1,

$$\limsup_{\partial\Omega\ni y\to x} \overline{P}f(y) \le f(x) \le f(x) + g(x) \quad \text{for } p\text{-regular } x \in \partial\Omega.$$

By Theorem 4.1, $\overline{P}f$ is p-harmonic or $-\infty$. If $\overline{P}f$ is p-harmonic, then $\overline{P}f \in \mathcal{L}_{f+g}$, and thus $\underline{P}(f+g) \ge \overline{P}f = \overline{P}(f+g) \ge \underline{P}(f+g)$. This is also true in the case when $\overline{P}f \equiv -\infty$. $\square$

Corollary 7.4 *Assume that Ω is p-regular. Then every upper semicontinuous function on $\partial\Omega$ bounded from above is resolutive.*

In the weighted Euclidean case, assuming that the function is bounded, this is Proposition 9.31 in Heinonen–Kilpeläinen–Martio [11].

Corollary 7.5 *Let $K \subset \partial\Omega$ be a compact set, and $E \subset \partial\Omega$ be a set with $C_p(E) = 0$ containing all p-irregular points. Then $\chi_{K\cup E}$ is resolutive and $P\chi_{K\cup E} = \overline{P}\chi_K$.*

Similarly, if $G \subset \partial\Omega$ is a relatively open set, then $\chi_{G\setminus E}$ is resolutive and $P\chi_{G\setminus E} = \underline{P}\chi_G$.

In particular, if Ω is p-regular, then χ_K and χ_G are resolutive.

8 Removability

As a corollary of Corollary 6.2 we have the following result.

Corollary 8.1 *Let $K \subset \Omega$ be a compact set with zero p-capacity and u be a bounded p-harmonic function in $\Omega \setminus K$. Then there is a bounded p-harmonic function U in Ω such that $U|_{\Omega \setminus K} = u$.*

PROOF. Let $\Omega' \Subset \Omega$ be an arbitrary domain containing K. Let $v = H_{\Omega'} u$. Observe that v is continuous on the boundary of $\Omega' \setminus K$ as $v = u$ on the boundary of Ω'. By the Kellogg property (Theorem 3.7) and the condition $C_p(K) = 0$, we have

$$\lim_{\Omega' \setminus K \ni y \to x} u(y) = \lim_{\Omega' \setminus K \ni y \to x} v(y) \quad \text{for } p\text{-q.e. } x \in \partial(\Omega' \setminus K).$$

Hence, by Corollary 6.2 applied to the open set $\Omega' \setminus K$, $u = v = H_{\Omega' \setminus K} v$ in $\Omega' \setminus K$.

Now we define

$$U(x) = \begin{cases} u(x), & x \in \Omega \setminus K, \\ v(x), & x \in \Omega'. \end{cases}$$

It follows that U is continuous, and since K has no interior, the construction of U is independent of the choice of Ω'.

Now let $\varphi \in \mathrm{Lip}_c(\Omega)$. Then there is a domain Ω' such that $\operatorname{supp} \varphi \subset \Omega' \Subset \Omega$. Since U is p-harmonic in Ω' it follows that

$$\int_{\operatorname{supp} \varphi} g_U^p \, d\mu \le \int_{\operatorname{supp} \varphi} g_{U+\varphi}^p \, d\mu.$$

We can now conclude that U is p-harmonic in Ω. $\square$

This proof cannot handle the case when $K \subset \Omega$ is merely relatively closed. Therefore, the following strengthening of Corollary 8.1 requires a different proof.

Proposition 8.2 *Let $E \subset \Omega$ be a relatively closed set with zero p-capacity and u be a bounded p-harmonic function in $\Omega \setminus E$. Then there is a bounded p-harmonic function U in Ω such that $U|_{\Omega \setminus E} = u$.*

In order to prove this result we use the following removability result for p-superharmonic functions.

Proposition 8.3 *Let $E \subset \Omega$ be a relatively closed set with zero p-capacity and u be a bounded p-superharmonic function in $\Omega \setminus E$. Then there is a bounded p-superharmonic function U in Ω such that $U|_{\Omega\setminus E} = u$.*

For u to be p-superharmonic in $\Omega \setminus E$ it is required that $\Omega \setminus E$ be an open set, hence the requirement that E is relatively closed.

Note also that to find the extension U we only need to find the unique semicontinuously regularized extension of u to Ω; $U(x) = \operatorname{ess\,lim\,inf}_{\Omega\setminus E \ni y \to x} u(y)$ (since $C_p(E) = 0$, E does not have any interior points).

PROOF. Since $C_p(E) = 0$, there exists a decreasing sequence $\{U_j\}_{j=1}^\infty$ of open sets such that $C_p(U_j) < 1/2^{jp}$ and $E \subset U_j \subset \Omega$; see Remark 3.3 of Kinnunen–Martio [19]. Without loss of generality we may assume that $\bigcap_{j=1}^\infty U_j = E$. By Lemma 5.3 we can find a decreasing sequence $\{\psi_j\}_{j=1}^\infty$ of nonnegative Newtonian functions such that $\|\psi_j\|_{N^{1,p}(X)}^p \leq 1/2^j$ and $\psi_j \geq 1$ in U_{j+1}. It follows that $\psi_j \to 0$ p-q.e., and by redefining ψ_j on a set of p-capacity zero outside U_{j+1} we can also require that $\lim_{j\to\infty} \psi_j(x) = 0$ for every $x \in \Omega \setminus E$.

Let $\varphi \in \operatorname{Lip}_c(\Omega)$ be nonnegative and $F := \operatorname{supp} \varphi \Subset \Omega$. We will now show that $\int_F g_u^p \, d\mu \leq \int_F g_{u+\varphi}^p \, d\mu$, i.e. that u is a p-superminimizer in Ω. Let $M = \sup \varphi$ and $\varphi_j = \max\{\varphi - M\psi_j, 0\}$. Then $\varphi_j \in N_0^{1,p}(\Omega \setminus E)$ with compact support contained in $F \setminus E$, and hence, by the p-superharmonicity of u in $\Omega \setminus E$,

$$\int_{F_j} g_u^p \, d\mu \leq \int_{F_j} g_{u+\varphi_j}^p \, d\mu,$$

where $F_j := \operatorname{supp} \varphi_j$. As $g_{u+\varphi_j} \leq g_{u+\varphi} + M g_{\psi_j}$ on F_j, we see that

$$\begin{aligned} \left(\int_{F_j} g_u^p \, d\mu \right)^{1/p} &\leq \left(\int_{F_j} g_{u+\varphi}^p \, d\mu \right)^{1/p} + M \left(\int_{F_j} g_{\psi_j}^p \, d\mu \right)^{1/p} \\ &\leq \left(\int_{F_j} g_{u+\varphi}^p \, d\mu \right)^{1/p} + M \|\psi_j\|_{N^{1,p}(X)}. \end{aligned}$$

Note that $\{F_j\}_{j=1}^\infty$ is an increasing sequence of sets whose union is $F \setminus E$. Thus, by letting $j \to \infty$, we see that the last term tends to zero and

$$\left(\int_{F\setminus E} g_u^p \, d\mu \right)^{1/p} \leq \left(\int_{F\setminus E} g_{u+\varphi}^p \, d\mu \right)^{1/p}.$$

As E has zero p-capacity and hence zero measure, we now have

$$\int_F g_u^p \, d\mu \leq \int_F g_{u+\varphi}^p \, d\mu$$

for all nonnegative Lipschitz functions $\varphi \in \mathrm{Lip}_c(\Omega)$. Hence u is a p-superminimizer in Ω, and its lower semicontinuously regularized extension to Ω is p-superharmonic in Ω. $\square$

Proof of Proposition 8.2 By Proposition 8.3 there exist bounded p-superharmonic functions V and W on Ω, such that $V = u$ and $W = -u$ in $\Omega \setminus E$.

By Corollary 7.8 in Kinnunen–Martio [21], V and W are p-superminimizers. As $W = -V$ p-q.e., we see that $-V$ is also a p-superminimizer. Hence V is a p-energy minimizer and there exists a p-harmonic function U such that $U = V = u$ p-q.e. in Ω, see the comment after Definition 5.4. Since both U and u are continuous in $\Omega \setminus E$, they coincide in $\Omega \setminus E$. $\square$

The following two propositions demonstrate the sharpness of Propositions 8.2 and 8.3. We have not been able to prove the sharpness for a general relatively closed subset $E \subsetneq \Omega$ with positive p-capacity.

Proposition 8.4 *Let $K \subset \Omega$ be compact with positive p-capacity and E be a relatively closed subset of Ω such that $\mu(E) = 0$. Then there is a bounded p-harmonic function $u : \Omega \setminus (K \cup E) \to \mathbf{R}$ with no p-superharmonic extension $U : \Omega \to (-\infty, +\infty]$ such that $U|_{\Omega \setminus (K \cup E)} = u$.*

Proposition 8.5 *Let $E \neq \Omega$ be a relatively closed proper subset of Ω so that $C_p(E) > 0$. If $C_p(\{x\}) = 0$ for each $x \in \partial E \cap \Omega$, then there exists a bounded p-harmonic function u in $\Omega \setminus E$ that has no p-harmonic extension to Ω that agrees with u on $\Omega \setminus E$.*

To prove these propositions we need the following potential theoretic lemma.

Lemma 8.6 *If $E \neq \Omega$ is a relatively closed proper subset of Ω such that $C_p(E) > 0$, then $C_p(\Omega \cap \partial E) > 0$.*

PROOF. One of our standing assumptions is that the measure of nonempty open subsets of X is positive. If $\mu(E) = 0$, then $E = \Omega \cap \partial E$, and the conclusion follows directly. Hence suppose that $\mu(E) > 0$. Then E has a point of density $x \in E$. Let $y \in \Omega \setminus E$; such a point exists because $E \neq \Omega$. Since Ω is a connected open set, there exists a finite collection of open balls $\{B_k\}_{k=1}^n$ with $2\lambda B_k \subset \Omega$ so that $B_k \cap B_{k+1} \neq \emptyset$ for $k = 1, \dots, n$, $x \in B_1$, and $y \in B_n$. Here λ is the dilation constant from the weak Poincaré inequality.

Let s be the smallest index such that $B_s \setminus E$ is nonempty. As $B_s \setminus E$ is open, we have $\mu(B_s \setminus E) > 0$. If $s \geq 2$, then the set $B_s \cap B_{s-1} \subset B_{s-1} \subset E$ is nonempty and open, and hence $\mu(B_s \cap E) \geq \mu(B_s \cap B_{s-1}) > 0$. If $s = 1$, then $\mu(B_s \cap E) > 0$ since x is a density point.

Suppose that $C_p(\Omega \cap \partial E) = 0$. Then the family of curves passing through ∂E has zero p-modulus (see Definition 2.1 and Lemma 6 in Shanmugalingam [34]) and hence the zero function is a p-weak upper gradient in $2\lambda B_s$ of the function χ_E. It follows that $u := \eta \chi_E \in N^{1,p}(X)$, where $\eta \in \operatorname{Lip}_c(2\lambda B_s)$ and $\eta = 1$ in λB_s.

The weak $(1,p)$-Poincaré inequality then implies, with r being the radius of B_s,

$$0 = Cr \biggl(\fint_{\lambda B_s} g_u^p \, d\mu \biggr)^{1/p} \geq \fint_{B_s} |u - u_{B_s}| \, d\mu = \frac{2\mu(B_s \cap E)}{\mu(B_s)} \biggl(1 - \frac{\mu(B_s \cap E)}{\mu(B_s)} \biggr) > 0,$$

a contradiction. Thus $C_p(\Omega \cap \partial E) > 0$. $\square$

Proof of Proposition 8.4 Observe that $f(x) = \min\{\operatorname{dist}(K,x)/\operatorname{dist}(K, X \setminus \Omega), 1\}$ is a Lipschitz function. Let $u = H_{\Omega \setminus K} f$. Then u is p-harmonic in $\Omega \setminus K$, $u = 0$ on K and $u = 1$ on $X \setminus \Omega$. Note that $u \geq 0$ in Ω.

Suppose there is a p-superharmonic function U on Ω such that $U = u$ in the open set $\Omega \setminus (K \cup E)$. As $\mu(E) = 0$, u is continuous in $\Omega \setminus K$, and U is lower semicontinuously regularized, we get directly that $U = u$ in $\Omega \setminus K$. By Theorem 3.12 it then follows that $U \geq 0$ in Ω.

Lemma 8.6 implies $C_p(\partial K) > 0$ and by the Kellogg property (Theorem 3.7), there exists a p-regular point $x_0 \in \partial K$, i.e.

$$\lim_{\Omega \setminus K \ni x \to x_0} U(x) = \lim_{\Omega \setminus K \ni x \to x_0} u(x) = u(x_0) = 0.$$

By the lower semicontinuity of U, $U(x_0) = 0$. This violates the fact that nonnegative p-superharmonic functions do not achieve their minima in their domains of p-superharmonicity; see Lemma 7.11 of Kinnunen–Martio [21]. Thus there is no p-superharmonic extension of u to Ω. $\square$

Proof of Proposition 8.5 Since by Lemma 8.6 we have $C_p(\Omega \cap \partial E) > 0$, there exists $\tau > 0$ so that $C_p(\overline{\Omega}_\tau \cap \partial E) > 0$, where $\Omega_\tau := \{x \in \Omega : \operatorname{dist}(x, X \setminus \Omega) > \tau\}$. By the Kellogg property (Theorem 3.7) and by the fact that finite subsets of $\partial E \cap \Omega$ have zero p-capacity, there exists a sequence $\{x_n\}_{n=1}^\infty$ of points in $\overline{\Omega}_\tau \cap \partial E$ that are p-regular for the open set $\Omega \setminus E$. Since $\overline{\Omega}_\tau$ is compact, without loss of generality we may assume that this sequence converges to a point $x_\infty \in \Omega \cap \partial E$ and has no other limit point, and moreover consists of distinct points. For each x_n in this sequence, let $B_n = B(x_n, r_n)$ be a ball so that $\overline{B}_n \subset \Omega$. We can also choose the balls B_n to be pairwise disjoint.

Let $\varphi_n \in \operatorname{Lip}_c(B_n)$ so that $\varphi_n = 1$ in $\frac{1}{2}B_n$ and $0 \le \varphi_n \le 1$, and we construct a lower semicontinuous function on $\overline{\Omega}$ as follows. We set

$$\Phi(x) = \begin{cases} \sum_{n=1}^{\infty} \varphi_{2n}(x), & x \neq x_\infty, \\ 0, & x = x_\infty. \end{cases}$$

It is easy to see that Φ is a bounded lower semicontinuous function on $\overline{\Omega}$ and is continuous on $\overline{\Omega} \setminus \{x_\infty\}$. Let $u = \overline{P}\Phi$ be the upper Perron solution of Φ on the set $\Omega \setminus E$. Clearly u is bounded and p-harmonic on $\Omega \setminus E$, by Theorem 4.1. We will show that u has no p-harmonic extension to Ω.

Since Φ is continuous at x_n for each n, we see by Corollary 7.2 that

$$\lim_{\Omega \setminus E \ni y \to x_n} u(y) = \Phi(x_n).$$

Note that $\Phi(x_n) = 1$ if n is even and $\Phi(x_n) = 0$ if n is odd. Hence as x_∞ is the limit point of the sequence $\{x_n\}_{n=1}^\infty$, we obtain a sequence $\{y_n\}_{n=1}^\infty$ in $\Omega \setminus E$ that converges to x_∞ so that $u(y_n) \ge \frac{3}{4}$ if n is even and $u(y_n) \le \frac{1}{4}$ otherwise. That is, $u|_{\Omega \setminus E}$ has no continuous extension to the point $x_\infty \in \Omega \cap \partial E$. Since p-harmonic functions are continuous, this implies that u has no p-harmonic extension to Ω. $\square$

9 The linear case; Cheeger two-harmonic functions

In this section we fix $p = 2$, and it is important that we use the Cheeger differential definition of Cheeger two-(super)harmonicity. This method does not work for two-harmonic functions, since it is not known whether the sum of two two-harmonic functions is always a two-harmonic function. We therefore consider the operators H and P to be defined using the Cheeger two-harmonic functions rather than standard two-harmonic functions in this section.

In Björn–Björn–Shanmugalingam [3] it was shown that for every $x \in \Omega$ there exists a harmonic measure ν_x on $\partial\Omega$ such that if $f \in N^{1,2}(X)$ or $f \in C(\partial\Omega)$ then $Hf(x) = \int_{\partial\Omega} f\, d\nu_x$, and if $f \in L^1(\partial\Omega, \nu_{x_0})$ for some $x_0 \in \partial\Omega$ then the function $x \mapsto \int_{\partial\Omega} f\, d\nu_x$ is Cheeger two-harmonic in Ω.

In this section we obtain the following result, where we have extended ν_x to be complete.

Theorem 9.1 *Let $x_0 \in \Omega$ and let $f : \partial\Omega \to \overline{\mathbf{R}}$ be a function. Assume that* *the Perron solutions have been defined with respect to Cheeger two-harmonic functions. Then the following are true:*

(a) *If $f \in L^1(\partial\Omega, \nu_{x_0})$, then f is resolutive and $Pf(x) = \int_{\partial\Omega} f \, d\nu_x$, $x \in \Omega$.*
(b) *If f is resolutive and Pf is not $\pm\infty$, then $f \in L^1(\partial\Omega, \nu_{x_0})$.*

The proof is more or less identical to the proof of Theorem 6.4.6 in Armitage–Gardiner [1]. Let us, however, make some comments.

First, Theorems 6.3.1 and 6.3.5 in [1] are proved in the same way in our case. The latter is the essential ingredient needed to obtain Lemma 6.4.4 in [1], which says that if $f : \partial\Omega \to (-\infty, +\infty]$ is lower semicontinuous, then f is resolutive. Using Theorem 5.1 in [3] and monotone convergence, we already know from Proposition 7.3, that $\underline{P}f = \int_{\partial\Omega} f \, d\nu_x$. From Theorem 6.3.5 in [1] we find that $\overline{P}f = \underline{P}f$, and Lemma 6.4.4 is obtained. After this the proof of Theorem 9.1 is the same as the proof of Theorem 6.4.6 in [1].

10 Open problems

We consider the following definition.

Definition 10.1 *Given a function $f : \partial\Omega \to \overline{\mathbf{R}}$, let $\widetilde{\mathcal{U}}_f$ be the set of all p-superharmonic functions u on Ω bounded below such that*

$$\liminf_{\Omega \ni y \to x} u(y) \geq f(x) \quad \textit{for } p\textit{-q.e. } x \in \partial\Omega.$$

Define

$$\overline{Q}f(x) = \inf_{u \in \widetilde{\mathcal{U}}_f} u(x), \quad x \in \Omega.$$

Similarly, let $\tilde{\mathcal{L}}_f$ be the set of all p-subharmonic functions u on Ω bounded above such that

$$\limsup_{\Omega \ni y \to x} u(y) \leq f(x) \quad \textit{for } p\textit{-q.e. } x \in \partial\Omega,$$

and define

$$\underline{Q}f(x) = \sup_{u \in \tilde{\mathcal{L}}_f} u(x), \quad x \in \Omega.$$

Note that the proof of Theorem 4.1 can also be used to show that $\overline{Q}f$ and $\underline{Q}f$ are p-harmonic functions or identically $\pm\infty$.

The operators $\overline{Q}$ and $\underline{Q}$ have been constructed to address a major shortcoming of the Perron solutions, namely, it is not known if Perron solutions are invariant under perturbation of the boundary function on a set of zero p-capacity. It is easy to see that $\overline{P}f \geq \overline{Q}f$ and $\underline{P}f \leq \underline{Q}f$ for all functions f on $\partial\Omega$. The important question in this case is whether $\underline{Q}f \leq \overline{Q}f$ for all functions f. Proposition 10.3 below shows the relation between these two open problems and several other open questions. Before stating Proposition 10.3 we state a result about semicontinuous functions.

Proposition 10.2 *Let f be a bounded upper semicontinuous function on $\partial\Omega$. Then $\overline{Q}f \leq \overline{P}f = \underline{Q}f$. Moreover the following are equivalent*:

(i) $\underline{Q}f = \overline{Q}f$;
(ii) $\underline{Q}f \leq \overline{Q}f$;
(iii) $\overline{P}f = \overline{Q}f$;
(iv) *if $g = f$ p-q.e. on $\partial\Omega$, then $\overline{P}g = \overline{P}f$.*

PROOF. The first inequality is clear. By Proposition 7.1 together with the Kellogg property (Theorem 3.7) $\overline{P}f \in \tilde{\mathcal{L}}_f$, and hence $\overline{P}f \leq \underline{Q}f$. On the other hand, let $u \in \tilde{\mathcal{L}}_f$, and let

$$g(x) = \begin{cases} +\infty, & \text{if } \limsup_{\Omega \ni y \to x} u(y) > f(x), \\ 0, & \text{otherwise.} \end{cases}$$

Then $u \in \mathcal{L}_{f+g}$ and $g = 0$ p-q.e. on $\partial\Omega$. By Proposition 7.3, $u \leq \underline{P}(f+g) \leq \overline{P}(f+g) = \overline{P}f$. Taking supremum over all $u \in \tilde{\mathcal{L}}_f$ shows that $\underline{Q}f \leq \overline{P}f$.

It follows directly that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

(iii) $\Rightarrow$ (iv) It is obvious that $\overline{Q}g = \overline{Q}f$. Hence using Proposition 7.3,

$$\overline{Q}f = \overline{Q}g \leq \overline{P}g \leq \overline{P}\max\{f, g\} = \overline{P}f = \overline{Q}f.$$

(iv) $\Rightarrow$ (iii) It is clear that $\overline{Q}f \leq \overline{P}f$. Let $u \in \tilde{\mathcal{U}}_f$ and let

$$g(x) = \begin{cases} -\infty, & \text{if } \liminf_{\Omega \ni y \to x} u(y) < f(x), \\ f(x), & \text{otherwise.} \end{cases}$$

Then $g = f$ p-q.e. and $u \in \mathcal{U}_g$, hence $u \geq \overline{P}g = \overline{P}f$. Taking infimum over all $u \in \tilde{\mathcal{U}}_f$ we see that $\overline{Q}f \geq \overline{P}f$. $\square$

Proposition 10.3 *Consider the following statements*:

(a) *If $f, g : \partial\Omega \to \overline{\mathbf{R}}$ are equal p-q.e., then $\overline{P}f = \overline{P}g$.*

(a′) *If* $f, g : \partial\Omega \to \overline{\mathbf{R}}$ *are equal* p*-q.e. and* f *is a nonnegative lower semicontinuous function, then* $\overline{P}f = \overline{P}g$.

(a″) *If* $f, g : \partial\Omega \to \overline{\mathbf{R}}$ *are equal* p*-q.e. and* f *is a bounded upper semicontinuous function, then* $\overline{P}f = \overline{P}g$.

(b) *If* $f : \partial\Omega \to \overline{\mathbf{R}}$*, then* $\overline{P}f = \overline{Q}f$.

(b′) *If* $f : \partial\Omega \to \overline{\mathbf{R}}$ *is a nonnegative Borel function, then* $\overline{P}f = \overline{Q}f$.

(b″) *If* $f : \partial\Omega \to \mathbf{R}$ *is a bounded upper semicontinuous function, then* $\overline{P}f = \overline{Q}f$.

(c) *If* $f : \partial\Omega \to \overline{\mathbf{R}}$*, then* $\underline{Q}f \le \overline{Q}f$.

(c″) *If* $f : \partial\Omega \to \mathbf{R}$ *is a bounded upper semicontinuous function, then* $\underline{Q}f \le \overline{Q}f$.

(d) *If* u *is a* p*-superharmonic function bounded below in* Ω*,* v *is a* p*-subharmonic function bounded above in* Ω*, and*

$$\limsup_{\Omega \ni y \to x} v(y) \le \liminf_{\Omega \ni y \to x} u(y) \quad \textit{for } p\textit{-q.e. } x \in \partial\Omega,$$

then $v \le u$ *in* Ω.

(d′) *If* u *is a bounded* p*-superharmonic function in* Ω*,* v *is a bounded* p*-subharmonic function in* Ω*, and*

$$\limsup_{\Omega \ni y \to x} v(y) \le \liminf_{\Omega \ni y \to x} u(y) \quad \textit{for } p\textit{-q.e. } x \in \partial\Omega,$$

then $v \le u$ *in* Ω.

(e) *If* u *and* v *are bounded* p*-harmonic functions in* Ω *and*

$$\lim_{\Omega \ni y \to x} u(y) = \lim_{\Omega \ni y \to x} v(y) \quad \textit{for } p\textit{-q.e. } x \in \Omega,$$

then $u = v$.

Then (a) $\Leftrightarrow$ (a′) $\Leftrightarrow$ (b) $\Leftrightarrow$ (b′) $\Rightarrow$ (a″) $\Leftrightarrow$ (b″) $\Leftrightarrow$ (c) $\Leftrightarrow$ (c″) $\Leftrightarrow$ (d) $\Leftrightarrow$ (d′) $\Rightarrow$ (e).

For Cheeger two-harmonic functions all of the statements in Proposition 10.3 are true. They are also true in the case when the empty set is the only subset of $\partial\Omega$ with zero p-capacity. In all other cases it is not known if these statements are true or false, even in the case $X = \mathbf{R}^n$, $n \ge 2$, equipped with the Lebesgue measure, and $1 < p \le n$, $p \ne 2$.

PROOF. That (a) $\Rightarrow$ (a′), (a) $\Rightarrow$ (a″), (b) $\Rightarrow$ (b′) $\Rightarrow$ (b″), (c) $\Rightarrow$ (c″) and (d) $\Rightarrow$ (d′) are immediate.

That (a″) $\Leftrightarrow$ (b″) $\Leftrightarrow$ (c″) follows directly from Proposition 10.2.

$\neg$(b) $\Rightarrow$ $\neg$(b′) There is $x \in \Omega$ and $u \in \widetilde{\mathcal{U}}_f$ such that $u(x) < \overline{P}f(x)$. Then there is a set E with $C_p(E) = 0$ such that $\liminf_{\Omega \ni y \to x} u(y) \ge f(x)$ for $x \in \partial\Omega \setminus E$.

Since C_p is a Choquet capacity we can find open sets $G_j \supset E$ with $C_p(G_j) < 1/j$. Letting $E' = \bigcap_{j=1}^{\infty} G_j \supset E$ gives a Borel set, in fact a G_δ set, with zero p-capacity. Now

$$g(x) = \begin{cases} \liminf\limits_{\Omega \ni y \to x} u(y), & x \in \partial\Omega \setminus E', \\ +\infty, & x \in E', \end{cases}$$

gives a Borel function such that $g \geq f$ and $u \in \widetilde{\mathcal{U}}_g$. It follows that $\overline{Q}g(x) \leq u(x) < \overline{P}f(x) \leq \overline{P}g(x)$. Moreover g is bounded from below and by adding a suitable constant to g we have a nonnegative counterexample.

$\neg$(b) $\Rightarrow$ $\neg$(a$'$) There is $x \in \Omega$ and $u \in \widetilde{\mathcal{U}}_f$ such that $u(x) < \overline{P}f(x)$. Let $h(z) = \liminf_{\Omega \ni y \to z} u(y)$, $z \in \partial\Omega$, a lower semicontinuous function bounded from below. Let $g = \max\{f, h\}$. Thus $\overline{P}h(x) \leq u(x) < \overline{P}f(x) \leq \overline{P}g(x)$. It follows that $\overline{P}g \neq \overline{P}h$. Since $f \leq h$ p-q.e., we see that $g = h$ p-q.e. By adding a suitable constant to g and h we have a counterexample with nonnegative functions.

(b) $\Rightarrow$ (a) This follows directly from the obvious fact that $\overline{Q}f = \overline{Q}g$ if $f = g$ p-q.e.

(d) $\Rightarrow$ (c) Let $u \in \widetilde{\mathcal{U}}_f$ and $v \in \tilde{\mathcal{L}}_f$. Then by (d), $v \leq u$. It follows that (c) holds.

$\neg$(d$'$) $\Rightarrow$ $\neg$(c$''$) Let u and v violate (d$'$) and let $f(x) = \limsup_{\Omega \ni y \to x} v(y)$ a bounded upper semicontinuous function. It follows that there is a point $y \in \Omega$ such that $u(y) < v(y)$, and that $u \in \widetilde{\mathcal{U}}_f$ and $v \in \tilde{\mathcal{L}}_f$. Hence $\overline{Q}f(y) \leq u(y) < v(y) \leq \underline{Q}f(y)$.

$\neg$(d) $\Rightarrow$ $\neg$(d$'$) Assume that u and v violate (d). Let $u' = \min\{u, \sup_{\partial\Omega} v\}$ and $v' = \max\{v, \inf_{\partial\Omega} u\}$. Then u' is p-superharmonic, v' is p-subharmonic, and both u' and v' are bounded. It follows that u' and v' violate (d$'$).

(d$'$) $\Rightarrow$ (e) This is immediate from the fact that in Ω both u and v are p-subharmonic, p-superharmonic and bounded. □

The following problems are open even in the case $X = \mathbf{R}^n$ when $n \geq 2$.

Problem 10.4 *If f is a bounded Borel function on $\partial\Omega$, is f then resolutive?*

A simpler problem, which we know is true in p-regular domains, is the following problem, see Corollary 7.4.

Problem 10.5 *If f is a bounded semicontinuous function, is f then resolutive?*

Problem 10.6 *Is it true that $|Hf(x) - Hg(x)| \le C(x)\|f-g\|_{N^{1,p}(X)}$ for some constant $C(x)$ independent of f and g?*

If this inequality holds, then it is a strong quantitative version of Proposition 3.8 and would strengthen some of the results in this paper.

Problem 10.7 *Is it true that $\lim_{m\to\infty} \overline{P}\min\{f,m\}(x) = \overline{P}f(x)$?*

A positive answer to this question would, in particular, make it possible to replace the word "nonnegative" by "bounded" in (a′) and (b′) in Proposition 10.3.

Problem 10.8 *Is it true that if u and v are p-harmonic functions on a bounded domain Ω such that $u \ge v$ in Ω and $u(x_0) = v(x_0)$ for some $x_0 \in \Omega$, then $u \equiv v$?*

For Cheeger two-harmonic functions all of the above questions have affirmative answers. In the unweighted Euclidean space $\mathbf{R}^2$ (considered with the standard derivative structure), when $1 < p < \infty$ the last question has an affirmative answer, see Manfredi [30], Theorem 2.

Let us pose one more question.

Problem 10.9 *For which spaces X is it true that if u is a p-harmonic function on Ω and $u \equiv 0$ in a nonempty open subset of Ω, then $u \equiv 0$ in Ω?*

For the unweighted Euclidean space $\mathbf{R}^n$ (with the standard derivative structure), two-harmonic functions are known to be real-analytic, see, e.g., Hörmander [15], Theorem 4.4.3, from which an affirmative answer to this problem follows. When $n = 2$ this question has an affirmative answer for every p with $1 < p < \infty$, see the discussion on p. 130 in Heinonen–Kilpeläinen–Martio [11].

Now consider the example of metric graphs. Let $G = (V, E)$ be a connected finite or infinite graph, where V stands for the set of vertices and E the set of edges. If x and y are endpoints of an edge we say that they are *neighbours* and write $x \sim y$. Consider an edge as a geodesic open ray of length 1 between its endpoints, and let $X = V \cup \bigcup_{e\in E} e$ be the metric graph equipped with the one-dimensional Hausdorff measure μ.

Let $\Omega \Subset X$ be a domain and assume for simplicity that $\partial\Omega \subset V$. Then u is a p-harmonic function in Ω if and only if it is linear on each edge in Ω and satisfies

$$\sum_{y\sim x} |u(y) - u(x)|^{p-2}(u(y) - u(x)) = 0 \quad \text{for all } x \in V \cap \Omega.$$

Such p-harmonic functions were considered by Holopainen–Soardi [13,14], and by Shanmugalingam in [36].

Assume that X also satisfies our standing assumptions. (See Section 4 in [14] for such examples.) Then nonempty sets have positive capacity and hence the statements in Proposition 10.3 are all true. Furthermore, any boundary function is continuous and hence Problems 10.4 and 10.5 trivially have positive answers, in view of Theorem 6.1.

A positive answer to Problem 10.6 is obtained by observing that since $\partial\Omega$ is finite and X is a metric graph, $|h(y)| \le C\|h\|_{N^{1,p}(X)}$ for $y \in \partial\Omega$ and $h \in N^{1,p}(X)$. Letting $h = f - g$ it follows that

$$Hf(x) = H(g+h)(x) \le H\Big(g + \sup_{\partial\Omega} h\Big)(x) \le Hg(x) + C\|h\|_{N^{1,p}(X)}.$$

Similarly, $Hf(x) \ge Hg(x) - C\|h\|_{N^{1,p}(X)}$, from which the desired inequality follows immediately.

A positive answer to Problem 10.7 is easily obtained, since if f is not bounded above on $\partial\Omega$, then $\overline{P}f \equiv \infty$. It is also straightforward to obtain a positive answer to Problem 10.8 (obtaining equality for any neighbour of a vertex with equality, leads to equality identically).

Consider the graph $G = (\{1,2,3,4\}, \{(1,2),(1,3),(1,4)\})$, let X be the corresponding metric graph, and let $\Omega = X \setminus \{3,4\}$. Note that X satisfies our standing assumptions. Let u be the continuous function on X which is linear on every edge and takes the values $u(1) = u(2) = 0$, $u(3) = 1$ and $u(4) = -1$. It is obvious that u is p-harmonic in Ω and thus provides a counterexample to Problem 10.9 for all $p \in (1,\infty)$, in particular for the linear case $p = 2$.

It is then easy to verify that a metric graph satisfying our standing assumptions has a positive answer to Problem 10.9 if and only if the degree of all vertices is at most two, i.e. the graph is linear. (The degree of a vertex is the number of its neighbours, which is always assumed to be finite for graphs.) Indeed, the above counterexample can be included in any graph containing a vertex with degree at least three.

Note that the measure in the above counterexample is one-dimensional. However, it can be modified to obtain higher dimensional counterexamples as follows. Let A, B and C be three n-dimensional closed solid unit cubes in $\mathbf{R}^n$. Choose one face s_a, s_b, s_c for each of the cubes and let X be the metric space obtained by gluing the cubes A, B, C along these three faces via an affine map. Let Ω be the domain obtained by removing from X the face s_b' opposite to s_b in the cube B and the face s_c' opposite to s_c in the cube C. Let u be the continuous function on X given by $u = 0$ on the cube A and on the common face of A, B and C, $u = 1$ on the face s'_b and linear in the cube B, and $u = -1$ on the face s'_c and linear in the cube C. It is easily seen that u is p-harmonic in Ω and provides an n-dimensional counterexample to the above problem.

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