Newtonian Spaces: An Extension of Sobolev Spaces to Metric Measure Spaces

Nageswari Shanmugalingam

Abstract

This paper studies a possible definition of Sobolev spaces in abstract metric spaces, and answers in the affirmative the question whether this definition yields a Banach space. The paper also explores the relationship between this definition and the Hajłasz spaces. For specialized metric spaces the Sobolev embedding theorems are proven. Different versions of capacities are also explored, and these various definitions are compared. The main tool used in this paper is the concept of moduli of path families.

1 Introduction

The theory of Sobolev spaces was originally developed for domains Ω in $\mathbb{R}^n$ and was based on the notion of distributional derivatives. For $1 \le p < \infty$ the Sobolev space $W^{1,p}(\Omega)$ is defined to be the collection of all functions u in $L^p(\Omega)$ such that the distributional derivatives $\partial_i u$, $i = 1, ..., n$, are in $L^p(\Omega)$, and is equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \sum_{i=1}^{n} \|\partial_i u\|_{L^p}.$$

See [EG], [M], and [Z] for details of Sobolev spaces for domains in $\mathbb{R}^n$. Since distributional derivatives are defined in terms of an action on smooth functions via integration by parts, an alternate way of defining Sobolev spaces needs to be found for general metric spaces.

It has been shown in [H1] that for $p > 1$ a p-integrable function u in $\mathbb{R}^n$ is in the Sobolev class $W^{1,p}(\mathbb{R}^n)$ if and only if there exists a non-negative

p-integrable function g such that for almost all points x and y in $\mathbb{R}^n$

$$|u(x) - u(y)| \leq |x - y|(g(x) + g(y)).$$

This inequality can be stated on any metric measure space X if the term $|x - y|$ is interpreted to be the metric distance between the points x and y, and therefore can be used to define Sobolev type spaces refered to in this paper as Hajłasz spaces.

Definition 1.1. Let X be a metric space with a metric d and a measure μ. For $1 \leq p < \infty$ the *Hajłasz space* $M^{1,p}(X)$ is the collection of L^p-equivalence classes of functions u that together with some p-integrable non-negative function g, called a *Hajłasz gradient* of u, satisfy the inequality

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y)) \tag{1}$$

for μ-almost all x, y in X. The corresponding norm for functions u in $M^{1,p}(X)$ is given by

$$\|u\|_{M^{1,p}} = \|u\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over all Hajłasz gradients g of u. With this norm, $M^{1,p}(X)$ is a Banach space.

See [H1], [H2], and Section 4 below for properties of Hajłasz spaces.

There is another equivalent definition of Sobolev functions for domains in $\mathbb{R}^n$ due to Ohtsuka, based on the notion of primitives of vector fields. Ohtsuka showed that a p-integrable function u is in the Sobolev space $W^{1,p}(\Omega)$ if and only if u is a generalized primitive of a p-integrable vector field, that is, there is a vector field V on the domain Ω such that $\rho(x) = |V(x)|$ is a p-integrable function and for p-modulus almost all rectifiable compact paths one has the equality

$$u(x) - u(y) = \int_\gamma V \cdot \frac{d\gamma}{ds} ds,$$

where x and y are the end point and the starting point of γ respectively. See [O, Section 4.3, Theorem 4.21] for details. See Definition 2.1 below for the definition of moduli of path families and the notion of a property holding for p-modulus almost all paths.

Definition 1.2. Let u be a real-valued function on a metric space X. A non-negative Borel-measurable function ρ is said to be an *upper gradient* of u if for all compact rectifiable paths γ the following inequality holds:

$$|u(x) - u(y)| \le \int_\gamma \rho ds, \tag{2}$$

where x and y are the endpoints of the path.

See [KM2] and [HeK1, Section 2.9] for a discussion on upper gradients; [HeK1] uses the term *very weak gradients* for this concept.

In the case of domains Ω of $\mathbb{R}^n$, it is easy to see in the light of [KM2, Lemma 2.4], Proposition 3.1, and [O, Theorem 4.16] that the existence of a p-integrable upper gradient is a necessary and sufficient condition for a p-integrable function to be a generalized primitive of a p-integrable vector field. Since the concept of upper gradient is definable on any metric space, for $1 \le p < \infty$ one can define a Sobolev type space on a metric measure space X to be the collection of p-integrable functions with p-integrable upper gradients. See Definitions 2.4 and 2.5. In the event that the p-modulus of the family of all compact rectifiable paths in the space is zero, for example if the metric measure space has no rectifiable curves, by Lemma 2.1 the corresponding definition of Sobolev type space would yield the space $L^p(X)$. If the metric space has an abundance of rectifiable curves an interesting theory of Sobolev spaces develops. In contrast, the Hajłasz space can be strictly smaller than $L^p(X)$ even when the space has no rectifiable curves.

The Sobolev type spaces obtained by using the above definition is referred to in this paper as Newtonian spaces in recognition of the fact that the idea behind their definition is a generalization of the fundamental theorem of calculus. The aim of this paper is to study the Newtonian spaces and their relationship to the Hajłasz spaces.

The Newtonian spaces are defined in the second section of this paper, and in the third section it is shown that these spaces are Banach spaces, and the relation between Newtonian spaces and Hajłasz spaces are explored in the fourth section. In the fifth section Sobolev type embedding theorems are proved. The final section contains some examples.

Acknowledgement. The author thanks Juha Heinonen for his patient guidance and helpful discussions. The author also thanks the referee of the paper for his or her careful perusal of the paper and for helpful suggestions.

2 Notations and Definitions

This section lists the notations and definitions used throughout the paper. The main focus of this section is to define the Newtonian spaces.

Throughout this paper (X, d, μ) is a metric, Borel measure space. Assume also that μ is positive and finite on balls in X. Throughout this paper constants are labeled C, and the value of C might change even from one line of the same proof to the next.

Throughout this paper it is assumed that p is a real number satisfying $1 \le p < \infty$ unless specifically stated otherwise.

Paths γ in X are continuous maps $\gamma : I \to X$, where I is some interval in $\mathbb{R}$; abusing terminology, the image $\gamma(I) =: |\gamma|$ of γ is also called a path. Let Γ_{rect} be the collection of all non-constant compact (that is, I is compact) rectifiable paths in X. For a discussion of rectifiable paths and path integration see [HeK1, Section 2] or [V, Chapter 1]. If A is a subset of X, then Γ_A is the family of all paths in Γ_{rect} that intersect the set A and Γ_A^+ is the family of all paths γ in Γ_{rect} such that the Hausdorff one-dimensional measure $\mathcal{H}_1(|\gamma| \cap A)$ is positive. The following definition is applicable to all families of paths, not necessarily only to collections of compact rectifiable paths. The rest of the paper however will only consider families of non-constant compact rectifiable paths.

Definition 2.1. Let Γ be a collection of paths in X. The *p-modulus* of the family Γ, denoted $\mathrm{Mod}_p\Gamma$, is defined to be the number

$$\inf_{\rho} \|\rho\|_{L^p}^p,$$

where the infimum is taken over the set of all non-negative Borel-measurable functions ρ such that for all rectifiable paths γ in Γ the path integral $\int_\gamma \rho\, ds$ is not smaller than 1. Such functions ρ used to define the p-modulus of Γ are said to be *admissible* for the family Γ.

It is known from [Fu1] that p-modulus is an outer measure on the collection of all paths in X. It is clear from the above definition that the p-modulus of the family of all non-rectifiable paths is zero. For additional information about p-moduli see [V], [AO], and [Fu1, Chapter 1].

A property relevant to paths in X is said to hold for *p-almost all paths* if the family of rectifiable compact paths on which the property does not hold has p-modulus zero. This is a slightly different definition from the standard definition used in other papers: the standard definition requires that the family of all compact rectifiable paths as well as non-compact locally rectifiable paths on which the property in question does not hold has zero p-modulus. The difference between these two definitions is immaterial in practice; for instance, all non-compact rectifiable paths can be completed to be compact rectifiable paths in the event that X is complete.

For any path $\gamma \in \Gamma_{rect}$ and for distinct points x and y in $|\gamma|$, choosing any two distinct numbers t_x and t_y from the domain of γ such that $\gamma(t_x) = x$ and $\gamma(t_y) = y$, denote γ_{xy} to be the subpath $\gamma|_{[t_x,t_y]}$. The subpath γ_{xy} is not a well-defined notion as there can be more than one choice of the related numbers t_x and t_y. Because of this ambiguity any property that is required for one choice of subpath γ_{xy}, is also required for all such choices of subpaths.

Definition 2.2. Let $l(\gamma)$ denote the length of γ. A function u is said to be ACC_p or *absolutely continuous on p-almost every curve* if $u \circ \gamma$ is absolutely continuous on $[0, l(\gamma)]$ for p-almost every rectifiable arc-length parametrized path γ in X. If X is a domain in $\mathbb{R}^n$ a function u is said to have the ACL property, or *absolute continuity on almost every line*, if on almost every line parallel to the coordinate axes with respect to the Hausdorff $(n-1)$-measure the function is absolutely continuous. An ACL function therefore has directional derivatives almost everywhere. An ACL function is said to have the property ACL_p if its directional derivatives are p-integrable.

The notation here is a slight modification of the notation used in [V], where an ACL_p-function is required to be continuous. It is shown in [Fu1, Theorem 11] and [V, Theorem 28.2] that for functions on domains in $\mathbb{R}^n$ the ACL_p property is equivalent to the ACC_p property. Recall that for domains Ω in $\mathbb{R}^n$, functions in $W^{1,p}(\Omega)$ have ACL_p representatives, and that conversely every p-integrable ACL_p-function is in $W^{1,p}(\Omega)$. See [EG] and [Z].

The following definition is due to [KM2], and is a weakening of the concept of upper gradient defined in Definition 1.2.

Definition 2.3. Let u be an arbitrary real-valued function on X, and let ρ be a non-negative Borel function on X. If there exists a family $\Gamma \subset \Gamma_{rect}$ such that $\mathrm{Mod}_p\Gamma = 0$ and inequality (2) is true for all paths γ in $\Gamma_{rect} \setminus \Gamma$, then ρ is said to be a *p-weak upper gradient* of u. If inequality (2) holds true for p-modulus almost all paths in a set $A \subset X$, then ρ is said to be a *p-weak upper gradient of u on A*. As the exponent p is usually fixed, in both cases ρ is simply called a weak upper gradient of u.

By [KM2, Lemma 2.4], the existence of a p-integrable weak upper gradient implies the existence of a p-integrable upper gradient which approximates the given weak upper gradient to any desired accuracy in the L^p-norm. This statement follows easily from the following lemma, which is a direct generalization of a theorem of Fuglede, [Fu1, Theorem 2], to metric measure spaces. The proof given in [Fu1] remains true even in this generality.

Lemma 2.1. *Let Γ be a collection of paths in X. Then $Mod_p \Gamma = 0$ if and only if there is a non-negative p-integrable Borel function ρ on X such that for all paths γ in Γ,*

$$\int_\gamma \rho ds = \infty.$$

Definition 2.4. Let the set $\tilde{N}^{1,p}(X, d, \mu)$ be the collection of all real-valued p-integrable functions u on X that have a p-integrable weak upper gradient.

Note that $\tilde{N}^{1,p}$ is a collection of functions and is also a vector space, since if $\alpha, \beta \in \mathbb{R}$ and $u_1, u_2 \in \tilde{N}^{1,p}$ with respective weak upper gradients ρ_1, ρ_2, then $|\alpha|\rho_1 + |\beta|\rho_2$ is a weak upper gradient of $\alpha u_1 + \beta u_2$. If u is a function in $\tilde{N}^{1,p}$, let

$$\|u\|_{\tilde{N}^{1,p}} = \|u\|_{L^p} + \inf_\rho \|\rho\|_{L^p},$$

where the infimum is taken over all p-integrable weak upper gradients of u. Again by [KM2, Lemma 2.4], the infimum could just as well be taken over all p-integrable upper gradients of u.

If u, v are functions in $\tilde{N}^{1,p}$, let $u \sim v$ if $\|u - v\|_{\tilde{N}^{1,p}} = 0$. It can be easily seen that $\sim$ is an equivalence relation, partitioning $\tilde{N}^{1,p}$ into equivalence classes. This collection of equivalence classes, under the norm of Definition 2.4, is a normed vector space.

Definition 2.5. The *Newtonian space* corresponding to the index p, $1 \leq p < \infty$, denoted $N^{1,p}(X)$, is defined to be the normed space $\tilde{N}^{1,p}(X, d, \mu)/\sim$, with norm $\|u\|_{N^{1,p}} := \|u\|_{\tilde{N}^{1,p}}$.

It will be shown in Corollary 3.3 that if two functions in $\tilde{N}^{1,p}$ agree almost everywhere, then they are in the same $N^{1,p}(X)$-equivalence class. However, it should be noted that if u is a function in $\tilde{N}^{1,p}$ and v is a function that agrees almost everywhere on X with u, it does not follow that v is also in $\tilde{N}^{1,p}$. Hence functions in the same equivalence class of $N^{1,p}(X)$ disagree on a smaller set than merely a measure zero set; see Section 4.

If u, v are functions in $\tilde{N}^{1,p}$, then it is easily verified that the functions $|u|$, $\min\{u, v\}$, and $\max\{u, v\}$ are also in $\tilde{N}^{1,p}$. Thus $N^{1,p}(X)$ enjoys all the lattice properties found in classical first order Sobolev spaces.

Definition 2.6. The space X is said to support a $(1,p)$-*Poincaré inequality* if there exists a constant $C > 0$ such that for all open balls B in X and all pairs of functions u and ρ defined on B, whenever ρ is an upper gradient of u on B and u is integrable on B the following inequality holds true:

$$\fint_B |u - u_B| \leq C diam(B) (\fint_B \rho^p)^{1/p}, \tag{3}$$

where, if f is a measurable function on X, then

$$f_B := \frac{1}{\mu(B)} \int_B f =: \fint_B f.$$

Note by the Hölder inequality that if a space supports a $(1,p)$-Poincaré inequality then it satisfies a $(1,q)$-Poincaré inequality for all $q > p$. For more discussion and examples of spaces with a Poincaré ineqality, see [HeK1], [HK2], [KM2], [S1], and [He]. These papers have a definition similar to the above, but requiring only that the inequality of Definition 2.6 be satisfied by continuous functions together with their upper gradients. The two definitions coincide under certain conditions; see [HeK2].

3 Newtonian Spaces Are Banach Spaces

This section explores some properties of Newtonian spaces, with the primary focus on proving in Theorem 3.7 that $N^{1,p}(X)$ is a Banach space. The difficulty in proving that the Cauchy sequences in $N^{1,p}(X)$ converge in its norm lies in taking care of the term involving upper gradients in the norm estimates. The problem lies in the fact that a difference of two $N^{1,p}$-functions does not necessarily have the difference of the respective weak upper gradients as a weak upper gradient.

Proposition 3.1. *If u is a function in $\tilde{N}^{1,p}$, then u is ACC_p.*

Proof. By the definition of $\tilde{N}^{1,p}$, u has a p-integrable weak upper gradient ρ. Let Γ be the collection of all paths in Γ_{rect} for which inequality (2) does not hold. Then by the definition of weak upper gradients, $\mathrm{Mod}_p\Gamma = 0$. Let Γ_1 be the collection of all paths in Γ_{rect} that have a sub-path in Γ. Then any admissible function used to estimate the modulus of Γ is an admissible function for Γ_1, and hence

$$\mathrm{Mod}_p\Gamma_1 \leq \mathrm{Mod}_p\Gamma = 0.$$

Let Γ_2 be the collection of all paths γ in Γ_{rect} such that $\int_\gamma \rho ds = \infty$. As ρ is p-integrable, $\mathrm{Mod}_p\Gamma_2$ is zero. Hence $\mathrm{Mod}_p(\Gamma_1 \cup \Gamma_2)$ is zero. If γ is a path in $\Gamma_{rect}\backslash(\Gamma_1 \cup \Gamma_2)$, then γ has no sub-path in Γ_1, and hence for all x, y in $|\gamma|$

$$|u(x) - u(y)| \leq \int_{\gamma_{xy}} \rho ds \ < \infty.$$

Therefore u is absolutely continuous on each path γ in $\Gamma_{rect}\backslash(\Gamma_1 \cup \Gamma_2)$.

□

Lemma 3.2. *Suppose u is a function in $\tilde{N}^{1,p}$ such that $\|u\|_{L^p} = 0$. Then the family*

$$\Gamma = \{\gamma \in \Gamma_{rect} \ : \ u(x) \neq 0 \textit{ for some } x \in |\gamma|\}$$

has zero p-modulus.

Proof. Since $\|u\|_{L^p} = 0$, the set $E = \{x \in X \ : \ u(x) \neq 0\}$ has measure zero. With the notation introduced in Section 2, one has $\Gamma = \Gamma_E$ and

$$\Gamma = \Gamma_E^+ \cup (\Gamma_E \backslash \Gamma_E^+).$$

The subfamily Γ_E^+ can be disregarded since

$$\mathrm{Mod}_p\Gamma_E^+ \leq \|\infty \cdot \chi_E\|_{L^p} = 0,$$

where χ_E is the characteristic function of the set E. The paths γ in $\Gamma_E\backslash\Gamma_E^+$ intersect E only on a set of linear measure zero, and hence with respect to linear measure almost everywhere on γ the function u takes on the value of zero. By the fact that γ also intersects E therefore, u is not absolutely continuous on γ. By Proposition 3.1,

$$\mathrm{Mod}_p(\Gamma_E \backslash \Gamma_E^+) = 0,$$

yielding that $\mathrm{Mod}_p\Gamma = 0$.

□

The above lemma yields the following:

Corollary 3.3. *If u_1 and u_2 are two functions in $\tilde{N}^{1,p}(X)$ such that $\|u_1 - u_2\|_{L^p} = 0$, then u_1 and u_2 belong to the same equivalence class in $N^{1,p}(X)$.*

The rest of the paper will not explicitly distinguish between the functions in $\tilde{N}^{1,p}$ and their equivalence classes in $N^{1,p}$.

The following lemma was first proved for $\mathbb{R}^n$ by Fuglede, [Fu1, Theorem 3 (f)]. The proof extends easily to metric measure spaces.

Lemma 3.4. *If $\{\rho_i\}_{i=1}^{\infty}$ is a sequence of Borel functions in $L^p(X)$ converging to zero in the L^p-norm, then there exists a subsequence $\{\rho_{i_k}\}_{k=1}^{\infty}$ and a zero p-modulus family $\Gamma \subset \Gamma_{rect}$ such that for all paths γ in $\Gamma_{rect} \setminus \Gamma$*

$$\lim_{k\to\infty} \int_\gamma \rho_{i_k} ds = 0.$$

Remark 3.5. By Lemma 3.4, if ρ_i is a Cauchy sequence of non-negative Borel functions in L^p converging to ρ in L^p, then there is a subsequence ρ_{i_k} such that for p-modulus almost every path γ in Γ_{rect},

$$\lim_{k\to\infty} \int_\gamma \rho_{i_k} ds = \int_\gamma \rho ds < \infty.$$

Different definitions for a capacity of a set can be found in literature. The definition of capacity used here is based on [KM1] and [AO].

Definition 3.1. The *p-capacity* of a set $E \subset X$ with respect to the space $N^{1,p}(X)$ is defined by

$$\mathrm{Cap}_p E = \inf_u \|u\|^p_{N^{1,p}}, \tag{4}$$

where the infimum is taken over all the functions u in $N^{1,p}$ whose restriction to E is bounded below by 1.

In the light of the following lemma, the discussion in [KM1] proving that the Hajłasz capacity is an outer measure is easily adaptable to show that Cap_p is indeed an outer measure. The papers [KM1] and [HeK2] explore the characteristics of sets of zero Hajłasz capacity. In particular, they discuss the Hausdorff measure properties of zero Hajłasz capacity sets.

Lemma 3.6. *If $F \subset X$ such that $Cap_p F = 0$, then $Mod_p \Gamma_F = 0$.*

Proof. Since $\mathrm{Cap}_p F = 0$, for each positive integer i there exists a function v_i in $N^{1,p}(X)$ such that $\|v_i\|_{N^{1,p}} \le 2^{-i}$ with $v_i|_F \ge 1$. Let

$$u_n = \sum_{i=1}^{n} |v_i|.$$

Then u_n is in $N^{1,p}(X)$ for each n, $u_n(x)$ is monotonic increasing for each $x \in X$, and

$$\|u_n - u_m\|_{N^{1,p}} \le \sum_{i=m+1}^{n} \|v_i\|_{N^{1,p}} \le 2^{-m} \to 0 \qquad \text{as } m \to \infty.$$

Therefore the sequence $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $N^{1,p}(X)$.

By the fact that the sequence is Cauchy in $N^{1,p}(X)$, the sequence is also Cauchy in L^p. Hence by passing to a subsequence if necessary, there is a function $\tilde{u}$ in L^p to which the subsequence converges both pointwise μ-almost everywhere and in the L^p-norm. Choose a further subsequence, also denoted $\{u_i\}_{i=1}^{\infty}$ for the sake of simplicity in notation, such that

$$\|u_i - \tilde{u}\|_{L^p} \le 2^{-i} \tag{5}$$

$$u_i \to \tilde{u} \text{ pointwise } \mu - a.e. \tag{6}$$

$$\|g_{i+1,i}\|_{L^p} \le 2^{-i}, \tag{7}$$

where g_{ij} is an upper gradient of $u_i - u_j$. If g_1 is an upper gradient of u_1, then $u_2 = u_1 + (u_2 - u_1)$ has as an upper gradient $g_2 = g_1 + g_{12}$. In general,

$$u_i = u_1 + \sum_{k=1}^{i-1} (u_{k+1} - u_k)$$

has as an upper gradient

$$g_i = g_1 + \sum_{k=1}^{i-1} g_{k+1,k}.$$

For $j < i$,

$$\|g_i - g_j\|_{L^p} \le \sum_{k=j}^{i-1} \|g_{k+1,k}\|_{L^p} \le \sum_{k=j}^{i-1} 2^{-k} \le 2^{-j+1} \to 0 \text{ as } j \to \infty.$$

Therefore $\{g_i\}_{i=1}^{\infty}$ is also a Cauchy sequence in L^p, and hence converges in the L^p-norm to a non-negative Borel function g.

Now let a function u be defined by

$$u(x) = \lim_{i\to\infty} u_i(x) \tag{8}$$

wherever the definition makes sense. Since $u_i \to \tilde{u}$ μ-almost everywhere by (6), $u(x) = \tilde{u}(x)$ μ-almost everywhere, and hence u is p-integrable. Let

$$E = \{x \ : \ \lim_{i\to\infty} u_i(x) = \infty\}.$$

The function u is well-defined outside of E. In order for u to be in the space $N^{1,p}(X)$ the function u has to be well-defined on almost all paths by Proposition 3.1. To this end it is shown that the p-modulus of the family Γ_E is zero.

Let Γ_1 be the collection of all paths γ from Γ_{rect} such that either $\int_\gamma g ds = \infty$ or

$$\lim_{i\to\infty} \int_\gamma g_i ds \neq \int_\gamma g ds.$$

Then by Lemma 3.4, $\mathrm{Mod}_p\Gamma_1 = 0$. Recall from Section 2 that

$$\Gamma_E^+ = \{\gamma \in \Gamma_{rect} \ : \ \mathcal{H}_1(|\gamma| \cap E) > 0\}.$$

As $\mu(E) = 0$ by (6), $\mathrm{Mod}_p\Gamma_E^+ = 0$. Therefore $\mathrm{Mod}_p(\Gamma_1 \cup \Gamma_E^+) = 0$. For any path γ in the family $\Gamma_{rect}\backslash(\Gamma_1 \cup \Gamma_E^+)$, by the fact that the path is not in Γ_E^+ there exists a point y in $|\gamma|$ such that y is not in E. For any point x in $|\gamma|$, since g_i is an upper gradient of u_i,

$$|u_i(x)| - |u_i(y)| \leq |u_i(x) - u_i(y)| \leq \int_\gamma g_i ds.$$

Therefore,

$$|u_i(x)| \leq |u_i(y)| + \int_\gamma g_i ds.$$

Taking limits on both sides and using the fact that γ is not in Γ_1,

$$\lim_{i\to\infty} |u_i(x)| \leq \lim_{i\to\infty} |u_i(y)| + \int_\gamma g ds < \infty,$$

and therefore x is not in E. Thus $\Gamma_E \subset \Gamma_1 \cup \Gamma_E^+$ and hence $\text{Mod}_p \Gamma_E = 0$.

Next, if γ is a path in $\Gamma_{rect} \backslash (\Gamma_1 \cup \Gamma_E^+)$, denoting the end points of γ as x and y and noting by the above argument that x and y are not in E, one has that

$$\begin{aligned} |u(x) - u(y)| &= |\lim_{i \to \infty} u_i(x) - \lim_{i \to \infty} u_i(y)| \\ &\leq \limsup_{i \to \infty} |u_i(x) - u_i(y)| \\ &\leq \lim_{i \to \infty} \int_\gamma g_i ds = \int_\gamma g ds \qquad \text{since } \gamma \notin \Gamma_1. \end{aligned} \tag{9}$$

Therefore g is a weak upper gradient of u, and hence u is in $N^{1,p}(X)$. For each x not in the set E one can write $u(x) = \lim_{i \to \infty} u_n(x)$, with $u(x)$ finite. If $F \backslash E$ is non-empty, then

$$u|_{F \backslash E} \geq u_n|_{F \backslash E} = \sum_{i=1}^{n} |v_i| \big|_{F \backslash E} \geq n$$

for arbitrarily large n, yielding that $u|_{F \backslash E}$ is infinite, which is not possible as x is not in the set E. Therefore $F \backslash E$ is empty, and hence $\Gamma_F \subset \Gamma_E$, and as it was shown above that the p-modulus of Γ_E is zero, the lemma follows.

$\square$

Theorem 3.7. *$N^{1,p}(X)$ is a Banach space.*

Proof. Let $\{u_i\}_{i=1}^\infty$ be a Cauchy sequence in $N^{1,p}(X)$. To show that this sequence is a convergent sequence in $N^{1,p}(X)$ it suffices to show that some subsequence is a convergent sequence in $N^{1,p}(X)$. Passing to a further subsequence if necessary, it can be assumed that

$$\|u_k - u_{k+1}\|_{N^{1,p}} \leq 2^{-k(p+1)/p} \tag{10}$$

and that

$$\|g_{i+1,i}\|_{L^p} \leq 2^{-i}, \tag{11}$$

where g_{ij} is an upper gradient of $u_i - u_j$ chosen to satisfy the above inequality. Let

$$E_k = \{x \in X \ : \ |u_k(x) - u_{k+1}(x)| \geq 2^{-k}\}.$$

Then $2^k|u_k - u_{k+1}|$ is in $N^{1,p}(X)$ and $2^k|u_k - u_{k+1}|\big|_{E_k} \geq 1$, and hence by inequality (10)

$$\mathrm{Cap}_p E_k \leq 2^{kp}\|u_k - u_{k+1}\|_{N^{1,p}}^p \leq 2^{kp}2^{-k(p+1)} \leq 2^{-k}.$$

Let $F_j = \bigcup_{k=j}^{\infty} E_k$. Then

$$\mathrm{Cap}_p F_j \leq \sum_{k=j}^{\infty} \mathrm{Cap}_p E_k \leq 2^{-j+1}.$$

Therefore the p-capacity of $F = \bigcap_{j\in\mathbb{N}} F_j$ is zero. If x is a point in $X\backslash F$, there exists j in $\mathbb{N}$ such that x is not in $F_j = \bigcup_{k=j}^{\infty} E_k$. Hence for all k in $\mathbb{N}$ such that $k \geq j$, x is not in E_k; for all k larger than j therefore $|u_k(x) - u_{k+1}(x)| \leq 2^{-k}$. Therefore whenever $l \geq k \geq j$ one has that

$$|u_k(x) - u_l(x)| \leq 2^{-k+1},$$

and thus the sequence $\{u_k(x)\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and therefore is convergent to a finite number. Hence if $x \in X\backslash F$, then

$$u(x) = \lim_{k\to\infty} u_k(x).$$

For $k < m$,

$$u_m = u_k + \sum_{n=k}^{m-1}(u_{n+1} - u_n).$$

Therefore for each x in $X\backslash F$,

$$\begin{aligned} u(x) &= u_k(x) + \sum_{n=k}^{\infty}(u_{n+1}(x) - u_n(x)), \\ u(x) - u_k(x) &= \sum_{n=k}^{\infty}(u_{n+1}(x) - u_n(x)). \end{aligned} \tag{12}$$

Noting by Lemma 3.6 that $\mathrm{Mod}_p\Gamma_F = 0$ and that for each path γ in $\Gamma_{rect}\backslash\Gamma_F$ for all points x in $|\gamma|$ equation (12) holds, conclude that $\sum_{n=k}^{\infty} g_{n+1,n}$ is a weak upper gradient of $u - u_k$. Therefore

$$\begin{aligned}
\|u - u_k\|_{N^{1,p}} &\le \|u - u_k\|_{L^p} + \sum_{n=k}^{\infty} \|g_{n+1,n}\|_{L^p} \\
&\le \|u - u_k\|_{L^p} + \sum_{n=k}^{\infty} 2^{-n} && \text{by condition (11)} \\
&\le \|u - u_k\|_{L^p} + 2^{-k+1} \to 0 && \text{as } k \to \infty.
\end{aligned}$$

Therefore the subsequence converges in the norm of $N^{1,p}(X)$ to u. The proof of the theorem is now complete.

□

Remark 3.8. The proof of the above theorem did not use the fact that the definition of (weak) upper gradients was based on all the compact rectifiable paths in X. One can therefore modify the definition of Newtonian spaces by modifying the definition of (weak) upper gradients by considering a particular family of compact rectifiable paths in X, and the modified spaces will also be Banach. This is useful in considering Example 3.10 below.

In the above proof it was shown that for each positive integer j there exists a set F_j of capacity no more than 2^{-j+1} such that the chosen subsequence converged uniformly outside of F_j. Hence the following corollary holds true:

Corollary 3.9. *Any Cauchy sequence $\{u_i\}_{i=1}^{\infty}$ in $N^{1,p}(X)$ has a subsequence that converges pointwise outside a set of zero p-capacity. Furthermore, the subsequence can be chosen so that there exist sets of arbitrarily small p-capacity such that the subsequence converges uniformly in the complement of each of these sets.*

The above corollary makes it possible to apply the machinery developed in [Fu2]. See also Remark 4.4.

In the following example, $P^{1,p}(X)$ is the vector space of p-integrable functions u that together with some p-integrable non-negative function ρ, not necessarily an upper gradient of u, satisfy the $(1,p)$-Poincaré inequality (3) on each open ball B.

Example 3.10. Let Ω be a domain in $\mathbb{R}^n$, and $X = \{X_1, ..., X_k\}$ be a collection of vector fields in Ω with real-valued locally Lipschitz coefficients. Such

X defines a differential operator on locally Lipschitz functions u on Ω:

$$Xu(x) = \sum_{j=1}^{k} X_j u(x) = \sum_{j=1}^{k} \langle X_j(x), \nabla u(x) \rangle \,,$$

where ∇u is defined almost everywhere on Ω by a theorem of Rademacher. Associated with such vector fields there is a Carnot-Carathéodory "metric" ρ: see [HK2, Section 11]. Suppose X satisfies the additional assumptions that the associated Carnot-Carathéodory metric ρ is indeed a metric on Ω, the metric space (Ω, ρ) satisfies a $(1,p)$-Poincaré inequality, and that the identity map from Ω equipped with the Euclidean metric to Ω equipped with the Carnot-Carathéodory metric is a homeomorphism: that is, the two induced topologies are equivalent. Vector fields satisfying Hörmander's condition, in particular the vector fields generating the tangent planes of a Carnot group, satisfy these conditions. Under these assumptions [HK2, Proposition 11.6] shows that if one restricts attention to the class of compact rectifiable paths γ whose tangent vectors are spanned by X, then $|Xu|$ is an upper gradient for each locally Lipschitz function u on Ω. In this structure, there is a natural definition of Newtonian spaces, namely the space $N^{1,p}_X(\Omega)$ of p-integrable functions u that have p-integrable upper gradients g, that is, for each compact rectifiable path γ whose tangent vectors are spanned by X inequality (2) is satisfied. The papers [FHK, Theorems 10, 11, and 12] and [HK2, Section 11] show that in this situation, if (Ω, ρ) supports a $(1,p)$-Poincaré inequality, then

$$N^{1,p}_X(\Omega) \subset P^{1,p}(\Omega) \subset H^{1,p}_X(\Omega),$$

where $H^{1,p}_X(\Omega)$ is the closure of the collection of all locally Lipschitz p-integrable functions on Ω such that $|Xu|$ is p-integrable, the closure being taken in the norm

$$\|u\| = \|u\|_{L^p} + \| |Xu| \|_{L^p}.$$

Hence in this situation, by Remark 3.8, $N^{1,p}_X(\Omega) = H^{1,p}_X(\Omega) = P^{1,p}(\Omega)$. For more discussion of Carnot-Carathéodory metric and Sobolev spaces generated by vector fields, see [GN]. The paper [HK2] contains further references to this topic.

4 $N^{1,p}(X)$ and $M^{1,p}(X)$

In the third section it was shown that even in the most general setting of the metric measure space $N^{1,p}(X)$ is a Banach space. However, if X does not have many rectifiable paths, then $N^{1,p}(X)$ reduces to the space $L^p(X)$. This section attempts to answer the question: when is $N^{1,p}(X)$ a reasonable space to consider.

Throughout the rest of the paper the open ball of radius r centered at x is denoted $B(x, r)$.

Definition 4.1. A metric measure space X is said to be a *doubling space* if there exists a constant $C \geq 1$ so that for all x in X and all radii $r > 0$,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Note that $\mathbb{R}^n$, together with Lebesgue measure, is a doubling space.

It is a classical result that smooth functions form a dense set in $W^{1,p}(\Omega)$ whenever Ω is a domain in $\mathbb{R}^n$. The following theorem is an analogue of this result for metric measure spaces supporting a $(1, p)$-Poincaré inequality as in Definition 2.6. The proof of the theorem is a modification of an idea due to Semmes, [S2].

Theorem 4.1. *If X is a doubling space such that*

$$C_1 = \inf_{x \in X} \mu(B(x, 1)) > 0,$$

and X supports a $(1, p)$-Poincaré inequality, then Lipschitz functions are dense in $N^{1,p}(X)$.

The proof of this theorem uses the following lemma, whose proof, obtained easily by an application of standard covering arguments, is omitted here.

Lemma 4.2. *Let X be as in Theorem 4.1, and let M^* be the non-centered maximal operator defined by*

$$M^* f(x) := \sup_B \fint_B |f| d\mu, \tag{13}$$

where the supremum is taken over balls B in X containing the point x. Then if g is a function in L^1, then

$$\lim_{\lambda \to \infty} \lambda \mu(\{x \in X \ : \ M^* g(x) > \lambda\}) = 0.$$

Lemma 4.3. *Suppose u is an ACC_p function on X such that there exists an open set $O \subset X$ with the property that on $X \backslash O$ the function $u = 0$ μ-almost everywhere. Then if g is an upper gradient of u, then $g\chi_O$ is also a weak upper gradient of u.*

Proof. Let $E = \{x \in X \backslash O \; : \; u(x) \neq 0\}$. Then by assumption $\mu(E) = 0$. Hence $\mathrm{Mod}_p(\Gamma_E^+)$ is also zero (since $\infty\chi_E$ is then an admissible function for this collection of paths). Let Γ_0 be the collection of paths on which u is not absolutely continuous. Let $\gamma \in \Gamma_{rect} \backslash (\Gamma_E^+ \cup \Gamma_0)$ connecting two points $x, y \in X$. If γ lies entirely in $O \cup E$, then clearly $|u(x) - u(y)| \leq \int_\gamma g = \int_\gamma g\chi_O$ because γ intersects E only on a set of Hausdorff 1-measure zero. If x and y are not in $O \cup E$, then $u(x) = u(y) = 0$, and hence again $|u(x) - u(y)| \leq \int_\gamma g\chi_O$. If x is a point in $O \cup E$ and γ does not lie entirely in $O \cup E$, noting that $(u \circ \gamma)^{-1}(0)$ is a compact subset of the domain $I = [a, b]$ of γ, the set $(u \circ \gamma)^{-1}(0)$ has a lower bound a_0 and an upper bound b_0 in I with $u \circ \gamma(a_0) = u \circ \gamma(b_0) = 0$ (it is possible that $b_0 = b$, but that does not create a problem here). Thus,

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u \circ \gamma(a_0)| + |u \circ \gamma(a_0) - u \circ \gamma(b_0)| + |u \circ \gamma(b_0) - u(y)| \\ &\leq \int_{\gamma|_{[a,a_0]}} g + \int_{\gamma|_{[b_0,b]}} g \\ &\leq \int_\gamma g\chi_O \end{aligned}$$

since the subpaths $\gamma\big|_{[a,a_0]}$ and $\gamma\big|_{[b_0,b]}$ intersect $E \cup (X \backslash O)$ only on a set of Hausdorff 1-measure zero. By the above three cases the result follows. $\square$

Proof of Theorem 4.1. If u is a function in $N^{1,p}(X)$, let

$$E_\lambda = \{x \in X \; : \; M^* g^p(x) > \lambda^p\},$$

where g is a p-integrable upper gradient of u. Then by Lemma 4.2,

$$\lambda^p \mu(E_\lambda) \to 0 \qquad \text{as} \qquad \lambda \to \infty. \tag{14}$$

If x is a point in $X \backslash E_\lambda$, then for all $r > 0$ one has that

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \leq Cr(\fint_{B(x,r)} g^p)^{1/p} \leq Cr(M^* g^p(x))^{1/p} \leq Cr\lambda.$$

Hence for $s \in [r/2, r]$ one has that

$$\begin{aligned}
|u_{B(x,s)} - u_{B(x,r)}| &\leq \fint_{B(x,s)} |u - u_{B(x,r)}| \\
&\leq \frac{\mu(B(x,r))}{\mu(B(x,s))} \fint_{B(x,r)} |u - u_{B(x,r)}| \\
&\leq C\lambda r
\end{aligned}$$

whenever x is in $X \backslash E_\lambda$. By a chaining argument for any positive $s < r$ (i.e. bounding s to be in an interval $[r/2^n, r/2^{n-1}]$ and then using triangle inequalities to move up to the radius r), for x in $X \backslash E_\lambda$ it is seen that

$$|u_{B(x,s)} - u_{B(x,r)}| \leq C\lambda r$$

and hence any sequence $u_{B(x,r_i)}$ is a Cauchy sequence in $\mathbb{R}$ and therefore is convergent. Therefore on $X \backslash E_\lambda$ the following function can be defined:

$$u_\lambda(x) := \lim_{r \to 0} u_{B(x,r)}.$$

Note that at Lebesgue points of u in $X \backslash E_\lambda$ it is true that $u_\lambda = u$, and that E_λ is an open set. For $x, y \in X$ consider the chain of balls $\{B_i\}_{i=-\infty}^{\infty}$, where

$$B_1 = B(x, d(x,y)) \qquad \text{and} \qquad B_{-1} = B(y, d(x,y))$$

and inductively for $i > 1$ obtain

$$B_i = (1/2)B_{i-1} \qquad \text{and} \qquad B_{-i} = (1/2)B_{-i+1}.$$

If x and y are in $X \backslash E_\lambda$, then they are also Lebesgue points of u_λ by construction, and hence

$$|u_\lambda(x) - u_\lambda(y)| \leq \sum_{i=-\infty}^{\infty} |u_{B_i} - u_{B_{i+1}}| \leq C\lambda d(x,y).$$

Hence u_λ is $C\lambda$-Lipschitz on $X \backslash E_\lambda$. Extend u_λ as a $C\lambda$-Lipschitz extension to the entire X; see [MS] for existence of such extensions. Choose an extension such that u_λ is bounded by $2C\lambda$. This can be done by truncating any Lipschitz extension at $C\lambda$. Such truncation will not affect the values of u_λ on the set

$X \backslash E_\lambda$ whenever λ is large enough so that $\mu(E_\lambda) \leq C_1/100$, since the original function u is bounded on its Lebesgue points of $X \backslash E_\lambda$, as seen by the following argument.

Suppose u is not bounded by $C\lambda$ on $X \backslash E_\lambda$ for any constant C independent of λ. Then one can find a sequence of points x_m in the set of Lebesgue points of $X \backslash E_\lambda$ so that $|u(x_m)|$ is larger than $m\lambda$, and by the fact that u is $C\lambda$-Lipschitz on that set, $|u|$ is bounded below by $\frac{1}{2}m\lambda$ on the balls $B(x_m, 1) \backslash (E_\lambda \cup L)$ where L is the set of non-Lebesgue points of u. Choosing $\lambda \geq 1$ large enough so that $\mu(E_\lambda) \leq C_1/100$, and choosing an infinite subsequence of x_m so that the unit balls centered around them are disjoint, therefore

$$\|u\|_{L^p}^p \geq \int_{B(x_m,1)\backslash E_\lambda} |u|^p \geq (\lambda/2)^p m^p \mu(B(x_m, 1) \backslash E_\lambda) \geq \lambda^p m^p C_1/50 \to \infty,$$

which would contradict the fact that u is p-integrable. Hence u is bounded by $C\lambda$ where $C^p = \frac{50}{C_1}\|u\|_{L^p}^p$ is independent of λ.

Now,

$$\begin{aligned} \int_X |u - u_\lambda|^p = \int_{E_\lambda} |u - u_\lambda|^p &\leq C \int_{E_\lambda} |u|^p + C \int_{E_\lambda} |u_\lambda|^p \\ &\leq C \int_{E_\lambda} |u|^p + C\lambda^p \mu(E_\lambda). \end{aligned}$$

By the fact that $\mu(E_\lambda) \to 0$ as $\lambda \to \infty$, by the fact that u is p-integrable, and by (14), it can be observed that the two terms on the right hand side of the above inequality tends to zero as λ tends to infinity. Hence u_λ converges to u as λ tends to infinity, the convergence occuring in the L^p-norm. The non-zero values of $u - u_\lambda$ are obtained only at points in the open set E_λ and on the set L whose measure is zero, and by Lemma 3.1 u is ACC_p and by the Lipschitz property so is u_λ. Therefore by Lemma 4.3 the function $(C\lambda + g)\chi_{E_\lambda}$ is a weak upper gradient of $u - u_\lambda$. Hence $u - u_\lambda$ is in $N^{1,p}(X)$, and therefore so is u_λ. Since by (14),

$$\int_X |\lambda \chi_{E_\lambda}|^p = \lambda^p \mu(E_\lambda) \to 0$$

and

$$\int_X |g\chi_{E_\lambda}|^p = \int_{E_\lambda} |g|^p \to 0$$

as $\lambda \to \infty$, the sequence u_λ converges to u in $N^{1,p}(X)$.

$\square$

Remark 4.4. By Corollary 3.9 and the above theorem, for all functions u in $N^{1,p}(X)$ there are open sets of arbitrarily small capacity such that u is continuous in the complement of these sets, provided the space X is doubling and satisfies the condition $\inf_{x\in X} \mu(B(x,1)) > 0$, and supports a $(1,p)$-Poincarè inequality. These sets are open since in the proof of Theorem 3.7 the sets E_k are open if the functions u_k are taken to be these Lipschitz approximations. Such continuity property is called quasicontinuity. The classical Sobolev spaces and Hajłasz spaces are composed of L^p-equivalece classes of functions, with each equivalence class containing a quasicontinuous function; see [EG, Section 4.2.1]. Due to the approach taken in this paper in defining Newtonian spaces, the equivalence classes in the Newtonian spaces consist solely of quasicontinuous functions whenever X supports a $(1,p)$-Poincaré inequality. In other words, if X is doubling and supports a $(1,p)$-Poincaré inequality, then functions in $\tilde{N}^{1,p}(X)$ are automatically quasicontinuous.

Theorem 4.5. *If $X = \Omega$ is a domain in $\mathbb{R}^n$, $d(x,y) = |x-y|$, and μ is Lebesgue n-measure, then as Banach spaces $N^{1,p}(X) = W^{1,p}(\Omega)$.*

Proof. Ohtsuka proved in [O] that $W^{1,p}(\Omega) \subset N^{1,p}(\Omega)$. See also [Fu1] and [V].

Suppose $u \in N^{1,p}(X)$. Then by Proposition 3.1, u has property ACC_p and has a p-integrable weak upper gradient ρ in L^p. Therefore u is ACL with principal directional gradient matrix ∇u such that by applying the fundamental theorem of calculus and a Lebesgue point argument one easily sees that $|\nabla u| \le \rho$ *a.e.* Hence u has property ACL_p and hence by [Z, Theorem 2.1.4], $u \in W^{1,p}(\Omega)$.

$\square$

The following Lemma has an easily verifiable proof.

Lemma 4.6. *If $u \in M^{1,p}(X)$, $p \ge 1$, with a Hajłasz gradient g, then there exists two functions $\tilde{u}, \tilde{g}$ such that $u = \tilde{u}$ a.e., and $\|g\|_p = \|\tilde{g}\|_p$, and for all points x, y in X*

$$|\tilde{u}(x) - \tilde{u}(y)| \le d(x,y)(\tilde{g}(x) + \tilde{g}(y)).$$

Furthermore, if u is a continuous function in $M^{1,p}(X)$, then only its Hajłasz gradient needs to be altered.

Inequality (1) defining the space $M^{1,p}(X)$ in Definition 1.1 is required to hold only almost everywhere. Hence $M^{1,p}(X)$ is a collection of equivalence classes of functions, with two functions belonging to the same equivalence class if and only if they differ only on a set of measure zero.

The idea for the proof of the following lemma is from [H2,Proposition 1].

Lemma 4.7. *The set of all equivalence classes of continuous functions u in $M^{1,p}(X)$ embeds into $N^{1,p}(X)$, with*

$$\|u\|_{N^{1,p}(X)} \leq 4\|u\|_{M^{1,p}(X)}.$$

Proof. Suppose u is a continuous representative of its equivalence class in $M^{1,p}(X)$. Then by Lemma 4.6, for each Hajłasz gradient of u there exists a function g in L^p with the same L^p-norm such that inequality (1) holds true everywhere. Let $x, y \in X$ and γ be an arc-length parametrizing rectifiable path connecting x to y. If $\int_\gamma g = \infty$, then we have that $|u(x) - u(y)| \leq \int_\gamma g$. So suppose the integral of g over γ is finite. For each number n in $\mathbb{N}$ let σ_n be the partition of the domain of γ into n pieces of equal length. On each partition $\gamma_i = \gamma\big|_{\sigma_n(i),\sigma_n(i+1)}$, $0 \leq i \leq n-1$, there exists x_i in $|\gamma_i|$ such that

$$g(x_i) \leq \fint_{\gamma_i} g ds.$$

Note that $d(x_i, x_{i+1}) \leq 2l(\gamma_i)$. Using these points x_i, one has that

$$\begin{aligned}
|u(x_0) - u(x_n)| \leq \sum_{i=0}^{n-1} |u(x_i) - u(x_{i+1})| &\leq \sum_{i=0}^{n-1} d(x_i, x_{i+1})(g(x_i) + g(x_{i+1})) \\
&\leq 4 \sum_{i=0}^{n} l(\gamma_i) \fint_{\gamma_i} g ds \\
&\leq 4 \sum_{i=0}^{n} \int_{\gamma_i} g ds = 4 \int_\gamma g ds.
\end{aligned}$$

Now, as u is continuous, by letting $n \to \infty$ the following inequality is obtained:

$$|u(x) - u(y)| \leq 4 \int_\gamma g ds.$$

Therefore the continuous representative u of its equivalence class in $M^{1,p}(X)$ belongs to an equivalence class in $N^{1,p}(X)$, with $\|u\|_{N^{1,p}} \le 4\|u\|_{M^{1,p}}$. By Lemma 3.2, if any representative in the equivalence class of u in $M^{1,p}(X)$ belongs to an equivalence class in $N^{1,p}(X)$, then it belongs to the same equivalence class as u in $N^{1,p}(X)$. Hence the embedding is well-defined.

$\square$

Theorem 4.8. *The Hajłasz space $M^{1,p}(X)$ continuously embeds into the space $N^{1,p}(X)$.*

Proof. By Theorem 3.7 the space $N^{1,p}(X)$ is a Banach space. Hence the closure of the subspace of equivalence classes of continuous functions in $M^{1,p}$ in the norm of $M^{1,p}$ yields a subspace of $N^{1,p}(X)$ by Lemma 4.7. By [H1] Lipschitz functions and therefore continuous functions are dense in $M^{1,p}(X)$ and hence such closure is the Hajłasz space $M^{1,p}(X)$, yielding the required result.

$\square$

The author does not know whether the embedding norm 4 in the above theorem can be improved.

The following theorem is obtained by Theorem 4.8, [KM2, Theorem 4.5], and the fact that if X supports a $(1,q)$-Poincaré inequality for some q in $[1,p)$ then $N^{1,p}(X) \subset P^{1,p}(X)$. The better Poincaré inequality ($q < p$) is required in order to apply [KM2, Theorem 4.5]. While [KM2] assumes X to be proper (that is, closed balls are compact), their proof of Theorem 4.5 does not need this assumption, for they consider a modified version of the Korevaar and Schoen space, [KS]. Here $P^{1,p}(X)$ is the vector space of p-integrable functions u that together with some p-integrable non-negative function ρ, not necessarily an upper gradient of u, satisfy the $(1,p)$-Poincaré inequality (3) on each open ball B.

Theorem 4.9. *If X is a metric measure space equipped with a doubling measure, and X supports a $(1,q)$- Poincaré inequality for some $q \in (1,p)$, then as sets*

$$M^{1,p}(X) = N^{1,p}(X) = P^{1,p}(X).$$

Moreover, $N^{1,p}(X) = M^{1,p}(X)$ isomorphically as Banach spaces.

For examples of spaces X where the Hajłasz spaces do not coincide with the Newtonian spaces, see Example 6.8.

[1] After this paper was submitted the author received a copy of a paper of Cheeger, [C], which gives another definition of Sobolev spaces. It turns out that this definition yields the same space as $N^{1,p}$ when $p > 1$.

Definition 4.2. For $p \geq 1$, the Sobolev type space $H_{1,p}(X)$ is the subspace of $L^p(X)$ consisting of functions f for which the norm

$$|f|_{1,p} = \|f\|_{L^p} + \inf_{\{f_i\}} \liminf_{i\to\infty} \|g_i\|_{L^p} \tag{15}$$

is finite. Here the limit infimum is taken over all upper gradients (or equivalently, weak upper gradients) g_i of the functions f_i, where the sequence f_i converges in the L^p-norm to the function f.

Theorem 4.10. *The above defined function space $H_{1,p}(X)$ is isometrically equivalent to $N^{1,p}(X)$ when $p > 1$.*

The following lemma is needed in the proof of the above theorem.

The proof of Lemma 3.6, inequalities (8) and (9), which remain valid for general Cauchy sequences of functions in $N^{1,p}(X)$, yields the following result: given a Cauchy sequence of functions in $L^p(X)$ and a corresponding Cauchy sequence in L^p of respective upper gradients, the two functions that the respective sequences converge to are related as a function-weak upper gradient pair. The following lemma from [KSh] proves a stronger version of this result. This result can be used most of the time in place of Mazur's lemma.

Lemma 4.11. *Let Y be a metric measure space and let $p > 1$. If $\{f_j\}_{j\in\mathbb{N}}$ is a sequence of functions in $L^p(Y)$ with upper gradients $\{g_j\}_{j\in\mathbb{N}}$ in $L^p(Y)$, such that f_j weakly converges to f in L^p and g_j weakly converges to g in L^p, then g is a weak upper gradient of f and there is a convex combination sequence $\tilde{f}_j = \sum_{k=j}^{n_j} \lambda_{kj} f_k$ and $\tilde{g}_j = \sum_{k=j}^{n_j} \lambda_{kj} g_k$ with $\sum_{k=j}^{n_j} \lambda_{kj} = 1$, $\lambda_{kj} > 0$, so that $\tilde{f}_j$ converges in L^p to f and $\tilde{g}_j$ converges in L^p to the function g.*

Proof of Theorem 4.10. Clearly functions in $N^{1,p}(X)$ satisfy the above definition: the sequence f_i could be taken to be the function itself. By Lemma 4.11, it is also clear that functions satisfying the above definition have an L^p-representative in $N^{1,p}(X)$. Moreover, the $N^{1,p}(X)$-norm is equal to the norm (15). □

[1] October 1998

When $p = 1$, it is still true that $N^{1,1}(X)$ embeds continuously into $H_{1,1}(X)$ by a norm non-increasing embedding, but it is no longer clear that $H_{1,1}(X)$ embeds into $N^{1,1}(X)$.

The paper [C] states that when X is doubling in measure and supports a $(1,p)$-Poincaré inequality and $p > 1$, the space $H_{1,p}(X)$ is reflexive. Hence by the above theorem, in this situation $N^{1,p}(X)$ is also reflexive.

5 Classical Sobolev Embedding Theorem

When $X = \mathbb{R}^n$, d the Euclidean metric, and μ the Lebesgue n-measure, one has the following classical embeddings:

$$W^{1,p}(X) \hookrightarrow L^{\frac{np}{n-p}} \qquad \text{if } \ p < n,$$
$$W^{1,p}(X) \hookrightarrow C^{0,1-n/p} \qquad \text{if } \ p > n,$$

where, for positive numbers $\alpha < 1$

$$C^{0,\alpha} = \{u : X \to \mathbb{R} \ : \ \exists C > 0 \text{ s.t. } \forall x, y \in X \ |u(x) - u(y)| \leq Cd(x,y)^{\alpha}\}.$$

Under certain conditions on the space X this section looks at the possiblity of obtaining similar embedding theorems. See [HK1] and [HK2] for similar results for the Hajłasz spaces.

Theorem 5.1. *Let $Q > 0$. If X is a doubling space satisfying $\mu(B(x,r)) \geq Cr^Q$, with C independent of $x \in X$ and $0 < r < 2diamX$, and supporting a (1,p)-Poincaré inequality for some $p > Q$, then $N^{1,p}(X)$ continuously embeds into the space $C^{0,1-Q/p}$.*

In other words, every $N^{1,p}$-equivalence class has a representative that is Hölder continuous with exponent $1 - Q/p$, with the Hölder norm bounded by its $N^{1,p}$-norm.

Proof. For $x, y \in X$ consider the chain of balls $\{B_i\}_{i=-\infty}^{\infty}$, where

$$B_1 = B(x, d(x,y)) \qquad \text{and} \qquad B_{-1} = B(y, d(x,y))$$

and inductively for $i > 1$ obtain

$$B_i = (1/2)B_{i-1} \qquad \text{and} \qquad B_{-i} = (1/2)B_{-i+1}.$$

If x and y are also Lebesgue points of u, the following is obtained:

$$
\begin{aligned}
|u(x) - u(y)| &\leq \sum_{i=-\infty}^{\infty} |u_{B_i} - u_{B_{i+1}}| \\
\text{with } |u_{B_i} - u_{B_{i+1}}| &\leq C\ diam(2B_i)(\fint_{2B_i} \rho^p)^{1/p} \\
&\leq C(diam(2B_i))^{1-Q/p}(\int_{2B_i} \rho^p)^{1/p} \\
&\leq C(diam(2B_i))^{1-Q/p}\|\rho\|_{L^p}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|u(x) - u(y)| &\leq C d(x,y)^{1-Q/p}(\sum_{i=-\infty}^{\infty} 2^{-|i|(1-Q/p)})\|\rho\|_{L^p} \\
&\leq C(Q,p)\|\rho\|_{L^p} d(x,y)^{1-Q/p}.
\end{aligned}
$$

Let L be the set of non-Lebesgue points of u. Since X is doubling, $\mu(L) = 0$. By the above argument $u\big|_{X\setminus L}$ is Hölder continuous with index $1 - Q/p$, and hence by [MS], can be extended as a Hölder continuous function $\tilde{u}$ to all of X. Note that the p-modulus of the collection Γ_L^+ is zero. If Γ_0 is the collection of curves on which u is not absolutely continuous, then $\text{Mod}_p\Gamma_0 = 0$. If γ is a path in the collection $\Gamma_{rect}\setminus(\Gamma_L^+ \cup \Gamma_0)$, then on $|\gamma|$ almost everywhere with respect to the one dimensional Hausdorff measure it is true that $\tilde{u} = u$. As u and $\tilde{u}$ are both continuous on $|\gamma|$, the two functions u and $\tilde{u}$ must agree on all of $|\gamma|$. Therefore if E is the collection of all points on which the two functions do not agree, then the p-modulus of the collection Γ_E is zero. Hence $\tilde{u}$ is in $N^{1,p}(X)$ and belongs to the same equivalence class as u.

□

Definition 5.1. Let $Q > 0$. A metric measure space X is said to be *Ahlofrs Q-regular* or Q-regular if there exists a constant $C \geq 1$ so that for each point x in X and for each positive $r < 2\text{diam}X$,

$$\frac{1}{C} r^Q \leq \mu(B(x,r)) \leq C r^Q.$$

Theorem 5.2. *If X is bounded, Q-regular, $Q > 1$, and supports a (1,q)-Poincaré inequality for some q such that $1 < q < p$ and $1 < p/q < Q$, then if u is in $N^{1,p}(X)$ and ρ is an upper gradient of u, then*

$$\left(\int_X |u(x) - u_X|^t d\mu(x)\right)^{1/t} \leq C\, diam X^{1-1/q} \|\rho\|_{L^p},$$

where $t = Qpq/(Qq - p)$.

The condition $1 < q$ is a technical requirement. If the space supports a $(1,1)$-Poincaré inequality, then it supports a $(1,q)$-Poincaré inequality for each $q > 1$, and the theorem remains true in this case as well. The condition $p/q < Q$ is easy to satisfy as one can always increase q while keeping the validity of $(1,q)$-Poincaré inequality. The non-trivial requirement here is the condition $q < p$.

Proof. For Lebesgue points x in the space X consider the collection of balls $\{B_i\}_{i=0}^{\infty}$ such that $B_0 = B(x, \mathrm{diam} X)$ and for each $i > 0$ the ball $B_i = B(x, 2^{-i}\mathrm{diam} X)$. Then,

$$\begin{aligned}
|u(x) - u_X| \leq \sum_{i=0}^{\infty} |u_{B_i} - u_{B_{i+1}}| \leq C \sum_{i=0}^{\infty} 2 \fint_{B_i} |u(z) - u_{B_i}| dz \\
\leq \sum_{i=0}^{\infty} C diam(B_i) (\fint_{B_i} \rho^q)^{1/q} \\
\leq \sum_{i=0}^{\infty} C 2^{-i(1-1/q)} \mathrm{diam} X^{1-1/q} (\int_{B_i} \frac{\rho(z)^q}{d(x,z)^{Q-1}} dz)^{1/q} \\
\leq C \mathrm{diam} X^{1-1/q} (\int_X \frac{\rho^q(z)}{d(x,z)^{Q-1}} dz)^{1/q}.
\end{aligned}$$

The last integral is the Riesz potential estimate $I_1(\rho^q)(x)$, and since in Q-regular spaces the Riesz kernel is a bounded map from L^s to $L^{Qs/(Q-s)}$ for $s < Q$, the last integral yields a function in $L^{Qpq/(Qq-p)}$; see [Z] for properties of Riesz potentials. The discussion in [Z] goes through even in this general setting. For further details see [HK2] and [He]. Thus the theorem is proved. $\square$

6 More Properties of $N^{1,p}(X)$ and Examples

Definition 6.1. An alternative definition for p-capacities of subsets E of X is as follows:

$$\mathrm{Cap}_p^* E = \inf_u \|u\|_{N^{1,p}}^p,$$

where the infimum is taken over all functions u in $N^{1,p}$ such that for p-almost all paths γ intersecting E the limit of $u \circ \gamma(t)$ along γ as $\gamma(t)$ and converges to any intersecting point in E exists and is not smaller than 1. This definition in Euclidean spaces was used in [AO].

Another definition of capacity, $\mathrm{Cap}_p^{**}E$, is obtained when the corresponding infimum is taken over all the functions u in $N^{1,p}$ that are bounded below by 1 in a neighbourhood of E.

Aikawa and Ohtsuka show in [AO, Theorem 5] that under certain conditions on the measure the last two definitions of capacity agree for subsets of bounded domains in $\mathbb{R}^n$. By the easily provable fact that if $\lambda \in \mathbb{R}$ and $u \in N^{1,p}(X)$, then the function $v = \min\{u, \lambda\}$ is also in $N^{1,p}(X)$ with any weak upper gradient of u also being a weak upper gradient of v, the condition "≥ 1" can be replaced with the condition "$= 1$" in the above definitions of capacity.

Lemma 6.1. *If $E \subset X$, then $Cap_p^{**}E \geq Cap_p E \geq Cap_p^* E$.*

Proof. Let u be any function in $N^{1,p}(X)$ such that $u|_E \geq 1$. Then as u is ACC_p by Proposition 3.1, it is also an admissible test function in determining $\mathrm{Cap}_p^* E$. Also, any admissible test function used in calculating $\mathrm{Cap}_p^{**}E$ is an admissible test function for $\mathrm{Cap}_p E$.

□

The rest of the section will assume that the measure is also an inner measure: that is, for every subset A of X, the measure of A is the supremum of the measures of closed subsets of A.

The following definition for functions in $\mathbb{R}^n$ is due to Ohtsuka, [O].

Definition 6.2. Let ρ be a p-integrable non-negative Borel function in X. Such a function defines an equivalence relation $\sim_\rho$ as follows: For $x, y \in X$, $x \sim_\rho y$ if either $y = x$ or there exists a path γ in Γ_{rect} connecting x to y such that $\int_\gamma \rho ds < \infty$. It is easy to see that this is indeed an equivalence relation, and $\sim_\rho$ partitions X into equivalence classes.

A metric measure space X is said to admit the *Main Equivalence Class* property with respect to p, or MEC_p, if each p-integrable non-negative Borel function ρ generates an equivalence class G_ρ, hereafter referred to as the *main equivalence class* of ρ, such that $\mu(X \backslash G_\rho) = 0$. It has been shown in [O] that $\mathbb{R}^n$ has the MEC_p-property for all p.

Note that in general equivalence classes need not be measurable sets. However, in MEC_p spaces, the main equivalence class, being of full measure, is necessarily measurable, and so are the other equivalence classes.

Definition 6.3. Let $Q > 1$. The space X is said to be a *Q-Loewner space* if X is path-connected and there is a monotonic decreasing function $\varphi : (0, \infty) \to (0, \infty)$ such that for all disjoint non-degenerate continua E and F the family $\Gamma(E, F)$ of all paths connecting E to F in X satisfies:

$$\mathrm{Mod}_Q(\Gamma(E, F)) \geq \varphi(\Delta(E, F)),$$

where

$$\Delta(E, F) = \frac{\mathrm{dist}(E, F)}{\min\{\mathrm{diam}(E), \mathrm{diam}(F)\}}.$$

See [HeK1, Section 3] for details. In particular, [HeK1] shows that under certain mild geometric conditions on a Q-regular space X, the space X supports a $(1, Q)$-Poincaré inequality if and only if it is Q-Loewner.

Theorem 6.2. *If X is a Q-Loewner space such that for almost all points x in X the mass density $\limsup_{r \to 0} \frac{\mu(B(x,r))}{r^Q} < \infty$, then X is a MEC_Q-space.*

Proof. Fix $\rho \in L^Q(X)$ and let $\Gamma_0 = \{\gamma \in \Gamma_{rect} \ : \ \int_\gamma \rho ds = \infty\}$. Then $\mathrm{Mod}_Q \Gamma_0 = 0$. Let

$$G = \bigcup_{\gamma \in \Gamma_{rect} \backslash \Gamma_0} |\gamma|.$$

Clearly if $x, y \in |\gamma|$ for some $\gamma \in \Gamma_{rect} \backslash \Gamma_0$, then $x \sim_\rho y$. If $x \in |\gamma_1|,\ y \in |\gamma_2|$, with $\gamma_1, \gamma_2 \in \Gamma_{rect} \backslash \Gamma_0$ and γ_1 and γ_2 do not intersect, then as γ_1 and γ_2 are compact sets, by the Loewner property there exists $\gamma_3 \in \Gamma_{rect} \backslash \Gamma_0$ intersecting both γ_1 and γ_2, and hence $x \sim_\rho y$. Therefore all elements of G belong to the same equivalence class with respect to ρ. Furthermore, if $x \in G$ and $y \notin G$, then there does not exist $\gamma \in \Gamma_{rect} \backslash \Gamma_0$ such that $x \in |\gamma|$ and $y \in |\gamma|$ and therefore $x \not\sim_\rho y$. Thus G is an equivalence class with respect to ρ.

Let $A_0 = X \backslash G$. It remains to show that $\mu(A_0) = 0$. Suppose $\mu(A_0) > 0$. The set A_0 may not be measurable. However, by the assumption made at the beginning of this section the measure is an inner measure, measure of arbitrary sets E are supremum of measures of closed subsets of E. Hence there is a closed set $A \subset A_0$ such that $\mu(A) > 0$. Since μ is a Borel measure, A is measurable. This set A has a point of density x_0 such that $\limsup_{r \to 0} \mu(B(x_0, r))/r^Q \le C_{x_0}/2 > 0$:

$$\limsup_{r \to 0} \frac{\mu(B(x_0, r) \bigcap A)}{\mu(B(x_0, r))} = 1.$$

Therefore for each positive number ϵ there exists a positive number r_ϵ such that

$$\frac{\mu(B(x_0, r_\epsilon) \backslash A)}{\mu(B(x_0, r_\epsilon))} \le \epsilon.$$

Consider $E, F \subset B(x_0, r_\epsilon/2)$ such that E and F are non-degenerate continua with the relative distance

$$\Delta(E, F) = \frac{\operatorname{dist}(E, F)}{\min\{\operatorname{diam}(E), \operatorname{diam}(F)\}}$$

comparable to a constant, and $dist(E, F) \ge kr_\epsilon$ where $k \le 1/2$ is some positive constant independent of ϵ. Such E, F exist because X is path-connected. For example, take E to be a path connecting the boundary of $B(x_0, r_\epsilon/16)$ to the boundary of the ball $B(x_0, r_\epsilon/8)$ without going outside the closure of $B(x_0, r_\epsilon/8)$, and take F to be a path connecting the boundary of $B(x_0, r_\epsilon/4)$ to the boundary of $B(x_0, r_\epsilon/3)$ without going into the ball $B(x_0, r_\epsilon/4)$ nor outside the closure of the ball $B(x_0, r_\epsilon/3)$. By the Loewner property of X the Q-modulus of all the paths joining E to F is bounded away from zero by a constant C independent of ϵ. Denote the collection of all such paths Γ_ϵ. Then

$$\Gamma_\epsilon \subset \Gamma_A \bigcup (\Gamma_\epsilon \backslash \Gamma_A),$$

where, recall that Γ_A is the collection of rectifiable paths intersecting the set A. The path $\gamma \in \Gamma_A$ implies that $\gamma \in \Gamma_0$ and hence $\mathrm{Mod}_Q \Gamma_A = 0$. Therefore,

$$\mathrm{Mod}_Q \Gamma_\epsilon = \mathrm{Mod}_Q (\Gamma_\epsilon \backslash \Gamma_A).$$

But the function

$$\rho = \chi_{B(x_0, r_\epsilon)\setminus A} \cdot 1/(kr_\epsilon)$$

is an admissible test function for $\Gamma_\epsilon \setminus \Gamma_A$, and hence by the fact that

$$\limsup_{r\to 0} \mu(B(x_0, r))/r^Q = C_{x_0} < \infty,$$

the Q-modulus of Γ_ϵ is less than or equal to $C_{x_0}\epsilon$ for some constant C_{x_0} independent of ϵ. This term converges to zero as ϵ tends to zero, contradicting the Loewner property. Hence the measure of A is zero, contradicting the choice of A. Therefore, $\mu(X\setminus G) = 0$.

□

The condition $\limsup_{r\to 0} \mu(B(x, r))/r^Q < \infty$ for almost every x in X is satisfied by the spaces having lower mass bounds of exponent Q. The lower mass bound condition is a global condition, whereas in the proof of the above theorem only the local version is needed. Manifolds such as infinitely long cylindrical surfaces are not 2-regular, but satisfy the above local limit property with $Q = 2$. These surfaces are 2-Loewner, and the above theorem shows that they are MEC_2 spaces. In fact, by the proof of the theorem above, all Riemannian manifolds of dimension n are MEC_n-spaces.

Remark 6.3. Under the assumption of MEC_p condition, it is easily seen by the following argument that $\mathrm{Cap}_p E = \mathrm{Cap}_p^* E$.

Suppose u is a function in $N^{1,p}(X)$ such that for p-almost every path γ in Γ_E

$$\lim_{\gamma(t)\to|\gamma|\cap E} (u \circ \gamma(t)) \geq 1.$$

Let E_1 be the set of all points x in E such that $u(x)$ is strictly less than 1. Then for each γ in Γ_{E_1} either

$$\lim_{\gamma(t)\to|\gamma|\cap E_1} (u \circ \gamma(t)) < 1$$

or else, either the limit does not exist or u is less than 1 at some point in $|\gamma| \cap E_1$; that is, u is not absolutely continuous on γ. By the choice of u and by Proposition 3.1 therefore $\mathrm{Mod}_p \Gamma_{E_1} = 0$. By Lemma 3.2 and by the fact that X is an MEC_p space and hence $\mu(E_1) = 0$, the value of u can be adjusted on E_1 to be greater than or equal to 1 to obtain a $N^{1,p}(X)$-function in the same $N^{1,p}(X)$-equivalence class as u but with the property of being greater than or equal to 1 on all of E. Hence $\text{Cap}_p E \leq \text{Cap}_p^* E$. By Lemma 6.1 the result follows.

Lemma 6.4. *Let X be a MEC_p-space containing two disjoint open sets. If $E \subset X$, then $Mod_p \Gamma_E = 0$ if and only if $Cap_p E = 0$.*

Proof. Suppose $E \subset X$ such that $\text{Mod}_p \Gamma_E = 0$. Then by Lemma 2.1 there exists a p-integrable non-negative Borel function ρ such that for all γ in Γ_E the integral $\int_\gamma \rho ds$ is infinite. By the MEC_p property of X, ρ has a main equivalence class G_ρ. Since X contains two disjoint open sets and open sets have positive measure, G_ρ has more than one element. If x is in E and $y \neq x$ is in G_ρ, one has that any path connecting x to y is in Γ_E and therefore by the choice of ρ one can see that $x \not\sim_\rho y$. Hence E is a subset of $X \backslash G_\rho$. Thus $\mu(E) = 0$. Therefore the function $u = \chi_E$ is in L^p and is absolutely continuous on all the paths in Γ_{rect} that are not in Γ_E. In addition, the zero function is a weak upper gradient of u, and hence u is in $N^{1,p}(X)$. Hence

$$\text{Cap}_p E \leq \|u\|_{N^{1,p}}^p = 0.$$

Now suppose that $E \subset X$ such that $\text{Cap}_p E = 0$. Then by Lemma 3.6 the p-modulus of Γ_E is zero.

□

The proof of the above lemma yields the following.

Lemma 6.5. *Let X be a MEC_p space containing two disjoint open sets. If $E \subset X$ and $Mod_p \Gamma_E = 0$, then the measure of E is zero.*

Corollary 6.6. *If X is a MEC_p space containing two disjoint open sets, then $Mod_p \Gamma_{rect}$ is strictly positive.*

Proof. Since the measure of X is positive, the p-capacity of X is not zero; for each admissible function u chosen in calculating $\text{Cap}_p X$,

$$\|u\|_{N^{1,p}}^p \geq \mu(X),$$

and hence taking the infimum over all such admissible functions,

$$\text{Cap}_p X \geq \mu(X) > 0.$$

Hence by Lemma 6.4 the p-modulus of Γ_{rect} is not zero. □

Remark 6.7. A similar result to Lemma 6.4 holds true for Q-Loewner spaces whose almost every point x satisfies the condition $\limsup_{r\to 0}\mu(B(x,r))/r^Q < \infty$. In such a space X, for any subset E of X the Q-modulus of the path family Γ_E^+ is zero if and only if $\mu(E)=0$. This can be proved by an argument similar to the proof of Theorem 6.2. Moreover, here it is sufficient to require the space X to be "locally Loewner" in a suitable sense.

Example 6.8. If X is an MEC_p-space such that there exists a ball B in X so that $\mu(B)>0$ and $\mu(X\backslash B)>0$, then there exists an equivalence class $[u]$ in L^p such that any function u in this equivalence class is not in any equivalence class of $N^{1,p}(X)$. In particular, $N^{1,p}(X)$ is strictly smaller than the space $L^p(X)$.

Let $\tilde{u}=\chi_B$ and $[u]$ its equivalence class in L^p; here $\|\tilde{u}\|_{L^p}=(\mu(B))^{1/p}<\infty$. Suppose u is a function in this equivalence class that also belongs to $\tilde{N}^{1,p}$. Then $u(x)=1$ for almost all x in B and $u(x)=0$ for almost all x in $X\backslash B$. Let

$$E=\{x\in X\ :\ u(x)\neq\tilde{u}(x)\}.$$

As u is in the same L^p-equivalence class as $\tilde{u}$ one can conclude that $\mu(E)=0$. Hence

$$\mathrm{Mod}_p(\Gamma_E^+\cup\Gamma_u)=0,$$

where Γ_u is the collection of paths on which u is not absolutely continuous (Proposition 3.1), and so by Lemma 2.1 there exists a non-negative Borel-measurable p-integrable function ρ so that for all paths γ in $\Gamma_E^+\cup\Gamma_u$ the integral $\int_\gamma \rho ds$ is infinite. As X is an MEC_p space ρ has a main equivalence class $G:\ \mu(X\backslash G)=0$. Thus there is a point x in B and a point y in $X\backslash B$ so that x and y are both in G: there is a rectifiable path γ connecting x to y so that $\int_\gamma \rho ds<\infty$. By the choice of ρ one then has that γ is in neither Γ_E^+ nor in Γ_u, and hence u is absolutely continuous on γ and

$$\mathcal{H}_1(|\gamma|\cap B\cap E)=0=\mathcal{H}_1(|\gamma|\cap(X\backslash B)\cap E).$$

Let x_0 be the point in $|\gamma|$ at which γ first leaves the open set B (such a point exists since $|\gamma|$ is a compact set). The function u however cannot be continuous at x_0 as every neighbourhood in $|\gamma|$ of x_0 contains points at which u is zero and also points at which u is 1. Thus u cannot be in $\tilde{N}^{1,p}$.

The following example shows that it is not always the case that $N^{1,p}(X)$ embeds into $M^{1,p}(X)$.

Example 6.9. In [K] for every $q \in (1, n]$ Koskela has an example of a space $X = \mathbb{R}^n \setminus E$, $E \subset \mathbb{R}^{n-1} \times \{0\}$, so that X supports a $(1, p)$-Poincaré inequality for every $p \geq q$ but does not support a $(1, p)$-Poincaré inequality for any $p < q$. In these spaces, by Theorem 4.1, one knows that Lipschitz functions are dense in $N^{1,p}(X)$ whenever $p \geq q$ Hence as Lipschitz functions are extendable uniquely (since $|E| = 0$) to all of $\mathbb{R}^n$, all $N^{1,p}(X)$ functions are extendable to all of $\mathbb{R}^n$:

$$N^{1,p}(X) = N^{1,p}(\mathbb{R}^n) = M^{1,p}(\mathbb{R}^n), \qquad p \geq q.$$

Since inequality (1) is needed to be satisfied only almost everywhere for $M^{1,p}$-functions and $|E| = 0$, it is true that $M^{1,p}(\mathbb{R}^n) = M^{1,p}(X)$ for all p, $1 \leq p < \infty$. Hence whenever $p \geq q$ one has that $N^{1,p}(X) = M^{1,p}(X)$. When $1 < p < q$, by [K, Theorem A] and Theorem 4.5 the space $N^{1,p}(X) \neq N^{1,p}(\mathbb{R}^n)$ and hence, as $M^{1,p}(\mathbb{R}^n) = M^{1,p}(X)$, in this case the space $N^{1,p}(X)$ does not embed into the space $M^{1,p}(X)$ if $p < q$.

Another question one could ask is whether in Theorem 4.9 one really needs $q < p$- i.e. does there exist an example of a space that supports a $(1, p)$-Poincaré inequality but does not support a $(1, q)$-Poincaré inequality for any $q < p$ and $N^{1,p}$ does not embed into $M^{1,p}$. In Example 6.8 the embedding was done by extending the $N^{1,p}$ functions to all of $\mathbb{R}^n$ and then embedding into $M^{1,p}$, which does not capture the essence of the effect of Poincaré inequalities. The following example answers the above question in the affirmative.

Example 6.10. Let $p = 1$ and X be a unit ball B in $\mathbb{R}^n$. Then $N^{1,p}(X) = W^{1,p}(X)$ is not the same space as $M^{1,p}(X)$ by the comments in [H2] and [HK2], and there is no number $q < p$ so that X supports a $(1, q)$-Poincaré inequality.

For $p > 1$ so far it is not known whether there are examples of spaces X supporting a $(1, p)$-Poincaré inequality but not a $(1, q)$-Poincaré inequality for any $q < p$ and $N^{1,p}(X)$ does not embed into $M^{1,p}(X)$.

References

[AO] H. Aikawa and M. Ohtsuka. *Extremal Length of Vector Measures,* Ann. Acad. Sci. Fenn. Math. **23** (1998) .

[C] J. Cheeger. *Differentiability of Lipschitz Functions on Measure Spaces,* preprint.

[EG] L. Evans and R. Gariepy. *Measure Theory and Fine Properties of Functions,* CRC Press, (1992).

[FHK] B. Franchi, P. Hajłasz, and P. Koskela. *Definitions of the Sobolev Space,* preprint.

[Fu1] B. Fuglede. *Extremal Length and Functional Completion,* Acta Math. **98** (1957), 171–218.

[Fu2] B. Fuglede. *The Quasi Topology Associated With a Countable Subadditive Set Function,* Ann. de l'Institut Fourier **21.1** (1971), 123–169.

[GN] N. Garofalo and D. M. Nhieu. *Isoperimetric and Sobolev Inequalities for Carnot-Carathéodory Spaces and the Existence of Minimal Surfaces,* Comm. Pure Appl. Math. **49** (1996) 1081–1144.

[H1] P. Hajłasz. *Sobolev Spaces on an Arbitrary Metric Space,* Potential Anal. **5** (1996), 403–415.

[H2] P. Hajłasz. *Geometric Approach to Sobolev Spaces and Badly Degenerated Elliptic Equations,* Nonlinear Analysis and Applications, Warsaw (1994), 141–168.

[HK1] P. Hajłasz and P. Koskela. *Sobolev meets Poincaré,* C.R. Acad. Sci., Paris **320** (1995), 1211–1215.

[HK2] P. Hajłasz and P. Koskela. *Sobolev Met Poincaré,* preprint.

[He] J. Heinonen. *Analysis on Metric Spaces,* Lecture Notes, University of Michigan, (1996).

[HeK1] J. Heinonen and P. Koskela. *Quasiconformal Maps in Metric Spaces with Controlled Geometry,* Acta Math. **181** (1998), 1–61.

[HeK2] J. Heinonen and P. Koskela. *A Note on Lipschitz Functions, Upper Gradients, and the Poincaré Inequality,* New Zealand Journal of Mathematics, to appear.

[KM1] J. Kinnunen and O. Martio. *The Sobolev Capacity on Metric Spaces,* Ann. Acad. Sci. Fenn. Math. **21** (1996), 367–382.

[KS] J. N. Korevaar and R. M. Schoen. *Sobolev Spaces and Harmonic Maps for Metric Space Targets,* Comm. Anal. Geom. **1** (1993), 561–659.

[K] P. Koskela. *Removable Sets for Sobolev Spaces,* Ark. Mat., to appear.

[KM2] P. Koskela and P. MacManus. *Quasiconformal Mappings and Sobolev Spaces,* Studia Math. **131, no.1** (1998), 1–17.

[KSh] S.Kallunki and N.Shanmugalingam. *Modulus and Continuous Capacity,* preprint.

[M] V. G. Maz'ya. *Sobolev Spaces,* Springer-Verlag, (1985).

[MS] E. J. McShane. *Extension of Range of Functions,* Bulletin, AMS **40** (1934) 837–842.

[O] M. Ohtsuka. *Extremal Lengths and Precise Functions in 3-Space,* manuscript.

[S1] S. Semmes. *Finding Curves on General Spaces Through Quantitative Topology with Applications to Sobolev and Poincarè Inequalities,* Selecta Math. **2** (1996) 155–295.

[S2] S. Semmes. Personal communication.

[V] J. Väisäla. *Lectures on n-dimensional Quasiconformal Mapping,* Lecture notes in Mathematics, Springer-Verlag **229** (1971).

[Z] W. Ziemer. *Weakly Differentiable Functions,* Springer-Verlag, Graduate Texts in Math. (1989).

Address: Department of Mathematics, University of Michigan, Ann Arbor, MI48109, U.S.A.

E-mail: lingam@math.lsa.umich.edu