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Carleson-Type Estimates for p-Harmonic Functions and the Conformal Martin Boundary of John Domains in Metric Measure Spaces

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Dedicated to Professor Saburou Saitoh on the occasion of his 60th birthday

1. Introduction

In the study of the local Fatou theorem for harmonic functions, Carleson [Ca] proved the following crucial estimate for positive harmonic functions, now referred to as the *Carleson estimate*. Given a bounded Lipschitz domain D in the Euclidean space $\mathbb{R}^n$, there exist constants $K, C > 1$, depending only on D, with the following property: If $\xi \in \partial D$, if $r > 0$ is sufficiently small, and if x_r is a point in D with $|x_r - \xi| = r$ and $\operatorname{dist}(x_r, \partial D) \ge r/C$, then

$$u \le Ku(x_r) \quad \text{on } D \cap B(\xi, r)$$

whenever u is a positive harmonic function in $D \cap B(\xi, Cr)$ vanishing continuously on $\partial D \cap B(\xi, Cr)$. Here $B(\xi, r)$ denotes the open ball with center ξ and radius r.

The Carleson estimate has been verified for more general Euclidean domains such as NTA domains, and it plays an important role in the study of harmonic analysis on nonsmooth domains. There are at least three different proofs of the Carleson estimate, based on (i) uniform barriers, (ii) the boundary Harnack principle, and (iii) the mean value inequality of subharmonic functions.

(i) Carleson's original proof, as well as the extension to NTA domains due to Jerison and Kenig [JK], are based on uniform barriers. This method was used also by the second author, together with Holopainen and Tyson, in [HoST] to study conformal Martin boundaries of bounded uniform domains in metric measure spaces of bounded geometry. This approach requires the notion of *uniform fatness* of the boundary introduced by Lewis in [Le].

(ii) In [A1], the first-named author proved the boundary Harnack principle directly and verified the Carleson estimate as a corollary. This method does not rely on uniform barriers and is applicable to uniform domains with a small boundary and, more generally, even to an irregular uniform domain. However, this method does not seem to be applicable to nonlinear equations.

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(iii) In the study of Denjoy domains, Benedicks [Be] employed Domar's argument [D] based on the mean value inequality of subharmonic functions. His approach was generalized to Lipschitz Denjoy domains by Chevallier [Ch]. The first author, together with Hirata and Lundh, utilized Domar's argument in [AHiL] to prove a version of the Carleson estimate for John domains in Euclidean spaces.

The first goal of this paper is to show that Domar's argument applies not only to harmonic functions on a Euclidean domain but also to solutions of certain nonlinear equations on metric measure spaces. Throughout we assume that (X, d, μ) is a proper metric measure space with at least two points and that μ is a doubling Borel measure. Here we say that X is *proper* if closed and bounded subsets of X are compact and that μ is *doubling* if there is a constant $C_d \geq 1$ such that

$$\mu(B(x, 2r)) \leq C_d \mu(B(x, r)),$$

where $B(x, r) = \{y \in X : d(x, y) < r\}$ is the open ball with center x and radius r. Moreover, we fix $1 < p < \infty$ and assume that X supports a $(1, p)$-Poincaré inequality (see Definition 2.1). We shall establish a Carleson-type estimate (Theorem 5.2) for John domains in the setting of such metric measure spaces by adapting the version of Domar's argument found in [AHiL].

Our second goal is the study of conformal Martin boundaries of bounded John domains whose boundaries may not be uniformly fat. Under the additional assumption that the measure μ is Ahlfors Q-regular (for some $Q > 1$), we will use our Carleson-type estimate to extend the results of [HoST] and [S3] to greater generality. One of our main results is Theorem 6.1, which describes the behavior of the conformal Martin kernels. The growth estimate (Theorem 6.1(ii)) is new.

In the general setting of metric measure spaces, it is not clear whether there exists even one bounded uniform domain in X. However, if X is a geodesic space then every ball in X is a John domain, with the center of the ball acting as a John center. It is a well-known fact that any doubling metric measure space supporting a $(1, p)$-Poincaré inequality is quasiconvex; that is, there is a constant $q \geq 1$ such that, for every pair of points $x, y \in X$, there exists a rectifiable curve γ_{xy} in X connecting x to y with the property that the length $\ell(\gamma_{xy})$ of γ_{xy} satisfies

$$\ell(\gamma_{xy}) \leq q\, d(x, y). \tag{1}$$

Thus, in our situation there are a plethora of bounded John domains in X even if X is not a geodesic space. It is therefore desirable to study the conformal Martin boundary of bounded John domains in X. The theory developed in [HoST] indicates that the conformal Martin boundary is conformally invariant. The results developed in this paper are therefore useful in the study of a Fatou-type property of conformal mappings between two bounded John domains in metric spaces; see [S3].

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2. Definitions and Notation

Unless otherwise stated, C denotes a positive constant whose exact value is unimportant, can change even within the same line, and depends only on fixed parameters such as X, d, μ, and p. If necessary, we will specify its dependence on other parameters.

In the setting of metric measure spaces that may not have a Riemannian structure, the following notion of upper gradients, first formulated by Heinonen and Koskela in [HeKo1], replaces the notion of distributional derivatives (in [HeKo1], upper gradients are referred to as "very weak" gradients). A Borel function g on X is an *upper gradient* of a real-valued function f on X if, for all nonconstant rectifiable paths $\gamma\colon [0,l_\gamma] \to X$ parameterized by arc length, we have

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \le \int_\gamma g\,ds,$$

where the inequality is interpreted as saying also that $\int_\gamma g\,ds = \infty$ whenever at least one of $|f(\gamma(0))|$ and $|f(\gamma(l_\gamma))|$ is infinite. See [HeKo1] and [KoMc] for more on this notion.

Definition 2.1. We say that X supports a $(1,p)$-*Poincaré inequality* if there are constants $\kappa \ge 1$ and $C_p \ge 1$ such that, for all balls $B(x,r) \subset X$, all measurable functions f on X, and all p-weak upper gradients g of f,

$$\fint_{B(x,r)} |f - f_{B(x,r)}|\,d\mu \le C_p r\biggl(\fint_{B(x,\kappa r)} g^p\,d\mu\biggr)^{1/p},$$

where

$$f_{B(x,r)} := \fint_{B(x,r)} f\,d\mu := \frac{1}{\mu(B(x,r))}\int_{B(x,r)} d\mu.$$

Following [S1], we consider a version of Sobolev spaces on X.

Definition 2.2. Let

$$\|u\|_{N^{1,p}} = \biggl(\int_X |u|^p\,d\mu\biggr)^{1/p} + \inf_g \biggl(\int_X g^p\,d\mu\biggr)^{1/p},$$

where the infimum is taken over all upper gradients of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}} = 0$.

The space $N^{1,p}(X)$ equipped with the norm $\|\cdot\|_{N^{1,p}}$ is a Banach space and a lattice (see [S1]). An alternative definition of Sobolev spaces given by Cheeger in [C] yields the same space as $N^{1,p}(X)$ whenever $p > 1$; see [S1, Thm. 4.10]. Cheeger's definition yields the notion of partial derivatives in the following theorem [C, Thm. 4.38].

THEOREM 2.3 (Cheeger). *Let X be a metric measure space equipped with a positive doubling Borel regular measure μ admitting a $(1, p)$-Poincaré inequality for some $1 < p < \infty$. Then there exists a countable collection (U_α, X^α) of measurable sets U_α and Lipschitz "coordinate" functions $X^\alpha\colon X \to \mathbb{R}^{k(\alpha)}$ such that $\mu\big(X \setminus \bigcup_\alpha U_\alpha\big) = 0$, and for each α the following conditions hold.*

The measure of U_α is positive and $1 \le k(\alpha) \le N$, where N is a constant depending only on the doubling constant of μ and the constant from the Poincaré inequality. If $f\colon X \to \mathbb{R}$ is Lipschitz, then there exist unique bounded measurable vector-valued functions $d^\alpha f\colon U_\alpha \to \mathbb{R}^{k(\alpha)}$ such that, for μ-a.e. $x_0 \in U_\alpha$,

$$\lim_{r\to 0^+} \sup_{x\in B(x_0,r)} \frac{|f(x) - f(x_0) - d^\alpha f(x_0)\cdot (X^\alpha(x) - X^\alpha(x_0))|}{r} = 0.$$

We can assume that the sets U_α are pairwise disjoint, and we extend $d^\alpha f$ by zero outside U_α. Regarding $d^\alpha f(x)$ as vectors in $\mathbb{R}^N$, let $df = \sum_\alpha d^\alpha f$. The differential mapping $d\colon f \mapsto df$ is linear, and it is shown in [C, p. 460] that there is a constant $C > 0$ such that, for all Lipschitz functions f and μ-a.e. $x \in X$,

$$\frac{1}{C}|df(x)| \le g_f(x) := \inf_g \limsup_{r\to 0^+} \fint_{B(x,r)} g\, d\mu \le C|df(x)|. \tag{2}$$

Here $|df(x)|$ is a norm coming from a measurable inner product on the tangent bundle of X created by the Cheeger derivative structure just described (see the discussion in [C]), and the infimum is taken over all upper gradients $g \in L^p(X)$ of f; observe that g_f is in some sense the *minimal* upper gradient of f (see [S2, Cor. 3.7]). Also, by [C, Prop. 2.2], $df = 0$ μ-a.e. on every set where f is constant.

By [C, Thm. 4.47] or [S1, Thm. 4.1], the Newtonian space $N^{1,p}(X)$ is equal to the closure in the $N^{1,p}$-norm of the collection of Lipschitz functions on X with finite $N^{1,p}$-norm. By [FHKo, Thm. 10], there exists a unique "gradient" du satisfying (2) for every $u \in N^{1,p}(X)$. Moreover, if $\{u_j\}_{j=1}^\infty$ is a sequence in $N^{1,p}(X)$, then $u_j \to u$ in $N^{1,p}(X)$ if and only if, as $j \to \infty$, $u_j \to u$ in $L^p(X,\mu)$ and $du_j \to du$ in $L^p(X,\mu;\mathbb{R}^N)$. Hence the differential structure extends to all functions in $N^{1,p}(X)$. We will use this structure throughout the paper; see for example Definition 2.5.

DEFINITION 2.4. The *p-capacity* of a Borel set $E \subset X$ is the number

$$\mathrm{Cap}_p(E) := \inf_u \left(\int_X |u|^p\, d\mu + \int_X |du|^p\, d\mu \right),$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E. A property is said to hold p-quasieverywhere in X if the set on which the property does not hold has zero p-capacity. The *relative p-capacity* $\mathrm{Cap}_p(K;\Omega)$ of a compact set K with respect to an open set $\Omega \supset K$ is given by

$$\mathrm{Cap}_p(K;\Omega) = \inf \int_\Omega |du|^p\, d\mu,$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$ for which $u|_K \ge 1$ and $u|_{X\setminus\Omega} = 0$. If no such function exists, then we set $\mathrm{Cap}_p(K;\Omega) = \infty$. For more on capacity, see [AO; HeKo2; KinM1; HeKiM, Chap. 2] and the references therein.

To compare the boundary values of Newtonian functions, we need a Newtonian space with zero boundary values. Let $\Omega \subset X$ be an open set and let

$$N_0^{1,p}(\Omega) = \{u \in N^{1,p}(X) : u = 0 \text{ } p\text{-quasieverywhere on } X \setminus \Omega\}.$$

Corollary 3.9 in [S1] implies that $N_0^{1,p}(\Omega)$ equipped with the $N^{1,p}$-norm is a closed subspace of $N^{1,p}(X)$. By [S2, Thm. 4.8], if Ω is relatively compact then the space $\mathrm{Lip}_c(\Omega)$ of Lipschitz functions with compact support in Ω is dense in $N_0^{1,p}(\Omega)$. In the rest of this paper, $\Omega \subset X$ will always denote a bounded domain in X with $\mathrm{Cap}_p(X \setminus \Omega) > 0$.

DEFINITION 2.5. Let $\Omega \subset X$ be a domain. A function $u \colon X \to [-\infty, \infty]$ is said to be *p-harmonic* in Ω if $u \in N_{\mathrm{loc}}^{1,p}(\Omega)$ and if, for all relatively compact subsets U of Ω and for every function $\varphi \in N_0^{1,p}(U)$,

$$\int_U |du|^p \, d\mu \le \int_U |d(u+\varphi)|^p \, d\mu. \tag{3}$$

We say that u is a p-subsolution in Ω if (3) holds for every nonpositive function $\varphi \in N_0^{1,p}(U)$. We say that u is a *p-quasiminimizer* if there is a constant $C_{qm} \ge 1$ such that, for all relatively compact subsets U of Ω and every function $\varphi \in N_0^{1,p}(U)$,

$$\int_U |du|^p \, d\mu \le C_{qm} \int_U |d(u+\varphi)|^p \, d\mu. \tag{4}$$

Furthermore, we say that u is a *p-quasisubminimizer* if (4) holds true whenever φ is a nonpositive function in $N_0^{1,p}(U)$.

REMARK 2.6. It is easily seen that p-harmonic functions are p-quasiminimizers and that p-subsolutions are p-quasisubminimizers. See [KinM3; KinS] for more on quasiminimizers.

DEFINITION 2.7. By $H_p^U f$ we denote the solution to the p-Dirichlet problem on the open set U with boundary data $f \in N^{1,p}(U)$; that is, $H_p^U f$ is p-harmonic in U and $H_p^U f - f \in N_0^{1,p}(U)$. An upper semicontinuous function u is said to be p-subharmonic in Ω if the comparison principle holds. That is, if $f \in N^{1,p}(U)$ is continuous up to ∂U and if $u \le f$ on ∂U, then $u \le H_p^U f$ on U for all relatively compact subsets U of Ω.

DEFINITION 2.8. Let Ω be a relatively compact domain in X and let $y \in \Omega$. An extended real-valued function $g = g(\cdot, y)$ on Ω is said to be a *p-singular function with singularity at y* if it satisfies the following four criteria:

(i) g is p-harmonic in $\Omega \setminus \{y\}$ and $g > 0$ on Ω;
(ii) $g|_{X\setminus\Omega} = 0$ p-q.e. and $g \in N^{1,p}(X \setminus B(y,r))$ for each $r > 0$;
(iii) y is a singularity (i.e., $\lim_{x\to y} g(x) = \infty$); and

(iv) whenever $0 \le a < b < \infty$,

$$\mathrm{Cap}_p(\Omega^b; \Omega_a) = (b-a)^{1-p}, \tag{5}$$

where $\Omega^b = \{x \in \Omega : g(x) \ge b\}$ and $\Omega_a = \{x \in \Omega : g(x) > a\}$.

In [HoS] it was shown that every relatively compact domain in a metric measure space equipped with a doubling measure supporting a $(1,q)$-Poincaré inequality with $q < p$ has a p-singular function that plays a role analogous to the Green function of the Euclidean p-Laplace operator. It was shown by Keith and Zhong [KeZ] that a complete metric space supporting a $(1,p)$-Poincaré inequality for some $p > 1$ also supports a $(1,q)$-Poincaré inequality for some $1 \le q < p$. Hence, we may apply the results of [KinS] and [HoS] without assuming a priori the better Poincaré inequality.

Note that we have the doubling property on the measure μ as a standing assumption. As a consequence of this doubling property, it can be shown that there are constants $Q > 0$ and C_1 such that, for all $x \in X$, $0 < \rho < R$, and $y \in B(x,R)$,

$$\frac{1}{C_1}\left(\frac{\rho}{R}\right)^Q \le \frac{\mu(B(y,\rho))}{\mu(B(x,R))}. \tag{6}$$

The book [He] has a proof of this fact. The measure μ is said to be *Ahlfors Q-regular* if there is a constant $C \ge 1$ such that, for every $x \in X$ and for every $r > 0$,

$$\frac{r^Q}{C} \le \mu(B(x,r)) \le Cr^Q. \tag{7}$$

For the rest of this section we will assume that $X = (X,d,\mu)$ is of Q-bounded geometry; that is, μ is Ahlfors Q-regular and X supports a $(1,Q)$-Poincaré inequality ([BoHeKo, Sec. 9] or [HoST]). It was shown in [HeKo1] that metric spaces of Q-bounded geometry possess a Loewner-type property related to the Q-modulus of curve families connecting compacta. Therefore, we can use the techniques of [Ho] to show that, for each $y \in \Omega$, there is exactly one Q-singular function for Ω with singularity at y satisfying equation (5). This enables us to define a boundary in a manner similar to the classical potential theoretic Martin boundary.

DEFINITION 2.9. Fix $x_0 \in \Omega$. Given a sequence (x_n) of points in Ω, we say that the sequence is *fundamental* (relative to x_0) if it has no accumulation point in Ω and if the sequence of normalized singular functions

$$M_{x_n}(x) = M(x,x_n) := \frac{g(x,x_n)}{g(x_0,x_n)}$$

is locally uniformly convergent in Ω. Here g is the Q-singular function for Ω. We set $M(x,x_0) = 0$ when $x \ne x_0$ and set $M(x_0,x_0) = 1$.

Given a fundamental sequence $\xi = (x_n)$, we shall denote the corresponding limit function

$$M(x) = M_\xi(x) := \lim_{n\to\infty} M(x,x_n)$$

and call it a *conformal Martin kernel* function. We say that two fundamental sequences ξ and ζ are equivalent (relative to x_0), $\xi \sim \zeta$, if $M_\xi = M_\zeta$. It is worth noting that M_ξ is a nonnegative Q-harmonic function in Ω, with $M_\xi(x_0) = 1$. Hence $M_\xi > 0$ in Ω by local Harnack's inequality (see [KinS, Cor. 7.3]). Note that if $\tilde{x}_0$ is another point in Ω then $g(x, x_n)/g(\tilde{x}_0, x_n) = M(x, x_n)/M(\tilde{x}_0, x_n)$. Hence the property of being a fundamental sequence is independent of the particular choice of x_0. Furthermore, fundamental sequences ξ and ζ are equivalent relative to x_0 if and only if they are equivalent relative to any $\tilde{x}_0 \in \Omega$. Thus the following definition is independent of the fixed point x_0.

Given a point $\chi \in \partial\Omega$, we say that a function M is a conformal martin kernel associated with χ if there is a fundamental sequence (y_n) in Ω such that $y_n \to \chi$ and the sequence of singular functions $M(\cdot, y_n)$ with singularity at y_n converges locally uniformly to M.

Definition 2.10. The collection of all equivalence classes of fundamental sequences in Ω (or, equivalently, the collection of all conformal Martin kernel functions) is the *conformal Martin boundary* $\partial_{cM}\Omega$ of the domain Ω. This collection is endowed with the local uniform topology: a sequence ξ_n in this boundary is said to converge to a point ξ if the sequence of functions M_{ξ_n} converges locally uniformly to M_ξ.

The classical Martin boundary theory can be extended to general domains in metric measure spaces under certain circumstances. However, there are examples of metric measure spaces with Ahlfors Q-regular measure, $Q > 2$, supporting a $(1, Q)$-Poincaré inequality but not $(1, 2)$-Poincaré inequality. For domains in such a metric space, 2-singular functions of Definition 2.8 may not exist and it is not easy to say what kind of ideal boundary should correspond to the classical Martin boundary, whereas the *conformal* Martin boundary can be constructed immediately as in [HoST].

3. Domar's Argument

Recall that X is a proper metric space and that μ is doubling and supports a $(1, p)$-Poincaré inequality. In this section we assume that $\Omega \subset X$ is a bounded open set. Increasing the value of C_1 in (6) to absorb the terms involving $B(x, 2\operatorname{diam}(\Omega))$ and $(2\operatorname{diam}(\Omega))^Q$, we obtain the lower mass bound:

$$\mu(B(y, r)) \geq \frac{r^Q}{C_1} \quad \text{for } y \in \bar{\Omega} \text{ and } 0 < r \leq 2\operatorname{diam}(\Omega). \tag{8}$$

Let u be a nonnegative, locally bounded p-subharmonic function or a p-quasisubminimizer in Ω. Then u is a p-quasisubminimizer [KinM2, Cor. 7.8] and hence u is in the De Giorgi class $DG_p(\Omega)$ (see [KinM3, Lemma 5.1]). This means that if $B(x, R) \subset \Omega$ then

$$\int_{\{y \in B(x,\rho)\,:\,u(y)>k\}} g_u^p \, d\mu \leq \frac{C}{(r-\rho)^p} \int_{\{y \in B(x,r)\,:\,u(y)>k\}} (u-k)^p \, d\mu$$

for every $k \in \mathbb{R}$ and $0 < \rho < r < R/\kappa$, where κ is the scaling constant from the Poincaré inequality. It therefore follows, from [KinS, Thm. 4.2] (with $k_0 = 0$) and the doubling property of the measure μ, that if $B(x, R) \subset \Omega$ then

$$u(x) \leq C_s \left(\fint_{B(x,R)} u^p \, d\mu \right)^{1/p}, \tag{9}$$

where $C_s \geq 1$ is independent of x, R, and u but depends on the quasisubminimizing constant C_{qm}. If a function u on an open set U satisfies (9) for every ball $B(x, R) \subset U$, then we say that u enjoys the *weak sub–mean value property in U.* Using the weak sub–mean value property, we shall give the following modification of Domar's theorem (see [D] and [AHiL]). Observe that the weak sub–mean value property (9) holds for more general classes of functions than the class of p-harmonic functions. Indeed, [KinS] (see also [KinM3]) proved this property for p-quasiminimizers and more generally for functions in the De Giorgi class. Hence the following lemma is phrased for the class of *all* functions satisfying the weak sub–mean value property, though in this paper it will later be applied only to p-quasisubminimizers. For $u > 0$ we write

$$\log^+ u = \begin{cases} \log u & \text{if } u \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 3.1. *Let Ω be a bounded open set in X and let $\delta_\Omega(x) = \operatorname{dist}(x, X \setminus \Omega)$. Suppose u is a locally bounded nonnegative function on Ω satisfying the weak sub–mean value property* (9) *in Ω. If there is a positive real number ε with*

$$I := \int_\Omega (\log^+ u)^{Q-1+\varepsilon} \, d\mu < \infty,$$

where Q is the lower mass bound in (8), *then there exists a constant $C > 0$ independent of u such that*

$$u(x) \leq 4C_s^2 \exp(C I^{1/\varepsilon} \delta_\Omega(x)^{-Q/\varepsilon}) \quad \textit{for all } x \in \Omega. \tag{10}$$

Note that in the Euclidean setting we have $\delta_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$, but in the general setting of metric measure spaces this may not be the case.

Proof. We first prove the following estimate.

Let $x \in X$ and $R > 0$ be such that u satisfies the p-sub–meanvalue property on $B(x, R)$. If $u(x) \geq t > 0$, $a \geq 2C_s$, and

$$\mu\left(\left\{ y \in B(x, R) : \frac{t}{a} < u(y) \leq at \right\}\right) \leq \frac{\mu(B(x, R))}{a^{2p}}, \tag{11}$$

then there exists an $x_2 \in B(x, R)$ such that $u(x_2) > at$. Here C_s is the constant in (9). To see this, suppose that $u(y) \leq at$ for every $y \in B(x, R)$. Then, by (9),

$$\begin{aligned}
t &\le u(x) \\
&\le C_s\left(\fint_{B(x,R)} u^p\,d\mu\right)^{1/p} \\
&\le C_s\left(\frac{1}{\mu(B(x,R))}\left[\left(\frac{t}{a}\right)^p \mu(B(x,R)) + \int_{\{y\in B(x,R)\,:\,u(y)>t/a\}} u^p\,d\mu\right]\right)^{1/p} \\
&\le \frac{2^{1/p}C_s}{a}t.
\end{aligned}$$

Since $C_s/a \le 1/2$ and $p > 1$ by assumption, we have $1 \le 2^{1/p}C_s/a < 1$, which is not possible. Hence there must be a point $x_2 \in B(x, R)$ such that $u(x_2) > at$.

Whatever the value of C might be, we have $C > 0$. Hence

$$\exp(CI^{1/\varepsilon}\delta_\Omega(x)^{-Q/\varepsilon}) \ge 1.$$

Therefore, in order to prove (10), it suffices to show that there exists a constant $C > 0$ such that, whenever $u(x) > 4C_s^2$,

$$u(x) \le \exp(CI^{1/\varepsilon}\delta_\Omega(x)^{-Q/\varepsilon}). \tag{12}$$

Fix $x \in \Omega$ such that $u(x) > 4C_s^2$. Then $u(x) > 1$ and hence $\log^+ u(x) = \log u(x)$. Therefore, demonstrating (12) is equivalent to showing that there is a constant $C > 0$ independent of x such that

$$\delta_\Omega(x) \le CI^{1/Q}(\log^+ u(x))^{-\varepsilon/Q}. \tag{13}$$

Toward this end, let us choose $a = 2C_s$. This sets us up to use (11). For $j \in \mathbb{N}$ let

$$R_j = \left[C_1 a^{2p}\mu(\{y \in \Omega : a^{j-2}u(x) < u(y) \le a^j u(x)\})\right]^{1/Q},$$

where C_1 is the constant in (8). We claim that

$$\delta_\Omega(x) \le 2\sum_{j=1}^{\infty} R_j. \tag{14}$$

In order to prove (14), we now construct a sequence of points in Ω, finite or infinite depending on the situation, as follows. Let $x_1 = x$. If $\delta_\Omega(x_1) < R_1$, then consider the singleton sequence (x_1). Suppose $\delta_\Omega(x_1) \ge R_1$. Since $B(x_1, R_1) \subset \Omega$, it follows that

$$\left[C_1 a^{2p}\mu(\{y \in B(x_1, R_1) : a^{-1}u(x_1) < u(y) \le au(x_1)\})\right]^{1/Q} \le R_1.$$

From (8) we then have that

$$\mu(\{y \in B(x_1, R_1) : a^{-1}u(x_1) < u(y) \le au(x_1)\}) \le \frac{R_1^Q}{C_1 a^{2p}} \le \frac{\mu(B(x_1, R_1))}{a^{2p}}.$$

Now, by (11) with $t = u(x_1)$, there is a point $x_2 \in B(x_1, R_1)$ such that $u(x_2) > au(x_1)$. If $\delta_\Omega(x_2) < R_2$ then consider the sequence (x_1, x_2). Otherwise, we have $B(x_2, R_2) \subset \Omega$ and, as before, we can apply (11) with $t = au(x_1)$ to obtain

$x_3 \in B(x_2, R_2)$ such that $u(x_3) > au(x_2) > a^2u(x_1)$. Inductively, we may construct x_J given $(x_1, x_2, \dots, x_{J-1})$ such that, for $j = 1, \dots, J-1$,

$$\delta_\Omega(x_j) \ge R_j, \quad d(x_j, x_{j-1}) < R_{j-1}, \quad u(x_j) > au(x_{j-1}) > a^{j-1}u(x_1).$$

If $\delta_\Omega(x_{J-1}) < R_{J-1}$, then we stop here. Otherwise, we may use (11) to find a point $x_J \in B(x_{J-1}, R_{J-1})$ such that $u(x_J) > au(x_{J-1}) > a^{J-1}u(x_1)$. We will now show that $\delta_\Omega(x_1) \le 2\sum_{j=1}^{\infty} R_j$. In order to do so, we consider two cases.

Case 1: the sequence is finite. Then there is a positive integer J such that $\delta_\Omega(x_J) < R_J$. Hence

$$\delta_\Omega(x_1) \le \sum_{j=1}^{J-1} d(x_j, x_{j+1}) + \delta_\Omega(x_J) < \sum_{j=1}^{J-1} R_j + R_J \le \sum_{j=1}^{\infty} R_j.$$

Case 2: the sequence is infinite. Then, for every $j \in \mathbb{N}$, we have $u(x_j) > a^{j-1}u(x_1)$; that is, $\lim_{j\to\infty} u(x_j) = \infty$. But then, as u is locally bounded on Ω, the infinite sequence $(x_j)_j$ has no accumulation point in Ω. Since (by assumption) X is proper and so $\bar{\Omega}$ is compact, there is a subsequence converging to a point in $\partial\Omega$. Hence there exists some $J \in \mathbb{N}$ for which $\delta_\Omega(x_J) < \frac{1}{2}\delta_\Omega(x_1)$. As a result,

$$\delta_\Omega(x_1) \le \sum_{j=1}^{J-1} d(x_j, x_{j+1}) + \delta_\Omega(x_J) < \sum_{j=1}^{J} R_j + \frac{1}{2}\delta_\Omega(x_1),$$

and we can then conclude that

$$\delta_\Omega(x_1) \le 2\sum_{j=1}^{J} R_j \le 2\sum_{j=1}^{\infty} R_j.$$

Hence (14) follows whether the sequence just constructed is finite or not.

Therefore, to show (13) it suffices to prove that

$$\sum_{j=1}^{\infty} R_j \le CI^{1/Q}(\log^+ u(x))^{-\varepsilon/Q}. \tag{15}$$

Let j_0 be the unique positive integer such that $a^{j_0} < u(x) \le a^{j_0+1}$. Then $j_0 \ge 2$, since we have assumed that $u(x) > 4C_s^2 = a^2$. Recall that, for $j \in \mathbb{N}$,

$$R_j = \left[C_1 a^{2p}\mu(\{y \in \Omega : a^{j-2}u(x) < u(y) \le a^j u(x)\})\right]^{1/Q}.$$

We obtain from Hölder's inequality that

$$\begin{aligned}
\sum_{j=1}^{\infty} R_j &\le C\sum_{j=1}^{\infty}\left[\mu(\{y \in \Omega : a^{j_0+j-2} < u(y) \le a^{j_0+j+1}\})\right]^{1/Q} \\
&\le C\sum_{j=j_0-1}^{\infty}\left[\mu(\{y \in \Omega : a^j < u(y) \le a^{j+3}\})\right]^{1/Q} \\
&= C\sum_{j=j_0-1}^{\infty}\frac{j^{(Q-1+\varepsilon)/Q}}{j^{(Q-1+\varepsilon)/Q}}\left[\mu(\{y \in \Omega : a^j < u(y) \le a^{j+3}\})\right]^{1/Q}
\end{aligned}$$

$$\leq C\Bigg[\sum_{j=j_0-1}^{\infty} \frac{1}{j^{(Q-1+\varepsilon)/(Q-1)}}\Bigg]^{(Q-1)/Q}\Bigg[\sum_{j=j_0-1}^{\infty} \int_{\{y\in\Omega\,:\,a^j<u(y)\leq a^{j+3}\}} j^{Q-1+\varepsilon}\,d\mu\Bigg]^{1/Q}$$

$$\leq Cj_0^{-\varepsilon/(Q-1)}\Bigg[\sum_{j=j_0-1}^{\infty} \int_{\{y\in\Omega\,:\,a^j<u(y)\leq a^{j+3}\}} \left(\frac{\log^+ u}{\log a}\right)^{Q-1+\varepsilon} d\mu\Bigg]^{1/Q}.$$

Note that each $y \in \Omega$ belongs to at most three of the sets $\{y \in \Omega : a^j < u(y) \leq a^{j+3}\}$. Hence we see that

$$\sum_{j=1}^{\infty} R_j \leq Cj_0^{-\varepsilon/Q} I^{1/Q}.$$

By the choice of j_0 it can be seen that $a^{j_0} < u(x) \leq a^{j_0+1}$ and $j_0 \geq 2$. We therefore have $j_0 \leq \log u(x)/\log a \leq j_0 + 1 \leq 2j_0$, and so

$$\sum_{j=1}^{\infty} R_j \leq C\left(\frac{\log^+ u(x)}{2\log a}\right)^{-\varepsilon/Q} I^{1/Q},$$

which is (15). Note that C is independent of x and u. This completes the proof of Lemma 3.1. □

It should be emphasized that in Lemma 3.1 we do not require that Ω be a domain. It is sufficient to assume merely that Ω is a bounded open subset of X such that $X \setminus \Omega \neq \emptyset$.

4. Geometry of Bounded John Domains in X

Let $\Omega \subset X$ be a domain. For $0 < c < 1$, a rectifiable curve γ connecting $x, y \in \Omega$ is said to be a *c-John curve* in Ω if $\delta_\Omega(z) \geq c\ell(\gamma_{xz})$ for every $z \in \gamma$, where γ_{xz} is the subcurve of γ having x and z as its two endpoints. We say that Ω is a *John domain* with *John center* $x_0 \in \Omega$ and *John constant* c if every point $x \in \Omega$ can be connected to x_0 by a c-John curve in Ω. For $A > 1$, a rectifiable curve γ connecting $x, y \in \Omega$ is said to be a *A-uniform curve* in Ω if $\ell(\gamma) \leq Ad(x, y)$ and $\min\{\ell(\gamma_{xz}, \ell(\gamma_{zy})\} \leq A\delta_\Omega(z)$ for every $z \in \gamma$. We say that Ω is an *A-uniform domain* if every pair of distinct points $x, y \in \Omega$ can be joined by an A-uniform curve γ in Ω. Obviously, a uniform domain is a John domain; but the converse is not necessarily true.

Given $x \in X$ and $R > 0$, we let $\bar{B}(x, R) = \{y \in X : d(x, y) \leq R\}$. Note that in general this may be a larger set than the closure of the open ball $B(x, R)$ itself. By $S(x, R)$ we mean the sphere centered at x of radius R: $S(x, R) = \{y \in X : d(x, y) = R\}$. This set contains $\partial B(x, R)$ but in general could be a larger set.

For domains $V \subset X$, we let k_V denote the following quasihyperbolic "metric" on V. If $x, y \in V$, then

$$k_V(x, y) = \inf_\gamma \int_\gamma \frac{ds(t)}{\delta_V(\gamma(t))},$$

where the infimum is taken over all rectifiable curves connecting x to y in V. If no such curve exists in V, then we set $k_V(x,y)=\infty$. It is worth noting that if V is a John domain then k_V is always finite-valued; that is, it is indeed a metric. In fact, if γ is a c-John curve connecting x to y in Ω, then

$$k_\Omega(x,y) \le \int_\gamma \frac{ds(t)}{\delta_\Omega(\gamma(t))} \le \int_0^{\delta_\Omega(x)/2} \frac{ds}{\delta_\Omega(x)/2} + \int_{\delta_\Omega(x)/2}^{\ell(\gamma)} \frac{ds}{cs}$$
$$\le 1 + \frac{\log 2/c}{c} + \frac{1}{c}\log\left(\frac{\delta_\Omega(y)}{\delta_\Omega(x)}\right). \tag{16}$$

Suppose for a moment that Ω is a uniform domain. Let $\xi \in \partial\Omega$ and $x \in \Omega \cap \bar{B}(\xi, R/2)$. By the uniformity we find a point $y \in S(\xi, R)$ with $\delta_\Omega(y) \ge R/(3A)$ and a uniform curve $\gamma \subset \Omega$ connecting x and y. Observe that $\gamma \subset \Omega \cap B(\xi, CR)$ with $C > 1$ depending only on the uniformity. For this point y we have

$$k_\Omega(x,y) = k_{\Omega\cap B(\xi,2CR)}(x,y) \le C\left[\log\left(\frac{R}{\delta_\Omega(x)}\right) + 1\right].$$

This property can be generalized to a John domain. Following [AHiL], we give the following definition.

DEFINITION 4.1. Let $N \in \mathbb{N}$. We say that $\xi \in \partial\Omega$ has a *system of local reference points of order N* if there exist constants $R_\xi > 0$, $\lambda_\xi > 1$, and $A_\xi > 1$ such that, whenever $0 < R < R_\xi$, we can find $y_1, \dots, y_N \in \Omega \cap S(\xi, R)$ with the following properties:

(i) $A_\xi^{-1} R \le \delta_\Omega(y_i) \le R$ for each $i = 1, \dots, N$; and

(ii) for every $x \in \Omega \cap \bar{B}(\xi, R/2)$, there exist some $i \in \{1, \dots, N\}$ for which

$$k_\Omega(x, y_i) = k_{\Omega\cap B(\xi,\lambda_\xi R)}(x, y_i) \le A_\xi\left[\log\left(\frac{R}{\delta_\Omega(x)}\right) + 1\right].$$

REMARK 4.2. If Ω is a uniform domain, then every $\xi \in \partial\Omega$ has a system of local reference points of order 1. The constants R_ξ, λ_ξ, and A_ξ depend only on the uniformity of Ω.

LEMMA 4.3. *Let $\Omega \subset X$ be a John domain with John center x_0 and John constant c. Then there exists $N_0 \in \mathbb{N}$, depending only on x_0, c, and the data of X, such that every $\xi \in \partial\Omega$ has a system of local reference points of order at most N_0 with constants $R_\xi = R_0 := \min\{\delta_\Omega(x_0)/2, \operatorname{diam}(\Omega)/10\}$, $\lambda_\xi = 8$, and $A_\xi = A_0 :=$ $\max\{2/c,\ 3/2 + c^{-1}\log 2/c\}$.*

Proof. Let $\xi \in \partial\Omega$ and $0 < R < R_0$ and suppose that $x \in \Omega \cap \bar{B}(\xi, R/2)$. Let γ_x be a c-John curve connecting x to x_0 and let y_x be the point of γ_x at which γ_x leaves the ball $B(\xi, R)$ for the first time. Then

$$R \ge \delta_\Omega(y_x) \ge c\ell(\beta_x) \ge cR/2 \ge R/A_0, \tag{17}$$

where β_x is the subcurve of γ_x that terminates at y_x. Let $E = \{y_x : x \in \Omega \cap \bar{B}(\xi, R/2)\}$. By the 5-covering lemma (see e.g. [He]) and the doubling property of μ, we find finitely many points $y_1 = y_{x_1}, \dots, y_N = y_{x_N} \in E$ such that $N \le N_0$,

$$E \subset \bigcup_{i=1}^{N} B\left(y_i, \frac{cR}{10q}\right) \subset B(\xi, R),$$

and $\{B(y_i, cR/(50q))\}_{i=1}^{N}$ is pairwise disjoint, where $q \geq 1$ is the quasiconvexity constant in (1).

Let us demonstrate that $y_1, \ldots, y_N$ satisfy the conditions of Definition 4.1. Obviously, (i) holds true by (17). To prove (ii), for $x \in \Omega \cap \bar{B}(\xi, R/2)$ take the point y_x and the subcurve β_x as described after (17). Then $\delta_{\Omega \cap B(\xi, 8R)}(z) = \delta_\Omega(z)$ for $z \in \beta_x$, so that (16) gives

$$k_{\Omega \cap B(\xi, 8R)}(x, y_x) \leq \int_{\beta_x} \frac{ds(t)}{\delta_\Omega(\beta_x(t))} \leq 1 + \frac{\log 2/c}{c} + \frac{1}{c} \log\left(\frac{R}{\delta_\Omega(x)}\right).$$

Since $y_x \in E \subset \bigcup_i B(y_i, cR/(10q))$, we can find a positive integer $i \leq N$ such that $y_x \in B(y_i, cR/(10q))$. By the quasiconvexity of X there is a rectifiable curve γ connecting y_x to y_i with $\ell(\gamma) \leq q\, d(y_x, y_i) \leq cR/10$. By (17),

$$\delta_{\Omega \cap B(\xi, 8R)}(z) = \delta_\Omega(z) \geq \delta_\Omega(y_x) - cR/10 > cR/5 \quad \text{for } z \in \gamma,$$

so that

$$k_{\Omega \cap B(\xi, 8R)}(y_x, y_i) \leq \int_\gamma \frac{ds(t)}{\delta_\Omega(\gamma(t))} \leq \frac{cR/10}{cR/5} = \frac{1}{2}.$$

Hence

$$\begin{aligned} k_{\Omega \cap B(\xi, 8R)}(x, y_i) &\leq k_{\Omega \cap B(\xi, 8R)}(x, y_x) + k_{\Omega \cap B(\xi, 8R)}(y_x, y_i) \\ &\leq A_0\left[\log\left(\frac{R}{\delta_\Omega(x)}\right) + 1\right]. \end{aligned}$$

This completes the proof of Lemma 4.3. □

LEMMA 4.4. *Let $\Omega \subset X$ be a bounded John domain with John center x_0 and John constant c. Then there exist positive constants C and τ depending only on the data of X and of Ω such that, for each $\xi \in \partial\Omega$ and $0 < R < 2c\delta_\Omega(x_0)/(10q)$,*

$$\int_{\Omega \cap B(\xi, R)} \left(\frac{R}{\delta_\Omega(x)}\right)^\tau d\mu(x) \leq C\mu(B(\xi, R)).$$

Hajłasz and Koskela [HKo1, Lemma 6] demonstrated the lemma for Euclidean domains by an indirect proof using Sobolev extension and embedding theorems (see [HKo1, Remark 11]). The following proof (a modification of the proof from [AHiL]) is more geometric, and it holds true for all relatively compact John domains in quasiconvex metric measure spaces equipped with a doubling measure irrespective of whether X supports a Poincaré inequality.

Proof. For each nonnegative integer j, let $E_j = \bigcup_{i=j+1}^{\infty} V_i$ with

$$V_i = \{x \in \Omega \cap B(\xi, R + (q+1)2^{2-i}R/c) : 2^{-(i+1)}R \leq \delta_\Omega(x) < 2^{-i}R\}.$$

We claim that for every $x \in E_j$ there exist two points $y_x, y'_x \in \Omega$ such that

(i) $x \in B(y_x, (q+1)2^{-j}R/c)$ and
(ii) $B(y'_x, 2^{-(j+2)}R) \subset V_j \cap B(y_x, (q+1)2^{-j}R/c)$.

To see this, let $x \in E_j$. Let γ_x be a c-John curve connecting x and the John center x_0. Observe that we can choose $y_x \in \gamma_x$ such that $\delta_\Omega(y_x) = 2^{-j}R \ge c\,d(x, y_x)$, thus satisfying (i). Because X is proper, there is a point $y_x^* \in X \setminus \Omega$ with $\delta_\Omega(y_x) = d(y_x, y_x^*)$. Since X is quasiconvex, there is a curve β connecting y_x and y_x^* such that $\ell(\beta) \le q2^{-j}R$. Let $y_x' \in \beta \cap \Omega$ be a point satisfying $\delta_\Omega(y_x') = (2^{-(j+1)} + 2^{-j})R/2$. Then, for $z \in B(y_x', 2^{-(j+2)}R)$:

$$d(y_x, z) \le d(y_x, y_x') + d(y_x', z) < \ell(\beta) + 2^{-(j+2)}R < \frac{(q+1)2^{-j}R}{c};$$

$$\begin{aligned} d(\xi, z) &\le d(\xi, x) + d(x, y_x) + d(y_x, z) \\ &< R + \frac{(q+1)2^{1-j}R}{c} + \frac{2^{-j}R}{c} + \frac{(q+1)2^{-j}R}{c} \\ &= R + \frac{(3q+4)2^{2-j}R}{4c}; \end{aligned}$$

$$2^{-(j+1)}R < \delta_\Omega(y_x') - d(y_x', z) \le \delta_\Omega(z) \le \delta_\Omega(y_x') + d(y_x', z) < 2^{-j}R.$$

These three inequalities together yield (ii).

In view of (i), the collection $\{B(y_x, (q+1)2^{-j}R/c)\}_{x \in E_j}$ forms a covering of E_j. By the 5-covering lemma, it follows that we have a pairwise disjoint subcollection $\{B(y_k, (q+1)2^{-j}R/c)\}_{k \in \mathbb{N}}$ such that $E_j \subset \{B(y_k, 5(q+1)2^{-j}R/c)\}_{k \in \mathbb{N}}$. Since μ is a doubling measure, we can find $C_2 > 1$, which depends solely on C_d, such that

$$\begin{aligned} \mu(B(y_k, 5(q+1)2^{-j}R/c)) &\le C_2 \mu(B(y_k', 2^{-(j+2)}R)) \\ &\le C_2 \mu(V_j \cap B(y_k, (q+1)2^{-j}R/c)). \end{aligned}$$

Here y_k' is associated with y_k in the same manner that y_x' is paired with y_x, and the second inequality follows from (ii). Therefore,

$$\begin{aligned} \sum_{i=j+1}^{\infty} \mu(V_i) = \mu(E_j) &\le \sum_{k \in \mathbb{N}} \mu(B(y_k, 5(q+1)2^{-j}R/c)) \\ &\le C_2 \sum_{k \in \mathbb{N}} \mu(V_j \cap B(y_k, (q+1)2^{-j}R/c)) \le C_2 \mu(V_j). \end{aligned}$$

Let $1 < t < 1 + 1/C_2$ and $\tau = \log_2 t$. Then, proceeding as in [AHiL, Proof of Lemma 4.1] or [A1, Lemma 5], we can deduce from the preceding inequality that

$$\begin{aligned} \int_{\Omega \cap B(\xi, R)} \left(\frac{R}{\delta_\Omega(z)}\right)^\tau d\mu(z) &\le \sum_{j=0}^{\infty} \int_{V_j} \left(\frac{R}{\delta_\Omega(z)}\right)^\tau d\mu(z) \\ &\le \sum_{j=0}^{\infty} t^{j+1} \mu(V_j) \le C\mu(B(\xi, R)). \qquad \square \end{aligned}$$

DEFINITION 4.5. A finite collection $\{B(x_i, r_i)\}_{i=1}^k$ of balls is said to be a *Harnack chain,* connecting two points $x, y \in \Omega$ with *length* k, if:

(i) for $i = 1, \dots, k-1$, $B(x_i, r_i) \cap B(x_{i+1}, r_{i+1}) \neq \emptyset$;
(ii) for $i = 1, \dots, k$, $B(x_i, 2\kappa r_i) \subset \Omega$;
(iii) $x \in B(x_1, r_1)$ and $y \in B(x_k, r_k)$.

Here κ is the scaling constant from the Poincaré inequality (see Definition 2.1).

In light of the definition of $k_\Omega(x, y)$, we observe that x and y are connected by a Harnack chain of length no more than $1 + Ck_\Omega(x, y)$. Hence the Harnack inequality for p-quasiminimizers (see [KinS]) gives the following lemma.

Lemma 4.6. *If h is a positive p-quasiminimizer on Ω, then*

$$\frac{1}{A_H} \exp(-A_H k_\Omega(x, y)) \leq \frac{h(x)}{h(y)} \leq A_H \exp(A_H k_\Omega(x, y)) \quad \textit{for } x, y \in \Omega,$$

where $A_H > 1$ is independent of h, x, and y.

5. A Carleson-Type Estimate for p-Harmonic Functions on a John Domain in Ahlfors Regular Spaces

We will henceforth assume that μ is Ahlfors Q-regular (see (7)). We will also assume from now on that $\Omega \subset X$ is a bounded John domain with John center $x_0 \in \Omega$ and John constant $0 < c < 1$ and that $X \setminus \Omega \neq \emptyset$. If $x \in \Omega$ and γ_x is a John curve connecting x to a local reference point y_i associated with x as in the proof of Lemma 4.3, then

$$k_\Omega(x, y_i) \leq C_3 \left[\log\left(\frac{\delta_\Omega(y_i)}{\delta_\Omega(x)} \right) + 1 \right].$$

Therefore, by Lemma 4.6, for every positive p-quasiminimizer h on Ω and $x \in \Omega$ we have

$$\frac{h(x)}{h(y_i)} \leq A_H \exp\left(A_H C_3 \left[\log\left(\frac{\delta_\Omega(y_i)}{\delta_\Omega(x)} \right) + 1 \right] \right) \leq C \left(\frac{\delta_\Omega(y_i)}{\delta_\Omega(x)} \right)^{\lambda},$$

where $C, \lambda > 0$ depend only on A_H and C_3 but not on x nor on h. Thus we have a weak estimate

$$\frac{h(x)}{h(y_i)} \leq C \left(\frac{\delta_\Omega(y_i)}{\delta_\Omega(x)} \right)^{\lambda}. \tag{18}$$

Let N_0 be as in Lemma 4.3 and R_0 as in the proof of Lemma 4.3. We recall again the standing assumption that X is a proper metric space.

Proposition 5.1. *Let $\xi \in \partial\Omega$, let $0 < R < R_0/16$, and let h be a positive p-quasiminimizer in $\Omega \cap B(\xi, 16R)$ vanishing p-quasieverywhere on $\partial\Omega \cap B(\xi, 16R)$. If h is bounded in $\Omega \cap B(\xi, R/2) \setminus \bar{B}(\xi, R/16)$, then for all $x \in \Omega \cap S(\xi, R/4)$ we have*

$$h(x) \leq C \sum_{i=1}^{N} h(y_i),$$

where $y_1, \dots, y_N$ is a system of local reference points for ξ ($N \leq N_0$).

Proof. By (18), for all $x \in \Omega \cap \bar{B}(\xi, R/2)$ there is an integer $i \in \{1, \dots, N\}$ such that

$$\frac{h(x)}{h(y_i)} \le C\left(\frac{R}{\delta_\Omega(x)}\right)^\lambda.$$

Hence, for every $x \in \Omega \cap \bar{B}(\xi, R/2)$,

$$h(x) \le C\left(\frac{R}{\delta_\Omega(x)}\right)^\lambda \sum_{i=1}^N h(y_i). \tag{19}$$

Let

$$u := \frac{1}{C\sum_{i=1}^N h(y_i)} h.$$

Then u is nonnegative and locally bounded on the bounded open set Ω_0 given by $\Omega_0 := B(\xi, R/2) \setminus \bar{B}(\xi, R/16)$. Moreover, u is a p-quasisubminimizer in Ω_0 because it is a p-quasiminimizer in $\Omega_0 \cap \Omega$ and vanishes in $\Omega_0 \cap \partial\Omega$; see [BBS1, Lemma 3.11]. By (19),

$$u(x) \le \left(\frac{R}{\delta_\Omega(x)}\right)^\lambda \quad \text{for } x \in \Omega_0 \cap \Omega.$$

Let $\tau > 0$ be as in Lemma 4.4. Choose $\varepsilon > 0$ such that $Q - 1 + \varepsilon > 1$ and apply the elementary inequality

$$(\log t)^{Q-1+\varepsilon} \le \left(\frac{Q-1+\varepsilon}{\tau}\right)^{Q-1+\varepsilon} t^\tau \quad \text{whenever } t \ge 1$$

to the quantity $t = R/\delta_\Omega(x) \ge 1$ whenever $x \in \Omega \cap \Omega_0$ to obtain the following estimate:

$$\begin{aligned} I &= \int_{\Omega\cap\Omega_0} (\log^+ u(x))^{Q-1+\varepsilon}\, d\mu(x) \le \int_{\Omega\cap\Omega_0} \left[\lambda \log^+\left(\frac{R}{\delta_\Omega(x)}\right)\right]^{Q-1+\varepsilon} d\mu(x) \\ &\le \int_{\Omega\cap\Omega_0} C\left(\frac{R}{\delta_\Omega(x)}\right)^\tau d\mu(x). \end{aligned}$$

By Lemma 4.4 and the Ahlfors regularity of μ,

$$I \le C_{\lambda,Q,\varepsilon} \int_{\Omega\cap B(\xi,R/2)} \left(\frac{R}{\delta_\Omega(x)}\right)^\tau d\mu(x) \le CR^Q < \infty. \tag{20}$$

Therefore, (10) of Lemma 3.1 yields

$$u(x) \le C\exp(CI^{1/\varepsilon}\delta_{\Omega_0}(x)^{-Q/\varepsilon}) \quad \text{whenever } x \in \Omega \cap \Omega_0.$$

On the other hand, if $x \in \Omega \cap S(\xi, R/4)$ then $\delta_{\Omega_0}(x) \approx R$. Hence, by (20) we have

$$u(x) \le C\exp(CI^{1/\varepsilon}R^{-Q/\varepsilon}) \le C.$$

Now, in view of the definition of u, the desired result follows. $\square$

As a corollary to Proposition 5.1 we have the Carleson estimate of Theorem 5.2. Such an estimate follows from Proposition 5.1 and the strong maximum principle (see [KinS]) for p-quasiminimizers.

THEOREM 5.2. *Let $\Omega \subset X$ be a bounded John domain, $\xi \in \partial\Omega$, and $0 < R < R_0/16$ where R_0 is as in the proof of Lemma 4.4. If h is a positive p-quasiminimizer in $\Omega \cap B(\xi, 16R)$ that is vanishing p-quasieverywhere on $\partial\Omega \cap B(\xi, 16R)$ and is bounded in $\Omega \cap B(\xi, R/2)$, then*

$$h(x) \leq C \sum_{i=1}^{N} h(y_i) \quad \textit{for every } x \in \Omega \cap B(\xi, R/4),$$

where $y_1, \dots, y_N \in \Omega \cap S(\xi, R)$ are a system of local reference points for ξ. The constant $C > 1$ is independent of x, ξ, R, and h.

COROLLARY 5.3. *Let $\Omega \subset X$ be a uniform domain, $\xi \in \partial\Omega$, and $0 < R < R_0/16$ where R_0 is as in the proof of Lemma 4.4. If h is a positive p-quasiminimizer in $\Omega \cap B(\xi, 16R)$ that is vanishing p-quasieverywhere on $\partial\Omega \cap B(\xi, 16R)$ and is bounded in $\Omega \cap B(\xi, R/2)$, then*

$$h(x) \leq Ch(y) \quad \textit{for every } x \in \Omega \cap B(\xi, R/4),$$

where $y \in \Omega \cap S(\xi, R)$ is such that $\delta_\Omega(y) \geq R/C$. The constant $C > 1$ is independent of x, ξ, R, and h.

REMARK 5.4. In light of Remark 2.6, Theorem 5.2 and Corollary 5.3 hold if h is p-harmonic.

6. The Conformal Martin Boundary of a Bounded John Domain in X

In this section we assume that X is of Q-bounded geometry and apply the results of the previous sections with $p = Q$. Given a Cheeger derivative structure on X, the corresponding conformal Martin boundary has been defined in [HoST] (see Definition 2.9). In the following, we let $\lambda > 0$ be as in (18).

THEOREM 6.1. *Let $\Omega \subset X$ be a bounded John domain such that* $\mathrm{Cap}_Q(X \setminus \Omega) > 0$. *Let M be a conformal Martin kernel for Ω with fundamental sequence (y_k). Then (y_k) converges to a point $\xi_M \in \partial\Omega$ and the following statements hold*:

- (i) *M is bounded in a neighborhood of ξ for each $\xi \in \partial\Omega \setminus \{\xi_M\}$;*
- (ii) *there is a constant $C \geq 1$ such that $M(x) \leq Cd(x, \xi_M)^{-\lambda}$ for all $x \in \Omega$;*
- (iii) *M vanishes continuously Q-quasieverywhere on $\partial\Omega$;*
- (iv) *there is a sequence (x_n) in Ω converging to ξ_M such that* $\lim_n M(x_n) = \infty$, *and hence the point ξ_M is uniquely determined from M but does not depend on the fundamental sequence that gave rise to it.*

A crucial result needed to prove part (iv) of the theorem is Corollary 6.2 of [BBS2]. Observe that this corollary is invalid if $\mathrm{Cap}_Q(X \setminus \Omega) = 0$. The proof of this result uses the fact that solutions to p-Dirichlet problems with boundary data belonging to $N^{1,Q}(X)$ satisfy a comparison theorem; see [S2] for a proof of this fact assuming $\mathrm{Cap}_Q(X \setminus \Omega) > 0$. However, if $\mathrm{Cap}_Q(X \setminus \Omega) = 0$ then this comparison theorem obviously fails and, as a consequence, [BBS2, Cor. 6.2] also fails to hold.

Let U be an open subset of X. Recall that $H_Q^U f$ stands for the solution to the Q-Dirichlet problem on the open set U with boundary data $f \in N^{1,Q}(X)$. Given a continuous function $f\colon \partial U \to \mathbb{R}$, the authors of [BBS2] construct a Perron solution in U with boundary data f (in fact, [BBS2] also shows that continuous functions are resolutive); let $P_Q^U f$ denote such a solution. If such a function f also happens to belong to $N^{1,Q}(X)$, then $P_Q^U f = H_Q^U f$. We say that $\xi \in \partial U$ is a Q-regular boundary point for U if $\lim_{U \ni x \to \xi} P_Q^U f(x) = f(\xi)$ whenever f is a bounded and continuous function on ∂U.

In order to prove Theorem 6.1, we need the following lemma.

LEMMA 6.2. *Let $\Omega \subset X$ be a bounded domain such that* $\mathrm{Cap}_Q(X \setminus \Omega) > 0$. *Then Q-quasievery point $\xi \in \partial\Omega$ is a Q-regular boundary point for* $\Omega \cap B(\xi, r)$ *whenever* $r > 0$.

Proof. Let (r_n) be an enumeration of all positive rational numbers. Moreover, let (x_k) be a sequence of points in Ω that is dense in Ω. By [BBS1, Thm. 3.9], the set $J_{k,n}$ of all points in $B(x_k, r_n) \cap \partial\Omega$ that are Q-irregular points for $\Omega \cap B(x_k, r_n)$ is of zero Q-capacity. Hence the set $J := \bigcup_{k,n} J_{k,n}$ is a zero Q-capacity subset of $\partial\Omega$. Let $\xi \in \partial\Omega \setminus J$. We will demonstrate that ξ satisfies the requirements of Lemma 6.2.

Suppose ξ does not satisfy the requirements of the lemma; that is, suppose there is an $r < \mathrm{diam}(\Omega)$ such that ξ is not a Q-regular boundary point for $\Omega \cap B(\xi, r)$. Then, following the notation set up in the proof of [BBS1, Thm. 3.9], we can find a ball $B_{l,m}$ centered at some point in $\partial\Omega$ and of radius $\rho_{l,m}$ such that $\xi \in B_{l,m}$, $\overline{2B_{l,m}} \subset B(\xi, r)$, and there exists a Lipschitz function $\varphi_{l,m}$ that is compactly supported in $2B_{l,m}$ and takes on the value of 1 in $B_{l,m}$ such that

$$\liminf_{\Omega \cap B(\xi,r) \ni y \to \xi} H_Q^{\Omega \cap B(\xi,r)} \varphi_{l,m}(y) < 1.$$

On the other hand, since $2B_{l,m} \subset B(\xi, r)$ and $\xi \in B_{l,m}$ we can find k and n such that $\xi \in B(x_k, r_n) \subset \bar{B}(x_k, r_n) \subset B(\xi, r)$. Because $\xi \notin J_{k,n}$, we know that ξ is a Q-regular boundary point of the open set $\Omega \cap B(x_k, r_n)$. By the comparison theorem it then follows that

$$\liminf_{\Omega \cap B(\xi,r) \ni y \to \xi} H_Q^{\Omega \cap B(\xi,r)} \varphi_{l,m}(y) \geq \liminf_{\Omega \cap B(x_k,r_n) \ni y \to \xi} H_Q^{\Omega \cap B(x_k,r_n)} \varphi_{l,m}(y) = 1,$$

contradicting the previous inequality. Hence such an r does not exist and consequently ξ satisfies the lemma. □

Proof of Theorem 6.1. Let (y_k) be a fundamental sequence in Ω giving rise to the kernel M; see Definition 2.9. We will show that there is a point $\xi_M \in \partial\Omega$ such that $\lim_k y_k = \xi_M$. Since (y_k) is a fundamental sequence, it has no accumulation point in Ω. Since $\bar{\Omega} \subset F$ is compact, there exist a point $\xi_M \in \partial\Omega$ and a subsequence $(y_{k_n})_n$ such that $\lim_n y_{k_n} = \xi_M$. We will demonstrate that, for every $\xi \in \partial\Omega \setminus \{\xi_M\}$, the function M is bounded in a neighborhood of ξ and that M satisfies parts (iii) and (iv) of the theorem. As a consequence, this observation concludes that $\lim_k y_k = \xi_M$. Moreover, if (w_n) is another fundamental sequence giving rise to M, then $\lim_n w_n = \xi_M$.

Let us begin with the proof of part (i). Fix $\xi \in \partial\Omega \setminus \{\xi_M\}$. For ease of discourse let us denote the subsequence of (y_k) that converges to ξ_M also by (y_k). Then there exists $r = 4r_\xi > 0$ such that, for all $k \in \mathbb{N}$, $y_k \notin \Omega \cap B(\xi, 16(q+1)r/c)$ and $16(q+1)r/c < R_0$, where R_0 is as in the proof of Lemma 4.3. Let M_k be the function given by $M_k(x) = g(x, y_k)/g(x_0, y_k)$. Then M_k is positive and Q-harmonic in $\Omega \cap B(\xi, 16(q+1)r/c)$ but vanishes Q-quasieverywhere on $\partial\Omega$. Moreover, M_k is bounded on $\Omega \cap B(\xi, r)$ and so, by Theorem 5.2, for $x \in \Omega \cap B(\xi, r/2)$ we have

$$M_k(x) \le C \sum_{i=1}^{N} M_k(y_i),$$

where $y_1, \ldots, y_N \in \Omega \cap S(\xi, r/2)$ is a system of local reference points for ξ. Note that $\delta_\Omega(y_i) \ge r/A$. Thus we have

$$k_\Omega(y_i, x_0) \le \frac{A}{cr} d(y_i, x_0) \le \frac{A}{cr} \operatorname{diam}(\Omega);$$

since $M_k(x_0) = 1$, by Lemma 4.6 we have $M_k(y_i) \le \exp(A_H (A/cr) \operatorname{diam}(\Omega))$. Consequently, for $x \in \Omega \cap B(\xi, r/4)$,

$$M_k(x) \le \sum_{i=1}^{N} \exp\left(A_H \frac{A}{cr} \operatorname{diam}(\Omega)\right) \le N_0 \exp\left(A_H \frac{A}{cr} \operatorname{diam}(\Omega)\right) =: C_\xi.$$

Since $M = \lim_k M_k$, it follows that M is bounded in a neighborhood of ξ for each $\xi \in \partial\Omega \setminus \{\xi_M\}$. Thus (i) in the theorem is satisfied.

Let us next prove part (iii). Since by assumption $\operatorname{Cap}_Q(X \setminus \Omega) > 0$, by the Poincaré inequality we see that $\operatorname{Cap}_Q(\partial\Omega)$ is positive. By Lemma 6.2 we may assume that $\xi \in \partial\Omega$ is Q-regular for $\Omega \cap B(\xi, r)$ for every $r > 0$. We can easily see that, whenever $\rho > 0$, we have $\operatorname{Cap}_Q(B(\xi, \rho) \setminus \Omega) > 0$. Let $r = r_\xi/4$. We have by the foregoing argument that $\{M_k\}_k$ is a uniformly bounded family of Q-harmonic functions on $\Omega \cap B(\xi, r_\xi)$, where $M_k \le C_\xi$. Let f be a compactly supported Lipschitz continuous function on X such that $f = C_\xi$ on $S(\xi, r)$ and $f = 0$ on $B(\xi, r/2)$. Then we easily see that

$$M_k(y) \le f(y) \quad \text{for every } \; y \in \partial(\Omega \cap B(\xi, r)),$$

since $M_k = 0$ on $X \setminus \Omega$ by the construction of M_k in [HoS]. Observe that $\operatorname{Cap}_Q(\bar{B}(\xi, 2r) \setminus \Omega) > 0$ and that both f and M_k are in the class $N^{1,Q}(\bar{B}(\xi, 2r))$ and are Q-harmonic in $B(\xi, r) \cap \Omega$ for sufficiently large k. By the regular comparison theorem (see [S2]), we therefore have $M_k \le H_Q^{\Omega \cap B(\xi, r)} f$ on $\Omega \cap B(\xi, r)$. Since ξ is a Q-regular boundary point for $\Omega \cap B(\xi, r)$ and since f is a continuous boundary data, it follows that for every $\varepsilon > 0$ there can be found $\rho_\varepsilon > 0$ such that $0 < H_Q^{\Omega \cap B(\xi, r)} f < \varepsilon$ on $\Omega \cap B(\xi, \rho_\varepsilon)$. Hence $0 < M_k(x) < \varepsilon$ on $\Omega \cap B(\xi, \rho_\varepsilon)$ for every k. Thus, as M is the pointwise (and locally uniform) limit of M_k in Ω, we see that $M \le \varepsilon$ on $\Omega \cap B(\xi, \rho_\varepsilon)$. This means that M tends to zero continuously Q-quasieverywhere in $\partial\Omega$. This proves (iii) of the theorem.

Now we prove part (iv). Suppose there is no sequence (x_n) in Ω converging to ξ_M for which $\lim_n M(x_n) = \infty$. Then M is also bounded in a neighborhood of ξ_M; consequently (by part (i), just proven), M is a positive Q-harmonic function in Ω that is bounded on Ω and, in addition, vanishes Q-quasieverywhere in $\partial\Omega$. It then follows from [BBS2, Cor. 6.2] that M is identically zero on Ω, contradicting the fact that $M(x_0) = 1$. Thus part (iv) of the theorem is also true.

It now only remains to prove (ii). We have already shown that, whenever $R > 0$, the function M is bounded in $\Omega \cap B(\xi_M, R/2) \setminus \bar{B}(\xi_M, R/16)$ and vanishes on $\partial\Omega \setminus \{\xi_M\}$. Hence, by Proposition 5.1, if $0 < R < R_0/32$ then M satisfies

$$M(x) \le C \sum_{i=1}^{N} M(y_i) \quad \text{for } x \in S(\xi_M, R) \cap \Omega,$$

where $y_1, \dots, y_N \in \Omega \cap S(\xi_M, 2R)$ is a system of local reference points of order N for ξ_M. By the comparison theorem we have

$$M(x) \le C \sum_{i=1}^{N} M(y_i)$$

whenever $x \in \Omega \setminus B(\xi_M, R)$. Now an application of (18) to each of the reference points y_i, together with the estimate $\delta_\Omega(y_i) \ge R/C$, gives

$$M(y_i) \le M(x_0)\left(\frac{\delta_\Omega(x_0)}{R}\right)^{\lambda} = \left(\frac{\delta_\Omega(x_0)}{R}\right)^{\lambda} = CR^{-\lambda}.$$

Therefore,

$$M(x) \le CNR^{-\lambda} \le CN_0\, d(x, \xi_M)^{-\lambda} \quad \text{for } x \in \Omega \cap B(\xi_M, 2R) \setminus B(\xi_M, R)$$

whenever $0 < R < R_0/32$. The desired result now follows from the fact that Ω is bounded.

This completes the proof of Theorem 6.1. □

To obtain Theorems 5.2 and 6.1 (as well as other results in this and the previous sections), it suffices to know only that the Q-quasievery boundary point of Ω is a "John point". More specifically, a point $\xi \in \partial\Omega$ is said to be a *John point* if there is a radius $R_\xi > 0$, a point $x_\xi \in \Omega$, and a constant $C_\xi > 0$ such that, for every $x \in B(\xi, R_\xi) \cap \Omega$, there is a compact rectifiable curve γ connecting x to x_ξ with $\gamma \subset \Omega \cap B(\xi, C_\xi R_\xi)$ and such that, for every $y \in \gamma$, we have $\delta_\Omega(y) \ge \ell(\gamma_{x,y})/C_\xi$ (see [S3]). Examples of domains satisfying this weak condition include Euclidean domains obtained by pasting outward-pointing cusps to a ball.

7. Compactness of X

The proof of the results discussed in this paper required that the closure of Ω be compact. In this section we demonstrate that, instead of merely assuming the closure of Ω be compact, we may assume the stronger condition that X is a complete metric space. Since a metric space supporting a doubling measure is totally bounded, if X is complete then it is proper as well and hence bounded domains in X are relatively compact.

If X is not a complete metric space, let $\hat{X}$ denote the completion of X, $\hat{d}$ the extension of d, and $\hat{\mu}$ the extension of μ to $\hat{X}$: $\hat{\mu}(A) = \mu(A \cap X)$ whenever $A \subset \hat{X}$.

PROPOSITION 7.1. *If (X, d, μ) is a metric measure space with doubling measure μ and supporting a $(1, p)$-Poincaré inequality, then so is $(\hat{X}, \hat{d}, \hat{\mu})$. In this case, $N^{1,p}(X) = N^{1,p}(\hat{X})$. In addition, if μ is Ahlfors Q-regular, then so is $\hat{\mu}$.*

Proof. We first prove the doubling property of $\hat{\mu}$. Let $B(x,r)$ be a ball in $\hat{X}$. If $x \in X$, then the doubling property of μ itself guarantees the doubling inequality for $\hat{\mu}$: $\hat{\mu}(B(x, 2r)) = \mu(B(x, 2r) \cap X) \le C_d\mu(B(x,r) \cap X) = C_d\hat{\mu}(B(x,r))$. Suppose $x \in \hat{X} \setminus X$. Then, since $\hat{X}$ is the completion of X, there is a point $x' \in X$ such that $d(x, x') < r/8$. Thus, by the doubling property of μ we have

$$\begin{aligned}\hat{\mu}(B(x,2r)) &\le \hat{\mu}(B(x',3r)) = \mu(B(x',3r) \cap X) \le C_d^2\mu(B(x',3r/8) \cap X)\\ &\le C_d^2\mu(B(x,r) \cap X) = C_d^2\hat{\mu}(B(x,r)).\end{aligned}$$

That is, $\hat{\mu}$ is doubling with doubling constant C_d^2. A similar argument demonstrates that if μ is Ahlfors Q-regular then so is $\hat{\mu}$.

Now we demonstrate that if μ is doubling and supports a $(1, p)$-Poincaré inequality then $\hat{\mu}$ also supports a $(1, p)$-Poincaré inequality. To this end, let $u \in N^{1,p}(\hat{X})$ and let g be an upper gradient of u in $\hat{X}$. Note that g is also an upper gradient of u in X since $X \subset \hat{X}$. Suppose $B(x,r) \subset \hat{X}$. If $x \in X$ then, by the definition of $\hat{\mu}$ and because μ itself supports a $(1, p)$-Poincaré inequality, we have the Poincaré inequality with respect to $\hat{X}$. If $x \in \hat{X} \setminus X$, then taking $x' \in X$ such that $d(x, x_0) < r/2$ yields

$$\begin{aligned}\inf_{c\in\mathbb{R}} \fint_{B(x,r)} |u-c|\,d\hat{\mu} &\le \inf_{c\in\mathbb{R}} \fint_{B(x',2r)} |u-c|\,d\hat{\mu} \le \fint_{B(x',2r)\cap X} |u-u_{B(x',2r)\cap X}|\,d\mu\\ &\le C_p 4r \fint_{B(x',2\kappa r)\cap X} g^p\,d\mu \le Cr \fint_{B(x,3\kappa r)\cap X} g^p\,d\mu\\ &= Cr \fint_{B(x,3\kappa r)} g^p\,d\hat{\mu}.\end{aligned}$$

Hence the pair u, g satisfies a $(1, p)$-Poincaré inequality on $\hat{X}$.

Finally, we prove that $N^{1,p}(\hat{X})|_X = N^{1,p}(X)$. Observe that $N^{1,p}(\hat{X})|_X \subset N^{1,p}(X)$. On the other hand, since X supports a $(1, p)$-Poincaré inequality, we know from [S1] that Lipschitz functions are dense in $N^{1,p}(X)$. We may extend a Lipschitz function $u \in N^{1,p}(X)$ to $\hat{X}$, for example via a McShane extension. Because the minimal p-weak upper gradient g_u of such a function has the property that $g_u(x) \approx \operatorname{Lip} u(x)$ for μ-a.e. $x \in X$ (see e.g. [C]) and hence for $\hat{\mu}$-a.e. $x \in \hat{X}$, we see that $u \in N^{1,p}(\hat{X})$ with the $N^{1,p}(\hat{X})$-norm of u equaling the $N^{1,p}(X)$-norm of u. Now the density of Lipschitz functions in $N^{1,p}(X)$ guarantees the coincidence of the two function spaces. □

The preceding result demonstrates that the requirement that X be proper is not as stringent as it appears to be, especially since we do not assume much about $X \setminus \Omega$ apart from the condition that $\mathrm{Cap}_Q(X \setminus \Omega) > 0$. The following proposition further strengthens this claim. Note that, for every domain $\Omega \subset X$, there is a domain $\Omega_0 \subset \hat{X}$ such that $\Omega = \Omega_0 \cap \hat{X}$. Let Ω_0 denote the largest such domain in $\hat{X}$.

PROPOSITION 7.2. *In the situation of Proposition 7.1, if* $\mathrm{Cap}_Q(\hat{X} \setminus X) = 0$ *then* $\partial_{cM}\Omega = (\Omega_0 \setminus \Omega) \cup \partial_{cM}\Omega_0$.

Proof. In this proof, balls $B(x,r)$ denote balls in X;

$$B(x,r) = \{y \in X : d(y,x) < r\}.$$

By $\hat{B}(x,r)$ we then denote the closure of $B(x,r)$ in $\hat{X}$.

Let (r_i) be a sequence of positive real numbers such that $\lim_i r_i = 0$. If $y \in \Omega \subset \Omega_0$, the construction of a Q-singular function on Ω with singularity at y is via a sequence of functions $u_i \in N_0^{1,Q}(\Omega)$ that are Q-energy minimizers in $\Omega \setminus \bar{B}(y, r_i)$, taking on the value 1 in $B(y, r_i)$ and value $0 \le u_i \le 1$ on X; see [HoS]. By a Q-energy minimizer we mean that, for every $\phi \in N_0^{1,Q}(\Omega \setminus \bar{B}(y, r_i))$,

$$\int_{\Omega \setminus \bar{B}(y,r_i)} |du_i|^Q \, d\mu \le \int_{\Omega \setminus \bar{B}(y,r_i)} |d(u_i + \phi)|^Q \, d\mu.$$

Since by Proposition 7.1 we have $N_0^{1,Q}(\Omega) \subset N^{1,Q}(X) = N^{1,Q}(\hat{X})$, we see that u_i is extendable to Ω_0 so that $u_i \in N_0^{1,Q}(\Omega_0)$. We claim that u_i is Q-harmonic in $\Omega_0 \setminus \hat{B}(y, r_i)$. Indeed, if $\varphi \in N_0^{1,Q}(\Omega_0 \setminus \hat{B}(y, r_i))$, then $\varphi\big|_X \in N_0^{1,Q}(\Omega)$ and hence

$$\begin{aligned}\int_{\Omega_0 \setminus \hat{B}(y,r_i)} |du_i|^Q \, d\hat{\mu} &= \int_{\Omega \setminus \bar{B}(y,r_i)} |du_i|^Q \, d\mu \\ &\le \int_{\Omega \setminus \bar{B}(y,r_i)} |d(u_i + \varphi)|^Q \, d\mu = \int_{\Omega_0 \setminus \hat{B}(y,r_i)} |d(u_i + \varphi)|^Q \, d\hat{\mu}.\end{aligned}$$

Therefore, the singular function $g(\cdot, y)$ on Ω can be extended to be a singular function on all of Ω_0 with singularity at y.

Now let $M \in \partial_{cM}\Omega$ and let (y_n) be an associated fundamental sequence in Ω. Since this sequence can have no accumulation point in Ω and since $\widehat{\Omega_0}$ is compact, it follows that this sequence has accumulation points in $(\Omega_0 \setminus \Omega) \cup \partial_{cM}\Omega_0$. If it has an accumulation point in $\Omega_0 \setminus \Omega$, then M is the normalized singular function on Ω_0 with singularity at that accumulation point; hence the sequence (y_n) must converge to this unique accumulation point. Otherwise, all of the accumulation points of the sequence lie in the set $\partial\Omega_0$ and hence $M \in \partial_{cM}\Omega_0$. We then have $\partial_{cM}\Omega \subset (\Omega_0 \setminus \Omega) \cup \partial_{cM}\Omega_0$.

To obtain equality, note that every point in $\Omega_0 \setminus \Omega$ corresponds to a unique Martin kernel function in $\partial_{cM}\Omega$. Moreover, the restriction of every function in $\partial_{cM}\Omega_0$ to Ω lies in $\partial_{cM}\Omega$. To see this, note that every point $y_0 \in \Omega_0$ is a limit of a sequence of points from Ω; hence every singular function in Ω_0 with singularity at y can be approximated to any desired accuracy by a singular function in Ω with a singularity in Ω. This is because (by our previous discussion) (a) singular functions in Ω with singularity in Ω are extendable to be a singular function in Ω_0 with singularity in Ω, (b) the limit of such a sequence of singular functions is a singular function in Ω_0 with singularity at y, and (c) only one such singular function exists. Now a diagonalization argument yields that $\partial_{cM}\Omega_0 \subset \partial_{cM}\Omega$. $\square$

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