Singular behavior of conformal Martin kernels, and non-tangential limits of conformal mappings *

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Abstract

The Martin boundary corresponding to Q-Laplacian operator (where Q is the Ahlfors regularity dimension of the space) was constructed in [HST]. In particular, it was shown in [HST] that if the domain is bounded, uniform, and has uniformly fat complement, the (conformal) Martin kernel functions in the conformal Martin boundary of the domain vanish Hölder continuously at the metric boundary points of the domain that do not arise as accumulation points of the corresponding fundamental sequence of points in the domain. The aim of this note is to extend the study of these Martin kernel functions to Q-almost locally uniform domains and by exploring their behavior near the metric boundary points of the domain that are accumulation points of any fundamental sequence associated with the Martin kernel function. We show that the kernel function exhibits singular behavior near such boundary points, that is, they converge to infinity along quasihyperbolic geodesic curves terminating at such boundary points.

We use this singular behavior of conformal Martin kernel functions to establish that conformal mappings between two bounded locally uniform domains whose complements are uniformly fat have non-tangential limits at every metric boundary of the domain of the mapping.

1 Introduction

Corresponding to certain degenerate elliptic operators it is possible to construct singular solutions and use them to obtain a Martin compactification pertaining to the operator; some recent papers such as [Ho], [HS], and [HST] have initiated this study. In particular, Holopainen constructed singular functions corresponding to the p-Laplacian ($p \geq 1$) on bounded domains in Riemannian manifolds and used them to classify manifolds that admit global singular functions. The paper [HS] extends this construction of singular functions to certain metric measure spaces equipped with a locally doubling measure supporting a

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Poincaré inequality. The paper [HST] continued this study by constructing the conformal Martin boundary (the Martin boundary corresponding to the operator that is conformally invariant) for relatively compact domains in metric spaces of Q^--bounded geometry. It was shown in [HST] that when the domain is uniform and its complement is uniformly q-fat for some $1 \le q < Q$, every conformal Martin boundary point corresponds to a *unique* metric boundary point of the domain. It is still unknown whether in this case there is atmost one conformal Martin boundary point corresponding to every metric boundary point. In the case of the classical Martin boundary (corresponding to the Laplacian) for relatively compact uniform domains in Riemannian manifolds, it was shown by Anderson-Schoen, Ancona, and Aikawa in [AS], [An], and [Ai] that the metric boundary and the Martin boundary are homeomorphic.

The aim of this note is to expand further on the singular behaviour of Martin kernel functions constructed in [HST] for domains in metric measure spaces. The domain studied in this note is a relatively compact locally uniform domain in a metric space of Q^--bounded geometry such that the complement of the domain is uniformly q-fat for some $1 \le q \le Q$. For the definition of Q^--bounded geometry as well as other relevant definitions of notions refered to in this secion, see Section 2. For uniform domains with uniformly q-fat complements, it was shown in [HST] that if M_χ is a Martin kernel corresponding to a boundary point χ of the domain in question and ζ is a different boundary point of the domain, then at $\partial\Omega$ near ζ the kernel M_χ converges in a Hölder continuous manner to zero; see Theorem 3.1 below for a restatement of this result. The behaviour of M_χ near the exceptional point χ was not studied in [HST]; the purpose of this note is to do so. It is shown in Proposition 1.1 that whenever γ is a quasihyperbolic geodesic ray in a locally uniform domain Ω (see Definition 2.11 in Section 2) with uniformly Q-fat boundary (that is, the complement of the domain is dense in the capacitary sense; see Definition 2.13 in Section 2) terminating at the singular point $\chi \in \partial\Omega$, and M_χ is a conformal Martin kernel corresponding to χ, then M_χ tends to infinity along γ. Indeed, it is demonstrated in this note that along non-tangential sequences converging to χ the Martin kernel M_χ tends to infinity.

Readers unfamiliar with the terminology used in this section such as Q^--bounded geometry, Q-almost locally uniform domains, and Martin kernels M_χ associated with the boundary point $\chi \in \partial\Omega$, should see Section 2.

Proposition 1.1. *Let X be a metric measure space of Q^--bounded geometry, Ω be a relatively compact Q-almost locally uniform domain in X with uniformly Q-fat complement, $\chi \in \partial\Omega$ be a point of local uniformity for Ω, and γ be a quasihyperbolic geodesic ray in Ω that terminates at χ. Then* $\lim_{t\to\infty} M_\chi(\gamma(t)) = \infty$.

We conclude by using this result to show a Fatou type result for conformal maps between two such domains. Namely, we will demonstrate the following theorem. In the study of quasiconformal mappings it is well-known that quasiconformal maps between two discs in $\mathbb{R}^2$ have radial limits along all radii. Many extensions of this Fatou type result have appeared in literature for certain types of quasiconformal maps from balls in $\mathbb{R}^n$, see [Jen], [Str], [V], and the references therein. Recently a strong version of a Fatou type theorem has been proven by Bonk, Heinonen, and Koskela in [BHK] for certain types of Gromov hyperbolic spaces and uniform domains. The following theorem provides a partial Fatou type result for more general domains than mere uniform domains, and claims that conformal maps between two Q-almost locally uniform domains have nontangential limits at almost every boundary point of the domain, that is, the conformal map achieves a limit as the point in the domain of the map moves along any r-cone towards the boundary of the domain. The notion of Q-almost locally uniform domains and related definitions will be given in the next section. For the definition of an r-cone $C(r)$ see Definition 2.12.

Theorem 1.2. *Let (X, d_X, μ_X) and (Y, d_Y, μ_Y) be two metric spaces of Q^--bounded geometry, and $U \subset X$ and $V \subset Y$ be two relatively compact Q-almost locally uniform domains such that $X \setminus U$ and $Y \setminus V$ are uniformly Q-fat, the set of boundary points that are not points of local uniformity for V are separated by ∂V, and let $f : U \to V$ be a conformal map. Then whenever $\chi \in \partial U$ is a point of local uniformity for U and $C(r)$ is a cone in U terminating at χ and $(x_n)_{n\in\mathbb{N}}$ is a sequence in $C(r)$ converging to χ, the limit $\lim_{n\to\infty} f(x_n)$ exists and the limit is independent of r.*

While the above theorem does not give details on the behavior of the conformal mapping near boundary points that are not points of local uniformity for Ω, the proof of the theorem suggests the following type of behavior provided the set of points of non-local uniformity for U is separated by ∂U. If ζ is a boundary point that is not of local uniformity for Ω, and $(x_n)_n$ is a sequence in U converging to ζ such that there is a Martin kernel function $M \in \partial_{cM} U$ with the property that $\lim_{n\to\infty} M(x_n) = \infty$, then $\lim_{n\to\infty} f(x_n)$ exists, though perhaps not independent of such sequence, or, strictly speaking, of the kernel M.

In the case that U, V are in addition a globally uniform domains, Theorem 1.2 can be substantially strengthened by using the techniques of [BHK]; in this case, all quasiconformal mappings f between two such domains extend as global quasiconformal mappings between the closures of the two domains. To see this note that the inverse of the uniformization described in [BHK] is a quasiconformal mapping from the uniform domain to a gromov hyperbolic space, and the lifting of the quasiconformal map f by this uniformization results in a quasiconformal mapping between two gromov hyperbolic spaces. A slight modification of the proof of Theorem 9.8 of [BHK] (using the generalization outlined in [Her] together with the fact that the boundary of the domains are uniformly fat) yields the quasiconformal extension of the mapping to the Gromov compactification of the corresponding hyperbolic spaces, which in turn, when we use the uniformization procedure, yields a quasiconformal extension of f to the metric closure of the two uniform domains.

The strength of Theorem 1.2 over the above observation lies in the flexibility of its proof. Not only do we relax the requirement on the domains to be merely Q-almost locally uniform, but the proof can be easily modified to apply to more general domains that have slits and so are not even Q-almost locally uniform as well as to domains with isolated boundary points (for which the uniform Q-fatness condition fails) and combinations thereof. See Definition 2.5 for the definition of conformal mappings between two metric spaces admitting a derivative structure.

This note is organized as follows. Section 2 consists of definitions and notations used throughout this paper and in Proposition 1.1 and Theorem 1.2, while Section 3 gives the preliminary results needed in the proof of the main results Proposition 1.1 and Theorem 1.2 of this note. The final section gives the proof of these two main results.

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2 Definitions and Notations

We assume throughout this note that X is a locally compact metric space endowed with a metric d and a non-trivial Borel regular measure μ so that bounded sets have finite measure and non-empty open sets have positive measure.

In the setting of metric measure spaces with no Riemannian structure, the following notion of upper gradients, formulated by Heinonen and Koskela in [HeK1], plays the role of derivatives. A Borel function g on X is an *upper gradient* of a real-valued function f on X if for all non-constant rectifiable paths $\gamma : [0, l_\gamma] \to X$ parameterized by arc length,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \le \int_\gamma g\, ds,$$

where the above inequality is interpretted as saying also that $\int_\gamma g\, ds = \infty$ whenever $|f(\gamma(0))|$ is infinite or $|f(\gamma(l_\gamma))|$ is infinite. If the above inequality fails only for a curve family with zero p-modulus (see e.g. Section 2.3 in [HeK1] for the definition of the p-modulus of a curve family), then g is a *p-weak upper gradient* of u. It is known that the L^p-closed convex hull of the set of all upper gradients of u that are in $L^p(X)$ is precisely the set of all p-weak upper gradients of u in $L^p(X)$; see Lemma 2.4 in Koskela–MacManus [KoMc].

Definition 2.1. We say that X supports a $(1,p)$-*Poincaré inequality* if there are constants $\tau, C > 0$ such that for all balls $B \subset X$, all measurable functions f on X, and all p-weak upper gradients g of f,

$$\fint_B |f - f_B|\, d\mu \le Cr\left(\fint_{\tau B} g^p\, d\mu\right)^{1/p},$$

where r is the radius of B and

$$f_B := \fint_B f\, d\mu := \frac{1}{\mu(B)} \int_B f\, d\mu.$$

Definition 2.2. As a metric measure space, X is said to be of *locally Q^--bounded geometry*, $Q > 1$; if the measure μ is (locally) Ahlfors Q-regular (that is, the measure of balls of radii r is comparable to the quantity r^Q) and supports a local $(1,q)$-Poincaré inequality for some $1 \le q < Q$ in the sense of Definition 2.1 above.

Following [Sh1], we consider a version of Sobolev spaces on the metric space X.

Definition 2.3. Let

$$\|u\|_{N^{1,p}} = \left(\int_X |u|^p\, d\mu\right)^{1/p} + \inf_g \left(\int_X g^p\, d\mu\right)^{1/p},$$

where the infimum is taken over all upper gradients of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}} = 0$.

The space $N^{1,p}(X)$ equipped with the norm $\|\cdot\|_{N^{1,p}}$ is a Banach space and a lattice, see [Sh1]. In the seminal paper [C], Cheeger gives an alternative definition of Sobolev spaces which leads to the same space whenever $p > 1$, see Theorem 4.10 in [Sh1]. Cheeger's definition yields the notion of partial derivatives in the following theorem (Theorem 4.38 in [C]).

Theorem 2.4 (Cheeger). *Let X be a metric measure space equipped with a positive doubling Borel regular measure μ. Assume that X admits a $(1,p)$-Poincaré inequality for some $1 < p < \infty$. Then there exists a countable collection (U_α, X^α) of measurable sets U_α and Lipschitz "coordinate" functions $X^\alpha = (X_1^\alpha, \ldots, X_{k(\alpha)}^\alpha) : X \to \mathbb{R}^{k(\alpha)}$ such that $\mu\big(X \setminus \bigcup_\alpha U_\alpha\big) = 0$, and for all α the following hold.*

The measure of U_α is positive, and the functions $X_1^\alpha, \ldots, X_{k(\alpha)}^\alpha$ are linearly independent on U_α and $1 \le k(\alpha) \le N$, where N is a constant depending only on the doubling constant of μ and the constant from the Poincaré inequality. If $f : X \to \mathbb{R}$ is Lipschitz, then there exist unique bounded measurable vector-valued functions $d^\alpha f : U_\alpha \to \mathbb{R}^{k(\alpha)}$ such that for μ-a.e. $x_0 \in U_\alpha$,

$$\lim_{r\to 0^+} \sup_{x\in B(x_0,r)} \frac{|f(x) - f(x_0) - d^\alpha f(x_0)\cdot(X^\alpha(x) - X^\alpha(x_0))|}{r} = 0.$$

We can assume that the sets U_α are pairwise disjoint and extend $d^\alpha f$ by zero outside U_α. Regard $d^\alpha f(x)$ as vectors in $\mathbb{R}^N$ and let $df = \sum_\alpha d^\alpha f$. The differential mapping $d : f \mapsto df$ is linear and it is shown in page 460 of [C] that there is a constant $C > 0$ such that for all Lipschitz functions f and μ-a.e. $x \in X$,

$$\frac{1}{C}|df(x)| \le g_f(x) := \inf_g \limsup_{r\to 0^+} \fint_{B(x,r)} g \, d\mu \le C|df(x)|. \tag{1}$$

Here $|df(x)|$ is a norm coming from a measurable inner product on the tangent bundle of X created by the above Cheeger derivative structure (see the discussion in [C]), g_f is the *minimal p-weak upper gradient* of f (see Corollary 3.7 in [Sh2] and Lemma 2.3 in Björn [Bj]), and the infimum is taken over all upper gradients g of f. Also, by Proposition 2.2 in [C], $df = 0$ μ-a.e. on every set where f is constant.

By Theorem 4.47 in [C] or Theorem 4.1 in [Sh1], the Newtonian space $N^{1,p}(X)$ is equal to the closure in the $N^{1,p}$-norm of the collection of Lipschitz functions on X with finite $N^{1,p}$-norm. By Theorem 10 in Franchi–Hajłasz–Koskela [FHK], there exists a unique "gradient" du satisfying (1) for every $u \in N^{1,p}(X)$. Moreover, if $\{u_j\}_{j=1}^\infty$ is a sequence in $N^{1,p}(X)$, then $u_j \to u$ in $N^{1,p}(X)$ if and only if as $j \to \infty$, $u_j \to u$ in $L^p(X,\mu)$ and $du_j \to du$ in $L^p(X,\mu;\mathbb{R}^N)$. Hence the differential structure extends to all functions in $N^{1,p}(X)$. Throughout this note we will use this structure, see for example Definition 2.7 below.

It was shown in [HKST] that if (Y_1, d_1, μ_1) and (Y_2, d_2, μ_2) are two metric measure spaces of locally Q-bounded geometry, then a homeomorphism $f : Y_1 \to Y_2$ is quasiconformal if and only if $f \in N^{1,Q}_{\mathrm{loc}}(Y_1; Y_2)$ and there exists a constant $K \geq 1$ so that

$$\operatorname{Lip} f(x)^Q \leq K\, J_f(x)$$

for μ_1-almost every $x \in Y_1$, see [HKST, Theorem 9.8]. Here

$$\operatorname{Lip} f(x) = \limsup_{r\to 0} \left(\operatorname{ess\,sup}_{d_1(x,y)\leq r} \frac{d_2(f(x), f(y))}{r} \right),$$

and

$$J_f(x) = \limsup_{r\to 0} \frac{\mu_2(fB(x,r))}{\mu_1(B(x,r))}$$

denotes the *infinitesimal volume distortion* of f at x. This Radon-Nikodym derivative exists because of the fact that f satisfies Lusin's condition (N); see [HKST]. For the definition of the metric space-valued Sobolev space $N^{1,Q}_{\mathrm{loc}}(Y_1; Y_2)$; see [HKST, Section 3].

Under the standing assumptions of Q^--bounded geometry on X, every relatively compact domain in X is of locally Q-bounded geometry. Therefore, if $U \subset X$ and $V \subset Y$ are relatively compact subdomains of metric measure spaces X and Y of locally Q-bounded geometry, then U and V are themselves of locally Q-bounded geometry and hence the results of [HKST, Section 9] apply to quasiconformal maps from U to V. Let $f : U \to V$ be such a map. By the discussion in [HKST, Section 10], there exists a matrix-valued map df, the *transposed Jacobian,* on U so that for every Lipschitz function φ on V,

$$d(\varphi \circ f)(x) = df(x)\, d\varphi(f(x))$$

for μ-a.e. $x \in U$.

Definition 2.5. Let U, V be two relatively compact subdomains of two metric measure spaces (X, d_X, μ_X) and (Y, d_Y, μ_Y) respectively, both of Q^--bounded geometry. Recall that a homeomorphism $f : U \to V$ is a conformal map if f is a quasiconformal mapping so that for μ_X-a.e. $x \in U$,

$$\|df(x)\|^Q \leq J_f(x) \text{ and } \|df^{-1}(x)\|^Q \leq j_{f^{-1}}(x);$$

see for example Section 4.1 of [HST]. Here $\|df(x)\|$ denotes the operator norm of the matrix $df(x)$.

By Lemma 4.4 of [HKST], conformal mappings preserve the class of Q-harmonic functions; that is, a function $u : U \to \mathbb{R}$ is Q-harmonic if and only if $u \circ f^{-1}$ is Q-harmonic on V.

It is worthwhile noting that the conformality of the mapping depends on the Cheeger derivative structure of X and of Y. It is clear from the construction given in [C] that a given metric measure space of Q^--bounded geometry will support more than one Cheeger derivative structure; hence, the conformality property of a quasiconformal map $f : U \to V$ depends quite heavily on the two Cheeger derivative structures, as do the conformal Martin boundaries. It should also be noted that even between two Euclidean domains, if the derivative structure considered is different from the standard Euclidean structure (and there are infinitely many of them), then the corresponding conformal mappings between the domains need not be restrictions of Möbius maps; therefore Theorem 1.2 is applicable to a wider class of mappings than just the class of Möbius maps.

Definition 2.6. The *p-capacity* of a Borel set $E \subset X$ is the number

$$\mathrm{Cap}(E) := \inf_u \left(\int_X |u|^p \, d\mu + \int_X |du|^p \, d\mu \right),$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E.

To be able to compare the boundary values of Newtonian functions we need a Newtonian space with zero boundary values. Let $\Omega \subset X$ be an open set and let

$$N_0^{1,p}(\Omega) = \{u \in N^{1,p}(X) : u = 0 \text{ } p\text{-q.e. on } X \setminus \Omega\}.$$

Corollary 3.9 in [Sh1] implies that $N_0^{1,p}(\Omega)$ equipped with the $N^{1,p}$-norm is a closed subspace of $N^{1,p}(X)$. By Theorem 4.8 in [Sh2], if Ω is relatively compact, the space $\mathrm{Lip}_c(\Omega)$ of Lipschitz functions with compact support in Ω is dense in $N_0^{1,p}(\Omega)$.

Unless otherwise stated, C denotes a positive constant whose exact value is unimportant, can change even within the same line, and depends only on fixed parameters such as X, d, μ and p. If necessary, we will specify its dependence on other parameters.

In the rest of this paper, $\Omega \subset X$ will always denote a bounded domain in X such that $\mathrm{Cap}(X \setminus \Omega) > 0$. Furthermore, we assume that Ω is a relatively compact domain.

Definition 2.7. Let $\Omega \subset X$ be a domain. A function $u : X \to [-\infty, \infty]$ is said to be *p-harmonic* in Ω if $u \in N_{\mathrm{loc}}^{1,p}(\Omega)$ and for all relatively compact subsets U of Ω and for every function $\varphi \in N_0^{1,p}(U)$,

$$\int_U |du|^p \, d\mu \le \int_U |d(u + \varphi)|^p \, d\mu,$$

or equivalently,

$$\int_U |du|^{p-2} du \cdot d\varphi \, d\mu = 0.$$

Here $\mathrm{Cap}_Q(K;\Omega)$ denotes the *relative Q-capacity* of a compact set K with respect to an open set $\Omega \supset K$; recall that this is equal to $\inf \int_\Omega |du|^Q \, d\mu$, the infimum being taken over all functions $u \in N^{1,Q}(X)$ for which $u|K \ge 1$ and $u|X \setminus \Omega = 0$. If such functions do not exist, we set $\mathrm{Cap}_Q(K;\Omega) = \infty$. For more on capacity, see [HeK2], [KiMa], [KaSh], [HKM, Chapter 2], and the references therein. It should be observed that the relative capacity $\mathrm{Cap}_Q(K;\Omega)$ is not the same as $\mathrm{Cap}(K)$; however, $\mathrm{Cap}_Q(K;\Omega) = 0$ if and only if $\mathrm{Cap}(K) = 0$.

Definition 2.8. Let Ω be a relatively compact domain in X and let $y \in \Omega$. An extended real-valued function $g = g(\cdot, y)$ on Ω is said to be a *Q-singular function with singularity at y* if it satisfies the following four criteria:

(i) g is Q-harmonic in $\Omega \setminus \{y\}$ and $g > 0$ on Ω;

(ii) $g\big|_{X\setminus\Omega} = 0$ p-q.e. and $g \in N^{1,Q}(X \setminus B(y,r))$ for all $r > 0$;

(iii) y is a singularity, i.e., $\lim_{x\to y} g(x) = \infty$;

(iv) whenever $0 \le a < b < \infty$,

$$\operatorname{Cap}_Q(\Omega^b; \Omega_a) = (b-a)^{1-Q}, \tag{2}$$

where $\Omega^b = \{x \in \Omega\colon g(x) \ge b\}$ and $\Omega_a = \{x \in \Omega\colon g(x) > a\}$.

In [HoSh] it was shown that every relatively compact domain in a space of locally Q^--bounded geometry supports a Q-singular function which plays a role analogous to the Green function of the Euclidean Q-Laplacian operator.

Since we have fixed the regularity exponent Q of the measure μ in this discussion, we shall simply call such functions singular functions, suppressing the reference to the index. Following the arguments given by Holopainen in [Ho], it can be seen that given $y \in \Omega$ there is precisely one Q-singular function satisfying equation (2). This observation enables us to define a Martin boundary in a manner similar to the classical potential theoretic Martin boundary.

Definition 2.9. Fix $x_0 \in \Omega$. Given a sequence (x_n) of points in Ω, we say that the sequence is *fundamental* (relative to x_0) if the sequence has no accumulation point in Ω and the sequence of normalized singular functions

$$M_{x_n}(x) = M(x, x_n) := \frac{g(x, x_n)}{g(x_0, x_n)}$$

is locally uniformly convergent in Ω. Above we set $M(x, x_0) = 0$ when $x \ne x_0$, and $M(x_0, x_0) = 1$.

Given a fundamental sequence $\xi = (x_n)$, we denote the corresponding limit function

$$M_\xi(x) := \lim_{n\to\infty} M(x, x_n),$$

and call it the *conformal Martin kernel* for ξ. We say that two fundamental sequences ξ and ζ are equivalent (relative to x_0), $\xi \sim \zeta$, if $M_\xi = M_\zeta$. It is worth noting that M_ξ is a non-negative Q-harmonic function in Ω, with $M_\xi(x_0) = 1$. Hence $M_\xi > 0$ in Ω by local Harnack's inequality (see [KiSh] for a proof of the local Harnack inequality). Note that if $\tilde{x}_0$ is another point in Ω, then $g(x, x_n)/g(\tilde{x}_0, x_n) = M(x, x_n)/M(\tilde{x}_0, x_n)$. Therefore, the property of being a fundamental sequence is independent of the particular choice of x_0. Furthermore, fundamental sequences ξ and ζ are equivalent relative to x_0 if and only if they are equivalent relative to any $\tilde{x}_0 \in \Omega$. Thus the following definition is independent of the fixed point x_0.

Given a point $\chi \in \partial\Omega$ we say that the function M_χ is a conformal martin kernel associated with χ if there is a fundamental sequence $(y_i)_i$ in Ω so that the sequence of singular functions $M(y_i, \cdot)$ with singularity at y_i converge locally uniformly to M_χ and $y_i \to \chi$.

Definition 2.10. The collection of all equivalence classes of fundamental sequences in Ω (or equivalently, the collection of all conformal Martin kernel functions) is the *conformal Martin boundary* $\partial_{cM}\Omega$ of the domain Ω. This collection is endowed with the local uniform topology: a sequence ξ_n in this boundary is said to converge to a point ξ if the sequence of functions M_{ξ_n} converges locally uniformly to M_ξ.

Definition 2.11. Let $\Omega \subsetneq X$ be a proper subdomain and let $A \geq 1$. We say that Ω is an *A-uniform* domain if every pair of distinct points $x, y \in \Omega$ can be joined by a rectifiable curve γ lying in Ω for which $l(\gamma) \leq A\, d(x, y)$ and

$$\min\{l(\gamma_{xz}), l(\gamma_{zy})\} \leq A\, \delta(z)$$

for all points z on γ. Here $\delta(z) = \delta_\Omega(z) = \operatorname{dist}(z, X \setminus \Omega)$ denotes the distance from z to the complement of Ω and γ_{ab} denotes the portion of the curve γ which lies between a and b. A curve γ in Ω which satisfies both of these conditions is said to be an *A-uniform curve.* We say that Ω is *uniform* if it is A-uniform for some A.

We say that Ω is *locally uniform at $x_0 \in \partial\Omega$* if x_0 has a neighborhood U_{x_0} so that $U_{x_0} \cap \Omega$ is uniform in Ω; that is, every pair of distinct points $x, y \in U_{x_0} \cap \Omega$ can be joined by a rectifiable curve γ lying in Ω for which $l(\gamma) \leq A_{x_0}\, d(x, y)$ and for all points z on γ,

$$\min\{l(\gamma_{xz}), l(\gamma_{zy})\} \leq A_{x_0}\, \delta(z).$$

We call a domain *Q-almost locally uniform* if the set of all points $x \in \partial\Omega$ at which Ω is not locally uniform is a zero Q-capacity set.

We call a set $A \subset \partial\Omega$ to be *separable by $\partial\Omega$* if for every $x \in A$ and $y \in \partial\Omega \setminus \{x\}$ there exists a compact set $K \subset \overline{\Omega}$ so that $A \cap K$ is empty and x and y lie in two different components of $\overline{\Omega} \setminus K$.

Clearly uniform domains are Q-almost locally uniform domains. A bounded Euclidean domain in $\mathbb{R}^n$ obtained by attaching an external cusp to a ball in $\mathbb{R}^n$ is an n-almost locally uniform domain for which the set of all points of local non-uniformity is separable by the boundary of the domain.

Definition 2.12. Let $r > 0$. A set $C(r)$ is said to be an r-cone terminating at a point $\chi \in \partial\Omega$ if there exists a quasihyperbolic geodesic γ in Ω terminating in χ so that

$$C(r) = \{x \in \Omega \,:\, \operatorname{dist}_\Omega(x, \gamma) < r\}.$$

Here $\operatorname{dist}_\Omega(x, \gamma)$ denotes the distance from x to the curve γ in the quasihyperbolic metric of Ω.

Definition 2.13. We say that Ω has *uniformly Q-fat complement* if there exist constants $c > 0$ and $r_0 > 0$ so that for every $x \in X \setminus \Omega$ and $r \in (0, r_0)$,

$$\frac{\operatorname{Cap}_Q(\bar{B}(x, r) \setminus \Omega; B(x, 2r))}{\operatorname{Cap}_Q(\bar{B}(x, r); B(x, 2r))} \geq c.$$

Recall that $\operatorname{Cap}_Q(E, U)$ denotes the Q-capacity of E in U, see the remarks preceding Definition 2.8. See [HKM], [Le], [Mi], and [BMS] for additional information on the uniform fatness condition.

3 Some preliminary results

In this section we give some preliminary results needed in the proof of the main results of this note. The proof of some of these results follow verbatim the corresponding proofs given in the paper [HST] and hence are omitted here; interested readers are encouraged to read [HST] (also available on the website `http://www.math.jyu.fi/research/report83.html`).

In [HST] attention was restricted to bounded uniform domains with uniformly Q-fat complement. However, most of the results contained in [HST] hold true even for locally uniform domains with uniformly Q-fat complement.

The following theorem is a modification of Lemma 3.13 of [HST], and characterizes the behavior of Martin kernels near boundary points of local uniformity for Ω that are not associated with the kernel function. The proof of this modified theorem is obtained by using the same proof found in [HST, Lemma 3.13], and therefore will not be included here.

Theorem 3.1. *Let Ω be a Q-almost locally uniform domain with uniformly Q-fat complement. Let $x_\infty \in \partial\Omega$ and let $\chi = (x_n)$ be a fundamental sequence with $\lim_n x_n = x_\infty$. Then M_χ vanishes continuously on the points of local uniformity in $\partial\Omega \setminus \{x_\infty\}$: for each $y_\infty \in \partial\Omega \setminus \{x_\infty\}$ that is a point of local uniformity of Ω there exists $r_{y_\infty} > 0$ so that*

$$\sup_{y \in B(y_\infty, r)\cap\Omega} M_\chi(y) \le Cr^s$$

for every $0 < r \le r_{y_\infty}$, where C and s are constants which are independent of r.

The following lemma is a consequence of the method of construction of singular functions (see [HS]) together with Theorem 3.1 above.

Lemma 3.2. *If Ω is a Q-almost locally uniform domain with uniformly Q-fat boundary, then whenever $x_1 \in \Omega$, $x_2 \in \partial\Omega$, and K is a compact subset of $\overline{\Omega}$ that separates x_1 and x_2 in the sense that x_1 and x_2 belong to different components of $\overline{\Omega} \setminus K$, and $K \cap \partial\Omega$ consists solely of points of local uniformity for Ω (if non-empty), then M_{x_2} is bounded on K and $M_{x_2}(x_1)$ is majorized by the same bound.*

Proof. We first prove that for every $y \in \Omega$ that is not in the component of $\overline{\Omega} \setminus K$ containing x_1, the singular function $g(\cdot, y)$ is bounded on K by a bound that depends solely on the data of K and $g(x_0, y)$ for a fixed $x_0 \in K$. Indeed, if u_i is the Q-potential of $\overline{B(y, r_i)}$ with respect to Ω and $\overline{B(y, r_i)}$ is a subset of the complement of the component of $\overline{\Omega} \setminus K$ that contains x, then by the construction in [HS, Theorem 3.4] we have that $g(x, y)$ is a locally uniform limit of the sequence of functions $g_i = u_i/\mathrm{Cap}_Q(\overline{B(y, r_i)}; \Omega)^{1/(Q-1)}$ as $i \to \infty$; here $r_i \to 0$. So for some i_0 and $i > i_0$, $|g_i(x_0, y) - g(x_0, y)| < g(x_0, y)$. Since the set $K \cap \partial\Omega$ consists solely of points of local uniformity for Ω, by Theorem 3.1 above $K \cap \partial\Omega$ can be covered by an open set U such that $g_i \le g(x_0, y)$ on $U \cap \Omega$; here U is chosen to be independent of $i > i_0$; see the proof of [HST, Lemma 3.13]. Since $K \setminus U$ is a compact subset of Ω, by the Harnack inequality we have that $\sup_{z \in K\setminus U} g_i(z, y) \le C_K g_i(x_0, y)$, and hence for sufficiently large i, $\sup_{z\in K} g_i(z, y) \le 2C_K g(x_0, y) =: M$ (note that we used the local uniform convergence of g_i to g to conclude that for sufficiently large i, $g_i(x_0, y) \le 2g(x_0, y)$).

As g_i converges locally uniformly to g, and $K \setminus U$ is a compact subset of Ω, for sufficiently large i we have $|g_i(z) - g(z, y)| < \epsilon$ on $K \setminus U$; that is, $g_i < M + \epsilon$ on K, which in turn means that $u_i \le (M+\epsilon)\mathrm{Cap}_Q(\overline{B(y,r_i)};\Omega)$ on K. Since u_i is a global (in $\Omega\setminus\overline{B(y,r_i)}$) energy minimizer (that is, it is an energy minimizer amongst all functions in $N^{1,2}(X)$ that have the same boundary data as u_i in the boundary set $\overline{B(y,r_i)}\cup(X\setminus\Omega)$), a truncation argument together with the uniqueness of solutions to Dirichlet problem (see [Sh2]) demonstrates that $g_i \le M+\epsilon$ on the component of $\overline{\Omega}\setminus K$ that contains x; that is, $g_i(x)\le M+\epsilon$ for sufficiently large i. Therefore $g(x,y)\le M+\epsilon$ for every $\epsilon>0$; in other words, $g(x,y)\le M = 2C_K g(x_0,y)$. It follows that $M(x,y)\le 2C_K$ whenever x is in a component of $\overline{\Omega}\setminus K$ that does not contain x_2.

Now if $(x_n)_n$ is a fundamental sequence that gives rise to M_{x_2}, for sufficiently large n we have that x_n does not lie in the same component as x_1 in $\overline{\Omega}\setminus K$. Since M_{x_2} is a local uniform limit of the normalized functions $M(\cdot,x_n)$, a repetition of the above argument with $M(\cdot,x_n)$ playing the role of g_i now yields the desired result. □

Observe that if Ω is a Q-almost locally uniform domain and $\chi\in\partial\Omega$ such that χ is a point of local uniformity for Ω, then for every $y\in\partial\Omega\setminus\{\chi\}$ there exists a compact set $K\subset\overline{\Omega}$ such that χ and y belong to different components of $\overline{\Omega}\setminus K$ and $K\cap\partial\Omega$ consists solely of points of local uniformity for Ω. To see this, note that there exists a neighbourhood U_χ of χ such that $U_\chi\cap\Omega$ is uniform in Ω, and hence all $\zeta\in U_\chi\cap\partial\Omega$ are points of local uniformity for Ω. If $y\in\partial\Omega\setminus\{\chi\}$, we can find $r>0$ such that $B(\chi,2r)\subset U_\chi$ and $2r<d(\chi,y)$. Then the compact set $K=\{x\in\overline{\Omega}\,:\,d(x,\chi)=r\}$ satisfies the conditions given above.

The following result follows from the above theorem and Lemma 3.2.

Corollary 3.3. *If Ω is as in Theorem 3.1, $x_\infty\in\partial\Omega$, and if x_∞ is a point of local uniformity for Ω or the set of all points of local non-uniformity of $\partial\Omega$ separable by $\partial\Omega$, and if $\chi=(x_n)$ a fundamental sequence with $\lim_n x_n = x_\infty$, then M_χ vanishes continuously on the points of local uniformity in $\partial\Omega\setminus\{x_\infty\}$ and is bounded in some neighbourhood of every $y_\infty\in\partial\Omega\setminus\{x_\infty\}$.*

The shape of level sets for the Martin kernel functions was also explored in [HST], resulting in the following proposition which states that the level sets are almost convex in the quasihyperbolic metric of Ω; see Proposition 3.14 of [HST]. While Proposition 3.14 of [HST] was formulated only for the singular functions in globally uniform domains, the proof of this proposition given in [HST] also applies to the situation considered in this note near points of local uniformity. Hence we have the following proposition.

Proposition 3.4. *Let Ω be a Q-almost uniform domain whose complement is uniformly Q-fat. Let g be a Martin kernel function on Ω associated with a point $\chi\in\partial\Omega$ of local uniformity for Ω. For $\tau\in(0,\infty]$, set*

$$E_\tau := \{z\in\Omega\,:\,g(z)\ge\tau\}.$$

Then every quasihyperbolic geodesic γ connecting two points $x,y\in E_\tau\cap U_\chi$ lies entirely in the set $E_{c\tau}$, where c is a positive constant which is independent of x, y, τ and g.

Here U_χ is a neighbourhood of χ such that $\Omega\cap U_\chi$ is uniform in Ω. The proof of the above proposition can be obtained by a trivial modification of the proof of Proposition 3.14 of [HST] and is left to the reader.

4 The singular behavior of Martin kernels and a proof of Theorem 1.2

In the following discussion, the quasihyperbolic metric of Ω plays a crucial role. It follows from the Gehring–Hayman theorem (see [BHK]) that the "cigar" type curves of uniform domains are closely associated with the quasihyperbolic geodesic curves pertinent to that domain. Recall that the *quasihyperbolic metric* k_Ω in a domain $\Omega \subsetneq X$ is defined to be

$$k_\Omega(x,y) := \inf_\gamma \int_\gamma \frac{ds(z)}{\delta(z)}, \qquad x,y \in \Omega, \tag{3}$$

where the infimum is taken over all rectifiable curves γ joining x to y in Ω and the integral denotes the line integral of the weight $\delta(z)^{-1}$ over γ, evaluated by using the arc length parametrization; see [GP]. If Ω is a Q-almost locally uniform domain, then for any point $w \in \partial\Omega$ of local uniformity for Ω, any two points in $\Omega \cap U_w$ can always be joined by (at least) one *quasihyperbolic geodesic*, i.e., a curve γ which achieves the infimum in (3) (see Lemma 1 of [GO] or Section 2 of [BHK]). See [K] for an overview of the quasihyperbolic metric.

Lemma 4.1. *Let Ω be a Q-almost locally uniform domain, and let $\chi \in \partial\Omega$ and M_χ be a conformal martin kernel associated with χ. If χ is a point of local uniformity of Ω or if the set of boundary points that are points of local non–uniformity is separable by $\partial\Omega$, then there exists a sequence $(x_i)_i$ in Ω so that $\lim_i x_i = \chi$ and $\lim_i M_\chi(x_i) = \infty$.*

Proof. Suppose not. Then M_χ is bounded in a neighbourhood of χ, and hence by Corollary 3.3 we have that M_χ is a bounded Q-harmonic function in Ω. In addition, as Ω is Q-almost locally uniform, for Q-almost every boundary point $w \in \partial\Omega$ we have

$$\lim_{\Omega \ni y \to w} M_\chi(y) = 0,$$

and hence by Corollary 6.2 of [BBS] we have that M_χ is the zero function, a contradiction. $\square$

We now are in a position to prove Proposition 1.1.

Proof of Proposition 1.1. By Lemma 4.1 there is a sequence $(y_i)_i$ in Ω converging to χ so that $M_\chi(y_i) \geq i$. Fix $\omega \in \Omega \cap U_\chi$ and let γ be a quasihyperbolic geodesic in Ω connecting χ to ω. Here U_χ is a neighbourhood of χ such that $\Omega \cap U_\chi$ is uniform in Ω.

For $n \in \mathbb{N}$ let $\beta_{i,n}$ be a quasihyperbolic geodesic in Ω connecting y_i to y_n. We assume here that $\beta_{i,n}$ are parametrized by arc-length (in the quasihyperbolic geodesic metric of Ω). By the Arzela-Ascoli theorem and by the fact that $\Omega \cap U_\chi$ is a proper (i.e., closed and bounded subsets are compact) metric space in its quasihyperbolic metric, the sequence $(\beta_{i,n})_i$ has a subsequence that converges locally uniformly to a quasihyperbolic geodesic ray β_n connecting y_n to χ. By Proposition 3.4 (see [HST, Proposition 3.14]), $M_\chi \circ \beta_{i,n} \geq n/C$ whenever $i \geq n$. Since M_χ is continuous on Ω and $\beta_{i,n}$ converges locally uniformly to β_n, we see that $M_\chi \circ \beta_n \geq n/C$.

Let γ be another quasihyperbolic geodesic ending at χ, and let $\omega \in \gamma \cap \Omega$. Observe that β_n and γ are quasihyperbolic geodesic rays ending at χ and emanating from y_n and

ω respectively. By [BHK, Theorem 3.6] and by [BS, inequality (3.5)] the quasihyperbolic geodesic triangle constructed using β_n, γ, and any quasihyperbolic geodesic α connecting y_n to ω is δ-thin for some $\delta > 0$ which is independent of ω, y_n (but may depend on χ) provided $\omega \in U_\chi \cap \Omega$. Since α is bounded in the quasihyperbolic metric of Ω, there must exist a neighbourhood U_n of χ so that $\beta_n \cap U_n \subset \bigcup_{x\in\gamma\cap U_n} B_k(x,\delta)$ and $\gamma \cap U_n \subset \bigcup_{x\in\beta_n\cap U_n} B_k(x,\delta)$. Here $B_k(x,r) := \{y \in \Omega : k_\Omega(x,y) < r\}$, k_Ω being the quasihyperbolic metric of Ω. Thus by the Harnack inequality, $M_\chi \circ \gamma\big|_{U_n} \geq \frac{e^{-C\delta}}{C} n$. Letting $n \to \infty$ yields the desired result. □

On the other hand, note by [HST, Lemma 3.13] or Theorem 3.1 that if γ is a quasihyperbolic geodesic ray that ends at any point in $\partial\Omega \setminus \{\chi\}$ of local uniformity of Ω, then

$$\lim_{t\to\infty} M_\chi \circ \gamma(t) = 0,$$

and if the endpoint of γ in $\partial\Omega \setminus \{\chi\}$ is not a point of local uniformity for Ω, then $M_\chi \circ \gamma$ is bounded.

Proposition 1.1 also indicates that along the non-tangential cones with vertices ending at a boundary point χ of Ω, the function M_χ exhibits a singularity behaviour. The following definition makes the notion of "non-tangential cones ending at a boundary point" more precise.

Corollary 4.2. *Let Ω satisfy the conditions of Theorem 1.1, $\chi \in \partial\Omega$ be a point of local uniformity of Ω, and let $C(r)$ be an r-cone in Ω terminating at χ. Then whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in $C(r)$ converging to χ in the metric topology of X, we have* $\lim_{n\to\infty} M_\chi(x_n) = \infty$.

This corollary is an easy consequence of Theorem 1.1 and the Harnack inequality applied to M_χ; we leave the proof to the reader.

Now we are ready to prove the Fatou type theorem 1.2 for comformal maps between two domains in two metric measure spaces of Q^--bounded geometry. In the following proof and in Theorem 1.2 a fixed Cheeger derivative structure is considered for the metric spaces X and Y.

Proof of Theorem 1.2. Since f is conformal, composition of f with Q-harmonic functions on the target space are Q-harmonic on the domain space. In addition, the relative Q-capacities are conformal invariants.

Clearly the independence of the limit from r follows from the fact that given two sequences in the cones $C(r_1)$ and $C(r_2)$, both terminating at the same boundary point of U, then the new sequence obtained by interleaving the two sequences is a sequence in the cone $C(r_1+r_2)$ converging to the same terminal point. Hence only the existence of the limit needs to be proven here.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a cone $C(r)$ with terminal point $\chi \in \partial U$ so that the limit $\lim_{n\to\infty} x_n = \chi$. Suppose the limit of the sequence $(f(x_n))_{n\in\mathbb{N}}$ does not exist. As $\overline{V}$ is compact, we can find two subsequences $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ of the original sequence $(x_n)_{n\in\mathbb{N}}$ so that the two limits $y_0 = \lim_{n\to\infty} f(y_n)$ and $z_0 = \lim_{n\to\infty} f(z_n)$ exist, but are not equal. Let M_{y_0} denote a conformal Martin kernel associated with the point $y_0 \in \partial V$. Then as the two boundary points $y_0 \neq z_0$, we see by Corollary 3.3 that $(M_{y_0}(f(z_n)))_{n\in\mathbb{N}}$ is a bounded sequence.

On the other hand, we know that the pull-back of a Q-harmonic function is Q-harmonic in U; hence it is easy to see that $M_{y_0} \circ f$ is indeed a Martin kernel corresponding to the endpoint χ in U. Thus by Corollary 4.2, $\lim_{n\to\infty} M_{y_0} \circ f(x_n) = \infty$. In particular, we must have $\lim_{n\to\infty} M_{y_0}(f(z_n)) = \infty$, contradicting the above conclusion. Thus it must be true that $y_0 = z_0$. Now the proof is complete. $\square$

Remark 4.3. It is possible that such a Fatou type theorem can be proven for quasiconformal maps by a similar method. Such a proof would first require the construction and study of Martin kernels for the operator obtained as the push-forward of the Q-Laplacian operator by the quasiconformal mapping.

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