The ∞-Poincaré inequality in metric measure spaces

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1 Introduction

A useful feature of the Euclidean n-space, $n \geq 2$, is the fact that every pair of points x and y can be joined not only by the line segment $[x, y]$, but also by a large family of curves whose length is comparable to the distance between the points. Once one has found such a "thick" family of curves, the deduction of important Sobolev and Poincaré inequalities is an abstract procedure in which the Euclidean structure no longer plays a role.

The classical Poincaré inequality allows one to obtain integral bounds on the oscillation of a function using integral bounds on its derivatives. In this type of inequalities the derivative itself is not needed, but only the size of the gradient of the function is really used; a nice discussion of this can be found in [17]. This is the idea behind generalizations of Poincaré inequalities in spaces where we may not have a linear structure. Heinonen and Koskela ([8],[9]) introduced a notion of "upper gradients" which serves the role of derivatives in a metric space X. A non-negative Borel function g on X is said to be an *upper gradient* for an extended real-valued function u on X if $|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g$ for every rectifiable curve $\gamma : [a, b] \to X$. The following Poincaré inequality is now standard in literature on analysis in metric measure spaces.

Definition 1.1. Let $1 \leq p < \infty$. We say that (X, d, μ) supports a *weak p-Poincaré inequality* if there exist constants $C_p > 0$ and $\lambda \geq 1$ such that for every Borel measurable function $u : X \to \mathbb{R} \cup \{\infty\}$ and every upper gradient $g : X \to [0, \infty]$ of u, the pair

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(u, g) satisfies the inequality

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \leq C_p\, r \Big(\fint_{B(x,\lambda r)} g^p d\mu \Big)^{1/p}$$

for each ball $B(x, r) \subset X$. The word *weak* refers to the possibility that λ may be strictly greater than 1.

Here $B(x, r)$ is an open ball with center at x and radius $r > 0$. For arbitrary $A \subset X$ with $0 < \mu(A) < \infty$ we write

$$u_A = \fint_A u = \frac{1}{\mu(A)} \int_A u\, d\mu.$$

There is a long list of metric spaces supporting a Poincaré inequality, including some standard examples such as $\mathbb{R}^n$, Riemannian manifolds with non-negative Ricci curvature, Carnot groups (in particular the Heisenberg group), but also other non-Riemannian metric measure spaces of fractional Hausdorff dimension, see for example [14], [7] and references therein. Metric spaces equipped with a p-Poincaré inequality support a nontrivial potential theory and geometric theory even without a priori smoothness structure of the metric space. Metric spaces with doubling measure and p-Poincaré inequality admit a first order differential calculus theory akin to that in Euclidean spaces. One surprising fact is that some geometric consequences of this condition seem to be independent of the parameter p and the picture is not yet clear.

It follows from Hölder's inequality that if a space admits a p-Poincaré inequality, then it admits a q-Poincaré inequality for each $q \geq p$. Recently Keith and Zhong [11] proved a self-improving property for Poincaré inequalities, that is, if X is a complete metric space equipped with a doubling measure satisfying a p-Poincaré inequality for some $1 < p < \infty$, then there exists $\varepsilon > 0$ such that X supports a q-Poincaré inequality for all $q > p - \varepsilon$. The strongest of all these inequalities would be a 1-Poincaré inequality, and it is well known that the 1-Poincaré inequality is equivalent to the relative isoperimetric property ([16], [1]). On the other hand, even for $p > 1$ the p-Poincaré inequality has strong links with the geometry of the underlying metric measure space. For instance, the Poincaré inequality implies that any pair of points in the space can be connected by curves that are not too long; this property is called quasiconvexity. A natural question is what would be the weakest version of p-Poincaré inequality that would still give reasonable information on the geometry of the metric space. One of the goals of this paper is to answer this question, by studying the following version of ∞-Poincaré inequality:

Definition 1.2. We say that (X, d, μ) supports a *weak ∞-Poincaré inequality* if there exist constants $C > 0$ and $\lambda \geq 1$ such that for every Borel measurable function u :

$X \to \mathbb{R} \cup \{\infty\}$ and every upper gradient $g : X \to [0, \infty]$ of u, the pair (u, g) satisfies the inequality

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq C \, r \|g\|_{L^\infty(B(x,\lambda r))}$$

for each ball $B(x, r) \subset X$.

The main result of this paper is a characterization of spaces supporting a ∞-Poincaré inequality; this is given in Theorem 4.7. A metric measure space is said to be *thick quasiconvex* if, loosely speaking, every pair of sets of positive measure, which are a positive distance apart, can be connected by a "thick" family of quasiconvex curves in the sense that the ∞-modulus of this family of curves is positive. The first aim of this paper is to show that a connected complete doubling metric measure space supports a weak ∞-Poincaré inequality if and only if it is thick quasiconvex, which is a purely geometric condition. We will also prove that this condition is equivalent to the purely analytic condition that $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ with comparable energy seminorms, in the sense described before Example 4.5.

The paper is organized as follows. In Section 2 we recall some standard notation and relevant notions regarding metric spaces supporting a doubling measure, ∞-modulus of curves, and Newtonian-Sobolev spaces $N^{1,\infty}(X)$. In Section 3 we introduce ∞-Poincaré inequality and present an example (Example 3.3) of a non-doubling metric space which supports an ∞-Poincaré inequality but does not support any p-Poincaré inequality for $p < \infty$. We do not know whether there is a metric space with a doubling measure which supports an ∞-Poincaré inequality but does not support any p-Poincaré inequality for $p < \infty$. Furthermore, we give some geometric implications of the ∞-Poincaré inequality, namely, that the space is quasiconvex. However, as one can appreciate in Corollary 4.15, quasiconvexity is not a sufficient condition for a space to support an ∞-Poincaré inequality. In Section 4 we will introduce the stronger notion of thick quasiconvexity (Definition 4.1), which leads us in Theorem 4.7 to obtain the desired analytic and geometric characterization of ∞-Poincaré inequality.

Unless otherwise stated, the letter C denotes various positive constants whose exact values are not important, and the value might change even from line to line.

2 Notation and Preliminaries

We assume throughout the paper that (X, d, μ) is a metric measure space, that is, a metric space equipped with a metric d and a Borel measure μ such that $0 < \mu(B) < \infty$ for each open ball $B \subset X$.

A measure μ is *doubling* if there is a constant $C_\mu > 0$ such that for all $x \in X$ and

$r > 0$,

$$\mu(B(x, 2r)) \le C_\mu\, \mu(B(x, r)).$$

Here $B(x, r) := \{y \in X : d(x, y) < r\}$. Also $\overline{B}(x, r) := \{y \in X : d(x, y) \le r\}$ and $\lambda B(x, r) := \{y \in X : d(x, y) < \lambda r\}$. We point out here that in the abstract metric setting, while $\overline{B}(x, r)$ contains the closure of $B(x, r)$, it might be larger.

An iteration of the above inequality shows that μ is also s_1-*homogeneous* for some $s_1 > 0$, that is, there are constants C and s depending only on C_μ such that, whenever B is a ball in X, $x \in B$ and $r > 0$ with $B(x, r) \subset B$,

$$\frac{\mu(B(x, r))}{\mu(B)} \ge \frac{1}{C} \left(\frac{r}{\mathrm{rad}(B)} \right)^{s_1}. \tag{1}$$

If in addition X is connected and has at least two points, then the doubling property also implies the existence of a constant $s_2 > 0$ such that for all balls $B \subset X$ and $B(x, r) \subset B$,

$$\frac{\mu(B(x, r))}{\mu(B)} \le \frac{1}{C} \left(\frac{r}{\mathrm{rad}(B)} \right)^{s_2}. \tag{2}$$

Because of the above inequality, letting $r \to 0$ we see that for all $x \in X$ we have $\mu(\{x\}) = 0$, that is, μ has no atoms.

In a complete metric space X, the existence of a doubling measure which is finite on balls and not trivial implies that X is separable and *proper*. The latter means that closed bounded subsets of X are compact. In particular, X is locally compact.

Some of the classical theorems in analysis in the Euclidean setting can be extended to doubling metric measure spaces. The Lebesgue differentiation theorem is such an example: if u is a locally integrable function on a doubling metric space X, then

$$u(x) = \lim_{r \to 0} \fint_{B(x,r)} u \, d\mu,$$

for μ-a.e. point in X. In other words, almost every point in X is a *Lebesgue point* for u, see for example [7].

Remark 2.1. The hypothesis of completeness is not so restrictive. The completion $(\hat{X}, \hat{d})$ of a metric space (X, d) is unique up to isometry. Note that (X, d) is a subspace of $(\hat{X}, \hat{d})$ and X is dense in $\hat{X}$. For our purposes, the crucial observation is that the essential features of X are inherited by $\hat{X}$. Indeed, if X is locally complete and there is a doubling Borel measure μ which is non-trivial and finite on balls, we may extend this measure to $\hat{X}$ so that $\hat{X} \setminus X$ has zero measure and the extended measure has the same properties as the original one. Also, if X supports a weak p-Poincaré inequality for some $1 \le p \le \infty$, then so does $\hat{X}$. See also [10] for further discussions on this topic.

By a *curve* γ we will mean a continuous mapping $\gamma : [a, b] \to X$. Recall that the *length* of a continuous curve $\gamma : [a, b] \to X$ in a metric space (X, d) is defined as

$$\ell(\gamma) = \sup \Big\{ \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) \Big\}$$

where the supremum is taken over all finite partitions $a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a, b]$. We will say that a curve γ is *rectifiable* if $\ell(\gamma) < \infty$. The integral of a Borel function g over a rectifiable path γ is usually defined via the path length parametrization γ_0 of γ in the following way:

$$\int_\gamma \rho ds = \int_0^{\ell(\gamma)} g \circ \gamma_0(t) dt.$$

Recall here that every rectifiable curve γ admits a parametrization by the arc-length; that is, with $\gamma_0 : [0, \ell(\gamma)] \to X$, for all t_1, t_2 with $t_1 \le t_2$, we have $\ell(\gamma_0|_{[t_1,t_2]}) = t_2 - t_1$. Hence from now on we only consider curves that are arc-length parametrized.

We denote by $\mathrm{LIP}^\infty(X)$ the space of bounded Lipschitz functions on X. In what follows, $\| \cdot \|_{L^\infty}$ will denote the essential supremum norm, provided we have a measure on X. In addition, $\mathrm{LIP}(\cdot)$ will denote the Lipschitz constant:

$$\mathrm{LIP}(u) := \sup_{\substack{x,y \in X \\ x \neq y}} \frac{|u(y) - u(x)|}{d(y, x)}.$$

The norm on $\mathrm{LIP}^\infty(X)$ is given by

$$\|u\|_{\mathrm{LIP}^\infty(X)} := \sup_{x \in X} |u(x)| + \mathrm{LIP}(u).$$

We recall the definition of ∞-*modulus* , an outer measure on the collection of all paths in X. In what follows let $\Upsilon \equiv \Upsilon(X)$ denote the family of all non-constant rectifiable curves in X. It may happen that Υ is empty, but we will be mainly interested in finding out when metric spaces have large enough Υ.

Definition 2.2. For $\Gamma \subset \Upsilon$, let $F(\Gamma)$ be the family of all Borel measurable functions $\rho : X \to [0, \infty]$ such that

$$\int_\gamma \rho \ge 1 \quad \text{for all} \quad \gamma \in \Gamma.$$

We define the ∞-*modulus of* Γ by

$$\mathrm{Mod}_\infty(\Gamma) = \inf_{\rho \in F(\Gamma)} \{ \|\rho\|_{L^\infty} \}.$$

If some property holds for all curves $\gamma \notin \Gamma$ for some $\Gamma \subset \Upsilon$ that satisfies $\mathrm{Mod}_\infty \Gamma = 0$, then we say that the property holds for ∞-*a.e. curve.*

It can be easily checked that Mod_∞ is an outer measure as it is for $1 \le p < \infty$, see for example [5, Theorem 5.2].

Remark 2.3. Notice that, if we have two measures μ and λ defined on X with the same zero measure sets, then the ∞-modulus of Γ is the same, independent of the measure we use to compute it.

Definition 2.4. Let $E \subset X$. Γ_E^+ is the family of curves γ such that $\mathscr{L}^1(\gamma^{-1}(\gamma \cap E)) > 0$, where $\mathscr{L}^1$ denotes the one-dimensional Lebesgue measure.

Recall that we only consider curves that are arc-length parametrized.

Lemma 2.5. *Let $E \subset X$. If $\mu(E) = 0$, then $\mathrm{Mod}_\infty(\Gamma_E^+) = 0$.*

Proof. Since μ is a Borel measure, by enlarging E if necessary, we may assume that E is a Borel set. Let $g = \infty \cdot \chi_E$. For $\gamma \in \Gamma_E^+$, we have that $\mathscr{L}^1(\gamma^{-1}(\gamma \cap E)) > 0$ and so

$$\int_\gamma g ds = \int_{\gamma \cap E} g ds = \infty.$$

Hence, by the definition of modulus

$$\mathrm{Mod}_\infty(\Gamma_E^+) \le \|g\|_{L^\infty(X)} = 0.$$

□

A related generalization of Sobolev spaces to general metric spaces are the so-called *Newtonian Spaces* $N^{1,p}$ introduced in [18, 19]. Its definition is based on the notion of upper gradients of Heinonen and Koskela. In this work, we will focus on the case $p = \infty$ studied in [3].

Definition 2.6. A non-negative Borel function g on X is an ∞*-weak upper gradient* of an extended real-valued function u on X if for ∞-a.e. curve $\gamma \in \Upsilon$,

$$|u(\gamma(a)) - u(\gamma(b))| \le \int_\gamma g$$

when both $u(\gamma(a))$ and $u(\gamma(b))$ are finite, and $\int_\gamma g = \infty$ otherwise. If the family of curves for which the above requirement is not satisfied is an empty family, then we say that g is an *upper gradient* of u.

Let $\widetilde{N}^{1,\infty}(X, d, \mu) = \widetilde{N}^{1,\infty}(X)$ be the class of all Borel functions $u \in L^\infty(X)$ for which there exists an ∞-weak upper gradient g in $L^\infty(X)$. For $u \in \widetilde{N}^{1,\infty}(X, d, \mu)$ we set

$$\|u\|_{\widetilde{N}^{1,\infty}} = \|u\|_{L^\infty} + \inf_g \|g\|_{L^\infty},$$

where the infimum is taken over all ∞-weak upper gradients g of u.

Definition 2.7. We define an equivalence relation in $\widetilde{N}^{1,\infty}(X)$ by $u \sim v$ if and only if $\|u - v\|_{\widetilde{N}^{1,\infty}} = 0$. The space $N^{1,\infty}(X, d, \mu) = N^{1,\infty}(X)$ denotes the quotient $\widetilde{N}^{1,\infty}(X, d, \mu)/\sim$ and it is equipped with the norm

$$\|u\|_{N^{1,\infty}} = \|u\|_{\widetilde{N}^{1,\infty}}.$$

It was shown in [3] that $N^{1,\infty}(X)$ is a Banach space. Note that if $u \in \widetilde{N}^{1,\infty}(X)$ and $v = u$ μ-a.e., then it is not necessarily true that $v \in \widetilde{N}^{1,\infty}$. Nevertheless, the following lemma shows that if $u, v \in \widetilde{N}^{1,\infty}$ and $v = u$ μ-a.e., then $\|u - v\|_{\widetilde{N}^{1,\infty}} = 0$.

Lemma 2.8. [3, 5.13] *Let $u_1, u_2 \in \widetilde{N}^{1,\infty}(X, d, \mu)$ such that $u_1 = u_2$ μ-a.e. Then $u_1 \sim u_2$, that is, both functions define exactly the same element in $N^{1,\infty}(X, d, \mu)$.*

If g is an ∞-weak upper gradient of f, then one can find a sequence $\{g_j\}_{j=1}^{\infty}$ of upper gradients of f such that $g_j \longrightarrow g$ in $L^{\infty}(X)$. It follows from the Lebesgue's differentiation Theorem that, if μ is doubling, then μ-a.e. $x \in X$ is a Lebesgue point of $N^{1,\infty}(X, d, \mu)$. Observe also that if $u \in \mathrm{LIP}^{\infty}(X)$, then the Lipschitz constant $\mathrm{LIP}(u)$ is an upper gradient for u. Therefore, $\|\cdot\|_{N^{1,\infty}} \leq \|\cdot\|_{\mathrm{LIP}^{\infty}}$ for every $u \in \mathrm{LIP}^{\infty}(X)$.

3 ∞-Poincaré inequality in metric measure spaces

We recall here again the definition of ∞-Poincaré inequality referred to in Section 1.

Definition 3.1. We say that (X, d, μ) supports a *weak ∞-Poincaré inequality* if there exist constants $C > 0$ and $\lambda \geq 1$ such that for every Borel measurable function $u : X \to \mathbb{R} \cup \{\infty\}$ and every ∞-weak upper gradient $g : X \to [0, \infty]$ of u, the pair (u, g) satisfies the inequality

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq C \, r \|g\|_{L^{\infty}(B(x,\lambda r))}$$

for each ball $B(x, r) \subset X$.

Remark 3.2. Let us observe that

$$\begin{aligned}\fint_B |u(x) - u_B| \, d\mu(x) &= \fint_B \left| \fint_B (u(x) - u(y)) d\mu(y) \right| d\mu(x) \\ &\leq \fint_B \fint_B |u(x) - u(y)| d\mu(y) d\mu(x),\end{aligned}$$

and so, when we want to check that (X, d, μ) supports a weak ∞-Poincaré inequality, it is enough to prove that each pair (u, g) satisfies

$$\fint_B \fint_B |u(x) - u(y)| d\mu(y) d\mu(x) \leq C \, r \|g\|_{L^{\infty}(\lambda B)} \tag{3}$$

for each ball $B \subset X$ with radius r. On the other hand, the inequality (3) is necessary to verify ∞-Poincaré inequality as well. To see this, note that

$$\begin{aligned} \fint_B \fint_B |u(x) - u(y)| d\mu(y) d\mu(x) &= \fint_B \fint_B |u(x) - u_B + u_B - u(y)| d\mu(y) d\mu(x) \\ &\leq 2 \fint_B |u(x) - u_B| d\mu(x). \end{aligned}$$

The next example shows that there exist spaces with a weak ∞-Poincaré inequality which do not admit a weak p-Poincaré inequality for any finite p.

Example 3.3. Let T be a non-degenerate triangular region in $\mathbb{R}^2$ and let T' be an identical copy of T. Let X be the metric space obtained by identifying a vertex V of T with a vertex V' of T' ($V = V' = \{0\}$) and the metric defined by

$$d(x,y) = \begin{cases} |x - y| & \text{if } x, y \in T \text{ or } x, y \in T', \\ |x - V| + |V' - y| & \text{if } x \in T \text{ and } y \in T'. \end{cases}$$

The space is equipped with the weighted measure μ given by $d\mu(x) = \omega(x) d\mathscr{L}^2(x)$, where $\omega(x) = e^{-\frac{1}{|x|^2}}$. Note that μ and the Lebesgue measure $\mathscr{L}^2$ have the same zero measure sets. It is already known that this space equipped with the Lebesgue measure $\mathscr{L}^2$ admits a p-Poincaré inequality for $p > 2$ (see for example [18]). Let us see that (X, d, μ) does not admit a weak p-Poincaré inequality for any finite p but admits a weak ∞-Poincaré inequality.

First, let us notice that given a measurable function u in X,

$$\fint_B |u - u_B| \, d\mu \leq 2 \inf_{c \in \mathbb{R}} \fint_B |u - c| d\mu, \tag{4}$$

where $u_B = \fint_B u d\mu$. Indeed, let $c \in \mathbb{R}$ and suppose $c \geq u_B$ (the case $c < u_B$ is analogous). Then,

$$\fint_B |c - u_B| \, d\mu = c - u_B = \fint_B c - \fint_B u = \fint_B (c - u) \leq \fint_B |c - u| d\mu.$$

Since $|u(x) - u_B| \leq |u(x) - c| + |c - u_B|$ for each $x \in X$, we have that

$$\fint_B |u - u_B| \, d\mu \leq \fint_B |u - c| \, d\mu + \fint_B |c - u_B| \, d\mu \leq 2 \fint_B |u - c| d\mu.$$

If we take the infimum over c on the right hand of the previous inequality, we get inequality (4). Let us consider an upper gradient g of u.

Now, we obtain the following chain of inequalities by using Hölder's inequality for $2 < p < q$. If g is an upper gradient for u,

$$\begin{aligned}
\fint_B |u - u_B|\, d\mu &\overset{(4)}{\leq} 2 \inf_{c\in\mathbb{R}} \fint_B |u - c| d\mu \leq 2 \fint_B |u - u_{B,\mathscr{L}^2}| d\mu \\
&\leq 2\|u - u_{B,\mathscr{L}^2}\|_{L^\infty(\mu)} = 2\|u - u_{B,\mathscr{L}^2}\|_{L^\infty(\mathscr{L}^2)} \\
&\leq C_p r \Big(\fint_{5\lambda B} g^p d\mathscr{L}^2 \Big)^{1/p} \leq C_p r \Big(\fint_{5\lambda B} g^q d\mathscr{L}^2 \Big)^{1/q},
\end{aligned}$$

where $u_{B,\mathscr{L}^2} = \fint_B u d\mathscr{L}^2$. In the third line of the previous chain of inequalities we have applied [6, Theorem 5.1]. If we let q tend to infinity we get

$$\fint_B |u - u_B|\, d\mu \leq C_p r \|g\|_{L^\infty(\mathscr{L}^2, 5\lambda B)} = C_p r \|g\|_{L^\infty(\mu, 5\lambda B)},$$

and so, (X, d, μ) admits a weak ∞-Poincaré inequality.

Let us see now that (X, d, μ) does not admit a p-Poincaré inequality for any finite p. Indeed, consider the function $u = 1$ in T and $u = 0$ in T' and in the vertex. It is not difficult to check that the function $g_\alpha(x) = \frac{\alpha}{|x|}$ is an upper gradient for u for each $\alpha > 0$. Taking the ball $B = X$, we have that $u_X > 0$ and therefore $\fint_X |u - u_X|\, d\mu > 0$. Nevertheless, $\int_X g_\alpha^p d\mu$ tends to zero when α tends to zero for $1 < p < \infty$, and so X does not admit a weak p-Poincaré inequality for any finite p.

Observe that the measure μ in the above example is *not* doubling.

One of the most useful geometric implications of the p-Poincaré inequality for finite p is the fact that if a complete doubling metric measure space supports a p-Poincaré inequality then there exists a constant such that each pair of points can be connected with a curve whose length is at most the constant times the distance between the points (see [17] or [6]), that is, the space is *quasiconvex*. If X is only known to support an ∞-Poincaré inequality, the same conclusion holds as demonstrated by Proposition 3.4 below.

Proposition 3.4. *Suppose that (X, d, μ) is a complete metric measure space with μ a doubling measure. If X supports a weak ∞-Poincaré inequality, then X is quasiconvex with a constant depending only on the constants of the Poincaré inequality and the doubling constant.*

Proof. Let $\varepsilon > 0$. We say that $x, z \in X$ lie in the same ε-component of X if there exists an ε-chain joining x with z, that is, there exists a finite chain $z_0, z_1, \ldots, z_n$ such that $z_0 = x$, $z_n = z$ and $d(z_i, z_{i+1}) \leq \varepsilon$ for all $i = 0, \ldots, n-1$. If x and y lie in different ε-components, then it is obvious that there does not exist a rectifiable curve joining x and y. Thus, the function $g \equiv 0$ is an upper gradient for the characteristic function of any of the components. Note that for every x in one of the components, the ball $B(x, \varepsilon/2)$ is a subset of that component; that is, each component is open and hence is a measurable set. By applying the weak ∞-Poincaré inequality to the characteristic function of any component, it follows that all the points of X lie in the same ε-component.

Now, let us fix $x, y \in X$ and prove that there exists a curve γ joining x and y such that $\ell(\gamma) \leq Cd(x,y)$, where C is a constant which depends only on the doubling constant and the constants involved in the Poincaré inequality. We define the ε-distance of x to z to be

$$\rho_{x,\varepsilon}(z) := \inf \sum_{i=0}^{N-1} d(z_i, z_{i+1}),$$

where the infimum is taken over all finite ε-chains $\{z_i\}$. Note that $\rho_{x,\varepsilon}(z) < \infty$ for all $z \in X$. In addition, if $d(z,w) \leq \varepsilon$ then $|\rho_{x,\varepsilon}(z) - \rho_{x,\varepsilon}(w)| \leq d(z,w)$. Hence, $\rho_{x,\varepsilon}$ is a locally 1-Lipschitz function, in particular, every point is a Lebesgue point of $\rho_{x,\varepsilon}$ and in addition, for all $\varepsilon > 0$, the function $g \equiv 1$ is an upper gradient of $\rho_{x,\varepsilon}$. For each $i \in \mathbb{Z}$, define $B_i = B(x, 2^{1-i}d(x,y))$ if $i \geq 0$, and $B_i = B(y, 2^{1+i}d(x,y))$ if $i \leq -1$. Thus, a telescopic argument, together with weak ∞-Poincaré inequality, gives us the following chain of inequalities:

$$\begin{aligned}
|\rho_{x,\varepsilon}(y)| =& |\rho_{x,\varepsilon}(x) - \rho_{x,\varepsilon}(y)| \\
\leq& \sum_{i\in\mathbb{Z}} \Big| \fint_{B_i} \rho_{x,\varepsilon} d\mu - \fint_{B_{i+1}} \rho_{x,\varepsilon} d\mu \Big| \\
\leq& C_\mu \sum_{i\in\mathbb{Z}} \frac{1}{\mu(B_i)} \fint_{B_i} \Big| \rho_{x,\varepsilon} - \fint_{B_{i+1}} \rho_{x,\varepsilon} d\mu \Big| d\mu \\
\leq& C_\mu C d(x,y) \sum_{i\in\mathbb{Z}} 2^{-|i|} \|g\|_{L^\infty(\lambda B_i)} \\
\leq& \, Cd(x,y) \qquad (5)
\end{aligned}$$

where C is a constant that depends only on X.

Since X is complete, the existence of a non trivial doubling measure implies that closed balls are compact. Using a standard limiting argument, which involves Arzela-Ascoli's theorem and inequality (5), we can construct a 1-Lipschitz rectifiable curve connecting x and y with length at most $Cd(x,y)$. Since x and y were arbitrary this completes the proof. For further details about the construction of the curve we refer the reader to [13, Theorem 3.1]. □

The following technical lemma will be useful in the sequel.

Lemma 3.5. *Let X be a complete separable metric space equipped with a σ-finite Borel measure μ, and let $g : X \longrightarrow [0, \infty]$ be a Borel function. Then, for each $x_0 \in X$, the function*

$$u(z) = \inf_{\gamma \text{ connects } z \text{ to } B(x_0,r)} \int_\gamma g\, ds,$$

is μ-measurable. Moreover, whenever $k \in \mathbb{R}$ the function g is an upper gradient for $v = \min\{u, k\}$.

Proof. Following the lines of [10, Corollary 1.10], one can prove that u is μ-measurable.

To see that g is an upper gradient of v on X, we argue as follows. Fix $z_1, z_2 \in X$ and β be a rectifiable curve in X connecting z_1 to z_2. There are three possible cases:

1. $v(z_1) = u(z_1)$ and $v(z_2) = u(z_2)$,
2. $v(z_1) = u(z_1)$ and $v(z_2) = k$,
3. $v(z_1) = k = v(z_2)$.

In the first case, both $u(z_1)$ and $u(z_2)$ are finite. Fix $\varepsilon > 0$; then we can find a rectifiable curve connecting z_1 to $B(x, \varepsilon)$ such that $u(z_1) \geq \int_\gamma g ds - \varepsilon$, and so

$$u(z_2) - u(z_1) \leq \int_{\gamma \cup \beta} g\, ds - \int_\gamma g\, ds + \varepsilon = \int_\beta g\, ds + \varepsilon,$$

where we can cancel $\int_\gamma g\, ds$ because it is a finite value. A similar argument gives

$$u(z_1) - u(z_2) \leq \int_\beta g\, ds + \varepsilon,$$

and the combination of the above two inequalities followed by letting $\varepsilon \to 0$ gives

$$|v(z_1) - v(z_2)| = |u(z_1) - u(z_2)| \leq \int_\beta g\, ds.$$

In the second case, $u(z_1) = v(z_1) \leq v(z_2) \leq u(z_2)$. In this case again, $u(z_1)$ is finite. For $\varepsilon > 0$ we can find a rectifiable curve γ connecting z_1 to $B(x, \varepsilon)$ such that $u(z_1) \geq \int_\gamma g\, ds - \varepsilon$, and so

$$\begin{aligned}|v(z_1) - v(z_2)| = v(z_2) - v(z_1) \leq u(z_2) - u(z_1) &\leq \int_{\gamma \cup \beta} g\, ds - \int_\gamma g\, ds + \varepsilon \\ &= \int_\beta g\, ds + \varepsilon,\end{aligned}$$

where again we were able to cancel the term $\int_\gamma g\,ds \le u(z_1)+\varepsilon$ because it is finite. Letting $\varepsilon \to 0$ we again obtain

$$|v(z_1)-v(z_2)| \le \int_\beta g\,ds.$$

In the third case we easily obtain the above inequality again, because in this case $v(z_1)-v(z_2)=0$. □

The following example shows one of the difficulties in working with $p=\infty$ as opposed to finite values of p.

Example 3.6. Let X be a complete metric space that supports a doubling Borel measure μ which is non-trivial and finite on balls, and suppose that X supports a weak ∞-Poincaré inequality. Denote by $\Gamma_{x_0,r,R}$ the family of curves that connect $B(x_0,r)$ to the complement of the ball $B(x_0,R)$ with $0<r<R/2<\operatorname{diam}(X)/4$.

We will prove that there is a constant $C>0$, independent of R,r and x_0, such that

$$\operatorname{Mod}_\infty(\Gamma_{x_0,r,R}) \ge \frac{C}{R}.$$

To see this, let g be a non-negative Borel measurable function on X such that for all $\gamma \in \Gamma_{x_0,r,R}$, the integral $\int_\gamma g\,ds \ge 1$. Notice here that by Proposition 3.4, X is quasiconvex. We then set

$$\tilde{u}(z) = \inf_{\gamma \text{ path connecting } z \text{ to } B(x_0,r)} \int_\gamma g\,ds,$$

and consider $u=\min\{\tilde{u},2\}$. Then it follows that $u=0$ on $B(x_0,r)$ and by the choice of g, $u\ge 1$ on $X\setminus B(x_0,R)$. By [10, Corollary 1.10] it follows that u is measurable and from Lemma 3.5 it follows that g is an upper gradient of u; that is, $u\in N^{1,\infty}(X)$.

If $x\in B(x_0,r)$ and $y\in B(x_0,R+r)\setminus B(x_0,R)$, for each $i\in\mathbb{Z}$ define $B_i = B(x,2^{1-i}d(x,y))$ if $i\ge 0$, and $B_i=B(y,2^{1+i}d(x,y))$ if $i\le -1$. By the weak ∞-Poincaré inequality and the doubling property of μ, we get for Lebesgue points $x\in B(x_0,r)$ and $y\in X\setminus B(x_0,R)$,

$$\begin{aligned}
1 \le |u(x)-u(y)| &\le \sum_{i\in\mathbb{Z}} \Big| \fint_{B_i} u\,d\mu - \fint_{B_{i+1}} u\,d\mu \Big| \\
&\le C_\mu \sum_{i\in\mathbb{Z}} \fint_{B_i} \Big| u - \fint_{B_i} u\,d\mu \Big| d\mu \\
&\le C_\mu C d(x,y) \sum_{i\in\mathbb{Z}} 2^{-|i|} \|g\|_{L^\infty(\lambda B_i)} \\
&\le C d(x,y) \|g\|_{L^\infty(X)}.
\end{aligned}$$

Hence

$$\|g\|_{L^\infty(X)} \geq \frac{1}{C\,d(x,y)} \geq \frac{1}{C\,(R+r)} \geq \frac{1}{2CR}.$$

Taking the infimum over all such g we obtain the desired inequality for the ∞-Modulus. An analogous statement holds for $\mathrm{Mod}_p(\Gamma_{x_0,r,R})$ if X supports a weak p-Poincaré inequality for sufficiently large finite p (that is, with p larger than the lower mass bound exponent s_1 obtained from the doubling property of the measure μ). For such finite p, we can approximate test functions g from above in $L^p(X)$ by lower semi-continuous functions (it follows from Vitali-Caratheodory theorem [4, pp. 209–213]), and so we would see as in [9] that the p-Modulus of the collection of all curves that connect x_0 itself to $X \setminus B(x_0,R)$ is positive. Unfortunately such an approximation by lower semi-continuous functions in the L^∞-norm does not hold true, and so we cannot conclude from the above computation that the ∞-modulus of the collection of all curves connecting x_0 to $X \setminus B(x_0,R)$ is positive *if X is only known to support a weak ∞-Poincaré inequality.*

The previous example highlights the difficulties when working with the L^∞-norm, namely, the L^∞-norm is insensitive to local changes, and we do not have Vitali-Caratheodory theorem.

4 Geometric characterization of weak ∞-Poincaré inequality

The connection between isoperimetric and Sobolev-type inequalities in the Euclidean setting is well-understood (see [16], [1]). In the context of metric spaces supporting a doubling measure, Miranda proved in [16] that a 1-weak Poincaré inequality implies a relative isoperimetric inequality for sets of finite perimeter. Recently, in [12] Kinnunen and Korte gave further characterizations of Poincaré type inequalities in the context of Newtonian spaces in terms of isoperimetric and isocapacitary inequalities.

In what follows, we will prove that ∞-Poincaré inequality also has a geometric characterization, namely, it is equivalent to *thick quasiconvexity* .

Definition 4.1. (X,d,μ) is a *thick quasiconvex* space if there exists $C \geq 1$ such that for all $x,y \in X$, $0 < \varepsilon < \frac{1}{4}d(x,y)$, and all measurable sets $E \subset B(x,\varepsilon)$, $F \subset B(y,\varepsilon)$ satisfying $\mu(E)\mu(F) > 0$ we have that

$$\mathrm{Mod}_\infty(\Gamma(E,F,C)) > 0,$$

where $\Gamma(E,F,C)$ denotes the set of curves $\gamma_{p,q}$ connecting $p \in E$ and $q \in F$ with $\ell(\gamma_{p,q}) \leq Cd(p,q)$. Here we do not require quantitative control on the modulus of the curve family.

Remark 4.2. Note that every complete thick quasiconvex space X supporting a doubling measure is quasiconvex. Indeed, let $x, y \in X$ and choose a sequence ε_j which tends to zero. Since X is thick quasiconvex, there exists a constant $C \geq 1$ such that for every ε_j there exists $x_j \in B(x, \varepsilon_j)$ and $y_j \in B(y, \varepsilon_j)$ and a curve γ_j connecting x_j to y_j with $\ell(\gamma_j) \leq Cd(x_j, y_j)$. Thus, we obtain a sequence $\{\gamma_j\}$ of curves such that

$$\ell(\gamma_j) \leq Cd(x_j, y_j) \leq 2Cd(x, y),$$

that is, a sequence of curves with uniformly bounded length. Since X is a complete doubling metric space and therefore proper, we may use Arzela-Ascoli's theorem to obtain a subsequence, also denoted $\{\gamma_j\}$, which converges uniformly to a curve γ which connects x and y with

$$\ell(\gamma) = \lim_{j\to\infty} \ell(\gamma_j) \leq C \lim_{j\to\infty} d(x_j, y_j) = Cd(x, y).$$

However, the converse is not true. In example 4.14 we will give a quasiconvex space endowed with a doubling measure which is not thick quasiconvex.

Standard assumptions : In what follows, we will assume that X is a connected complete metric space supporting a doubling Borel measure μ which is non-trivial and finite on balls.

We have already proved in Proposition 3.4 that weak ∞-Poincaré inequality for Lipschitz functions implies quasiconvexity. However, in the following proposition we prove that weak ∞-Poincaré inequality for Newtonian functions implies the stronger property of thick quasiconvexity.

Proposition 4.3. *If X supports a weak ∞-Poincaré inequality for functions in $N^{1,\infty}(X)$ with upper gradients in $L^\infty(X)$, then X is thick quasiconvex.*

We wish to point out here that $N^{1,\infty}(X)$ consists precisely of functions in $L^\infty(X)$ that have an upper gradient in $L^\infty(X)$.

Proof. Let $x, y \in X$ such that $x \neq y$, and let $0 < \varepsilon < d(x, y)/4$. Fix $n \in \mathbb{N}$ and let $\Gamma_n = \Gamma(B(x, \varepsilon), B(y, \varepsilon), n)$ be the collection of all rectifiable curves connecting $B(x, \varepsilon)$ to $B(y, \varepsilon)$ such that $\ell(\gamma) \leq n\, d(x, y)$. Observe that by the choice of ε, if p, q are the end points of γ, then $d(p, q)/4 \leq d(x, y) \leq 4d(p, q)$.

Suppose that $\mathrm{Mod}_\infty(\Gamma_n) = 0$. By [3, Lemma 5.7] there exists a non-negative Borel measurable function $g \in L^\infty(X)$ such that $\|g\|_{L^\infty(X)} = 0$ and for all $\gamma \in \Gamma_n$, the path integral $\int_\gamma g\, ds = \infty$. In this case we define

$$u(z) = \inf_{\gamma \text{ connecting } z \text{ to } B(x,\varepsilon)} \int_\gamma (1 + g)\, ds.$$

Observe that $\|1+g\|_{L^\infty(X)} = 1$ and $u = 0$ on $B(x,\varepsilon)$. If $z \in B(y,\varepsilon)$ and γ is a rectifiable curve connecting z to $B(x,\varepsilon)$, then either $\gamma \in \Gamma_n$ in which case $\int_\gamma (1+g)\,ds \ge \int_\gamma g\,ds = \infty$, or else $\gamma \notin \Gamma_n$, in which case $\ell(\gamma) > nd(x,y)$ and so $\int_\gamma (1+g)\,ds \ge \int_\gamma 1\,ds > nd(x,y)$, and so $u(z) \ge n\,d(x,y)$. It follows that the function $v = \min\{u, 2n\,d(x,y)\}$ has the properties that

1. $v = 0$ on $B(x,\varepsilon)$,
2. $v \ge nd(x,y)$ on $B(y,\varepsilon)$,
3. $v \in N^{1,\infty}(X)$,
4. $1+g$ is an upper gradient of v on X (see Lemma 3.5), with $\|g\|_{L^\infty(X)} = 0$.

Let $y_0 \in B(y,\varepsilon/2)$ be a Lebesgue point of v; then by using the chain of balls $B_i = B(x, 2^{1-i}d(x,y))$ if $i \ge 0$ and $B_i = B(y_0, 2^{1+i}d(x,y))$ if $i \le -1$ and using the weak ∞-Poincaré inequality, we get

$$\begin{aligned} n\,d(x,y) \le v(y_0) = |v(x) - v(y_0)| &\le \sum_{i\in\mathbb{Z}} |v_{B_i} - v_{B_{i+1}}| \\ &\le C \sum_{i\in\mathbb{Z}} \fint_{2B_i} |v - v_{B_i}|\,d\mu \\ &\le C \sum_{i\in\mathbb{Z}} 2^{-|i|} d(x,y) \|1+g\|_{L^\infty(\lambda B_i)} \\ &= Cd(x,y) \sum_{i\in\mathbb{Z}} 2^{-|i|} \le Cd(x,y). \end{aligned}$$

Observe that x is a Lebesgue point of v since $v = 0$ on $B(x,\varepsilon)$. Thus we must have $n \le C$, with C depending solely on the doubling constant and the constant of the Poincaré inequality. Hence if $n > C$ then the curve family $\Gamma_n = \Gamma(B(x,\varepsilon), B(y,\varepsilon), n)$ must have positive ∞-Modulus, completing the proof in the simple case that $E = B(x,\varepsilon)$ and $F = B(y,\varepsilon)$. The proof for more general E, F is very similar, where we modify the definition of u by looking at curves that connect z to E, and then observing that almost every point in E and almost every point in F are Lebesgue points for the modified function v, with $v = 0$ on E and $v \ge nd(x,y)$ on F. This completes the proof of the proposition. □

The following result indicates an advantage of a thick quasiconvex space.

Lemma 4.4. *Let X be a thick quasiconvex space. If u is a measurable function (finite μ-a.e.) on X and g is an upper gradient of u, and if B is a ball in X such that*

$\|g\|_{L^\infty(2CB)} < \infty$, *then there is a set* $F \subset B$ *with* $\mu(F) = 0$ *such that* u *is* $C\|g\|_{L^\infty(2CB)}$-*Lipschitz continuous on* $B \setminus F$. *Here* C *is the constant appearing in the definition of thick quasiconvexity.*

Proof. Since u is measurable (and finite μ-almost everywhere), by Lusin's theorem ([4, pp. 61]) for every $n \in \mathbb{N}$ there is a measurable set $E_n \subset X$ such that $\mu(E_n) < 1/n$ and $u_{|B\setminus E_n}$ is continuous. Moreover, for each $n \geq 1$ we can choose G_n be an open set such that $E_n \subset G_n$, $\mu(G_n) < \frac{1}{n}$ (see Theorem 1.10 in [15]) and $u_{|X\setminus G_n}$ is continuous. Now, $V_n = G_1 \cap G_2 \cap \cdots \cap G_n$ is an open set with $\mu(V_n) < \frac{1}{n}$. Observe that $B \setminus V_n = (B \setminus G_1) \cup \cdots \cup (B \setminus G_n)$ and $u_{|B\setminus V_n}$ is continuous.

We will show that u is $C\|g\|_{L^\infty(2CB)}$-Lipschitz continuous on $B \setminus V_n$. Let $P = \{x \in 2CB : g(x) > \|g\|_{L^\infty(2CB)}\}$; then by assumption, $\mu(P) = 0$, and so it follows from Lemma 2.5 that $\mathrm{Mod}_\infty(\Gamma_P^+) = 0$. To prove that u is $C\|g\|_{L^\infty(2CB)}$-Lipschitz continuous on $B\setminus V_n$, we fix $x, y \in B\setminus V_n$ that are points of density for $B\setminus V_n$. Let $0 < \delta < d(x,y)/4$. By the thick quasiconvexity applied to the sets $E_\delta := B(x,\delta)\setminus V_n$ and $F_\delta := B(y,\delta)\setminus V_n$, there is a curve γ connecting a point $x_\delta \in E_\delta$ and $y_\delta \in F_\delta$ with $\ell(\gamma) \leq Cd(x_\delta, y_\delta)$ and $\mathscr{L}^1(\gamma^{-1}(\gamma \cap P)) = 0$. Notice that since x is a point of density for $B \setminus V_n$,

$$\lim_{\rho\to 0} \frac{\mu(B(x,\rho) \cap (B \setminus V_n)}{\mu(B(x,\rho))} = 1,$$

and so $\mu(E_\delta) > 0$. Analogously, we obtain that since y is a point of density for $B \setminus V_n$, $\mu(F_\delta) > 0$. Hence we can apply the thick quasiconvexity property to E_δ, F_δ.

Thus,

$$|u(x_\delta) - u(y_\delta)| \leq \int_\gamma g\, ds \leq \|g\|_{L^\infty(2CB)}\ell(\gamma) \leq C\|g\|_{L^\infty(2CB)}d(x_\delta, y_\delta). \tag{6}$$

Since u is continuous on $B \setminus V_n$, by letting $\delta \to 0$ in (6), we see that

$$|u(x) - u(y)| \leq C\|g\|_{L^\infty(2CB)}d(x,y)$$

as wanted.

Now, we set $F = \bigcap_n V_n$. Note that since $\{V_n\}_n$ is a decreasing sequence of sets, $\mu(F) = \lim_{n\to\infty} \mu(V_n) = 0$. To conclude, let $x, y \in B \setminus F$. Since $B \setminus V_n$ is an increasing sequence of sets, there exists $n \in \mathbb{N}$ such that $x, y \in B\setminus V_n$ and so $u_{|B\setminus F}$ is $C\|g\|_{L^\infty(2CB)}$-Lipschitz. □

In what follows we say that $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ *with comparable energy seminorms* if there is a constant $C > 0$ such that for all $u \in N^{1,\infty}(X)$, there exists $u_0 \in \mathrm{LIP}^\infty(X)$ with $u = u_0$ μ-a.e. and

$$\mathrm{LIP}(u_0) \leq C \inf_g \|g\|_{L^\infty},$$

where the infimum is taken over all ∞-weak upper gradients g of u.

The following example shows that the requirement that $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ as Banach spaces does not by itself imply that these two Banach spaces should have comparable energy seminorms. If however the two seminorms are comparable, then the two Banach space norms are equivalent.

Example 4.5. Consider the set $X = \mathbb{R}^2 \setminus \cup_{n=1}^\infty R_n$, where R_n is the open rectangle $R_n = (2n, 2n+1) \times (0, n)$. We endow X with the Euclidean distance and the 2-dimensional Lebesgue measure. It is clear that X is not quasiconvex. Nevertheless, X is uniformly locally thick quasiconvex, that is, for every $p \in X$, the ball $B(p,1)$ in X with center p and radius $1/2$ is thick quasiconvex with quasiconvexity constant 2. Indeed, if the ball does not contain any corner of the rectangles R_n, $n \in \mathbb{N}$, then it is thick quasiconvex with quasiconvexity constant 1, and if it contains a corner of one of the rectangles R_n then the ball is thick quasiconvex with quasiconvexity constant 2. Now we will see that each $u \in N^{1,\infty}(X)$ coincides a.e. with a function in $\mathrm{LIP}^\infty(X)$. The set $E = \{x \in X : u(x) > \|u\|_{L^\infty}\}$ has measure zero. If $x, y \in X \setminus E$ with $d(x,y) \geq 1/8$, then $|u(x) - u(y)| \leq 2\|u\|_{L^\infty(X)} \leq 16\|u\|_{L^\infty(X)}\, d(x,y)$.

Fix an upper gradient $g \in L^\infty(X)$ of u. Let (p_j) be an enumeration of the points in X having rational coordinates, and for each j consider the ball $B(p_j, 1/2)$. By Lemma 4.4, for each j there is a set F_j of measure zero such that u is $2\|g\|_{L^\infty(B(p_j,1))}$-Lipschitz on $B(p_j,1/2)\setminus F_j$ and hence is $2\|g\|_{L^\infty(X)}$-Lipschitz continuous on $B(p_j,1/2)\setminus F_j$. The set $F = \cup_{j=1}^\infty F_j \cup E$ is of measure zero. If $x, y \in X \setminus F$ such that $d(x,y) < 1/8$, then there is some j with $x, y \in B(p_j, 1/2)$, and so $|u(x) - u(y)| \leq 2\|g\|_{L^\infty(X)} d(x,y)$. It follows that for all $x, y \in X \setminus F$,

$$|u(x) - u(y)| \leq 2[\|u\|_{L^\infty(X)} + 8\|g\|_{L^\infty(X)}]\, d(x,y).$$

Now the restriction $u_{|X\setminus F}$ can be extended to a Lipschitz function on X (for example, via McShane extension, see e.g. [7, Theorem 6.2]). In this way we obtain the equality $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$. Finally, because X is not quasiconvex, it follows from Theorem 4.7 below that we do not have comparable energy seminorms for this case.

Proposition 4.6. *If X is a thick quasiconvex space, then* $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ *with comparable energy seminorms.*

Proof. Since we have always that given a Lipschitz function u on X, the constant function $\rho(x) = \mathrm{LIP}(u)$ is an upper gradient of u, we have a continuous embedding $\mathrm{LIP}^\infty(X) \subset N^{1,\infty}(X)$. Hence it suffices to check that we have a continuous embedding $N^{1,\infty}(X) \subset \mathrm{LIP}^\infty(X)$. This follows from Lemma 4.4, by exhausting X by balls of large radii and then modifying $f \in N^{1,\infty}(X)$ on the exceptional set of measure zero via McShane extension (see for example [7, Theorem 6.2]). $\square$

We are now ready to state the main result of this paper.

Theorem 4.7. *Suppose that X is a connected complete metric space supporting a doubling Borel measure μ which is non-trivial and finite on balls. Then the following conditions are equivalent:*

(a) *X supports a weak ∞-Poincaré inequality.*

(b) *X is thick quasiconvex.*

(c) $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ *with comparable energy seminorms.*

(d) *X supports a weak ∞-Poincaré inequality for functions in $N^{1,\infty}(X)$.*

The equivalence of Condition (c) with the other three conditions needs the additional assumption of connectedness of X since the example of the union of two disjoint planar discs satisfies (c) but fails the other three conditions. The other three conditions directly imply that X is connected.

The result $a \Rightarrow d$ is immediate, and so the proof of Theorem 4.7 is split in three parts:

- $d \Rightarrow b$: has been proven above as Proposition 4.3.
- $b \Rightarrow c$: has been proven above as Proposition 4.6.
- $c \Rightarrow a$: will be proved in Proposition 4.11 below.

Remark 4.8. We point out here that if X is complete, connected, and equipped with a non-trivial doubling measure, then the following are equivalent:

(i) X is quasiconvex.

(ii) X supports an ∞-Poincaré inequality for locally Lipschitz continuous functions with continuous upper gradients.

(iii) $\mathrm{LIP}^\infty(X) = D^\infty(X)$ with comparable energy seminorms.

Recall that $D^\infty(X)$ is the class of all bounded functions $u : X \longrightarrow \mathbb{R}$ for which the local Lipschitz constant function $\mathrm{Lip}\, u$ is uniformly bounded; see [3]. The norm on $D^\infty(X)$ is given by

$$\|u\|_{D^\infty(X)} := \sup_{x\in X} |u(x)| + \sup_{x\in X} \mathrm{Lip}\, u(x)$$

where

$$\operatorname{Lip} u(x) := \limsup_{\substack{y \to x \\ y \neq x}} \frac{|u(x) - u(y)|}{d(x,y)}.$$

So by $\mathrm{LIP}^\infty(X) = D^\infty(X)$ *with comparable energy seminorms* we mean that the two sets coincide and there is a constant $C > 0$ such that for all $u \in \mathrm{LIP}^\infty(X)$,

$$\mathrm{LIP}(u) \leq C \sup_{x \in X} \operatorname{Lip} u(x).$$

It is well known that $\mathrm{LIP}^\infty(X)$ is a Banach space. In general $D^\infty(X)$ is not a Banach space, as shown in [3]. But $\mathrm{LIP}^\infty(X) \subset D^\infty(X)$ as an isometric embedding, since if u is a Lipschitz function,

$$\operatorname{Lip} u(x) \leq \mathrm{LIP}(u) \text{ for every } x \in X.$$

In the case that Condition (iii) is satisfied $D^\infty(X)$ will also be a Banach space.

The implication of (ii)$\Rightarrow$ (i) is given by the proof of Proposition 3.4. We only need to apply the Poincaré inequality to the locally Lipschitz continuous function $\rho_{x,\varepsilon}$ and its continuous upper gradient 1. The implication (i)$\Rightarrow$ (ii) follows from the argument that if g is a continuous upper gradient of a locally Lipschitz continuous function u, then for $x, y \in X$, by choosing a quasiconvex path γ connecting x to y, we get

$$|u(x) - u(y)| \leq \int_\gamma g\, ds \leq C\, d(x,y) \sup_{z \in B(x, Cd(x,y))} g(z).$$

So if B is a ball in X and x, y are points in B, then

$$\fint_B \fint_B |u(x) - u(y)|\, d\mu(x)\, d\mu(y) \leq C\mathrm{rad}(B) \sup_{z \in CB} g(z) = C\mathrm{rad}(B) \|g\|_{L^\infty(CB)}.$$

The fact that Condition (i) implies Condition (iii) can be found in [3, Lemma 2.3, Corollary 2.4].

Now suppose that Condition (iii) holds. Then as in the proof of Proposition 3.4, for each $x \in X$ and $\varepsilon > 0$ we consider the function $\rho_{x,\varepsilon}$, and since X is connected we see that $\rho_{x,\varepsilon}$ is finite-valued everywhere and $|\rho_{x,\varepsilon}(z) - \rho_{x,\varepsilon}(w)| \leq d(z,w)$ when $d(z,w) < \varepsilon$; thus for all $w \in X$ we have $\operatorname{Lip} \rho_{x,\varepsilon}(w) \leq 1$. Hence $\rho_{x,\varepsilon}$ belongs to $D^\infty(X)$. Because (iii) holds, there is a constant $C > 0$ such that $\mathrm{LIP}(\rho_{x,\varepsilon}) \leq C$ with C independent of x, ε. It follows that for all $y \in X$ and all $\varepsilon > 0$,

$$|\rho_{x,\varepsilon}(y)| = |\rho_{x,\varepsilon}(y) - \rho_{x,\varepsilon}(x)| \leq \mathrm{LIP}(\rho_{x,\varepsilon}) d(x,y) \leq C d(x,y).$$

Therefore as in the proof of Proposition 3.4, there is a curve γ connecting x to y with length $\ell(\gamma) \leq C\, d(x,y)$, that is, X is quasiconvex. These two arguments prove that Conditions (i) and (iii) are equivalent.

Now we continue on to prove Theorem 4.7 as outlined before Remark 4.8.

The following two technical lemmas will be useful in the sequel.

Lemma 4.9. *Suppose $N^{1,\infty}(X) = \mathrm{LIP}^\infty(X)$ with comparable energy seminorms. Then there exists a constant $C \geq 1$ such that for every $E \subset X$ with $\mu(E) = 0$ and for every $x \in X$ and $r > 0$ there is a set $F \subset X$ with $\mu(F) = 0$ so that whenever $y \in X \setminus (B(x, 2r) \cup F)$, there is a rectifiable curve γ_y connecting y to $\overline{B}(x, r)$ such that $\ell(\gamma_y) \leq C\, d(x, y)$ and $\mathscr{L}^1(\gamma_y^{-1}(\gamma_y \cap E)) = 0$.*

Proof. Let $E \subset X$ such that $\mu(E) = 0$; since μ is a Borel measure, we may assume (by enlarging E if necessary) that E is a Borel set. Then $\rho = \infty \cdot \chi_E \in L^\infty(X)$ is a non-negative Borel measurable function. Let Γ_E^+ be the collection of all rectifiable curves γ for which $\mathscr{L}^1(\gamma^{-1}((\gamma \cap E))) > 0$. Then clearly for such curves γ we have $\int_\gamma \rho\, ds = \infty$, and so $\mathrm{Mod}_\infty(\Gamma_E^+) = 0$. As before, we define for $r > 0$,

$$u(z) = \inf_{\gamma \text{ connects } z \text{ to } B(x,r)} \int_\gamma (1 + \rho)\, ds,$$

where $\|1 + \rho\|_{L^\infty(X)} = 1$. For positive integers k we set $u_k = \min\{k, u\}$. Then $u_k \in N^{1,\infty}(X)$ with $1 + \rho$ as an upper gradient (see Lemma 3.5), and $u = 0$ on $B(x, r)$. Let F_k be the exceptional set on which u_k has to be modified in order to be Lipschitz continuous; we have $\mu(F_k) = 0$. Observe that since $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ with comparable energy seminorms,

$$\mathrm{LIP}(u_k) \leq C \inf_g \|g\|_{L^\infty} \leq C\|1 + \rho\|_{L^\infty(X)} = C,$$

where the infimum is taken over all ∞-weak upper gradients g of u_k.

Let $F = \cup_{k \in \mathbb{N}} F_k$. Thus for $y \in X \setminus (F \cup B(x, 2r))$, there exists a positive integer k such that $d(x, y) < k/2C$. In addition,

$$|u_k(y)| = |u_k(y) - u_k(x_1)| \leq C\, d(x_1, y) \leq C(d(x_1, x) + d(x, y)) \leq 2Cd(x, y),$$

for any $x_1 \in B(x, r) \setminus F_k$ and $u_k(y) = \tilde{u}(y)$ is finite. Thus, there exists a rectifiable curve γ_y such that

$$\ell(\gamma_y) + \int_{\gamma_y} \rho\, ds = \int_{\gamma_y} (1 + \rho)\, ds \leq Cd(x, y).$$

Hence, we have

$$\ell(\gamma_y) \leq C\, d(x, y) \qquad \text{and} \qquad \int_{\gamma_y} \rho < +\infty,$$

and so $\mathscr{L}^1(\gamma_y^{-1}(\gamma_y \cap E)) = 0$, as we wanted. □

Lemma 4.10. *Let $u \in N^{1,\infty}(X)$ and $g \in L^\infty(X)$ be an upper gradient of u. If v is a Lipschitz continuous function on X such that $u = v$ μ-a.e., then g is an ∞-weak upper gradient of v and so there is a Borel measurable function $0 \le \rho \in L^\infty(X)$ with $\rho = g$ μ-a.e. such that ρ is an upper gradient of v.*

Proof. Let $E = \{x \in X : u(x) \neq v(x)\}$; then $\mu(E) = 0$, and so $\mathrm{Mod}_\infty(\Gamma_E^+) = 0$. If $x, y \in X \setminus E$ and β a rectifiable curve connecting x to y in X, then

$$|u(x) - u(y)| = |v(x) - v(y)| \le \int_\beta g\, ds.$$

Let γ be a non-constant rectifiable compact curve with end points x and y, such that $\gamma \notin \Gamma_E^+$. Then we can find two sequences of points $\{z_i\}$ and $\{w_i\}$ from the trajectory of γ such that for i we have $z_i, w_i \in \gamma \setminus E$ and $z_i \to x$, $w_i \to y$ as $i \to \infty$. Letting γ_i be a subcurve of γ with end points z_i and w_i; then by the above discussion,

$$|v(z_i) - v(w_i)| \le \int_{\gamma_i} g\, ds \le \int_\gamma g\, ds.$$

Since v is Lipschitz continuous, by letting $i \to \infty$ in the above, we get

$$|v(x) - v(y)| \le \int_\gamma g\, ds.$$

It follows that g is an ∞-weak upper gradient of v. Since $\mathrm{Mod}_\infty(\Gamma_E^+) = 0$, by [3, Lemma 5.7], there is a non-negative Borel measurable function ρ_0 such that $\|\rho_0\|_{L^\infty(X)} = 0$ but for all $\gamma \in \Gamma_E^+$ the integral $\int_\gamma \rho_0\, ds = \infty$. It follows that $\rho = g + \rho_0$ is an upper gradient of v with the desired property. ☐

Proposition 4.11. *Suppose that X is connected and $N^{1,\infty}(X) = \mathrm{LIP}^\infty(X)$ with comparable energy seminorms. Then X supports a weak ∞-Poincaré inequality.*

Proof. Let $u \in N^{1,\infty}(X)$, $g \in L^\infty(X)$ be an upper gradient of u, and fix a ball $B \subset X$. By the assumption that $N^{1,\infty}(X) = \mathrm{LIP}^\infty(X)$ and by Lemma 4.10, we may assume that u is itself Lipschitz continuous on X. Let $E = \{w \in 2CB : g(w) > \|g\|_{L^\infty(2CB)}\}$, where C is the constant from Lemma 4.9. Then $\mu(E) = 0$. Fix $\varepsilon > 0$.

Observe that since μ is doubling and X is connected, we deduce that $\mu(\{x\}) = 0$ for all $x \in X$ (see condition (2)). So for $x \in B$, we can choose $r > 0$ sufficiently small so that

1. $B(x, 2r) \subset B$,
2. $\mu(B(x, 2r)) < \mu(B)/2$,

3. for all $w \in \overline{B}(x,r)$ we have $|u(w) - u(x)| < \varepsilon$ (possible because u is Lipschitz continuous),

4. $\fint_{\overline{B}(x,2r)} |u - u(x)|\, d\mu \leq \frac{1}{2} \fint_B |u - u(x)|\, d\mu$.

Then,

$$\fint_B |u - u(x)|\, d\mu \leq \frac{2}{\mu(B)} \int_{B \setminus B(x,2r)} |u - u(x)|\, d\mu \leq 2 \fint_{B \setminus B(x,2r)} |u(y) - u(x)|\, d\mu(y).$$

Let $F \subset X$ be the set given by Lemma 4.9 with respect to x and r, and for $y \in B \setminus (F \cup B(x,2r))$ let γ_y be the corresponding curve connecting y to $B(x,r)$. We denote the other end point of γ_y as $w_y \in \overline{B}(x,r)$. By the choice of r, we see that $|u(y) - u(x)| \leq |u(y) - u(w_y)| + |u(w_y) - u(x)| < |u(y) - u(w_y)| + \varepsilon$. It follows that $|u(y) - u(x)| \leq \varepsilon + \int_{\gamma_y} g\, ds \leq \varepsilon + C\|g\|_{L^\infty(2CB)} d(x,y)$, where we used the fact that $\mathscr{L}^1(\gamma_y^{-1}(\gamma_y \cap E)) = 0$. Therefore,

$$\begin{aligned}
\fint_B |u - u(x)|\, d\mu &\leq 2 \fint_{B \setminus (F \cup B(x,2r))} (\varepsilon + C\|g\|_{L^\infty(2CB)} d(x,y)) d\mu(y) \\
&\leq 4 \fint_{B \setminus (F \cup B(x,2r))} (\varepsilon + C\|g\|_{L^\infty(2CB)} \mathrm{rad}(B)) d\mu(y) \\
&= 4(\varepsilon + C\|g\|_{L^\infty(2CB)} \mathrm{rad}(B)).
\end{aligned}$$

Now integrating over x, we obtain

$$\fint_B \fint_B |u(y) - u(x)|\, d\mu(y)\, d\mu(x) \leq 4(\varepsilon + C\|g\|_{L^\infty(2CB)} \mathrm{rad}(B)).$$

Letting $\varepsilon \to 0$ we get the inequality

$$\fint_B \fint_B |u(y) - u(x)|\, d\mu(y)\, d\mu(x) \leq 2C \mathrm{rad}(B) \|g\|_{L^\infty(2CB)},$$

which in turn implies, by remark 3.2, the weak ∞-Poincaré inequality for the pair (u,g). Since the constants are independent of u, g, B, we have that (X,d,μ) supports a weak ∞-Poincaré inequality for Newtonian functions. It follows from Proposition 4.3 that X is thick quasiconvex.

To complete the proof, we have to check that (X,d,μ) admits a weak ∞-Poincaré inequality for every Borel measurable function $u : X \to \mathbb{R}$ and every upper gradient. Let u be a measurable function and let g be a measurable upper gradient for u. Fix B. If $\|g\|_{L^\infty(2CB)} = \infty$ we are done, so let us assume that $\|g\|_{L^\infty(2CB)} < \infty$. Since by above we have X is thick quasiconvex, we can invoke Lemma 4.4 to see that u is Lipschitz in $B \subset X$ up to a set of measure zero. By Lemma 4.10, we can assume that u is Lipschitz in all of B and that g is an upper gradient of u in B. Thus we can repeat the proof above for the pair u and g, and the proof is now complete. □

Example 4.12. The space (X, d, μ) considered in Example 3.3 with a measure that decays very fast to zero at the origin (the point where the two triangular regions are glued) is thick quasiconvex. We can prove it by the aid of theorem 4.7 despite the fact that μ is not doubling. Indeed, since $(X, d, \mathscr{L}^2)$ supports a p-PI for $p > 2$ ([18, 4.3.1.]), it also supports an ∞-PI. By theorem 4.7, it is also thick quasiconvex (observe that we can apply it since $\mathscr{L}^2$ is a doubling measure). Using the idea in Remark 2.3, we conclude that (X, d, μ) is also thick quasiconvex.

The rest of this section will be devoted to show that in Theorem 4.7 the thick quasiconvexity cannot be replaced with the weaker notion of quasiconvexity.

The next lemma is useful in verifying whether a metric space does not support any Poincaré inequality. Its proof is an adaptation of [2, Lemma 4.3] for the case $p = \infty$.

Lemma 4.13. *Let (X, d, μ) be a bounded doubling metric measure space admitting a weak ∞-Poincaré inequality, and let $f : X \longrightarrow I$ be a surjective Lipschitz function from X onto an interval $I \subset \mathbb{R}$. Then, $\mathscr{L}^1_{|I} \ll f_{\#}\mu$. Here $f_{\#}\mu$ denotes the pushforward measure of μ under f.*

Proof. Let us denote $L = \mathrm{LIP}(f)$. Suppose the contrary. Then, there exists a Borel set N in I such that $\mathscr{L}^1(N) > 0$ and $\mu(f^{-1}(N)) = f_{\#}\mu(N) = 0$. On X we consider the function

$$u(x) = \int_0^{f(x)} \chi_N(t) d\mathscr{L}^1(t).$$

This function is L-Lipschitz, because for $x, y \in X$ we have

$$|u(y) - u(x)| = \Big| \int_{f(x)}^{f(y)} \chi_N \, d\mathscr{L}^1 \Big| = \mathscr{L}^1([f(x), f(y)] \cap N) \leq |f(y) - f(x)| \leq L\, d(y, x).$$

Moreover, $g = L\ (\chi_N \circ f)$ is an upper gradient of u. Indeed, for each rectifiable curve $\gamma : [a, b] \longrightarrow X$ one has (without loss of generality we assume that $f(\gamma(a)) < f(\gamma(b))$)

$$|u(\gamma(a)) - u(\gamma(b))| = \Big| \int_{f(\gamma(a))}^{f(\gamma(b))} \chi_N(t) d\mathscr{L}^1(t) \Big| = \mathscr{L}^1([f(\gamma(a)), f(\gamma(b))] \cap N),$$

and

$$\int_\gamma g = \int_a^b L \cdot (\chi_N \circ f(\gamma(t))) d\mathscr{L}^1(t) = L\, \mathscr{L}^1([a, b] \cap (f \circ \gamma)^{-1}(N)).$$

Because γ is arclength-parametrized, $f \circ \gamma$ is L-Lipschitz. It follows that

$$\mathscr{L}^1([a,b] \cap (f \circ \gamma)^{-1}(N)) \geq L^{-1} \mathscr{L}^1([f(\gamma(a)), f(\gamma(b))] \cap N),$$

and hence,

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g d\mathscr{L}^1(t)$$

for each rectifiable curve γ in X. However, $\mu\{x \in X : f(x) \in N\} = f_\# \mu(N) = 0$ by hypothesis, and so $\chi_N \circ f(x) = 0$ μ-a.e. Therefore by the weak ∞-Poincaré inequality, $\int_X |u - u_X| \, d\mu = 0$, which means that u is constant μ-almost everywhere on X. Because u is Lipschitz continuous on X, it follows that u is constant on X, which contradicts the fact that u is non-constant on the set $f^{-1}(N)$ (this set is non-empty because f is surjective, and u is not constant here because $\mathscr{L}^1(N) > 0$). □

Example 4.14. Let $Q = [0,1] \times [0,1] \subset \mathbb{R}^2$ be the unit square. Divide Q in nine equal squares of side length 1/3 and remove the central one. In this way, we obtain a set Q_1, which is the union of 8 squares of side length 1/3. Repeating this procedure on each square we get a sequence of sets Q_j consisting of 8^j squares of side length $1/3^j$. We define the *Sierpinski carpet* to be $S = \bigcap Q_j$. If d is the distance in $\mathbb{R}^2$ given by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|,$$

then (S, d) is a complete geodesic metric space. Let μ be the Hausdorff measure on (S, d) of dimension s, where s is given by the formula, $3^s = 8$. It can be checked that μ is a doubling measure and that the metric d defined above is biLipschitz equivalent to the restriction of the Euclidean metric.

The Sierpinski carpet (S, d, μ) is clearly quasiconvex, and so the following corollary demonstrates that the quasiconvexity property is not sufficient to guarantee ∞-Poincaré inequality.

Corollary 4.15. *The Sierpinski carpet (S, d, μ) does not admit an ∞-Poincaré inequality.*

Proof. Let f be the projection on the horizontal axis. It can be checked that $f_\# \mu \perp \mathscr{L}^1$ (see [2, 4.5]). Indeed, as shown in [2], given a point $0 < x < 1$, by the way of ternary expansion of x we can see that the interval I_n centered at x of radius 3^{-n} has Lebesgue measure $\mathscr{L}^1(I_n) \approx 3^{-n}$, but $f_\# \mu(I_n) \approx \exp(-\psi(x, n))$ for appropriately chosen function ψ, with the property that

$$\lim_{n \to \infty} \frac{f_\# \mu(I_n)}{\mathscr{L}^1(I_n)} \approx \limsup_{n \to \infty} \frac{\exp(-\psi(x, n))}{3^{-n}}$$

which is for $\mathscr{L}^1$-a.e. x either 0 or ∞; which, in conjunction with the Radon-Nikodym theorem implies that $f_\# \mu$ is singular with respect to the Lebesgue measure $\mathscr{L}^1$.

The result now follows from Lemma 4.13. □

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