Sobolev extensions of Hölder continuous and characteristic functions on metric spaces

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Abstract. We study when characteristic and Hölder continuous functions are traces of Sobolev functions on doubling metric measure spaces. We provide analytic and geometric conditions sufficient for extending characteristic and Hölder continuous functions into globally defined Sobolev functions.

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1. Introduction

For Lipschitz domains in $\mathbf{R}^n$, the trace space of the Sobolev space $W^{1,p}(\Omega)$ is the Slobodeckiĭ space $W^{1-1/p,p}(\partial\Omega, H_{n-1})$, where H_{n-1} is the $(n-1)$-dimensional Hausdorff measure, see e.g. Gagliardo [9], Kufner–John–Fučík [24], Theorems 6.9.2 and 8.3.13, or Jonsson–Wallin [17], p. 212. Similar results for Sobolev spaces on Carnot–Carathéodory spaces have been obtained in Danielli–Garofalo–Nhieu [8]. A "nearly sharp" description of traces on fractal subsets of $\mathbf{R}^n$ has been given in Hajłasz–Martio [12].

In the last decade, there has been a lot of development in the theory of Sobolev spaces and p-harmonic functions on metric spaces, see e.g. Hajłasz [10], Cheeger [6] and Shanmugalingam [27]. In particular, the Dirichlet problem for p-harmonic functions has been solved for Sobolev-type boundary data in Shanmugalingam [28] and Kinnunen–Martio [22]. The fact that Sobolev spaces are natural spaces for solving the Dirichlet problem for p-harmonic functions is one of the motivations for studying traces of Sobolev functions. Unlike in Euclidean and Carnot–Carathéodory spaces, very little is known about traces of Sobolev functions in general metric spaces.

In this paper, we study the question of when characteristic and Hölder continuous functions are traces of Sobolev functions on metric measure spaces. We provide analytic and geometric conditions sufficient for extending characteristic and Hölder continuous functions into globally defined Sobolev functions. In particular, if X is a doubling metric measure space, then we prove the following:

- Let $F \subset X$ be closed and linearly locally connected, $E \subset F$ and the upper Minkowski dimension of the relative boundary $\partial_F E$ be strictly less then $Q-p$, where Q is the "dimension" of X. Then the characteristic function χ_E is a restriction to F of a globally defined Sobolev function, see Theorem 4.9.

- If $\Omega \subset X$ is open and satisfies the β-shell condition with $\beta > p(1-\kappa)$, then every κ-Hölder continuous function on $\partial\Omega$ is a trace on $\partial\Omega$ of a κ-Hölder continuous Sobolev function on Ω, see Theorem 5.4.

This makes it possible to solve the Dirichlet problem for such boundary data, which in turn has applications for p-harmonic measures and boundary regularity of p-harmonic functions.

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2. Notation and preliminaries

We assume throughout the paper that $X = (X, d, \mu)$ is a metric space endowed with a metric d and a doubling measure μ, i.e. there exists a constant $C > 0$ such that for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in X (we make the convention that balls are nonempty and open),

$$0 < \mu(2B) \le C\mu(B) < \infty,$$

where $\lambda B = B(x_0, \lambda r)$. We emphasize that the σ-algebra on which μ is defined is obtained by the completion of the Borel σ-algebra. We also assume that $1 < p < \infty$.

Note that we do not assume that X is complete, unless explicitly required. Since μ is doubling, X is proper (i.e. closed bounded sets are compact) if and only if it is complete.

A *curve* is a continuous mapping from an interval. We will only consider curves which are nonconstant, compact and rectifiable. A curve can thus be parameterized by its arc length ds.

Definition 2.1. A nonnegative Borel function g on X is an *upper gradient* of an extended real-valued function f on X if for all curves $\gamma : [0, l_\gamma] \to X$,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \le \int_\gamma g\,ds \tag{2.1}$$

whenever both $f(\gamma(0))$ and $f(\gamma(l_\gamma))$ are finite, and $\int_\gamma g\,ds = \infty$ otherwise. If g is a nonnegative measurable function on X and if (2.1) holds for p-almost every curve, then g is a *p-weak upper gradient* of f.

By saying that (2.1) holds for p-almost every curve we mean that it fails only for a curve family with zero p-modulus, see Definition 2.1 in Shanmugalingam [27]. It is implicitly assumed that $\int_\gamma g\,ds$ is defined (with a value in $[0, \infty]$) for p-almost every curve.

If $g \in L^p(X)$ is a p-weak upper gradient of f, then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of f such that $g_j \to g$ in $L^p(X)$, see Lemma 2.4 in Koskela–MacManus [23].

Following Shanmugalingam [27], we define a version of Sobolev spaces on the metric space X.

Definition 2.2. Whenever $u \in L^p(X)$, let

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

The space $N^{1,p}(X)$ is a Banach space and a lattice, see Shanmugalingam [27].

Definition 2.3. The *p-capacity* of a set $E \subset X$ is the number

$$C_p(E) = \inf \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E.

The capacity is countably subadditive. For this and other properties as well as equivalent definitions of the capacity we refer to Kilpeläinen–Kinnunen–Martio [19] and Kinnunen–Martio [20], [21].

The capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$ and v is a function on X, then $u \sim v$ if and only if $u = v$ outside a set of capacity zero. Moreover, Corollary 3.3 in Shanmugalingam [27] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ μ-a.e., then $u \sim v$. In particular, in the Euclidean setting, $N^{1,p}(\mathbf{R}^n)$ is the refined Sobolev space of quasicontinuous functions as defined on p. 96 of Heinonen–Kilpeläinen–Martio [14].

Definition 2.4. We say that X supports a *weak $(1,q)$-Poincaré inequality* if there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B \subset X$, all measurable functions f on X and all upper gradients g of f,

$$\fint_B |f - f_B| \, d\mu \leq C \operatorname{diam}(B) \left(\fint_{\lambda B} g^q \, d\mu \right)^{1/q}, \tag{2.2}$$

where $f_B := \fint_B f \, d\mu := \mu(B)^{-1} \int_B f \, d\mu$.

In the definition of Poincaré inequality we can equivalently assume that g is a q-weak upper gradient—see the comments above.

For some of the results in this paper we need to assume that X supports a $(1,p)$-Poincaré inequality; this will always be mentioned explicitly when needed.

Unless otherwise stated, C denotes a positive constant whose value is unimportant, may vary with each appearance, and depends only on the fixed parameters.

3. The geometry of Whitney coverings

In this section we construct Whitney coverings of bounded open sets in X and prove some of their properties. Whitney coverings in metric spaces were first constructed by Coifman–Weiss [7]. We have chosen to provide complete proofs in order to get the constants suitable for our purpose.

From now on, F always denotes a nonempty closed subset of X, $R>0$ is a fixed constant, and $V=\{x\in X: 0<\operatorname{dist}(x,F)\le 16R\}$. We also let $\mathcal{W}=\{B_{i,j}\}$ be a corresponding Whitney covering given by the following theorem.

Theorem 3.1. *There exists a countable family of balls*

$$\mathcal{W}=\{B_{i,j}=B(x_{i,j},r_i): i\in\mathbf{N}, j\in J_i\}$$

such that for all $i\in\mathbf{N}$ and $j\in J_i$,

(i) $V\subset\bigcup_{B\in\mathcal{W}}B\subset X\setminus F$,
(ii) $r_i=2^{-i}R$,
(iii) $8r_i<\operatorname{dist}(x_{i,j},F)\le 16r_i$,
(iv) *the balls* $\bigl\{\frac{1}{2}B: B\in\mathcal{W}\bigr\}$ *are pairwise disjoint.*

Remark 3.2. Note that if V is bounded then all index sets J_i are finite.

Proof. Let

$$S_0=\{x\in V:\operatorname{dist}(x,F)>8r_0\}.$$

Using the Hausdorff maximality principle we find a maximal pairwise disjoint collection of balls $B\bigl(x_{0,j},\frac{1}{2}r_0\bigr)$ with $x_{0,j}\in S_0$, $j\in J_0$. The doubling property of the measure μ implies that this collection is at most countable, with the collection finite if V is bounded. Let $B_{0,j}=B(x_{0,j},r_0)$ and $\mathcal{W}_0=\{B_{0,j}: j\in J_0\}$. Then $S_0\subset\bigcup_{B\in\mathcal{W}_0}B$.

Next, we inductively construct families $\mathcal{W}_i=\{B_{i,j}: j\in J_i\}$, $i=1,2,\dots$, and set $\mathcal{W}=\bigcup_{i=0}^\infty\mathcal{W}_i$. Assume that the families $\mathcal{W}_0,\ \dots,\ \mathcal{W}_{i-1}$ have already been constructed and let

$$S_i=\{x\in V: 8r_i<\operatorname{dist}(x,F)\le 16r_i\}\setminus\bigcup_{B\in\mathcal{W}_{i-1}}B.$$

Again, there exists a maximal, at most countable, pairwise disjoint collection of balls $B\bigl(x_{i,j},\frac{1}{2}r_i\bigr)$ with $x_{i,j}\in S_i$ when $j\in J_i$. Let $B_{i,j}=B(x_{i,j},r_i)$ and $\mathcal{W}_i=\{B_{i,j}: j\in J_i\}$. Then $S_i\subset\bigcup_{B\in\mathcal{W}_i}B$.

It follows from the construction that the family $\mathcal{W}=\bigcup_{i=0}^\infty\mathcal{W}_i$ satisfies (i)–(iii) in the statement of the theorem. To complete the proof it remains to verify the property (iv). By construction, the balls $\frac{1}{2}B_{i,j}$ and $\frac{1}{2}B_{k,l}$ are disjoint if $i=k$. If $i>k$, then $\frac{1}{2}r_i+\frac{1}{2}r_k<r_k\le d(x_{i,j},x_{k,l})$, as $x_{i,j}\notin B_{k,l}$ by the construction of S_i. $\square$

Remark 3.3. Let Ω be a bounded open set in X with nonempty boundary. By choosing $R=\frac{1}{16}\operatorname{diam}\Omega$ and $F=\partial\Omega$ we obtain a Whitney covering of Ω.

The following two lemmas give us simple estimates on the overlap of the Whitney balls.

Lemma 3.4. *Let $\lambda>0$. Then there is a constant M, depending only on λ and the doubling constant of μ, such that for each $B_{i,j}\in\mathcal{W}$,*

$$\#\{l:\lambda B_{i,j}\cap\lambda B_{i,l}\ne\varnothing\}\le M.$$

Proof. Without loss of generality we may assume that $\lambda\ge 1$. If $\lambda B_{i,j}\cap\lambda B_{i,l}\ne\varnothing$, then $\frac{1}{2}B_{i,l}\subset 3\lambda B_{i,j}$. The pairwise disjointness of the balls $\bigl\{\frac{1}{2}B: B\in\mathcal{W}\bigr\}$ and the doubling property of μ then give a bound on the number of such balls $B_{i,l}$. $\square$

Lemma 3.5. *Let $0<\lambda<8$. Then there exists a constant $M>0$ depending only on λ and the doubling constant of μ such that we have the following.*

(i) *If* $\lambda B_{i,j} \cap \lambda B_{k,l} \neq \varnothing$, *then* $r_i < (16+\lambda) r_k/(8-\lambda)$ *and hence*

$$i - \log_2\left(\frac{16+\lambda}{8-\lambda}\right) < k < i + \log_2\left(\frac{16+\lambda}{8-\lambda}\right).$$

(ii) *If* $B_{i,j} \in \mathcal{W}$, *then*

$$\#\{B \in \mathcal{W} : \lambda B_{i,j} \cap \lambda B \neq \varnothing\} \leq M,$$

and hence for each $x \in X$,

$$\sum_{B \in \mathcal{W}} \chi_{\lambda B}(x) \leq M.$$

Proof. (i) We have

$$16 r_k \geq \operatorname{dist}(x_{k,l}, F) \geq \operatorname{dist}(x_{i,j}, F) - d(x_{k,l}, x_{i,j}) > 8 r_i - \lambda(r_k + r_i),$$

from which the result follows.

(ii) For each of the balls $B_{k,l}$ such that $\lambda B_{i,j} \cap \lambda B_{k,l} \neq \varnothing$ we have $r_i/C < r_k < C r_i$, by (i). Hence $d(x_{k,l}, x_{i,j}) < \lambda(r_i + r_k) \leq C r_i$, i.e. $\frac{1}{2} B_{k,l} \subset C' B_{i,j}$. By the doubling property, $\mu\big(\frac{1}{2} B_{k,l}\big) \geq \mu(C' B_{i,j})/C$. Thus the pairwise disjointness of the balls $\big\{\frac{1}{2} B : B \in \mathcal{W}\big\}$ gives the desired bound. □

Theorem 3.6. *There exists a partition of unity* $\Phi = \{\varphi_{i,j} : i \in \mathbf{N}, j \in J_i\}$ *subordinate to the collection* $\{2B : B \in \mathcal{W}\}$ *and satisfying for all* $i \in \mathbf{N}$ *and all* $j \in J_i$,

(i) $\sum_{i,j} \varphi_{i,j} = 1$ *on* $\bigcup_{B \in \mathcal{W}} B$;
(ii) $0 \leq \sum_{i,j} \varphi_{i,j} \leq 1$ *on* X;
(iii) $\operatorname{supp} \varphi_{i,j} \subset 2B_{i,j}$;
(iv) $\varphi_{i,j}$ *is a nonnegative* C/r_i*-Lipschitz function.*

Proof. Let $\widehat{\varphi}_{i,j}$ be a nonnegative $2/r_i$-Lipschitz function with $\operatorname{supp} \widehat{\varphi}_{i,j} \subset 2B_{i,j}$ such that $\widehat{\varphi}_{i,j} \leq 1$ with $\widehat{\varphi}_{i,j} \equiv 1$ on $B_{i,j}$, e.g., let $\widehat{\varphi}_{i,j}(y) = \max\{0, \min\{3 - 2d(y, x_{i,j})/r_i, 1\}\}$. Set $h := \sum_{i,j} \widehat{\varphi}_{i,j}$. By Lemma 3.5 it follows that $1 \leq h \leq M$ on $\bigcup_{B \in \mathcal{W}} B$, and that h is CM/r_i-Lipschitz in $2B_{i,j}$. Thus also $1/\max\{1, h\}$ is CM/r_i-Lipschitz in $2B_{i,j}$, and it follows that $\varphi_{i,j} := \widehat{\varphi}_{i,j}/\max\{1, h\}$ satisfies the requirements. □

If $F \subset X$ is bounded, then the number of balls in each generation $\mathcal{W}_i$ of the Whitney covering $\mathcal{W}$ is related to the upper Minkowski dimension of F. Thus, the Whitney covering plays a crucial role in the geometry of subsets of X.

Definition 3.7. Let $A \subset X$ be bounded and let $P(A, \varepsilon)$ be the maximal number of pairwise disjoint balls with centers in A and radius ε. The *upper Minkowski dimension* of A is the nonnegative number

$$\dim_{\mathrm{M}} A := \inf\Big\{ s > 0 : \limsup_{\varepsilon \to 0+} P(A, \varepsilon) \varepsilon^s = 0 \Big\}. \tag{3.1}$$

Note that $P(\overline{A}, \varepsilon) \leq P(A, \varepsilon/2)$ and hence $\dim_{\mathrm{M}} \overline{A} = \dim_{\mathrm{M}} A$. For more on Minkowski dimensions see Sections 5.3–5.12 in Mattila [26].

Lemma 3.8. *Let* $A \subset X$ *be bounded and* $\alpha, \beta > 0$. *Then for each* $i \in \mathbf{N}$,

$$\#\{j : \alpha B_{i,j} \cap A \neq \varnothing\} \leq M P(A, \beta r_i),$$

where M *depends only on* α, β *and the doubling constant of* μ.

Proof. Let $\{B_n\}_{n=1}^{P(A,\beta r_i)}$ be a maximal pairwise disjoint collection of balls with centers in A and radii βr_i. Then $A \subset \bigcup_{n=1}^{P(A,\beta r_i)} 2B_n$, i.e. each ball $\alpha B_{i,j}$ intersecting A intersects at least one ball $2B_n$.

On the other hand, if both $\alpha B_{i,j}$ and $\alpha B_{i,l}$ intersect the same ball $2B_n$, then

$$(\alpha + 2\beta)B_{i,j} \cap (\alpha + 2\beta)B_{i,l} \neq \varnothing,$$

and Lemma 3.4 gives a bound on the number of balls $\alpha B_{i,j}$ intersecting each $2B_n$. $\square$

To prove the extension theorems of this note we shall need a projection map $\pi : X \to F$ such that $d(x, \pi(x)) \leq 2 \operatorname{dist}(x, F)$. In the case when X is complete we could have replaced the constant 2 by 1, but in the noncomplete case there would not always exist such a projection. Note also that there is no uniqueness for π (not even in the case when X is complete and we require that $d(x, \pi(x)) = \operatorname{dist}(x, F)$.)

Lemma 3.9. *Let* $\mathcal{C} = \{x_{i,j} : B_{i,j} \in \mathcal{W}\}$. *Then* $\pi|_{\mathcal{C}}$ *is a* 65*-Lipschitz map. Moreover, if* $0 < \alpha < 8$, $x \in \alpha B_{i,j}$ *and* $y \in F$, *then*

$$\begin{aligned} d(\pi(x_{i,j}), y) &\leq 3d(x_{i,j}, y), \\ d(\pi(x), x_{i,j}) &\leq (32 + 3\alpha) r_i. \end{aligned}$$

Proof. This is an easy exercise in triangle inequalities. If $x_{i,j}, x_{k,l} \in \mathcal{C}$, then by the fact that $\frac{1}{2}B_{i,j} \cap \frac{1}{2}B_{l,k}$ is empty, we have

$$d(x_{i,j}, x_{k,l}) \geq \tfrac{1}{2}(r_i + r_l),$$

and hence

$$\begin{aligned} d(\pi(x_{i,j}), \pi(x_{k,l})) &\leq d(\pi(x_{i,j}), x_{i,j}) + d(x_{i,j}, x_{k,l}) + d(x_{k,l}, \pi(x_{k,l})) \\ &\leq 32r_i + d(x_{i,j}, x_{k,l}) + 32r_l \leq 65\, d(x_{i,j}, x_{k,l}), \end{aligned}$$

i.e. $\pi|_{\mathcal{C}}$ is a 65-Lipschitz map. Next,

$$d(\pi(x_{i,j}), y) \leq d(\pi(x_{i,j}), x_{i,j}) + d(x_{i,j}, y) \leq 2 \operatorname{dist}(x_{i,j}, F) + d(x_{i,j}, y) \leq 3d(x_{i,j}, y).$$

Finally,

$$\begin{aligned} d(\pi(x), x_{i,j}) &\leq d(\pi(x), x) + d(x, x_{i,j}) \leq 2 \operatorname{dist}(x, F) + d(x, x_{i,j}) \\ &\leq 2 \operatorname{dist}(x_{i,j}, F) + 3d(x, x_{i,j}) \leq 32r_i + 3\alpha r_i. \quad \square \end{aligned}$$

4. Extending characteristic functions

In this section we give sufficient conditions for extending characteristic functions of bounded sets to obtain Newtonian functions on X. As a corollary, we generalize some results about p-harmonic measures from Björn–Björn–Shanmugalingam [4] to metric spaces.

Let $\{\varphi_{i,j}\}$ be the partition of unity associated with the Whitney covering $\mathcal{W}$ given by Theorem 3.6, and fix a projection map $\pi : X \to F$ such that $d(x, \pi(x)) \leq 2 \operatorname{dist}(x, F)$.

Given a function $f : F \to \mathbf{R}$, we define $\mathcal{E}f : X \to \mathbf{R}$ by

$$\mathcal{E}f(x) = \begin{cases} \sum_{i,j} f(\pi(x_{i,j}))\, \varphi_{i,j}(x) & \text{if } x \notin F, \\ f(x) & \text{if } x \in F. \end{cases}$$

The following lemma shows that the extension $\mathcal{E}f$ behaves well in a neighbourhood of F. Here $\partial_F E = \overline{E} \cap \overline{F \setminus E}$ is the boundary of E in the relative topology of F.

Lemma 4.1. *Let $Y := F \cup \bigcup_{B\in\mathcal{W}} B$. For $x_0 \in F$ we have*

$$\sup_{x\in Y\cap B(x_0,\delta)} |\mathcal{E}f(x) - f(x_0)| \le \sup_{y\in F\cap B(x_0,4\delta)} |f(y) - f(x_0)|.$$

In particular, if $E \subset F$ and $x_0 \in F \setminus \partial_F E$, then $\mathcal{E}\chi_E$ is constant in a neighbourhood of x_0.

Proof. Let $x \in Y \cap B(x_0,\delta)$ be arbitrary. If $x \in F$, then

$$|\mathcal{E}f(x) - f(x_0)| = |f(x) - f(x_0)| \le \sup_{y\in F\cap B(x_0,4\delta)} |f(y) - f(x_0)|.$$

Assume therefore that $x \notin F$. As $x \in \bigcup_{B\in\mathcal{W}} B$ we get that $\sum_{i,j} \varphi_{i,j}(x) = 1$. Then,

$$|\mathcal{E}f(x) - f(x_0)| \le \sum_{i,j} |f(\pi(x_{i,j})) - f(x_0)|\varphi_{i,j}(x), \tag{4.1}$$

where it suffices to sum only over those i and j for which $x \in 2B_{i,j}$. For such i and j, we have

$$8r_i < \operatorname{dist}(x_{i,j}, F) \le d(x_{i,j}, x_0) \le d(x_{i,j}, x) + d(x, x_0) < 2r_i + \delta,$$

i.e. $r_i < \frac{1}{6}\delta$. It follows from Lemma 3.9 that

$$d(\pi(x_{i,j}), x_0) \le 3d(x_{i,j}, x_0) < 3(2r_i + \delta) < 4\delta.$$

Inserting this into (4.1) we find that

$$\begin{aligned} |\mathcal{E}f(x) - f(x_0)| &\le \sum_{i,j} |f(\pi(x_{i,j})) - f(x_0)|\,\varphi_{i,j}(x) \\ &\le \Big(\sup_{y\in F\cap B(x_0,4\delta)} |f(y) - f(x_0)|\Big) \sum_{i,j} \varphi_{i,j}(x) \\ &= \sup_{y\in F\cap B(x_0,4\delta)} |f(y) - f(x_0)|. \quad \square \end{aligned}$$

Lemma 4.2. *Let $E \subset F$ and let $\mathcal{W}_E$ be the collection of all balls $B \in \mathcal{W}$ for which $\pi(5B)$ intersects both E and $F \setminus E$. Furthermore, let g be defined by*

$$g(x) = \begin{cases} 1/\operatorname{dist}(x,F) & \text{if } x \in \bigcup_{B\in\mathcal{W}_E} B, \\ 0 & \text{otherwise.} \end{cases}$$

Then Cg is an upper gradient in V of $\mathcal{E}\chi_E$, where χ_E is the characteristic function of E and the constant C is independent of E.

Note that $\pi(5B) \subset 47B$, by Lemma 3.9. Hence, the set $\pi(5B)$ in the definition of $\mathcal{W}_E$ can be replaced by $47B$ and the resulting function Cg will still be an upper gradient for u.

Proof. Let γ be a curve connecting x and y in V. By cutting γ into small segments if necessary we may assume that $\gamma \subset B_{k,l}$ for some k and l. Let $u := \mathcal{E}\chi_E$. We have

$$\begin{aligned} u(x) - u(y) &= \sum_{i,j} \chi_E(\pi(x_{i,j}))(\varphi_{i,j}(x) - \varphi_{i,j}(y)) \\ &= \sum_{i,j} \big(\chi_E(\pi(x_{i,j})) - \chi_E(\pi(x_{k,l}))\big)(\varphi_{i,j}(x) - \varphi_{i,j}(y)). \end{aligned} \tag{4.2}$$

The term $\varphi_{i,j}(x) - \varphi_{i,j}(y)$ is nonzero only if at least one of x and y is in $2B_{i,j}$, and hence we only need to sum over those i and j for which $2B_{i,j} \cap B_{k,l} \neq \varnothing$. For such i, we have by Lemma 3.5 (i) that $\frac{1}{3} r_k < r_i < 3r_k$ and thus as $r_k = 2^{-k}R$ and $r_i = 2^{-i}R$, we have $\frac{1}{2} r_k \le r_i \le 2r_k$. Hence, $d(x_{k,l}, x_{i,j}) < r_k + 2r_i \le 5r_k$, i.e. $x_{i,j} \in 5B_{k,l}$. Therefore if $\pi(5B_{k,l})$ intersects only one of the sets E and $F \setminus E$, then each term in the sum on the right-hand side in (4.2) is zero.

So, let us consider the case when $B_{k,l} \in \mathcal{W}_E$. In this case,

$$|u(x) - u(y)| \le \sum_{\{(i,j):2B_{i,j} \cap B_{k,l} \neq \varnothing\}} |\varphi_{i,j}(x) - \varphi_{i,j}(y)|.$$

By Lemma 3.5 (ii), there are at most M balls $B_{i,j}$ such that $2B_{i,j} \cap B_{k,l}$ is nonempty, and $|\varphi_{i,j}(x) - \varphi_{i,j}(y)| \le Cd(x,y)/r_i \le Cd(x,y)/r_k$ for such i and j. As $\operatorname{dist}(z,F) \le 17r_k$ for all $z \in B_{k,l}$, we see that

$$|u(x) - u(y)| \le CMd(x,y)/r_k \le C \int_\gamma \frac{ds}{\operatorname{dist}(\,\cdot\,,F)},$$

which completes the proof. $\square$

Definition 4.3. The measure μ is said to have a *Q-upper mass bound* if there is a constant $C > 0$ such that for all $x \in X$ and for all $r > 0$ we have $\mu(B(x,r)) \le Cr^Q$.

If $r^Q/C \le \mu(B(x,r)) \le Cr^Q$ for all $x \in X$ and for all $r > 0$, then X is *Ahlfors Q-regular.*

For our purpose it is enough to know that for every ball $B' \subset X$ there is a constant $C_{B'}$ such that $\mu(B(x,r)) \le C_{B'} r^Q$ for all balls $B(x,r) \subset B'$. It is known that doubling measures on uniformly perfect metric spaces have such a local version of a Q-upper mass bound for some Q; see, e.g., the exercises in Chapter 13 of Heinonen [13].

Theorem 4.4. *Assume that μ has a Q-upper mass bound, and let E be a bounded subset of F. Suppose that there exists a nonempty bounded subset E_0 of F with $\dim_{\mathrm{M}} E_0 < Q - p$ and that there exist constants $r > 0$ and $\lambda \ge 1$ such that for all $x \in F \setminus E$ with $\operatorname{dist}(x,E) \le r$,*

$$\operatorname{dist}(x,E_0) \le \lambda \operatorname{dist}(x,E). \tag{4.3}$$

Then $u := \mathcal{E}\chi_E \in N^{1,p}(V)$.

If moreover $C_p(\partial_F E) = 0$, then χ_E is the restriction to F of a function in $N^{1,p}(X)$.

Proof. Clearly, $0 \le u \le 1$ and $u(x) \neq 0$ for $x \notin F$ only if $x \in 2B_{i,j}$ for some $B_{i,j} \in \mathcal{W}$ with $\pi(x_{i,j}) \in E$. For such x we have

$$\operatorname{dist}(x,E) \le d(x,x_{i,j}) + d(x_{i,j}, \pi(x_{i,j})) \le 34r_i \le 34R.$$

As E is bounded, we see that $u \in L^p(X)$. Let $\mathcal{W}_E$ and g be as in Lemma 4.2. By Lemma 4.2 the function Cg is an upper gradient of u in V. We shall show that $g \in L^p(V)$.

It is clear that when computing the L^p-norm of g it suffices to consider only those balls $B_{i,j}$, for which $B_{i,j}$ lies in $\mathcal{W}_E$. For such balls we have

$$\int_{B_{i,j}} g^p \, d\mu \le \frac{C\mu(B_{i,j})}{r_i^p} \le Cr_i^{Q-p}. \tag{4.4}$$

For large i, we shall estimate the number of such balls $B_{i,j}$. Let i be large enough so that $94r_i < r$. Since $\pi(5B_{i,j})$ intersects both E and $F \setminus E$, we can find

$x, y \in 5B_{i,j}$ such that $\pi(x) \in E$ and $\pi(y) \in F \setminus E$. By Lemma 3.9, $d(\pi(x), \pi(y)) \leq d(\pi(x), x_{i,j}) + d(x_{i,j}, \pi(y)) \leq 94r_i < r$. By the assumption (4.3),

$$\operatorname{dist}(\pi(y), E_0) \leq \lambda d(\pi(x), \pi(y)) \leq 94\lambda r_i,$$

and thus there exists $z \in E_0$ such that $d(\pi(y), z) < 95\lambda r_i$. Hence

$$d(x_{i,j}, z) \leq d(x_{i,j}, \pi(y)) + d(\pi(y), z) < 47r_i + 95\lambda r_i < 142\lambda r_i,$$

i.e. $E_0 \cap 142\lambda B_{i,j}$ is nonempty. Lemma 3.8 implies that for each i, there are at most $MP(E_0, r_i)$ such balls $B_{i,j}$. As $\dim_{\mathrm{M}} E_0 < Q - p$, there exists $s < Q - p$ such that $P(E_0, r_i) \leq C r_i^{-s}$ for sufficiently large i. Hence, using (4.4),

$$\int_V g^p \, d\mu \leq C + \sum_{\{i,j:\, 142\lambda B_{i,j} \cap E_0 \neq \varnothing\}} C r_i^{Q-p} \leq C + \sum_{i=0}^{\infty} CM r_i^{Q-p-s} < \infty,$$

i.e. $g \in L^p(V)$ and hence $u \in N^{1,p}(V)$.

Assume now that $C_p(\partial_F E) = 0$. We shall show that g is a p-weak upper gradient of u in $V \cup F$. Let $\gamma : [0, L] \to V \cup F$ be a curve which does not intersect $\partial_F E$. As $C_p(\partial_F E) = 0$, p-almost every curve in X has this property. If γ does not intersect F, then we already know that $|u(\gamma(0)) - u(\gamma(L))| \leq \int_\gamma g \, ds$. Assume therefore that γ intersects F and let

$$a = \inf\{t \in [0, L] : \gamma(t) \in F\} \quad \text{and} \quad b = \sup\{t \in [0, L] : \gamma(t) \in F\}.$$

As F is a closed set, $\gamma(a)$ and $\gamma(b)$ belong to F, and Lemma 4.1 yields

$$\begin{aligned} |u(\gamma(0)) - u(\gamma(L))| \leq& |u(\gamma(0)) - u(\gamma(a))| + |u(\gamma(a)) - u(\gamma(b))| \\ &+ |u(\gamma(b)) - u(\gamma(L))| \\ \leq& \int_{\gamma_1} g \, ds + |u(\gamma(a)) - u(\gamma(b))| + \int_{\gamma_2} g \, ds, \end{aligned} \tag{4.5}$$

where γ_1 and γ_2 are the restrictions of γ to the intervals $[0, a]$ and $[b, L]$, respectively.

If $u(\gamma(a)) = u(\gamma(b))$, then by (4.5), $|u(\gamma(0)) - u(\gamma(L))| \leq \int_\gamma g \, ds$. Assume therefore that $\gamma(a) \in E$ and $\gamma(b) \in F \setminus E$, and let

$$\begin{aligned} a' &= \sup\{t \in [0, L] : \gamma(t) \in E\}, \\ b' &= \inf\{t \in [a', L] : \gamma(t) \in F \setminus E\}. \end{aligned}$$

Note that $\gamma(a') \in \overline{E}$ and $\gamma(b') \in \overline{F \setminus E}$. Moreover, as $\gamma \cap \partial_F E$ is empty, we have actually $\gamma(a') \in E \setminus \partial_F E$, $\gamma(b') \in F \setminus \overline{E}$ and $a' < b'$. Lemma 4.1 then implies that there exist a'', b'' such that $a' < a'' < b'' < b'$,

$$\begin{aligned} u(\gamma(a'')) &= u(\gamma(a')) = 1 = u(\gamma(a)), \\ u(\gamma(b'')) &= u(\gamma(b')) = 0 = u(\gamma(b)) \end{aligned}$$

and the restriction γ' of γ to the interval $[a'', b'']$ lies in V. Hence,

$$|u(\gamma(a)) - u(\gamma(b))| = 1 = |u(\gamma(a'')) - u(\gamma(b''))| \leq \int_{\gamma'} g \, ds.$$

Inserting this into (4.5) gives $|u(\gamma(0)) - u(\gamma(L))| \leq \int_\gamma g \, ds$, which shows that $u \in N^{1,p}(V \cup F)$.

To complete the proof, let η be a Lipschitz function with bounded support in $V \cup F$ such that $\eta = 1$ on E. Then χ_E is the restriction to F of the function $\eta u \in N^{1,p}(X)$. $\square$

In the proof of Theorem 4.4, it is essential to have $C_p(\partial_F E)=0$. On the other hand, Example 3 in Herron–Koskela [16] shows that there exist Cantor sets $E \subset \mathbf{R}^{n-1}$ with positive $(n-1)$-dimensional Hausdorff measure (and hence $C_p(\partial_{\mathbf{R}^{n-1}} E)$ is positive for all $p>1$), such that χ_E is the restriction to $\mathbf{R}^{n-1}$ of a function from $N^{1,p}(\mathbf{R}^n)$ for all $1<p<2$.

In our next theorems, we shall give some more explicit conditions under which the assumptions of Theorem 4.4 are satisfied. A crucial role will be played by the relative boundary $\partial_F E = \overline{E} \cap \overline{F \setminus E}$ of E with respect to F and by the following relative r-boundary. As far as we are aware, the notion of relative r-boundary is new. It is useful for us when X is complete.

Definition 4.5. Given $r>0$, the *relative r-boundary* of $E \subset F$ is the set

$$\partial_{F,r} E := \{x \in \overline{E} : d(y,x) = \operatorname{dist}(y,E) \le r \text{ for some } y \in \overline{F \setminus E}\}.$$

The following proposition clarifies the relationship between the relative boundary $\partial_F E$ and the relative r-boundary $\partial_{F,r} E$.

Proposition 4.6. *Let $E \subset F$ and $r>0$. Then*

$$\partial_F E = \bigcap_{r>0} \partial_{F,r} E. \tag{4.6}$$

Moreover, if X is complete, then $\partial_{F,r} E$ is closed and

$$\partial_{F,r} E = \bigcap_{\rho>r} \partial_{F,\rho} E.$$

Proof. If $x \in \partial_F E$, then $x \in \overline{E} \cap \overline{F \setminus E}$, and $d(x,x) = \operatorname{dist}(x,E) = 0$, so $x \in \partial_{F,r} E$.

Conversely, if $x \in \partial_{F,r} E$ for all $r>0$, then there exist $y_j \in \overline{F \setminus E}$ such that $d(y_j,x) = \operatorname{dist}(y_j,E) \le 1/j$, $j=1,2,\ldots$. Therefore $\lim_{j\to\infty} y_j = x$ and $x \in \overline{F \setminus E}$. On the other hand, as $x \in \partial_{F,r} E$ we see that $x \in \overline{E}$ and hence $x \in \partial_F E$ and (4.6) is proved.

Let next $x_j \in \partial_{F,r} E$ with $x_j \to x_0 \in \overline{E}$ and assume that X is complete (and thus proper). Then there exist points $y_j \in \overline{F \setminus E}$ such that $d(y_j,x_j) = \operatorname{dist}(y_j,E) \le r$. As X is proper, the sequence $\{y_j\}_{j=1}^\infty$ has a subsequence, also denoted $\{y_j\}_{j=1}^\infty$, converging to some $y_0 \in \overline{F \setminus E}$. Then

$$d(y_0,x_0) = \lim_{j\to\infty} d(y_j,x_j) = \lim_{j\to\infty} \operatorname{dist}(y_j,E) = \operatorname{dist}(y_0,E) \le r,$$

i.e. $x_0 \in \partial_{F,r} E$ and thus $\partial_{F,r} E$ is closed.

Finally, if $x \in \bigcap_{\rho>r} \partial_{F,\rho} E$, then for each $j \in \mathbf{N}$ we consider $y_j \in \overline{F \setminus E}$ such that $d(y_j,x) = \operatorname{dist}(y_j,E) \le r+1/j$. Again we extract a convergent subsequence, also denoted $\{y_j\}_{j=1}^\infty$, converging to a point $y_0 \in \overline{F \setminus E}$, and note that

$$d(y_0,x) = \lim_{j\to\infty} d(y_j,x) = \lim_{j\to\infty} \operatorname{dist}(y_j,E) = \operatorname{dist}(y_0,E) \le r,$$

and hence $x \in \partial_{F,r} E$. As $\partial_{F,r} E \subset \partial_{F,\rho} E$ for every $\rho > r$, we have the desired equality. $\square$

Note also that if $\partial_{F,r} E \ne \partial_F E$, then $\partial_{F,r} E \ne \partial_{F,r}(F \setminus E)$. It is also possible to have $\partial_{F,r} E = \partial_F E \ne \partial_{F,r}(F \setminus E)$; let

$$F = [-3,3]^2 \setminus \bigl\{(x,y) : 0 < y < 2\sqrt{1-x^2}\bigr\} \subset \mathbf{R}^2,$$

and $E = \{(x,y) \in F : y > 0\}$. Then $\partial_F E = \partial_{F,r} E = \{(x,0) : 1 \le |x| \le 3\}$ and $\partial_{F,r}(F \setminus E) = \bigl\{(x,0) : 1 - \sqrt{1-r^2/4} \le |x| \le 3\bigr\}$.

Following Heinonen–Koskela [15], we consider the following definition.

Definition 4.7. Given two sets $A \subset D \subset X$, we say that A is *linearly locally connected* in D if there are constants $C \geq 1$ and $r_0 > 0$, such that whenever $x, y \in A$ and $d(x,y) < r < r_0$, there is a curve in $B(x, Cr) \cap D$ connecting x and y.

A set is *linearly locally connected* if it is linearly locally connected in itself.

Connected components of boundaries of bounded Lipschitz domains in $\mathbf{R}^n$ are linearly locally connected. Boundaries of domains in $\mathbf{R}^n$ with cusps usually are linearly locally connected if $n \geq 3$, but not if $n = 2$. It was shown in Hajłasz–Koskela [11], Proposition 4.5, that Ahlfors Q-regular metric measure spaces, $Q > 1$, supporting a weak $(1, Q)$-Poincaré inequality are linearly locally connected. In general, it is not clear when the boundary of a domain is linearly locally connected.

Theorem 4.8. *Assume that μ has a Q-upper mass bound. Let $E \subset F$ be bounded and assume that one of the following conditions is satisfied*:

(i) $\operatorname{dist}(E, F \setminus E) > 0$ (*this does not require a Q-upper mass bound*);

(ii) $\dim_{\mathrm{M}} E < Q - p$;

(iii) $\dim_{\mathrm{M}} (F \setminus E) < Q - p$.

Then χ_E is the restriction to F of a function from $N^{1,p}(X)$.

Proof. (i) In this case we can find a Lipschitz function η with bounded support such that $\eta|_E = 1$ and $\eta|_{F \setminus E} = 0$.

(ii) Since the Hausdorff dimension $\dim_{\mathrm{H}} \partial_F E \leq \dim_{\mathrm{M}} \partial_F E < Q - p$, it follows that $C_p(\partial_F E) = 0$. Setting $E_0 = E$ we obtain the desired result by Theorem 4.4. (In the case when $E = \varnothing$ we cannot apply Theorem 4.4, since E_0 need to be nonempty, but in this case the conclusion is trivial.)

(iii) Since $\dim_{\mathrm{M}} (F \setminus E) < Q - p$, F must be bounded. Hence by (ii), $\chi_{F \setminus E}$ is the restriction to F of a function $v \in N^{1,p}(X)$. Now let η be a Lipschitz function with bounded support such that $\eta = 1$ on F. Then χ_E is the restriction to F of $\eta(1 - v) \in N^{1,p}(X)$. $\square$

In the case when X is complete, we can say more.

Theorem 4.9. *Assume that X is complete and that μ has a Q-upper mass bound. Let $E \subset F$ be bounded, $r > 0$ and assume that one of the following conditions is satisfied*:

(i) $\partial_F E = \varnothing$ (*this does not require a Q-upper mass bound*);

(ii) $\dim_{\mathrm{M}} \partial_{F,r} E < Q - p$;

(iii) $\dim_{\mathrm{M}} \partial_{F,r} (F \setminus E) < Q - p$;

(iv) *F is linearly locally connected and* $\dim_{\mathrm{M}} \partial_F E < Q - p$;

(v) *$\partial_{F,r} E \cup \partial_{F,r}(F \setminus E)$ is linearly locally connected in F and* $\dim_{\mathrm{M}} \partial_F E < Q - p$.

Then χ_E is the restriction to F of a function from $N^{1,p}(X)$.

Proof. (i) As X is proper, $\operatorname{dist}(E, F \setminus E) > 0$ (or $E = \varnothing$), and we can thus apply Theorem 4.8.

(ii) Set $E_0 = \partial_{F,r} E$. Note that as $\dim_{\mathrm{H}} \partial_F E \leq \dim_{\mathrm{M}} \partial_F E \leq \dim_{\mathrm{M}} \partial_{F,r} E < Q - p$, it follows that $C_p(\partial_F E) = 0$. To see that the condition (4.3) is satisfied, let $x \in F \setminus E$ be such that $\operatorname{dist}(x, E) \leq r$. Find $y \in \overline{E}$ such that $d(x,y) = \operatorname{dist}(x, E)$. Such a point y exists because X is proper. Then $y \in \partial_{F,r} E$ and hence

$$\operatorname{dist}(x, \partial_{F,r} E) \leq d(x,y) = \operatorname{dist}(x, E) \leq r,$$

i.e. (4.3) holds with $\lambda = 1$. Now the desired result follows from Theorem 4.4.

(iii) Let $\widetilde{F} = \{x \in F : \operatorname{dist}(x, E) \leq r\}$, which is a closed bounded set. Then $\partial_{\widetilde{F},r}(\widetilde{F} \setminus E) = \partial_{F,r}(F \setminus E)$, and therefore, by (ii), $\chi_{\widetilde{F} \setminus E}$ is the restriction to $\widetilde{F}$ of a function $v \in N^{1,p}(X)$. Let now η be a Lipschitz function with bounded support such that $\eta = 1$ on E and $\eta = 0$ on $F \setminus \widetilde{F}$. Then χ_E is the restriction to F of $\eta(1-v) \in N^{1,p}(X)$.

(iv) This follows from (v).

(v) We shall show that $E_0 = \partial_F E$ satisfies condition (4.3). Let $x \in F \setminus E$ be such that $\operatorname{dist}(x, E) \le r$, and find $y \in \overline{E}$ such that

$$d(x, y) = \operatorname{dist}(x, E).$$

Choose $z \in \overline{F \setminus E}$ that satisfies $d(y, z) = \operatorname{dist}(y, F \setminus E) \le d(y, x) \le r$. Note that $y \in \partial_{F,r} E$ and $z \in \partial_{F,r}(F \setminus E)$. As X is linearly locally connected, there exists a curve γ connecting y and z in $B(y, Cd(y, z)) \cap F$. We observe that $\gamma \cap \partial_F E$ is nonempty. Let $x_0 \in \gamma \cap \partial_F E$. Thus,

$$\begin{aligned} \operatorname{dist}(x, \partial_F E) &\le d(x, x_0) \le d(x, y) + d(y, x_0) \le d(x, y) + Cd(y, z) \\ &\le (1 + C) d(x, y) = (1 + C) \operatorname{dist}(x, E), \end{aligned}$$

i.e. $\partial_F E$ satisfies (4.3) with $\lambda = 1 + C$. Finally, as $\dim_{\mathrm{H}} \partial_F E \le \dim_{\mathrm{M}} \partial_F E < Q - p$, it follows that $C_p(\partial_F E) = 0$, and we can apply Theorem 4.4. $\square$

We shall now demonstrate how Theorems 4.4, 4.8 and 4.9 enable us to generalize some results from Björn–Björn–Shanmugalingam [4] to metric measure spaces.

Definition 4.10. Assume that X is complete and supports a weak $(1, p)$-Poincaré inequality. Let $\Omega \subset X$ be a nonempty bounded open set such that $C_p(X \setminus \Omega) > 0$. The *upper and lower p-harmonic measure* of $E \subset \partial\Omega$ evaluated at $x \in \Omega$ are respectively

$$\overline{\omega}_{x,p}(E) := \overline{P}\chi_E(x) \quad \text{and} \quad \underline{\omega}_{x,p}(E) := \underline{P}\chi_E(x),$$

where $\overline{P}f$ and $\underline{P}f$ are the upper and the lower Perron solutions of the Dirichlet problem for p-harmonic functions in Ω with the boundary data f.

For the definition of Perron solutions on metric spaces, see Björn–Björn–Shanmugalingam [3] . (Note that the results in [3] are stated for bounded domains but the proofs and results hold for nonempty bounded open sets.) If $\overline{\omega}_{x,p}(E) = \underline{\omega}_{x,p}(E)$, then we call $\omega_{x,p}(E) := \overline{\omega}_{x,p}(E)$ the *p-harmonic measure* of E. Note that despite its name, the p-harmonic measure is not a measure in general, merely a nonlinear analogue of the harmonic measure.

Corollary 4.11. *Assume that X is complete and that μ has a Q-upper mass bound. Assume further that X supports a weak $(1, p)$-Poincaré inequality for some $1 < p < Q$. Let $\Omega \subset X$ be a nonempty bounded open set such that $C_p(X \setminus \Omega) > 0$. If $E \subset F := \partial\Omega$ satisfy at least one of the conditions in Theorem* 4.4, 4.8 *or* 4.9, *then $\omega_{x,p}(E) = \omega_{x,p}(\overline{E})$ for all $x \in \Omega$.*

Proof. Note that by Keith–Zhong [18], X supports a weak $(1, q)$-Poincaré inequality for some $q \in [1, p)$, which was earlier a standard assumption. By Theorems 4.4, 4.8 or 4.9, the characteristic function χ_E is the restriction to $\partial\Omega$ of a function $u \in N^{1,p}(X)$. By Theorem 5.1 in Björn–Björn–Shanmugalingam [3], u is resolutive and

$$\omega_{x,p}(E) = P\chi_E(x) = Hu(x),$$

where Hu is the p-harmonic extension of u to Ω, i.e. Hu is p-harmonic in Ω and

$$Hu - u \in N_0^{1,p}(\Omega) := \{f|_\Omega : f \in N^{1,p}(X) \text{ and } f = 0 \text{ on } X \setminus \Omega\}.$$

Now, let $\bar{u}(x) = 1$ if $x \in \overline{E}$ and $\bar{u}(x) = u(x)$ otherwise. As $C_p(\partial_{\partial\Omega} E) = 0$, $\bar{u}$ belongs to the same equivalence class in $N^{1,p}(X)$ as u and

$$Hu(x) = H\bar{u}(x) = \omega_{x,p}(\overline{E}). \quad \square$$

Example 4.12. Let $\varphi : [0,1] \to \mathbf{R}$ be a positive increasing continuous function such that

$$\lim_{t\to 0+} \frac{\varphi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to 0+} \frac{\varphi(1)-\varphi(1-t)}{t} < \infty,$$

and let $\Omega \subset \mathbf{R}^2$ be the cuspidal domain

$$\Omega = \{(x_1,x_2) \in \mathbf{R}^2 : 0 < x_1 < 1 \text{ and } 0 < x_2 < \varphi(x_1)\}.$$

Let $0 < r < 1$ and

$$E = \{(x_1,x_2) \in \mathbf{R}^2 : 0 < x_1 < 1 \text{ and } x_2 = \varphi(x_1)\}.$$

Then $\partial_{\partial\Omega,r}(\partial\Omega \setminus E)$ contains the interval $[0,\varphi^{-1}(r)] \subset \mathbf{R}$ and has dimension one. On the other hand, the countable set

$$E_0 = \{(t_{i,j},0) : i,j \in \mathbf{N}\} \cup \{(1,\varphi(1))\}$$

with $t_{i,0} = \varphi^{-1}(2^{-i}r)$ and $t_{i,j} = t_{i,0} - 2^{-i}jr > 0$ satisfies (4.3) for some λ.

We shall estimate $\dim_{\mathrm{M}} E_0$. Let $\varepsilon = 2^{-k}r$. For $i < k$, we can place at most $\varphi^{-1}(2^{-i}r)/2^{-i}r$ pairwise disjoint balls with radius ε and centers in the points $(t_{i,j},0)$, $j \in \mathbf{N}$. Along the interval $[0,t_{k,0}] \subset \mathbf{R}$ we can put at most $t_{k,0}/\varepsilon = \varphi^{-1}(2^{-k}r)/2^{-k}r$ such balls. Hence

$$P(E_0,\varepsilon) \le 1 + \sum_{i=0}^{k} \frac{\varphi^{-1}(2^{-i}r)}{2^{-i}r}.$$

If $\varphi(t) \ge Ct^\alpha$ with $\alpha > 1$, then $P(E_0,\varepsilon) \le C\varepsilon^{1/\alpha-1}$ and $\dim_{\mathrm{M}} E_0 \le 1 - 1/\alpha$. It follows that for $p < 1 + 1/\alpha$, the characteristic function of E is the restriction to $\partial\Omega$ of a function from $N^{1,p}(\mathbf{R}^2)$ and the p-harmonic measure of E with respect to Ω equals the p-harmonic measure of $\overline{E}$.

5. Extending Hölder continuous functions

In this section we study when Hölder continuous functions on the boundary of an open set $\Omega \subset X$ can be extended as Newtonian functions in $N^{1,p}(\Omega)$ that are Hölder continuous on $\overline{\Omega}$.

Definition 5.1. Let $A \subset X$ and $f : A \to \mathbf{R}$. For $0 < \kappa \le 1$, the *κ-Hölder norm* of f is given by

$$\|f\|_{\kappa,A} = \sup_{x,y\in A} \frac{|f(x)-f(y)|}{d(x,y)^\kappa} + \sup_{x\in A} |f(x)|.$$

Under this norm, the vector space of all κ-Hölder continuous functions on A is a Banach space.

Definition 5.2. We say that $\Omega \subset X$ satisfies a *β-shell condition* with $\beta > 0$ if for all sufficiently small $t > 0$ the shell $S_t = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) \le t\}$ satisfies $\mu(S_t) \le Ct^\beta$.

It is clear that if Ω satisfies a β_1-shell condition and $0 < \beta_2 < \beta_1$, then Ω satisfies a β_2-shell condition as well. The following simple proposition shows a connection between the β-shell conditions and the upper Minkowski dimension in Ahlfors regular spaces. In particular, this generates many nontrivial examples in Carnot groups. For a similar result in Euclidean spaces and for related notions, see Martio–Vuorinen [25].

Proposition 5.3. *Let K be a nonempty compact subset of an Ahlfors Q-regular metric measure space X such that $\mu(K)=0$. Then*

$$\dim_{\mathrm{M}} K = \inf\{s : X\setminus K \textit{ satisfies a } (Q-s)\textit{-shell condition}\}.$$

In particular, a bounded open set $\Omega \subset X$ satisfies a β-shell condition for all $\beta < Q - \dim_{\mathrm{M}} \partial\Omega$.

Proof. Let $s > \dim_{\mathrm{M}} K$, $\rho > 0$ and $\{B_n\}_{n=1}^{P(K,\rho)}$ be a maximal pairwise disjoint collection of balls with centers in K and radii ρ. Then $K \subset \bigcup_{n=1}^{P(K,\rho)} 2B_n$ and hence $S_\rho := \{x \in X\setminus K : \operatorname{dist}(x,K) \le \rho\} \subset \bigcup_{n=1}^{P(K,\rho)} 3B_n$, which in turn yields

$$\mu(S_\rho) \le CP(K,\rho)\rho^Q = CP(K,\rho)\rho^s\rho^{Q-s} \le C\rho^{Q-s},$$

where the last inequality follows from (3.1).

Conversely, let $s < s' < \dim_{\mathrm{M}} K$. By the definition of Minkowski dimension, there exists $\delta > 0$ and a sequence $\rho_j \to 0+$ such that $P(K,\rho_j)\rho_j^{s'} \ge \delta$ for all $j = 1, 2\ldots$. It follows that

$$\mu(S_\rho) \ge P(K,\rho_j)\rho_j^Q \ge \delta\rho_j^{Q-s}\rho_j^{s-s'}$$

and letting $j \to \infty$ shows that $X\setminus K$ does not satisfy the $(Q-s)$-shell condition. $\square$

If X is a geodesic metric space and μ is doubling, then there exists $\beta_0 > 0$ such that balls in X satisfy a β-shell condition for every $\beta < \beta_0$; see (1.1) of Buckley [5].

It is easily verified that bounded Lipschitz domains in Euclidean spaces satisfy a 1-shell condition, while it follows from Theorem 4.22 in Martio–Vuorinen [25] that complements of many self-similar fractals in $\mathbf{R}^n$ (including the von Koch snowflake domain in $\mathbf{R}^2$) satisfy an $(n-\alpha)$-shell condition, where α is the dimension of the fractal.

Note that it is possible to have $\beta > 1$. The domain $\Omega = \mathbf{R}^n\setminus I$, where $I \subset \mathbf{R}$ is an interval, satisfies the $(n-1)$-shell condition. For an example with a bounded domain, let $\Omega = [0,1]^n \setminus \{(0,\ldots,0)\}$ and $X = \Omega \cup \{(x_1,\ldots,x_n) \in \mathbf{R}^n : x_j \le 0, j = 1,\ldots,n\}$.

Theorem 5.4. *Let Ω be a bounded open set in X with nonempty boundary. Assume that Ω satisfies a β-shell condition for some $\beta > 0$, and let $\max\{0, 1-\beta/p\} < \kappa \le 1$. Then there exists $C > 0$ such that every κ-Hölder continuous function $f : \partial\Omega \to \mathbf{R}$ has a κ-Hölder continuous extension $\mathcal{E}f : \overline{\Omega} \to \mathbf{R}$ with $\|\mathcal{E}f\|_{N^{1,p}(\Omega)} \le C\|f\|_{\kappa,\partial\Omega}$ and $\|\mathcal{E}f\|_{\kappa,\overline{\Omega}} \le C\|f\|_{\kappa,\partial\Omega}$.*

Proof. Let $F = \partial\Omega$, $R = \frac{1}{16}\operatorname{diam}\Omega$ and the notation be as in the previous sections. As $\partial\Omega$ is bounded, so is $\operatorname{supp}\mathcal{E}f$, and thus $\|\mathcal{E}f\|_{L^p(X)} \le C\|f\|_{\kappa,\partial\Omega}$. We will show that $\mathcal{E}f$ has an L^p-integrable upper gradient in $\Omega = \Omega \cap V \subset \bigcup_{B\in\mathcal{W}} B$. Let γ be a curve in Ω connecting x and y. By splitting γ into segments if necessary, we may assume that $\gamma \subset 2B_{k,l}$ for some k and l.

As in (4.2), we have

$$|\mathcal{E}f(x) - \mathcal{E}f(y)| \le \sum_{i,j} |f(\pi(x_{i,j})) - f(\pi(x_{k,l}))|\,|\varphi_{i,j}(x) - \varphi_{i,j}(y)|, \tag{5.1}$$

where we only need to sum over those i and j for which $2B_{i,j} \cap 2B_{k,l} \ne \varnothing$. Note that by Lemma 3.5, there are at most M such balls $B_{i,j}$ and that $r_i \le 2r_k$ for such i. Hence,

$$\begin{aligned} d(\pi(x_{i,j}),\pi(x_{k,l})) &\le d(\pi(x_{i,j}),x_{i,j}) + d(x_{i,j},x_{k,l}) + d(x_{k,l},\pi(x_{k,l})) \\ &\le 34(r_i + r_k) \le 102 r_k \end{aligned}$$

and it follows that

$$|f(\pi(x_{i,j})) - f(\pi(x_{k,l}))| \le C\|f\|_{\kappa,\partial\Omega} d(\pi(x_{i,j}),\pi(x_{k,l}))^\kappa \le C\|f\|_{\kappa,\partial\Omega} r_k^\kappa.$$

Inserting this into (5.1), together with the estimate $|\varphi_{i,j}(x)-\varphi_{i,j}(y)| \le Cd(x,y)/r_k$, implies

$$|\mathcal{E}f(x) - \mathcal{E}f(y)| \le CM\|f\|_{\kappa,\partial\Omega} r_k^{\kappa-1} d(x,y) \le C\|f\|_{\kappa,\partial\Omega} \int_\gamma \operatorname{dist}(\,\cdot\,,\partial\Omega)^{\kappa-1}\,ds, \quad (5.2)$$

i.e. the function $g(x) = C\|f\|_{\kappa,\partial\Omega} \operatorname{dist}(x,\partial\Omega)^{\kappa-1}$ is an upper gradient for $\mathcal{E}f$ in Ω. Hence, by the Cavalieri principle,

$$\begin{aligned}
\int_\Omega g^p\,d\mu &\le C\|f\|_{\kappa,\partial\Omega}^p \int_\Omega \operatorname{dist}(x,\partial\Omega)^{p(\kappa-1)}\,d\mu(x) \\
&\le C\|f\|_{\kappa,\partial\Omega}^p \int_0^{16R} t^{p(\kappa-1)-1}\mu(\{x\in\Omega : \operatorname{dist}(x,\partial\Omega) < t\})\,dt \\
&\le C\|f\|_{\kappa,\partial\Omega}^p \int_0^{16R} t^{p(\kappa-1)-1+\beta}\,dt \\
&\le C\|f\|_{\kappa,\partial\Omega}^p,
\end{aligned}$$

and thus $\|\mathcal{E}f\|_{N^{1,p}(\Omega)} \le C\|f\|_{\kappa,\partial\Omega}$.

We now show that $\mathcal{E}f$ is κ-Hölder continuous on $\overline{\Omega}$. Observe that by Lemma 4.1,

$$|\mathcal{E}f(x) - \mathcal{E}f(y)| \le C\|f\|_{\kappa,\partial\Omega} d(x,y)^\kappa$$

for all $x\in\partial\Omega$ and $y\in\overline{\Omega}$.

Let $x\in B_{k,l}$. Assume first that $y\in 2B_{k,l}$. Then by (5.2),

$$|\mathcal{E}f(x) - \mathcal{E}f(y)| \le C\|f\|_{\kappa,\partial\Omega} r_k^{\kappa-1} d(x,y) \le C\|f\|_{\kappa,\partial\Omega} d(x,y)^\kappa.$$

Assume next that $y\notin 2B_{k,l}$, but $y\in B_{i,j}$ for some i and j. Without loss of generality we can assume that $k\le i$. Then

$$d(x,\pi(x)) \le 34r_k, \quad d(y,\pi(y)) \le 34r_i \le 34r_k, \quad d(x,y)\ge r_k \quad \text{and}$$

$$d(\pi(x),\pi(y)) \le d(x,y) + 68r_k.$$

Hence by Lemma 4.1,

$$\begin{aligned}
|\mathcal{E}f(x) - \mathcal{E}f(y)| &\le |\mathcal{E}f(x) - f(\pi(x))| + |f(\pi(x)) - f(\pi(y))| + |f(\pi(y)) - \mathcal{E}f(y)| \\
&\le C\|f\|_{\kappa,\partial\Omega}(d(x,y)+r_k)^\kappa \le C\|f\|_{\kappa,\partial\Omega} d(x,y)^\kappa. \quad \square
\end{aligned}$$

Remark 5.5. The assumption that Ω be bounded in Theorem 5.4 is no restriction in general. Indeed, if Ω is unbounded with bounded boundary then we can apply Theorem 5.4 to the open set $\Omega\cap B$ for some large ball B (letting $f=0$ on ∂B). Multiplying the obtained extension by a Lipschitz continuous cut-off function η with compact support in B and $\eta=1$ in a neighbourhood of $\partial\Omega$ produces the desired extension.

Corollary 5.6. *Assume that X is complete and supports a weak $(1,p)$-Poincaré inequality. Let $\Omega\subset X$ be a nonempty bounded open set such that $C_p(X\setminus\Omega)>0$. Suppose that Ω satisfies a β-shell condition for some $\beta>0$ and let $\max\{0,1-\beta/p\} < \kappa \le 1$. If f is a κ-Hölder continuous function on $\partial\Omega$, then $Pf\in N^{1,p}(\Omega)$, where Pf is the Perron solution of the Dirichlet problem for p-harmonic functions in Ω with boundary data f.*

Proof. Note that by Keith–Zhong [18], X supports a weak $(1,q)$-Poincaré inequality for some $q \in [1,p)$. Let $\mathcal{E}f$ be the κ-Hölder continuous extension of f to $\overline{\Omega}$, as in Theorem 5.4, and let u be the p-harmonic extension of $\mathcal{E}f$ to Ω, i.e. u is p-harmonic in Ω and $u-\mathcal{E}f \in N_0^{1,p}(\Omega)$, see Kinnunen–Martio [22], Theorem 3.2, for the existence and uniqueness of the p-harmonic extension in this case. Theorem 5.2 in Björn–Björn [1] then shows that

$$\lim_{\Omega \ni y \to x} u(y) = f(x)$$

for all regular boundary points $x \in \partial\Omega$. By the Kellogg property (Theorem 3.9 in Björn–Björn–Shanmugalingam [2]), the set of irregular boundary points has p-capacity zero. Now Corollary 6.2 of Björn–Björn–Shanmugalingam [3] implies that $u = Pf$. $\square$

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