1 Zeroth Order Calculus.

Let (X, d, μ) be a metric measure space, and let the measure μ be Borel regular; that is, Borel sets are measurable, and for each $A \subset X$ there exists a Borel set B in X containing A so that $\mu(A) = \mu(B)$. μ is said to be *doubling* if there exists a positive constant C so that for each point $x \in X$ and for each $r > 0$ the following inequality holds:

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)). \tag{1}$$

Here and hereafter, $B(x, r)$ denotes the open ball of radius r centered at the point x.

Definition 1.1. (X, d, μ) is said to be a space of *homogeneous type* if d is a quasi-metric (not necessarily a metric) and the measure μ is doubling.

"Zeroth order calculus" can be done in spaces of homogeneous type. Zeroth order calculus is about maximal functions, Lebesgue differentiation theorem, BMO, etc.

Theorem 1.1. *Let (X, d, μ) be a doubling metric measure space. Let $f \in L^1_{loc}(X, \mu)$ be non-negative. Then for μ-almost every $x \in X$,*

$$f(x) = \lim_{r \to 0} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} f \, d\mu. \tag{2}$$

For a proof of this, see [He].

Definition 1.2. Let $f \in L^1_{loc}(X, \mu)$. The *Hardy-Littlewood maximal function Mf of f is defnied as follows:*

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f| \, d\mu. \tag{3}$$

If A is a measurable subset of X and f is a locally integrable function, we denote the quantity

$$\frac{1}{\mu(A)} \int_A f \, d\mu =: \fint_A f \, d\mu. \tag{4}$$

Theorem 1.2. *Let (X, μ) be a doubling metric measure space. Considering M above as a Hardy-Littlewood maximal operator on L^p functions, the following is true:*

$$M : L^p(X, d\mu) \to L^p(X, d\mu), \tag{5}$$
$$M : L^1(X, d\mu) \to wk - L^1(X, d\mu) \tag{6}$$

as bounded operators.

For a proof of this, see [He].

The space $\mathrm{wk} - L^1(X, d\mu)$ is the space of functions $f : X \to [-\infty, \infty]$ with the property that

$$\mu(\{x \in X \ : \ |f(x)| > \lambda\}) \le \frac{C}{\lambda}, \tag{7}$$

with $C > 0$ independent of λ. The smallest such value of C is the $\mathrm{wk}-L^1$-norm of f.

2 First Order Calculus.

In the previous section it was sufficient for d to be a quasi-metric: there is a constant $K > 0$ so that for every triple $x, y, z \in X$,

$$d(x, z) \le K(d(x, y) + d(y, z)). \tag{8}$$

This however is not sufficient if we want to measure lengths of curves and to look at covering theorems. Hence from now on, d is actually a metric.

Definition 2.1. A metric measure space X is said to be a *Vitali space* if

1. every ball has finite measure,

2. the Vitali covering theorem holds; that is, if $A \subset X$ and $\mathcal{B}$ is a collection of closed balls so that $A \subset \bigcup_{B \in \mathcal{B}} B$ with the property that for each $x \in A$, the infimum of the radii of all the balls in $\mathcal{B}$ that have x as their center is zero, then there is a countable, pairwise disjoint subcollection $\{B_i\}_{i \in \mathbb{N}}$ of $\mathcal{B}$ so that

$$\mu\left(A \setminus \bigcup_{i \in \mathbb{N}} B_i\right) = 0. \tag{9}$$

$\mathbb{R}^n$ together with Euclidean metric and any Radon measure is a Vitali space, and any metric space endowed with a doubling metric is a Vitali space. For concrete examples and proofs see [Ma] and [He].

Theorem 2.1. *If μ is a Radon measure on $\mathbb{R}^n$, then $(\mathbb{R}^n, \mu)$ is a Vitali space.*

Theorem 2.2. *Let (X, d, μ) be a Vitali space. Then the conclusion of the Lebesgue differentiation theorem 1.1 holds in X.*

A proof of this theorem can be found in [He].

Open Problem: Characterize Vitali spaces, or at least give good sufficient conditions for a space to be a Vitali space.

proof of **Theorem 2.1**. Since we are in $\mathbb{R}^n$ with the Euclidean metric, the Besicovich covering theorem holds (see [He]). Let $A \subset \mathbb{R}^n$ be a bounded set, and $\mathcal{B}$ be a covering of A by closed balls of arbitrarily small radii:

$$\inf\{\mathrm{rad} B \ : \ B \in \mathcal{B}, x \text{ is the center of } B\} = 0 \tag{10}$$

for each $x \in A$. Applying Besicovich's theorem to this collection, we get a partition $\mathcal{B}_1$,...,$\mathcal{B}_M$ of $\mathcal{B}$ so that each subcollection $\mathcal{B}_i$ is pairwise disjoint. Recall that M is a constant dependent only on the topological dimension n of $\mathbb{R}^n$.

Since the measure is a Radon measure, there is an open neighbourhood U of A so that

$$\mu(U) \leq \left(1 + \frac{1}{4M}\right) \mu(A). \tag{11}$$

By equation (10), we can suppose that $\mathcal{B}$ has the property that every $B \in \mathcal{B}$ is contained in the open set U. Note that

$$A \subset \bigcup_{i=1}^{M} \bigcup_{B \in \mathcal{B}_i} B. \tag{12}$$

Hence there is a positive integer $i \leq M$ so that

$$\mu(A) \leq M\mu\left(\bigcup_{B \in \mathcal{B}_i} B\right). \tag{13}$$

Since the measure of A is finite, there exists a pairwise disjoint finite sequence of balls B'_1,...,B'_k in the collection $\mathcal{B}_i$ so that

$$\mu(A) \leq 2M\mu\left(\bigcup_{j=1}^{k} B'_j\right). \tag{14}$$

Let $A_1 = A \setminus \cup_{j=1}^{k} B'_j$. Then

$$\begin{aligned} \mu(A_1) \leq & \mu(U) - \mu\left(\bigcup_{j=1}^{k} B'_j\right) \\ \leq & \left(1 + \frac{1}{4M}\right)\mu(A) - \frac{1}{2M}\mu(A) \\ \leq & \left(1 - \frac{1}{2M}\right)\mu(A) = \lambda\mu(A). \end{aligned} \tag{15}$$

Note that $0 < \lambda < 1$ with λ depending only on M, which in turn depends only on the dimension n. The finite collection B'_j is an "almost cover" of A.

Note that the collection

$$\mathcal{B}^1 = \{B \in \mathcal{B} \ : \ B \cap B'_j = \emptyset, j = 1, ..., k\} \tag{16}$$

is a Vitali type cover of A_1 because the balls B'_j are closed. Iterate the above process to get an "almost cover" of A_1 and the corresponding uncovered part A_2. Repeating this process countable number of times yields a countable disjoint cover of almost all of A. $\square$

Theorem 2.3 (Maximal Function Theorem). *Let μ and ν be Radon measures on $\mathbb{R}^n$. Let*

$$M_\mu\nu(x) = \sup_{r>0} \frac{\nu(\overline{B(x,r)})}{\mu(\overline{B(x,r)})}. \tag{17}$$

Then the following hold true:

1.

$$\mu(\{x \ : \ M_\mu\nu(x) > \lambda\}) \leq \frac{C_1}{\lambda}\nu(\mathbb{R}^n). \tag{18}$$

2. *If $d\nu = f d\mu$ with $f \in L^p(\mathbb{R}^n, \mu)$, where p is a finite number greater than 1, then*

$$\int_{\mathbb{R}^n} |M_\mu \nu|^p d\mu \leq C_{p,n} \int_{\mathbb{R}^n} |f|^p d\mu, \tag{19}$$

that is, M_μ is a bounded map from $L^p(\mathbb{R}^n, \mu)$ into $L^p(\mathbb{R}^n, \nu)$.

Proof. **Proof of (1):** Let R be a finite positive number, and

$$E_\lambda^R = \{x \in \mathbb{R}^n \ : \ M_\mu \nu(x) > \lambda\} \cap B(0, R). \tag{20}$$

For every $x \in E_\lambda^R$ there exists a positive number r_x so that $\nu(B(x, r_x)) > \lambda \mu(B(x, r_x))$. The set E_λ^R is covered by such balls. An application of the Besicovich covering theorem yields a countable subcollection $\mathcal{B}_0 = \{B_1, B_2, ...\}$ that covers E_λ^R so that $\sum_i \chi_{B_i} \leq Q(n)$. Therefore,

$$\mu(E_\lambda^R) \leq \sum_i \mu(B_i) \leq \frac{1}{\lambda} \sum_i \nu(B_i) \leq \frac{Q(n)}{\lambda} \nu(\mathbb{R}^n). \tag{21}$$

Now let $R \to \infty$.

Case (2) of the above theorem is proved from case (1) via the following proposition. □

The following proposition is known classically as the Marcinkiewicz interpolation theorem.

Proposition 2.4. *Let (X, μ) be any measure space and $1 < p < \infty$. If T is a sublinear operator acting as a bounded operator on the following spaces:*

$$\begin{aligned} T :& L^1 \to wk - L^1 \\ T :& L^\infty \to L^\infty, \end{aligned} \tag{22}$$

then

$$\int_X |Tf|^p d\mu \leq C \int_X |f|^p d\mu \tag{23}$$

for each $f \in L^p$. The constant C is independent of f.

The following theorem is the analogue of the classical Lebesgue differentiation theorem. See [He] or [Ma] for its proof.

Theorem 2.5. *Let (X, d, μ) be a Vitali space. Let f be a non-negative function in $L^1_{loc}(X, \mu)$. Then for μ-almost every $x \in X$,*

$$f(x) = \lim_{r \to 0} \fint_{B(x,r)} f d\mu. \tag{24}$$

In other words, almost every point is a Lebesgue point of f.

3 Sobolev Spaces.

Let $1 \leq p < \infty$, Ω be a domain in $\mathbb{R}^n$, and μ a Radon measure on $\mathbb{R}^n$. The Sobolev space $H^{1,p}(\Omega, \mu)$ of functions on Ω is defined to be the closure of the following set

$$\{u \in C^\infty(\Omega) \ : \ \|u\|_{L^p(\mu)} + \|\nabla u\|_{L^p(\mu)} < \infty\} \tag{25}$$

under the norm

$$\|u\|_{1,p} := \|u\|_{L^p(\mu)} + \|\nabla u\|_{L^p(\mu)}. \tag{26}$$

For $u \in H^{1,p}(\Omega, \mu)$ let Du be the set of all possible limits that arise from the derivatives of sequences of C^∞-functions converging to u. That is, if $\varphi_k \to u$ in L^p and $\nabla \varphi_k \to v$ in L^p, then $v \in Du$. It is easy to see that Du is convex and closed in $L^p(\Omega)$ whenever $p \geq 1$. If $p > 1$, there is a unique element $\overline{v}$ of Du so that $\|\overline{v}\|_{L^p}$ is the smallest of all the L^p-norms of elements in Du. Define $\nabla u := \overline{v}$.

Open problem: Characterize Radon measures for which Du is of cardinality 1. It is known that if μ is doubling and supports a p-Poincaré inequality (see [He] or below for definition of support of such inequality), then Du has cardinality 1.

Open problem: Given a Cantor set $K \subset \mathbb{R}^n$, when does there exist a doubling measure μ on $\mathbb{R}^n$ such that $\mu(K) = 0$?

The space $H^{1,p}(\Omega, \mu)$ has some standard properties of Sobolev spaces, for example the lattice properties: If u, v are two functions in this space, then so are $\max\{u, v\}$ and $\min\{u, v\}$. If λ is a non-negative number, then $\min\{u, \lambda\}$ is in this space, and if λ is not positive, then $\max\{u, \lambda\}$ is in this space.

Definition 3.1. A measure μ on $\mathbb{R}^n$ is said to support a $(1,p)$-*Poincaré inequality* if there are constants $C>0$, $\tau\geq 1$ so that for each ball $B\subset\mathbb{R}^n$ and for each function $u\in C^\infty(\tau B)$ the following inequality is satisfied:

$$\fint_B |u-u_B|d\mu \leq C\left(\fint_{\tau B}|\nabla u|^p d\mu\right). \tag{27}$$

From the classical theory of Sobolev functions, it is known that for each $p\geq 1$, the n-dimensional Lebesgue measure supports a $(1,p)$-Poincaré inequality with the constant $\tau=1$ and C depending only on n and p. There are other non-trivial examples of measures supporting a $(1,p)$-Poincaré inequality; see [HeKM].

Theorem 3.1 (Semmes). *Suppose*

1. *μ is a doubling measure,*
2. *$d\mu = w dx$ with $w>0$ almost everywhere and $w\in L^1_{loc}(\mathbb{R}^n,dx)$, and*
3. *$(\mathbb{R}^n,\mu)$ supports a $(1,p)$-Poincaré inequality.*

Then Du is of cardinality 1; that is, if $\varphi_k\in C_0^\infty(\Omega)$ is a sequence converging to the zero function in the L^p-norm and $\nabla\varphi_k\to\vec{v}$ in the L^p-norm, then $\vec{v}=0$.

Open problem: If μ supports a $(1,p)$-Poincaré inequality and μ is a doubling Radon measure, then is it true that μ is absolutely continuous with respect to the Lebesgue measure?

Observation: If μ is doubling and $u:\mathbb{R}^n\to\mathbb{R}$ is a smooth function in $H^{1,p}(\mathbb{R}^n,\mu)$, then

$$|u(x)-u_{B(x,R)}| \leq CR\left(M_R|\nabla u|^p(x)\right)^{1/p}, \tag{28}$$

$$|u(x)-u(y)| \leq CR\left((M_{2R}|\nabla u|^p(x))^{1/p}+(M_{2R}|\nabla u|^p(y))^{1/p}\right). \tag{29}$$

For a proof of these facts see [HeKM], [HaK1], and [HaK2]. Hence such functions u are in the Hajłasz space; see [Ha]. Here

$$M_R f(x) := \sup_{0<r<R}\fint_{B(x,r)} f\,d\mu. \tag{30}$$

Proof of **Semmes' theorem**. Call a compact set $K \subset \mathbb{R}^n$ *helpful* if there is a constant $C = C(K) \geq 1$ so that for every pair of points $x, y \in K$ with $|x - y| \leq 1$ and for all smooth real functions u on $\mathbb{R}^n$,

$$|u(x) - u(y)| \leq CR\left((\tilde{M}|\nabla u|^p(x))^{1/p} + (\tilde{M}|\nabla u|^p(y))^{1/p}\right), \tag{31}$$

where

$$\tilde{M}f(x) = \sup_{r>0}\left(r^{-n}\int_{B(x,r)} f\,d\mu\right). \tag{32}$$

The rest of the proof is done via the following two lemmata. □

Lemma 3.2. *In the situation of the above proof, if K is helpful, then $\vec{v} = 0$ almost everywhere on K.*

Lemma 3.3. *Given u as in the above theorem, there is a countable collection K_i of helpful sets corresponding to u so that $\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} K_i$ has zero measure.*

Proof of **Lemma 3.2**. Let K be a helpful set. Fix $\lambda > 0$. Without loss of generality,

$$\left(\int_{\mathbb{R}^n} |\varphi_j - \varphi_{j+1}|^p d\mu\right)^{1/p} \leq 10^{-j}, \tag{33}$$

and

$$\left(\int_{\mathbb{R}^n} |\nabla\varphi_j - \nabla\varphi_{j+1}|^p d\mu\right)^{1/p} \leq 10^{-j}. \tag{34}$$

Denote $\varphi_{j+1} - \varphi_j =: g_j$, and let

$$G_j = \{x \in \mathbb{R}^n \ : \ \tilde{M}|\nabla g_j|^p(x) \leq (2^{-j}\lambda)^p\}, \tag{35}$$
$$H_j = \{x \in \mathbb{R}^n \ : \ |g_j(x)|^p w(x) \leq (2^j\lambda)^{-p}\}, \tag{36}$$
$$D_\lambda = \{x \in \mathbb{R}^n \ : \ w(x) \geq \lambda^{-p}\}. \tag{37}$$

Now fix $k \in \mathbb{N}$ and consider

$$K_{k,\lambda} = K \cap \left(\bigcap_{j\geq k}(G_j \cap H_j)\right) \cap D_\lambda. \tag{38}$$

On these sets $K_{k,\lambda}$, for $j \geq k$ the function g_j is $C2^{-j}\lambda$-Lipschitz, where $C > 0$ is a constant depending only on the helpfulness constant. Using McShane extension, extend g_j to all of $\mathbb{R}^n$ to get $\tilde{g}_j$ as a Lipschitz function with the same Lipschitz constant as g_j. Also on $K_{k,\lambda}$, it is true that $|g_j(x)| \leq 2^{-j}$. Hence we can choose the extension $\tilde{g}_j$ so that on all of $\mathbb{R}^n$, $\tilde{g}_j(x) \leq 10 \cdot 2^{-j}$, by truncation of the McShane extension if necessary.

Let $j > k$ and $v_j := \varphi_k + \sum_{i=k}^{j-1} \tilde{g}_i$. Then v_j is locally Lipschitz with fixed local Lipschitz constant. On the set $K_{k,\lambda}$ we have that $v_j = \varphi_j$, and hence $\nabla v_j = \nabla \varphi_j$ almost everywhere on $K_{k,\lambda}$. Now,

$$|v_j - v_{j+1}| = |\tilde{g}_j| = O(2^{-j}), \tag{39}$$

$$|\nabla v_j - \nabla v_{j+1}| = |\nabla \tilde{g}_j| = O(2^{-j}). \tag{40}$$

Here equation (40) is because $\tilde{g}_j$ is $C2^{j}\lambda$-Lipschitz. By (39), the sequence v_j is Cauchy in L^∞_{loc}, and hence $v_j \to v_0$ locally uniformly in $\mathbb{R}^n$. Therefore v_0 is locally Lipschitz in $\mathbb{R}^n$. Also, $\nabla v_j \to \nabla v_0$ almost everywhere in $\mathbb{R}^n$, and $\nabla \varphi_j = \nabla v_j$ on $K_{k,\lambda}$. Since on $K_{k,\lambda}$ $v_j = \varphi_j$, and $\varphi_j \to 0$ almost everywhere, therefore $v_0 = 0$ on $K_{k,\lambda}$. Hence $\nabla v_0 = 0$ on $K_{k,\lambda}$.

We now complete the proof by showing that the Lebesgue measure of $K \backslash K_{k,\lambda}$ tends to zero as $k \to \infty$ and then $\lambda \to \infty$. Note that

$$K \backslash K_{k,\lambda} = \left(\bigcup_{j \geq k} G_j^C \bigcup_{j \geq k} H_j^C \bigcup D_\lambda^C \right) \cap K. \tag{41}$$

Note also that $\tilde{M} f$ is the classical Hardy-Littlewood maximal function $M(fw)$. Hence, if $f_j = |\nabla g_j|^p \in L^1(\mathbb{R}^n, \mu)$, then by inequality (34),

$$\begin{aligned} |G_j^C| &= |\{\tilde{M} f_j > (2^{-j}\lambda)^p\}| && (42)\\ &= |\{M f_j w > (2^{-j}\lambda)^p\}| && (43)\\ &\leq \frac{C}{(2^{-j}\lambda)^p} \int_{\mathbb{R}^n} |\nabla g_j|^p d\mu && (44)\\ &\leq C \frac{5^{-jp}}{\lambda^p}, && (45) \end{aligned}$$

and it follows that $|\bigcup_{j \geq k} G_j^C| \leq C 5^{-kp} \lambda^{-p} \to 0$ as $k \to \infty$. Also,

$$|H_j^C| = |\{|g_j(x)|^p w(x) > (2^j \lambda)^{-p}\}|, \tag{46}$$

and by using the fact that $\|g_j\|_{L^p(d\mu)} \le 10^{-j}$, we can see that $|H_j^C| \le 5^{-jp}\lambda^p$. In fact, by inequality (33),

$$
\begin{aligned}
|H_j^C| = \int_{H_j^C} dx \le & \int_{H_j^C} (2^j\lambda)^p |g_j|^p d\mu \\
\le & (2^j\lambda)^p \int_{\mathbb{R}^n} |g_j|^p d\mu \le 5^{-jp}\lambda^p.
\end{aligned} \tag{47}
$$

Therefore, $|\bigcup_{j\ge k} H_j^C| \to 0$ as $k \to \infty$. Since $w > 0$ almost everywhere, clearly $|D_\lambda^C| \to 0$ as $\lambda \to \infty$. Now it follows that $K \backslash K_{k,\lambda} \to 0$ as first $k \to \infty$ and then $\lambda \to \infty$. This completes the proof of the lemma. $\square$

Proof of **Lemma 3.3**. Let

$$
\Omega_j = \{x \in \mathbb{R}^n \ : \ \frac{1}{j} r^n \le \mu(B(x,r)) \le j r^n\} \tag{48}
$$

with r some fixed number such that $0 < r < 10$. The sequence of sets Ω_j exhausts $\mathbb{R}^n$ and it is easy to see that they are helpful, for

$$
\mu(B(x,r)) = \int_{B(x,r)} w dx \ge \frac{w(x)}{2} |B(x,r)| \tag{49}
$$

for all Lebesgue points $x \in \Omega_j$ of the function w (we already know that Lebesgue points are almost everywhere in $\mathbb{R}^n$) and for all $r < r_x$ for some positive number r_x. Here we denote the Lebesgue measure of $B(x,r)$ as $|B(x,r)|$. $\square$

Open problem: Is Semmes' theorem true if we remove the condition that μ be absolutely continuous with respect to the Lebesgue measure? Characterize the measures μ in $\mathbb{R}^n$ for which the conclusion of Semmes' theorem holds. See the following example for a measure for which non-uniqueness of derivatives occurs.

Example 3.4. Let us look at the unit interval $[0,1]$ in $\mathbb{R}$. Let α be a number so that $2\alpha + 1 > p$, and let

$$
w_n(x) = \begin{cases} |x - \frac{k}{2^n}|^\alpha & \text{if } x \in \overline{B(\frac{k}{2^n}, \frac{1}{2^{2n}})} \\ 1 & \text{otherwise,} \end{cases} \tag{50}
$$

and define

$$w(x) = \prod_{n=2}^{\infty} w_n(x). \tag{51}$$

Now we define a measure μ on $[0,1]$ by $d\mu(x) = w(x)dx$. It can be shown that the above construction of w yields a non-trivial measure, and in fact, $w(x) > 0$ almost everywhere. It can also be shown that μ is not a doubling measure; there is a sequence of intervals violating the doubling condition for each constant $k \in \mathbb{N}$. Now we define the sequence φ_n of Lipschitz functions according to the following figure.

It can be shown that $\int_0^1 \varphi_n d\mu \to 0$ and that $\int_0^1 |\varphi_n' - 1| d\mu \to 0$. Hence $\varphi = \lim_n \varphi_n$ has both 0 and 1 as derivatives.

Nontrivial fact: There are singular doubling measures on $\mathbb{R}$; see for example [AB]. In fact, these measures can also be supported on a set of Hausdorff dimension smaller than ϵ for any pre-assigned $\epsilon > 0$.

4 Analysis of Semmes' Proof and D-structure.

In proving Semmes' theorem, we needed the following tools:

1. Need (X, d, μ) metric measure space, with μ doubling.
2. $\mathcal{C}$, the collection of locally Lipschitz functions.
3. A measurable *Hilbert vector bundle*

   $$\begin{array}{c} E \\ \downarrow \\ X \end{array}$$

   that is, for μ-almost every point $x \in X$ there is an associated Hilbert space E_x with inner product $< \ , \ >_x$ so that the following criteria are satisfied.
4. There is an operator $D : \mathcal{C} \to \Gamma(E)$ (where $\Gamma(E)$ is the collection of all measurable sections of E) so that for every function u in $\mathcal{C}$ the map $Du(x) \in E_x$ satisfies
   - D is linear: $D(u + v) = Du + Dv$,
   - $|Du| \leq L$ a.e. if u is L-Lipschitz,
   - $Du = 0$ almost everywhere on the set $\{u = \text{const.}\}$, and
   - for each $u \in \mathcal{C}$ the map $x \mapsto |Du(x)| = \langle Du(x), Du(x)\rangle_x^{1/2}$ is measurable.
5. Need a $(1, p)$-Poincaré inequality for functions in $\mathcal{C}$ with respect to the measure μ and the function $|Du|$.
6. If u_j is a sequence in $\mathcal{C}$ tending to zero in its norm and $Du_j \to s$ locally uniformly, then $s = 0$ almost everywhere.

Conditions (1) through (4) are sufficient to define the space $H^{1,p}(X, \mathcal{C}, D)$ as the closure of the collection

$$\{u \in \mathcal{C} \ : \ \|u\|_{L^p} + \||Du|\|_{L^p} < \infty\} \tag{52}$$

under the norm

$$\|u\|_{1,p} := \|u\|_{L^p} + \||Du|\|_{L^p}. \tag{53}$$

The other criteria are needed to ensure that the domain of the operator D can be extended to all of $H^{1,p}$.

Theorem 4.1. *Under conditions* $(1)-(6)$*, the operator* D *can be extended to all of* $H^{1,p}$ *as a measurable linear operator. Furthermore, if* w *is a function on* X *such that* $w(x) > 0$ *for* μ*-almost every* x *and the measure* ν *defined by* $d\nu = wd\mu$ *is doubling and supports a* $(1,p)$*-Poincaré inequality, then* D *can be extended to* $H^{1,p}(d\nu)$*.*

Condition (6) is redundant: see Franchi, Hajłasz and Koskela [FHK]. It also suffices to have E_x be a Banach space with norm $\|\cdot\|_x$ for almost every $x \in X$.

Proof. If $u \in \mathcal{C}$, then for all pairs of points $x, y \in X$,

$$|u(x) - u(y)| \le C|x-y| \left(M|Du|^p(x)^{1/p} + M|Du|^p(y)^{1/p} \right). \tag{54}$$

This can easily be shown from the assumption of Poincaré inequality. If $u \notin \mathcal{C}$, then there is a sequence u_j of functions in $\mathcal{C}$ so that $u_j \to u$ in the norm of $H^{1,p}$. Without loss of generality, it can be assumed that

$$\int_X |u_j - u_{j+1}|^p d\mu + \int_X |Du_j - Du_{j+1}|^p d\mu \le 10^{-jp}. \tag{55}$$

Note that the above inequality also implies that $u_j \to u$ almost everywhere with respect to the μ-measure and hence with respect to the Lebesgue measure. Just as in the proof of Semmes' Theorem 3.1, let $v_j = u_{j+1} - u_j$. Then by inequality (54),

$$|v_j(x) - v_j(y)| \le C|x-y| \left(M|Dv_j|^p(x)^{1/p} + M|Dv_j|^p(y)^{1/p} \right), \tag{56}$$

and hence, if l and k_0 are positive integers with $l > k_0$, then

$$\begin{aligned} |(u_l - u_{k_0})(x) - &(u_l - u_{k_0})(y)| \\ &\le C|x-y| \left(\sum_{k=k_0}^{\infty} M|Dv_k|^p(x)^{1/p} + \sum_{k=k_0}^{\infty} M|Dv_k|^p(y)^{1/p} \right). \end{aligned} \tag{57}$$

For $z \in X$ let

$$g_{k_0}(z) := \sum_{k=k_0}^{\infty} M|Dv_k|^p(z)^{1/p}, \tag{58}$$

and look at the set $\{z \in X \ : \ g_{k_0}(z) > t\}$ for a fixed positive number t (compare with the set G_j^C in the proof of Theorem 3.1). Since $\sum_{n=k_0}^{\infty} 2^{-n}t \simeq 2^{-k_0}t$, it follows that for each z in this set,

$$\sum_{n=k_0}^{\infty} 2^{k_0-n}t \simeq t < g_{k_0} = \sum_{k=k_0}^{\infty} M|Dv_k|^p(z)^{1/p}. \tag{59}$$

Here $\simeq$ means that the two quantities are equivalent; $f(\vec{s}) \simeq g(\vec{s})$ iff there is a constant $C > 0$ so that for every $\vec{s}$ in the domain of f and g the two-sided inequality $f(\vec{s})/C \leq g(\vec{s}) \leq Cf(\vec{s})$ holds. By inequality (59), for each z in the set $\{z \in X \ : \ g_{k_0}(z) > t\}$ there is an integer n with $k_0 \leq n < \infty$, so that $2^{k_0-n}t/C \leq M|Dv_n|^p(z)^{1/p}$. Therefore,

$$\begin{aligned}
\mu(\{z \in X \ : \ g_{k_0}(z) > t\}) &\leq \sum_{j=k_0}^{\infty} \mu\left(\left\{z \in X \ : \ M|Dv_j|^p(z)^{1/p} > \frac{t}{2^{j-k_0}C}\right\}\right) \\
&\leq \sum_{j=k_0}^{\infty} C\frac{\||Dv_j|\|_{L^p}^p}{t^p/2^{(j-k_0)p}} \\
&\leq \sum_{j=k_0}^{\infty} C\frac{1}{t^p}\frac{5^{-jp}}{2^{k_0p}}.
\end{aligned} \tag{60}$$

Hence

$$\mu(\{z \in X \ : \ g_{k_0}(z) > t\}) \leq C\frac{1}{t^p}2^{-k_0p} \to 0 \text{ as } \ k_0 \to \infty. \tag{61}$$

If x, y are two points in $X \backslash \{z \in X \ : \ g_{k_0}(z) > t\}$ so that $u_l(x), u_l(y) \to 0$ as $l \to \infty$, then

$$|(u_l - u_{k_0})(x) - (u_l - u_{k_0})(y)| \to |u_{k_0}(x) - u_{k_0}(y)| \leq Ct|x-y|. \tag{62}$$

Therefore u_{k_0} is Ct-Lipschitz almost everywhere in $X \backslash \{z \in X \ : \ g_{k_0}(z) > t\}$. Therefore the Lipschitz constant of u_{k_0} restricted to this set is bounded above by Ct, and hence by comparing with the McShane extension and by Condition

(4), we conclude that $|Du_{k_0}| \leq Ct$. Without loss of generality, $Du_j \to s$ almost everywhere. Therefore, $|s| \leq Ct$. However, since $\mu(\{z \in X \;:\; g_{k_0}(z) > t\}) \to 0$ as $k_0 \to \infty$ and without loss of generality $|s| \leq 2g_{k_0}$ for all k_0 since

$$\sum_{k_0}^{\infty} |Dv_k| \geq |Du_{k_0}| \to |s| \text{ as } k_0 \to \infty, \tag{63}$$

therefore $|s| \leq Ct$ almost everywhere. Since $t > 0$ was arbitrarily chosen, it follows that $|s| = 0$ almost everywhere. □

Example 4.2. The following examples support conditions (1) – (6).

1. $\mathbb{R}^n$, equipped with Euclidean metric and Lebesgue n-measure, with $E_x = \mathbb{R}^n$ for each $x \in \mathbb{R}^n$, the D-operator being the gradient operator ∇.

2. $\mathbb{R}^n$, equipped with Euclidean metric and any doubling measure that supports a $(1,p)$-Poincaré inequality. Here again $E_x = \mathbb{R}^n$ for each $x \in \mathbb{R}^n$, the D-operator being the gradient operator ∇.

3. Let X be a compact polyhedron in $\mathbb{R}^n$, equipped with Euclidean metric and the localized Hausdorff measure with respect to the local Hausdorff dimension. Here $E_x = \mathbb{R}^{n_x}$ with n_x the local Hausdorff dimension of X at x, and Du is the ordinary derivative of u, which exists almost everywhere on X by the Rademacher-Stepanov theorem.

In this last example, the measure is doubling iff X is purely d-dimensional for some d; that is, $\lim_{r\to 0} \mathcal{H} - \dim(B(z,r)) = d$ for every $z \in X$. In this case, X supports a $(1,p)$-Poincaré inequality for every $p > d$.

The issue in the above proof is local: If $X = \bigcup_i X_i \cup$ a null set , where for each i the measure $\mu\big|_{X_i}$ is doubling and $(X_i, \mu\big|_{X_i})$ admits a $(1,p)$-Poincaré inequality, then the conclusion of the above theorem holds for all of X.

5 Rectifiable Sets.

The Banach bundle $E = \{(x, E_x) \;:\; x \in X\}$ in the above discussion corresponds to the Finsler bundle.

Example 5.1. The following examples are spaces with such bundles.

1. Finsler manifolds.

2. Polyhedra or simplicial complex X in $\mathbb{R}^n$. If Δ_d is a simplex of dimension d in X, then $\mu\big|_{\Delta_d} = \mathcal{H}_d\big|_{\Delta_d}$ is doubling. The set of points joining two different simplices in X is of μ-measure zero. We have $(1,p)$-Poincaré inequality on each simplex Δ_d.

3. Spaces that are n-rectifiable sets in $\mathbb{R}^N$.

We next define what we mean by a rectifiable set. Let $1 \leq n \leq N$ be an integer.

Definition 5.1. $X \subset \mathbb{R}^N$ is *n-rectifiable* if

1. The n-dimensional Hausdorff measure on X is locally finite, and

2. $X \subset \bigcup_{j\geq 1} f_j(\mathbb{R}^n) \cup$ a Hausdorff n-dimensional zero set where the map $f_j : \mathbb{R}^n \to \mathbb{R}^N$ is Lipschitz for each j.

The n-dimensional rectifiable sets have a natural D-structure.

Example 5.2. For each $q_n \in \mathbb{Q} \times \mathbb{Q} \subset \mathbb{R}^2$, let S_n be the sphere of radius 2^{-n} centered at q_n. Now, let $X = \bigcup_n S_n$.

Definition 5.2. Let Y be any subset of $\mathbb{R}^N$. The *tangent cone* of Y at a point $a \in Y$ is

$$\mathrm{Tan}(a,Y) = \mathbb{R} \bigcap_{\epsilon>0} \text{closure}\left(\left\{ \frac{x-a}{|x-a|} \ : \ x \in B(a,\epsilon) \cap Y, x \neq a \right\}\right), \tag{64}$$

where the closure is taken in the norm on the unit sphere in $\mathbb{R}^N$. The *approximate tangent cone* of dimension n is defined to be the set

$$\mathrm{apTan}_n(a,X) = \bigcap_S \mathrm{Tan}(a,S), \tag{65}$$

where $S \subset X$ such that the n-density of $X \backslash S$ at the point a is zero.

The n-density of S at $a \in S$ is the number

$$\lim_{r\to 0} \frac{\mathcal{H}_n(S \cap B(a,r))}{\alpha(n) r^n}, \tag{66}$$

whenever the limit exists. Here the balls $B(a,r)$ are balls in $\mathbb{R}^N$, and $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$.

Theorem 5.3. *Let $X \subset \mathbb{R}^N$ be a rectifiably connected set with dimension n. Then for $\mathcal{H}_n$-almost every $x \in X$ the approximate tangent cone $apTan_n(x, X)$ is an n-dimensional vector subspace of $\mathbb{R}^N$.*

Hence if X is n-rectifiable, then in the discussion about D-structures we can take F_x to be the space $\mathrm{Hom}(\mathrm{apTan}_n(x, X), \mathbb{R})$.

For a proof of the above theorem, see Federer, Morgan, Besicovich(n=1), Semmes, etc.

We can talk about n-rectifiable subsets of *any* metric measure space M, by replacing $\mathbb{R}^N$ with M in the above discussion. There is a theory about such sets, but it is too complicated to be covered in this course.

Open problem: Characterize metric spaces that are biLipschitz embeddable in some $\mathbb{R}^N$. In the classes of ultra-metric spaces this known: they are basically Cantor sets. A metric is called an *ultra-metric* if for every triple of points $x, y, z \in X$ the inequality $d(x, y) \leq \max\{d(x, z), d(y, z)\}$ holds.

Open problem: Are there D-structures that are extendable to $H^{1,p}$ for some p but not all p?

Definition 5.3. Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$, ν a measure on $\mathbb{R}^n$, and s be a positive integer. Let $a \in A$. Say f is (ν, s)-*approximately differentiable* at a if there is an open subset U of $\mathbb{R}^n$ and a function $g : U \to \mathbb{R}^m$ so that g is differentiable at a and

$$\limsup_{r \to 0} \frac{\nu\big|_A(\{x \in A \ : \ f(x) \neq g(x)\} \cap B(a, r))}{r^s} = 0. \tag{67}$$

It is verifiable that if A is s-rectifiable and $\nu = \mathcal{H}_s$, then

$$\mathrm{ap}Df(a) := \nabla g(a) : \mathbb{R}^n \to \mathbb{R}^m \tag{68}$$

is independent of the function g. Note that $\mathrm{ap}Df(a) \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$.

Theorem 5.4. *Let $X \subset \mathbb{R}^N$ be n-rectifiable with locally finite Hausdorff measure $\mathcal{H}_n$. Then with respect to this measure, for almost every point $x \in X$,*

1. $\Theta^n(x, X) := \lim_{r \to 0} \frac{\mathcal{H}_n(B(x,r) \cap X)}{\alpha(n) r^n} = 1$,
2. *$apTan_n(x, X)$ is an n-dimensional subspace of $\mathbb{R}^N$, and*
3. *if $f : X \to \mathbb{R}$ is Lipschitz, then f is approximately differentiable almost everywhere with respect to the index $s = n$.*

Here $\alpha(n)$ is the Lebesgue n-dimensional measure of the unit ball $B(0,1)$ in $\mathbb{R}^n$.

Sketch of Proof: Let

$$\overline{\Theta}^n(x,X) := \limsup_{r\to 0} \frac{\mathcal{H}_n(B(x,r)\cap X)}{\alpha(n)r^n}, \tag{69}$$

and

$$\underline{\Theta}^n(x,X) := \liminf_{r\to 0} \frac{\mathcal{H}_n(B(x,r)\cap X)}{\alpha(n)r^n}, \tag{70}$$

Without loss of generality, the n-dimensional Hausdorff measure of X is finite. In general, $\overline{\Theta}^n(x,X) = 0$ for almost every $x \in \mathbb{R}^N\backslash X$. Also, $\overline{\Theta}^n(x,X) \leq 1$ for every $x \in X$. These two claims follow from a covering argument.

To prove that $\Theta^n(x,X)$ exists and is equal to 1, the following claim is used. **Claim:** Given any $\lambda > 1$ there exists a countable number of compact sets $K_1, K_2, ... \subset \mathbb{R}^n$ and λ-biLipschitz functions $\psi_j : K_j \to X$ so that $\psi_i(K_i) \cap \psi_j(K_j) = \emptyset$ if $i \neq j$ and $X\backslash \bigcup_{j=1}^{\infty} \psi_j(K_j)$ has zero measure.

This claim is proved by doing several "countable processes". By the definition of n-rectifiability, we have Lipschitz functions f_j. By the Rademacher-Stepanov theorem, we have differentiability of the functions f_j almost everywhere in $\mathbb{R}^n$. By a generalization of Sard's theorem, $\mathcal{H}_n(f_j(\{y : J_{f_j}(y) = 0\})) = 0$. Therefore, we can consider only the points of $\mathbb{R}^n$ at which f_j are differentiable and $J_{f_j} > 0$. Call this collection of points corresponding to f_j as G_j. Now a countable process decomposes $\mathbb{R}^n$ into sets where each f_j is biLipschitz and the sets are compact. Thereby we find K_j as in the claim, except that the biLipschitz constant can be large, not the prescribed value λ. However, we can "correct" the functions f_j by a linear transformation of $\mathbb{R}^n$ to improve the biLipschitz constant to λ.

To prove the third condition we look at the Lipschitz function $f \circ \psi_j$ on $\psi_j^{-1}(K_j)$ and apply Rademacher's theorem. □

Note: If $X \subset \mathbb{R}^N$, $\mathcal{H}_s(X) < \infty$ for some $s > 0$, and $\Theta(x,X) = 1$ for $\mathcal{H}_s$-almost every $x \in X$, then s is an integer.

Note: If $(\mathbb{R}^n, \mu)$ is a Radon measure space, then it is a Vitali space. This is not true in arbitrary compact metric spaces: see [D] and [PT]. However, if the metric space is a separable Banach space endowed with a Radon measure, then it is a Vitali space.

The rest of this section is based on Whitney's book, [W]. Let $\mathcal{C}^d$ be the collection of oriented polyhedral d-chains in $\mathbb{R}^N$, with d non-negative integer (see below for formal definition). Then, by the fact that any polyhedral region is decomposable into finitely many elementary polyhedra, it is clear that $\mathcal{C}^d$ is a vector space. An element σ of this vector space can be written as

$$\sigma = \sum_{j=1}^{k} a_j \sigma_j^d, \tag{71}$$

where σ_j^d are oriented d-dimensional polyhedra with disjoint interiors. Note that $\sigma_j^d \cap \sigma_i^d$ is lower dimensional when $j \neq i$. We can equip $\mathcal{C}^d$ with the norm, called the *flat norm,* as follows: For $A \in \mathcal{C}^d$, let

$$|A|^\flat := \inf\{|A - \partial D| + |D| \ : \ D \in \mathcal{C}^{d+1}\}, \tag{72}$$

where

$$\partial : \mathcal{C}^{d+1} \to \mathcal{C}^d \tag{73}$$

is a linear map analogous to the algebraic topological construction. This is illustrated by the following example.

Let A be the 1-chain which is the perimeter of a planar rectangle of length 1 and height ϵ. Then, $|A| = 2(1+\epsilon)$. If we consider the 2-chain that is the planar rectangle A bounds, then $A = \partial D$, and in this case $|A - \partial D| = 0$ and $|D| = \epsilon$, and hence we have that $|A|^\flat \leq \epsilon$.

As another example, consider A to be the chain composed of two flat lines meeting at a vertex. Each line is of length 1. Hence $|A| = 2$. However, if we take D to be the planar triangle shown in the figure below, oriented clockwise, then $|A - \partial D| = 2(1-\epsilon_1) + \epsilon_2$, where ϵ_1 is the length of the side of the triangle lying on top of each of the lines comprising A, and ϵ_2 is the length of the third side of the triangle D. It is easy to see that $|A|^\flat < 2$. In fact, if we choose D to be a polygon made up of the two sides the triangle lying on the component lines of A together with some more lines that cut into the triangle considered above, we can reduce the estimate $|A - \partial D| + |D|$ further. We do not know what the best estimate in this case is.

The following proposition has been proved in [W, Chapter V Section 12 (Theorem 12A)].

Proposition 5.5. *The flat norm* $|\cdot|^\flat$ *is a norm on* $\mathcal{C}^d$.

The non-trivial fact in the above proposition is that if $A \neq 0$, then $|A|^\flat > 0$.

Definition 5.4. A *flat d-chain* is an element of the completion of $\mathcal{C}^d$ under the flat norm.

Fact: Let $X \subset \mathbb{R}^N$ be n-rectifiable with $\mathcal{H}_n(X)$ finite. Then X is a flat n-chain, that is, there exists a sequence of n-chains $A_k \in \mathcal{C}^n$ with "$A_k \to X$" in the flat norm. We will show what is meant by "$A_k \to X$" in a short while.

Now let us formalize what is meant by a d-chain.

Definition 5.5. Let E^d be an oriented affine d-plane in $\mathbb{R}^N$. Consider oriented d-dimensional polyhedral regions σ_i in such affine d-planes $E_i^d \subset \mathbb{R}^N$ so that $\mathrm{int}_{E_i^d}\sigma_i \cap \mathrm{int}_{E_j^d}\sigma_j = \emptyset$ if $i \neq j$. These regions σ_i come with an orientation (unless $d = 0$, in which case the "regions" are just points). A *polyhedral d-chain* is an algebraic sum of such regions: $A = \sum_{i=1}^k a_i\sigma_i$; $a_i \in \mathbb{R}$.

Note that we do not need $\bigcup_i \sigma_i$ to be connected here; the following figure is a possible choice of such polyhedral 1-chain.

We want a polyhedral region to be identified with the algebraic sum of its subdivisions into smaller polyhedral regions. This is Hypothesis (H2) in Whitney's book. So, when is $A \sim B$ if $B = \sum a'_j \sigma'_j$? We define equivalence via a function that is associated to each polyhedral d-chain as follows. Let $A = \sum_{i=1}^k a_i \sigma_i$. Then we define an associated function $A : \mathbb{R}^N \to \mathbb{R}$ by:

$$A(p) = \begin{cases} a_i & \text{if } p \in \mathrm{int}_{E^d}\sigma_i \text{ provided the orientation on} \\ & \sigma_i \text{ is the same as the orientation on } E^d \\ -a_i & \text{if } p \in \mathrm{int}_{E^d}\sigma_i \text{ provided the orientation on} \\ & \sigma_i \text{ is opposite to the orientation of } E^d \\ 0 & \text{otherwise.} \end{cases} \tag{74}$$

Definition 5.6. Let A and B be two polyhedral d-chains. We say that $B \sim A$ if the set $\{p \in \mathbb{R}^N \ : \ B(p) \neq A(p)\}$ has dimension smaller than d.

Definition 5.7. The space $\mathcal{C}_d^{pol}(\mathbb{R}^N)$ is the collection of equivalence classes of all polyhedral d-chains, under the equivalence $\sim$ defined above.

In general, a polyhedral d-chain in $\mathbb{R}^N$ is a sum $A = \sum_{i=1}^k a_i \sigma_i$. If A, B are in $\mathcal{C}_d^{pol}(\mathbb{R}^N)$ with $A = \sum_{i=1}^k a_i \sigma_i$ and $B = \sum_{j=1}^l b_j \tau_j$, then for $a \in \mathbb{R}$, we define $aA = \sum_{i=1}^k a a_i \sigma_i$. Whenever σ_i and τ_j intersect in their d-interiors, they intersect along a lower dimensional polyhedron, and we subdivide them to a common refinement to get $A = \sum_{i=1}^m a'_i \sigma'_i$, $B = \sum_{i=1}^m b'_i \sigma'_i$, and we define $A+B = \sum_{i=1}^m (a'_i + b'_i)\sigma'_i$. Some of the a'_i, b'_i can be zero. Under these operations, $\mathcal{C}_d^{pol}(\mathbb{R}^N)$ is a vector space.

Next, we need the "boundary map" $\partial : \mathcal{C}_d^{pol} \to \mathcal{C}_{d-1}^{pol}$ so that ∂ respects the given equivalence relation, is linear, and $\partial^2 = 0$. We define this map by stating what the image of d-dimensional polyhedral regions are, and then extending the map to all of $\mathcal{C}_d^{pol}$ by linearity. Let σ be such a oriented d-dimensional polyhedral region in E^d. Then the topological boundary of σ in E^d is a finite union of $d-1$-dimensional polyhedral regions, with orientations inherited from the orientation of σ. Let $\partial(\sigma)$ be the algebraic sum of these $d-1$-dimensional regions. If $d = 0$, then ∂ is the zero map.

Definition 5.8. Let $A \in \mathcal{C}_d^{pol}$. We can write $A = \sum_{i=1}^k a_i \sigma_i$ with σ_i oriented d-dimensional polyhedral regions. The *mass* of A is defined to be

$$|A| = \sum_{i=1}^k |a_i||\sigma_i| = \int_{\bigcup_i \sigma_i} |A(p)|, \tag{75}$$

where $|\sigma_i|$ is the Lebesgue d-dimensional area of the region σ_i. The mass of a 0-dimensional polyhedral region (a single point) τ is $|\tau| = 1$.

Now the flat norm $|A|^\flat$ is defined via equation (72). It is easy to see that $0 \leq |A|^\flat \leq |A|$.

Definition 5.9. The completion of $\mathcal{C}_d^{poly}$ under the flat norm is the *space of flat d-chains,* denoted $\mathcal{C}_d^\flat(\mathbb{R}^N)$.

Let us look at an example now. The following example is for $d = 1$, and the object we get in the limit is a snowflake.

Here $|A_n| \to \infty$, but in the flat norm, $\{A_i\}_i$ is a Cauchy sequence.

$$|A_n|^\flat \leq |A_n - \partial D_n| + |D_n| = |D_n| \leq C < \infty, \tag{76}$$

where D_n is the 2-polyhedron bounded by A_n and C is independent of n. Therefore,

$$|A_n - A_m|^\flat \leq |D_n - D_m| \to 0 \text{ as } m, n \to \infty. \tag{77}$$

Lemma 5.6. *The map ∂ extends to a bounded operator $\partial : \mathcal{C}_d^\flat \to \mathcal{C}_{d-1}^\flat$.*

Proof. It suffices to show that $|\partial A|^\flat \leq |A|^\flat$ when A is a polygonal region. To see this, note that

$$\begin{aligned} |\partial A|^\flat \leq& |\partial A - \partial(A - \partial D)| + |A - \partial D| \\ =& |\partial(A - (A - \partial D))| + |A - \partial D| \\ =& |A - \partial D| \leq |A - \partial D| + |D| \end{aligned} \tag{78}$$

for all D. $\square$

It is easy to see that for every flat d-chain A,

$$|A|^\flat = \inf\{|A - \partial D|^\flat + |D|^\flat \ : \ D \in \mathcal{C}^\flat_{d+1}\}. \tag{79}$$

Call the elements of the dual $(\mathcal{C}^\flat_d)^*$ of $\mathcal{C}^\flat_d$ the *flat d-co-chains.* This dual space is equipped with the operator norm.

The smooth d-forms in $\mathbb{R}^N$ are examples of such flat co-chains. Let ω be a C^∞ d-form in $\mathbb{R}^N$. Then we can write $\omega = \sum_I \omega_I dx_I$, where $\omega_I \in C^\infty(\mathbb{R}^N)$ and $I \in \mathbb{N}^d$. Let $A = \sum_i a_i \sigma_i$ be a d-chain. We can consider ω to be acting on A by the following action:

$$\langle \omega, A \rangle = \sum_i \int_{\sigma_i} a_i \omega. \tag{80}$$

If $A = \sigma$, a d-dimensional polygonal region, then by Stokes' theorem,

$$\begin{aligned} |\langle \omega, A \rangle| = |\langle \omega, \sigma \rangle| &= |\int_\sigma \omega| \\ &= |\int_{\sigma - \partial\tau} \omega + \int_{\partial\tau} \omega| \\ &\leq \|\omega\|_\infty |\sigma - \partial\tau| + \|d\omega\|_\infty |\tau| \\ &\leq \|\omega\|_{1,\infty}(|\sigma - \partial\tau| + |\tau|). \end{aligned} \tag{81}$$

Therefore,

$$|\langle \omega, A \rangle| \leq \|\omega\|_{1,\infty} |A|^\flat, \tag{82}$$

where $\|\omega\|_{1,\infty} = \|\omega\|_\infty + \|d\omega\|_\infty < \infty$. As polygonal regions are the bases of a dense subspace of $\mathcal{C}^\flat_d$, the flat norm (or the operator norm) on smooth d-forms is smaller than the Sobolev norm $\|\omega\|_{1,\infty}$. Now we have that the space of smooth d-forms with the flat norm embeds continuously into the dual space $(\mathcal{C}^\flat_d)^*$. Hence, $\mathcal{C}^\flat_d$ embeds continuously into the dual of the space of flat forms;

$$\mathcal{C}^\flat_d \hookrightarrow \{\text{flat } d\text{-forms}\}^*. \tag{83}$$

Definition 5.10. Let $\Omega \subset \mathbb{R}^N$ be open, and m be a non-negative integer. Let $D^m(\Omega)$ denote the collection of smooth m-forms with compact support in Ω. This has a topological vector space (TVS) structure. In this structure, its dual $D^m(\Omega)^* =: D_m(\Omega)$ is called the *space of m-currents.*

More precisely, $T \in D_m(\Omega)$ if

1. $T : D^m(\Omega) \to \mathbb{R}$ is linear,
2. for every compact set $K \subset \Omega$ there is a constant $M > 0$ and $i \geq 0$ so that

$$|T(\varphi)| \leq M \sup_{0 \leq |\alpha| \leq i} \|\partial^\alpha \varphi\|_{\infty,K} \tag{84}$$

whenever $\varphi = \sum_I \varphi_I dx_I$ and the support of φ is contained in K.

Here α, I are multi-indexes.

There is a linear map $\partial : D_m \to D_{m-1}$ so that $\partial^2 = 0$ and $\langle \partial T, \omega \rangle := \langle T, d\omega \rangle$ so that ∂T is also a current. By the above discussion, $\mathcal{C}_m^\flat \hookrightarrow D_m$.

Example 5.7. Let a, b be two points in $\mathbb{R}^N$, and let $T = [a, b]$ be the line segment joining a to b, equipped with the orientation given by the direction from a to b. Then T is an element of D_1. To see this, note that if ω is a 1-form, then we can define the action by T as follows:

$$\langle T, \omega \rangle = \int_{[a,b]} \omega. \tag{85}$$

If f is a function (that is, a 0-form), then

$$\langle \partial T, f \rangle = \langle T, df \rangle = \int_{[a,b]} df = f(b) - f(a). \tag{86}$$

Hence, $\partial T = \delta_b - \delta_a$, where δ_s is the Dirac delta distribution based at $s \in \mathbb{R}^N$.

Note that if $\omega \in D^d$ and $A \in \mathcal{C}_d^{pol}$, then

$$|\langle A, \omega \rangle| = |\sum_i \int_{\sigma_i} a_i \omega| \leq \max\{\|\omega\|_\infty, \|d\omega\|_\infty\} |A|^\flat. \tag{87}$$

Define an equivalent flat norm on ω to be

$$\|\omega\|_F := \max\{\|\omega\|_\infty, \|d\omega\|_\infty\}. \tag{88}$$

Then $|\langle A, \omega \rangle| \leq \|\omega\|_F |A|^\flat$. Recall that $\mathcal{C}_d^{pol}$ embeds continuously into D_d. Whitney showed that this embedding is isometric under the operator norm induced on D_d by the above flat norm on D^d.

The following theorem tells us what the dual of $\mathcal{C}_d^{pol}$ is.

Theorem 5.8. *The dual of $\mathcal{C}_d^{pol}$ is the collection of d-forms ω with bounded and measurable coefficients such that $d\omega$ also has bounded and measurable coefficients. Here $d\omega$ is the distributional derivative of ω.*

The following theorem tells us why rectifiable sets are limits of flat d-chains.

Theorem 5.9. *Suppose that $X \subset \mathbb{R}^N$ is d-rectifiable. Then X determines a d-current.*

Proof. For almost every $x \in X$ there is an approximate tangent cone $T_xX := \mathrm{apTan}_d(x, X)$. Choose an orthonormal orientation ξ_x on these tangent cones in a measurable way. Let $\xi = (\xi_x)_{x \in X}$. The pair (X, ξ) determines a current; if ω is a smooth d-form, then

$$\int_X \langle w(x), \xi_x \rangle d\mathcal{H}_d(x) =: \langle (X, \xi), \omega \rangle. \tag{89}$$

□

Definition 5.11. A current $T \in D_d$ is said to be a *rectifiable current* if it is represented by a d-rectifiable set as in the proof of the above theorem, with multiplicities allowed; that is, the orientation is given by $\xi_x = n(x)$orientation basis, where $n(x)$ is an integer, and

$$\int_{X \cap K} |\xi_x| d\mathcal{H}_d(x) < \infty \tag{90}$$

for every compact $K \subset \mathbb{R}^N$.

Multiplicities are allowed to take care of issues such as X crossing itself; for example, if X is a loop.

The action of (X, ξ) on D^d can be extended to all of $(\mathcal{C}_d^\flat)^*$ in the flat norm. This follows from the Hahn-Banach theorem or the fact that smooth d-forms are dense in $(\mathcal{C}_d^\flat)^*$. Note that if $d\omega \in L^\infty$, then ω "sees" d-dimensional spaces.

Theorem 5.10. *Given X as above, there exists a sequence $A_i \in \mathcal{C}_d^{pol}$ so that $\lim_j A_j = X$ where the limit is taken in the norm of $(\mathcal{C}_d^\flat)^{**}$; that is,*

$$\langle X, \omega \rangle = \lim_j \langle A_j, \omega \rangle. \tag{91}$$

This theorem is due to Federer [Fe, 4.1, page 379].

Definition 5.12. Let T be a d-rectifiable current associated with (X, ξ) where $X \subset \mathbb{R}^N$. Say X is a *Stokes current* if for all $p \in X$ there is a neighbourhood $B(p, \epsilon) \cap X =: U(p, \epsilon)$ so that whenever φ is a $d-1$-form with support in $U(p, \epsilon)$ we have

$$\langle \partial T, \varphi \rangle := \langle T, d\varphi \rangle = 0. \tag{92}$$

Open problem: Characterize Stokes currents.

6 Back to D-structures.

Recall the definition of D-structures from Section 4. Using the D-structure, we can define a type of Sobolev space

$$H^{1,p}(X) := \overline{\{u \in \mathrm{locLip}(X) \ : \ \|u\|_{L^p} + \|Du\|_{L^p} < \infty\}}, \tag{93}$$

where the closure is taken with respect to the norm

$$\|u\|_{1,p} := \|u\|_{L^p} + \|Du\|_{L^p}. \tag{94}$$

Definition 6.1. Let X, Y be two Banach spaces, and let T be a linear map from a subset of X, called the domain dom$T$, into Y. Let

$$G(T) := \{(x, Tx) \ : \ x \in \mathrm{dom}T\} \subset X \times Y \tag{95}$$

be the graph of T. We say T is *closed* if $G(T)$ is closed in the product topology of $X \times Y$. We says that T is *closable* if for each sequence x_n in dom$T$ such that $x_n \to 0 \in X$ and $Tx_n \to s \in Y$ it is necessary that $s = 0$.

The condition of T being closable ensures that we can extend T *uniquely* to the set $\{x \in X \ : \ \exists y, (x, y) \in \overline{G(T)}\}$. In this case, denoting the unique extension as $\overline{T}$, we see that $\overline{T}(x) = \lim_n T(x_n)$ whenever $x_n \to x$. This ensures that the extension $\overline{T}$ is closed; hence the terminology.

Definition 6.2. The D-structure on (X, F) is said to be *p-closable* if the operator D is closable in the L^p-norm; that is, if $x_n \to 0$ in the L^p-norm and $Dx_n \to s$ in the L^p-norm, then $s = 0$.

Open problem: Let $X = \mathbb{R}^n$ be endowed with the Euclidean metric, and let D be the usual gradient operator on locally Lipschitz functions. Are there measures μ on $\mathbb{R}^n$ so that D is p-closable for some values of p but not for others?
Question: If $p < q$, is it true that if $\mathbb{R}^n$ supports a $(1, q)$-Poincaré inequality with respect to a doubling Radon measure then $\mathbb{R}^n$ supports a $(1, p)$-Poincaré inequality with respect to that measure?

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