Poincaré inequalities, uniform domains and extension properties for Newton-Sobolev functions in metric spaces

Jana Björn *

Department of Mathematics, Linköpings universitet, SE-581 83 Linköping, Sweden; *jabjo@mai.liu.se*

Nageswari Shanmugalingam †

Department of Mathematical Sciences, University of Cincinnati, P.O. Box 210025, Cincinnati, OH 45221-0025, U.S.A.; *nages@math.uc.edu*

September 22, 2006

Abstract: In the setting of metric measure spaces equipped with a doubling measure supporting a weak p-Poincaré inequality with $1 \le p < \infty$, we show that any uniform domain Ω is an extension domain for the Newtonian space $N^{1,p}(\Omega)$ and that Ω, together with the metric and the measure inherited from X, supports a weak p-Poincaré inequality. For $p > 1$, we obtain a near characterization of $N^{1,p}$-extension domains with local estimates for the extension operator.

Key words and phrases: Boman chain condition, corkscrew condition, extension domain, measure density, Newtonian function, Poincaré inequality, shell condition, uniform domain.

Mathematics Subject Classification (2000): 46E35; Secondary: 49Q15.

1. Introduction

It is well known that uniform domains in Euclidean spaces are extension domains for the Sobolev spaces $W^{1,p}(\Omega)$, $1 \le p < \infty$. This fact was first proved by Jones in [20] for a wider class of domains called (ε, δ)-domains, and extended to weighted Sobolev spaces on Euclidean domains by Chua in [6], [7] and [8], and to Carnot–Carathéodory spaces by Garofalo-Nhieu [10].

In the last decade, Sobolev spaces have been extensively studied in the setting of metric measure spaces, see e.g. Cheeger [3], Hajłasz [15], Shanmugalingam [30], Hajłasz–Koskela [12] and the references therein. In particular, the study of quasiconformal mappings between metric spaces leads to the notion of Sobolev spaces on metric measure spaces, see e.g. Heinonen–Koskela [19]. Analysis on metric measure spaces is a natural generalization of analysis on Euclidean spaces and Riemannian manifolds, since Gromov-Hausdorff limits of manifolds need not be manifolds at all, see e.g. Cheeger–Colding [4] and Cheeger–Colding–Tian [5]. This general approach also includes analysis on e.g. graphs and simplicial complexes.

*The first author is supported by the Swedish Research Council.

†The second author is partially supported by NSF grant DMS-0355027.

The abstract approach of Sobolev spaces on metric measure spaces introduced by Hajłasz [15] has been used in Hajłasz–Martio [16] to show that a Euclidean domain Ω satisfying a corkscrew condition is a $W^{1,p}$-extension domain if and only if the Sobolev space $W^{1,p}(\Omega)$ coincides with the Hajłasz–Sobolev space $M^{1,p}(\Omega)$. This, in particular, recovers Jones's extension theorem [20] for uniform domains in $\mathbf{R}^n$. The assumption that Ω satisfies a corkscrew condition in [16] has been recently removed by Hajłasz–Koskela–Tuominen [13].

Extension domains for the Hajłasz–Sobolev spaces have been studied in Harjulehto [17] and Hajłasz–Koskela–Tuominen [14] in the setting of metric spaces equipped with a doubling measure. In [17] it was shown that a porosity condition is sufficient for the extension property. Recently, it was shown in [14] that a domain Ω in a doubling metric measure space is an extension domain for the Hajłasz spaces if and only if the following measure density condition holds: There is a constant $C > 0$ such that for all $x \in \overline{\Omega}$ and $0 < r < \min\{1, \operatorname{diam} \Omega\}$,

$$\mu(B(x,r) \cap \Omega) \geq C\mu(B(x,r)). \tag{1.1}$$

Domains satisfying the corkscrew condition or the porosity condition found in [17] certainly satisfy this measure density condition.

A feature of the Hajłasz spaces is that they *a priori* support a Poincaré inequality for the Hajłasz gradient and hence the validity of a Poincaré inequality gives no information about the geometry of the domain itself, see [14]. For example, the slit disc in $\mathbf{R}^2$ is a Hajłasz extension domain, but it is not an extension domain for the classical Sobolev space $W^{1,p}$. On the other hand, Theorem 4.5 in Shanmugalingam [30] shows that for all Euclidean domains Ω, the Sobolev space $W^{1,p}(\Omega)$ and the Newtonian space $N^{1,p}(\Omega)$ are isomorphic. In fact $N^{1,p}(\Omega)$ is obtained by considering the quasicontinuous representatives from $W^{1,p}(\Omega)$. Thus, extension domains for the classical Sobolev spaces are precisely the extension domains for the Newtonian spaces in $\mathbf{R}^n$.

In this paper we extend a result from Jones [20] to metric spaces equipped with a doubling measure supporting a weak p-Poincaré inequality, thus providing a result analogous to Harjulehto [17] for the Newtonian spaces and for $p = 1$. In particular, we show that uniform domains in these metric measure spaces are $N^{1,p}$-extension domains for all $p \geq 1$, see Theorem 5.6 and Proposition 5.9. For $1 < p < \infty$, amongst a class of domains satisfying a geometric condition, we characterize extension domains satisfying local estimates for the extension operator as those domains which support a weak p-Poincaré inequality, see Corollary 5.10.

Our proof uses the ideas of Jones [20] and Chua [6], where we have replaced the unavailable polynomial approximations by Lipschitz approximations, whose existence follows from the weak p-Poincaré inequality. The validity of the weak p-Poincaré inequality on uniform domains in metric spaces is also proved here, thus extending a recent result by Keith [21], see Theorem 4.4. As a byproduct of our investigations, in Section 2 we prove equivalences of several geometrical conditions, such as the corkscrew condition, the β-shell condition and a certain measure density condition.

A combination of results from Hajłasz–Koskela–Tuominen [14], Keith–Zhong [22] and Shanmugalingam [30] shows that the measure density condition (1.1) together with a weak p-Poincaré inequality on Ω are sufficient for Ω to be an $N^{1,p}$-extension domain for $p > 1$, see Proposition 5.1. By [14], $N^{1,p}$-extension domains in Ahlfors regular spaces necessarily satisfy the measure density condition (1.1), but the slit disc shows that (1.1) is not sufficient for the $N^{1,p}$-extension property.

A combination of Theorem A and Theorem 2.4 in Koskela [25] (see also [27]), together with the results in Hajłasz–Koskela [12] shows that if $p > n$, then a domain $\Omega \subset \mathbf{R}^n$ is a $W^{1,p}$-extension domain if and only if the measure density condition (1.1) is satisfied and Ω supports a weak p-Poincaré inequality. It is possible that this characterization holds true in metric measure spaces with a doubling measure provided p is sufficiently large. At the same time, Example 2.5 in Koskela [24] provides us with a $W^{1,p}$-extension domain (for sufficiently small p) which does not support a weak p-Poincaré inequality, see Remark 5.2. Is there a weaker condition than weak p-Poincaré inequality that together with (1.1) characterizes $N^{1,p}$-extension domains?

Acknowledgement: We wish to thank Pekka Koskela and Heli Tuominen for sharing their preliminary work [14] with us and for many valuable discussions. Part of the research for this paper was done while the second author was visiting University of Jyväskylä and Linköping University in the spring of 2005; she wishes to thank both institutions for their kind hospitality.

2. Density conditions for doubling measures

In this section we compare several density conditions, such as the corkscrew condition, the β-shell condition and the porosity condition from Harjulehto [17].

We assume throughout the paper that $X = (X, d, \mu)$ is a metric measure space equipped with a metric d and a complete Borel measure μ on X such that $0 < \mu(B) < \infty$ for all balls $B = B(x,r) := \{y \in X : d(x,y) < r\}$. At the end of this section we will add the assumption that μ is doubling on X and that X is proper.

Unless otherwise stated, C denotes a positive constant whose value is unimportant, may vary with each appearance, and depends only on the fixed parameters. We also write $a \simeq b$ if $a/C \le b \le Ca$. We denote by $\mathbf{N}$ the set of natural numbers $\{0, 1, 2, ...\}$. Throughout the paper, Ω will be an open subset of X. In this section, we let A denote either Ω or $\overline{\Omega}$ and let $\delta_A(x) = \operatorname{dist}(x, X \setminus A)$.

Definition 2.1. We say that μ is *doubling on* $A \subset X$ if for all $x \in \overline{A}$ and all $r > 0$,

$$\mu(B(x,2r) \cap A) \le C\mu(B(x,r) \cap A),$$

where C is independent of x and r (but may depend on A). We say that μ is doubling if it is doubling on X.

Note that in Definition 2.1, as well as Definitions 2.3, 2.4, 2.6 and 3.4 below, we could equivalently require $x \in A$. Since we will later work with both Ω and $\overline{\Omega}$, requiring $x \in \overline{A}$ is more convenient.

The following lemma is an easy consequence of the doubling property, see Lemma 4.2 in Harjulehto [17]. It will often be used without further notice.

Lemma 2.2. *Assume that μ is doubling on $A \subset X$. Let $B = B(x,r)$ and $B' = B(x',r')$ be two balls with $x, x' \in A$, $d(x,x') \le ar$ and $r/a \le r' \le ar$. Then $\mu(B' \cap A) \simeq \mu(B \cap A)$ with the comparison constant depending only on a and the doubling constant of μ.*

Definition 2.3. Let $0 < \varepsilon < 1$ and $0 < \delta \le 1$. We say that $A \subset X$ satisfies the (ε, δ)-*measure density condition* if for all $x \in \overline{A}$ and all $0 < r \le \operatorname{diam} A$,

$$\mu(\{y \in B(x,r) \cap A : \delta_A(y) > \varepsilon r\}) \ge \delta\mu(B(x,r) \cap A). \tag{2.1}$$

Definition 2.4. We say that A satisfies the *corkscrew condition* if there exists $\varepsilon > 0$ such that for all $x \in \overline{A}$ and $0 < r \le \operatorname{diam} A$, the set $B(x,r) \cap A$ contains a ball with radius εr.

Clearly, the (ε, δ)-measure density condition implies the corkscrew condition (since the set $\{y \in B(x,r) : \delta_A(y) > \varepsilon r\}$ must have positive measure and is thus nonempty). On the other hand, it is easily checked that if A satisfies the corkscrew condition and μ is doubling on A or on X, then A satisfies the $(\varepsilon/2, \delta)$-density condition for some $\delta > 0$.

Lemma 2.5. *If A satisfies the corkscrew condition and μ is doubling on A or on X, then the measure $d\nu(y) = \delta_A(y)^\alpha \, d\mu(y)$ is doubling on A for all $\alpha \ge 0$.*

Proof. First let $0 < r < \operatorname{diam} A$ and $x \in \overline{A}$. If $\delta_A(x) \ge 3r$, then $\delta_A(y) \simeq \delta_A(x)$ for all $y \in B(x, 2r)$ and the result follows from the doubling property of μ. If $\delta_A(x) \le 3r$, then $B(x,r) \cap A$ contains a ball B' with radius εr. As $\delta_A(y) \simeq r$ for all $y \in \frac{1}{2}B'$, the doubling property of μ yields

$$\begin{aligned}\nu(B(x,r) \cap A) &\ge \nu\big(\tfrac{1}{2}B'\big) \simeq r^\alpha \mu\big(\tfrac{1}{2}B'\big) \\ &\ge \frac{1}{C} r^\alpha \mu(B(x,2r) \cap A) \ge \frac{1}{C} \nu(B(x,2r) \cap A),\end{aligned}$$

which proves the lemma for $r < \operatorname{diam} A$. When $r \ge \operatorname{diam} A$, the validity of the doubling condition is obvious. $\square$

Definition 2.6. Let $\beta > 0$. We say that A satisfies the *local β-shell condition* if for all $x \in \overline{A}$ and $0 < \rho \le r \le \operatorname{diam} A$,

$$\mu(\{y \in B(x,r) \cap A : \delta_A(y) \le \rho\}) \le C\Big(\frac{\rho}{r}\Big)^\beta \mu(B(x,r) \cap A). \tag{2.2}$$

The β-shell condition is intimately connected with the Minkowski content of $\partial\Omega$. More information can be found in [29] and [2]. By choosing $(\rho/r)^\beta \le 1/2C$ in (2.2), it is easily verified that the local β-shell condition implies the (ε, δ)-measure density condition with $\varepsilon = 1/(2C)^{1/\beta}$ and $\delta = 1/2$, which in turn implies the corkscrew condition. Theorem 2.8 below shows that the converse implications hold under the assumption that μ is doubling on A or on X.

The following result extends Lemma 2.5 to negative exponents.

Lemma 2.7. *Assume that μ is doubling on A and that A satisfies the local β-shell condition. If $\alpha < \beta$, then the measure $d\nu(y) = \delta_A(y)^{-\alpha} \, d\mu(y)$ is doubling on A.*

Proof. It suffices to consider the case $\beta > \alpha > 0$. Let $x \in \overline{A}$ and $0 < r < \operatorname{diam} A$. If $\delta_A(x) \ge 3r$, then $\delta_A(y) \simeq \delta_A(x)$ for all $y \in B(x, 2r)$ and the result follows from the doubling property of μ. If $\delta_A(x) \le 3r$, then by (2.2) and the doubling property of μ,

$$\begin{aligned}\nu(B(x,2r) \cap A) &\le \sum_{j=1}^\infty (5 \cdot 2^{-j} r)^{-\alpha} \mu(\{y \in B(x,2r) : 5 \cdot 2^{-j} r \le \delta_A(y) < 5 \cdot 2^{1-j} r\}) \\ &\le C\mu(B(x,2r) \cap A) r^{-\alpha} \sum_{j=0}^\infty 2^{-j(\beta-\alpha)} \\ &\le C\nu(B(x,r) \cap A),\end{aligned}$$

since $0 < \alpha < \beta$. The case $r \ge \operatorname{diam} A$ is trivial. $\square$

Theorem 2.8. *Let μ be doubling on A or on X. Then the following are equivalent*:

(i) *A satisfies the (ε, δ)-measure density condition for some $\varepsilon > 0$ and $\delta > 0$;*

(ii) *A satisfies the corkscrew condition*;

(iii) *A satisfies the local β-shell condition for some $\beta > 0$;*

(iv) *there exists $\alpha_0 < 0$, such that for* all *$\alpha > \alpha_0$, all $0 < r \le \operatorname{diam} A$ and all $x \in \overline{A}$ with $\delta_A(x) \le 2r$,*

$$\mu(B(x,r) \cap A) \le C r^{-\alpha} \nu(B(x,r) \cap A), \tag{2.3}$$

where $d\nu(y) = \delta_A(y)^\alpha \, d\mu(y)$ and C depends only on Ω and α;

(v) *the inequality* (2.3) *holds for* some $\alpha > 0$, *all* $0 < r \le \operatorname{diam} A$ *and all* $x \in \overline{A}$ *with* $\delta_A(x) \le 2r$.

Moreover, under any of the above conditions, both μ *and* ν *are doubling on* A.

Remark 2.9. Note that the corkscrew condition does not depend on μ. Thus, the conditions in Theorem 2.8 hold simultaneously for all doubling measures μ.

Also, by switching μ and ν in Theorem 2.8(iv), we get that for all α sufficiently close to zero, $\mu(B(x,r)\cap A) \simeq r^{-\alpha}\nu(B(x,r)\cap A)$.

Proof of Theorem 2.8. If μ is doubling on X and (ii) holds, then Lemma 2.5 with $\alpha = 0$ shows that μ is doubling on A. The doubling property of ν on A is a consequence of Lemma 2.7 and the doubling property of μ on A.

That (iii)⇒(i)⇒(ii) follows from the comments above.

Now we show that (ii)⇒(iii), which in turn demonstrates the equivalence of the first three conditions. First, we show that if the corkscrew condition (2.1) holds, then there exists $\gamma < 1$ such that for all $x \in \overline{A}$ and $0 < \rho \le r \le \operatorname{diam} A$,

$$\mu(\{y \in B(x,r)\cap A : \delta_A(y) \le \varepsilon\rho\} \le \gamma\mu(\{y \in B(x,r+2\rho)\cap A : \delta_A(y) \le 3\rho\}. \tag{2.4}$$

To prove this, we use the doubling property of μ to find a maximal collection of pairwise disjoint balls $\{B_i\}_{i=1}^N$ with centres in $V = \{y \in B(x,r)\cap A : \delta_A(y) < \rho\}$ and radii 2ρ. By the maximality of this collection, $V \subset \bigcup_{i=1}^N 2B_i \cap A$. By the corkscrew condition, each $B_i \cap A$ contains a ball B_i' with radius $2\varepsilon\rho$. The doubling property of μ then yields

$$\begin{aligned}\sum_{i=1}^N \mu(\{y \in B_i \cap A : \delta_A(y) > \varepsilon\rho\}) &\ge \sum_{i=1}^N \mu\bigl(\tfrac12 B_i'\bigr) \ge C\sum_{i=1}^N \mu(2B_i\cap A)\\ &\ge C\mu(V) \ge C\mu(\{y\in V : \delta_A(y)\le\varepsilon\rho\}).\end{aligned}$$

It follows that

$$\begin{aligned}&\mu(\{y \in B(x,r+2\rho)\cap A : \delta_A(y) \le 3\rho\}\\ &\qquad\ge \mu(\{y\in V : \delta_A(y)\le\varepsilon\rho\}) + \sum_{i=1}^N \mu(\{y\in B_i\cap A : \delta_A(y) > \varepsilon\rho\})\\ &\qquad\ge (1+C)\mu(\{y\in V : \delta_A(y)\le\varepsilon\rho\}),\end{aligned}$$

which proves (2.4).

To obtain (2.2), let $\rho_j = (3/\varepsilon)^j\rho$, $j = 1,2,\ldots$. An iteration of (2.4) then yields

$$\begin{aligned}&\mu(\{y\in B(x,r)\cap A : \delta_A(y)\le\rho\})\\ &\qquad\le \gamma\mu(\{y\in B(x,r+2\rho_1/3)\cap A : \delta_A(y)\le\rho_1\})\\ &\qquad\le \ldots \le \gamma^j\mu(\{y\in B(x,r+2(\rho_1+\ldots+\rho_j)/3)\cap A : \delta_A(y)\le\rho_j\}).\end{aligned}$$

Choose j so that $\varepsilon r/3 < \rho_j \le r$. Then $r + 2(\rho_1+\ldots+\rho_j)/3 \le r + 2\rho_j/(3-\varepsilon) \le 2r$ and hence by the doubling property of μ on A,

$$\mu(\{y\in B(x,r)\cap A : \delta_A(y)\le\rho\}) \le \gamma^j\mu(B(x,2r)\cap A) \le C\gamma^j\mu(B(x,r)\cap A),$$

which proves (2.2) with $\beta = \log\gamma/\log(\varepsilon/3) > 0$.

Clearly, (iv)⇒(v). To prove (i)⇒(iv), let $\delta_A(x) \le 2r$. If $\alpha > 0$, then by (i),

$$\delta\mu(B(x,r)\cap A) \le \mu(\{y\in B(x,r) : \delta_A(y) > \varepsilon r\}) \le (\varepsilon r)^{-\alpha}\nu(B(x,r)\cap A),$$

i.e. (2.3) holds. For $\alpha \le 0$, we have directly

$$\mu(B(x,r)\cap A)=\int_{B(x,r)\cap A}\delta_A(y)^{-\alpha}\,d\nu(y)\le (Cr)^{-\alpha}\nu(B(x,r)\cap A).$$

We now prove that (v)$\Rightarrow$(i). Either $B(x,r)\subset A$ and the condition (2.1) is trivially fullfilled, or $\delta_A(x)\le r$. In the latter case, whenever $0<\varepsilon<1$,

$$\begin{aligned}C^{-1}r^{\alpha}\mu(B(x,r)\cap A)&\le \nu(B(x,r)\cap A)\\ &=\nu(\{y\in B(x,r):\delta_A(y)>\varepsilon r\})+\nu(\{y\in B(x,r)\cap A:\delta_A(y)\le\varepsilon r\})\\ &\le Cr^{\alpha}\mu(\{y\in B(x,r):\delta_A(y)>\varepsilon r\})+\varepsilon^{\alpha}r^{\alpha}\mu(B(x,r)\cap A).\end{aligned}$$

Subtracting the last term on the right-hand side from the left-hand side and choosing $\varepsilon=1/(2C)^{1/\alpha}$, we obtain the (ε,δ)-measure density condition (2.1) with $\delta=1/(2C^2)$. □

In particular, if $A=\overline{\Omega}$ and μ is doubling on X, then any of the conditions in Theorem 2.8 is equivalent to the density condition considered by Harjulehto [17], which is just the corkscrew condition for $\overline{\Omega}$. Note that the corkscrew condition for Ω is more restrictive than the corkscrew condition for $\overline{\Omega}$. It is also clear that if μ is doubling on X, then each of the conditions in Theorem 2.8 implies the measure density condition (1.1) considered by Hajłasz–Koskela–Tuominen in [14]. Hence, no $x\in\partial\Omega$ can be a point of density of $X\setminus\Omega$ and the Lebesgue differentiation theorem then shows that $\mu(\partial\Omega)=0$.

Throughout the rest of the paper we assume that the measure μ is doubling on X and that X is proper (i.e. that closed and bounded subsets of X are compact). Note that since X doubling, it is proper if and only if it is complete. Let also $p\ge 1$.

3. Newtonian spaces

In this section we define upper gradients and Newtonian spaces. A *curve* is a continuous mapping from an interval. We will only consider curves which are nonconstant, compact and rectifiable. A curve can thus be parameterized by its arc length ds.

Definition 3.1. A nonnegative Borel function g on X is an *upper gradient* of an extended real-valued function f on X if for all curves $\gamma:[0,\ell_\gamma]\to X$,

$$|f(\gamma(0))-f(\gamma(\ell_\gamma))|\le\int_\gamma g\,ds \tag{3.1}$$

whenever both $f(\gamma(0))$ and $f(\gamma(\ell_\gamma))$ are finite, and $\int_\gamma g\,ds=\infty$ otherwise. If g is a nonnegative measurable function on X and if (3.1) holds for p-almost every curve, then g is a *p-weak upper gradient* of f. By saying that (3.1) holds for p-almost every curve we mean that it fails only for a curve family with zero p-modulus, see Definition 2.1 in Shanmugalingam [30]. It is implicitly assumed that $\int_\gamma g\,ds$ is defined (with a value in $[0,\infty]$) for p-almost every curve.

By Lemma 2.4 in Koskela–MacManus [26], every p-weak upper gradient $g\in L^p(X)$ of f can be approximated in $L^p(X)$ by upper gradients of f. If f has a p-weak upper gradient in $L^p(X)$, then it has a *minimal p-weak upper gradient* $g_f\in L^p(X)$ in the sense that for every p-weak upper gradient $g\in L^p(X)$ of f, $g_f\le g$ μ-a.e., see Corollary 3.7 in Shanmugalingam [31] for $p>1$ and Theorem 7.16 in Hajłasz [11] for $p=1$.

Using the notion of upper gradients we define a version of Sobolev spaces on the metric space X.

Definition 3.2. Whenever $u \in L^p(X)$, let

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu + \int_X g_u^p \, d\mu \right)^{1/p},$$

where g_u is the minimal p-weak upper gradient of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

The Newtonian space is a Banach space, see Shanmugalingam [30].

The space $N^{1,p}(\Omega)$ is defined in the same way as $N^{1,p}(X)$: regard $(\Omega, d|_\Omega, \mu|_\Omega)$ as a metric measure space on its own. Note that if g is an upper gradient of a function u in X, then $g|_\Omega$ is an upper gradient of $u|_\Omega$ in Ω. As for the opposite direction, it is easily verified that if g is an upper gradient of u in Ω and $F \subset \Omega$ is closed, then the function

$$\tilde{g}(x) = \begin{cases} g(x), & \text{if } x \in F, \\ \infty, & \text{otherwise,} \end{cases}$$

is an upper gradient of u in X.

Definition 3.3. We say that Ω is an $N^{1,p}$*-extension domain* if there exists a bounded linear operator $E : N^{1,p}(\Omega) \to N^{1,p}(X)$ such that $Ef = f$ in Ω.

As in Section 2, in the following definition we let A denote either Ω or $\overline{\Omega}$.

Definition 3.4. We say that a measure ν supports a *weak q-Poincaré inequality on A* if there exist constants $C > 0$ and $\sigma \geq 1$ such that for all balls B with centre in $\overline{A}$, all $u \in N^{1,q}(A, \nu)$ and all upper gradients g of u in A,

$$\fint_{B \cap A} |u - u_{B \cap A}| \, d\nu \leq C \operatorname{diam} B \left(\fint_{\sigma B \cap A} g^q \, d\nu \right)^{1/q},$$

where $u_E = \fint_E u \, d\nu = \nu(E)^{-1} \int_E u \, d\nu$.

Theorem 4.1 in Shanmugalingam [30] shows that if μ is doubling on Ω and supports a p-Poincaré inequality on Ω, then Lipschitz functions are dense in $N^{1,p}(\Omega)$. Moreover, by Proposition 7.1 in Aikawa–Shanmugalingam [1], μ is doubling on $\overline{\Omega}$, supports a weak p-Poincaré inequality on $\overline{\Omega}$ and $N^{1,p}(\Omega) = N^{1,p}(\overline{\Omega})$.

4. Poincaré inequality for uniform domains

In this section we show that uniform domains in "nice" metric spaces support a weak p-Poincaré inequality.

Definition 4.1. A domain $\Omega \subset X$ is A-*uniform*, $A \geq 1$, if for every pair $x, y \in \Omega$ there is a curve γ in Ω connecting x and y so that its length $\ell(\gamma)$ satisfies $\ell(\gamma) \leq A\, d(x, y)$ and for all $z \in \gamma$,

$$\delta_\Omega(z) \geq A^{-1} \min\{\ell(\gamma_{x,z}), \ell(\gamma_{y,z})\}.$$

Here $\gamma_{x,z}$ and $\gamma_{y,z}$ are subcurves of γ connecting z to x and y, respectively. A curve satisfying these conditions is called an A-*uniform curve*.

Uniform domains in Euclidean spaces were first studied by Martio–Sarvas [28] in the context of quasiconformal mappings. It is well known that uniform domains satisfy the corkscrew condition. To see this, note that if $y \in \Omega \cap B(x, r/3)$ and $y' \in \Omega \setminus B(x,r)$, then we can connect y and y' by an A-uniform curve in Ω and find a point z where this curve intersects the sphere $\partial B(x, 2r/3)$. Then $B(z, r/3A) \subset \Omega \cap B(x,r)$. We state this as a lemma.

Lemma 4.2. *Let $\Omega \subset X$ be an A-uniform domain, $x \in \overline{\Omega}$ and $r > 0$. If $\Omega \setminus B(x,r)$ is nonempty, then there exists a ball $B \subset \Omega \cap B(x,r)$ with radius $r/3A$.*

The following lemma is a localization of the version of Boman chain condition studied in Hajłasz–Koskela [12] and in Chua [6], [7]. Theorem 9.3 in [12] shows that weak John domains, and in particular uniform domains, satisfy the chain condition. Here we strengthen this result, showing that the intersections of balls centred in the uniform domain with the domain itself satisfy a generalized chain condition, where the chain is allowed to leave the ball but stays within an enlarged ball. This property will be used to prove that uniform domains support a Poincaré inequality.

Lemma 4.3. *Let $\Omega \subset X$ be an A-uniform domain, $z \in \Omega$, $\sigma \geq 1$, $\rho_0 \leq \delta_\Omega(z)/16\sigma A$, and $r > 0$. Then every $x \in B(z,r) \cap \Omega$ can be connected to the ball $B_{0,0} := B(z, \rho_0)$ by a chain of balls $\{B_{i,j} : i = 0, 1, \ldots ; j = 0, 1, \ldots, m_i\}$ with the following properties:*
(a) *For all balls B in the chain, we have $4\sigma B \subset \Omega$.*
(b) *For all i and j, the ball $B_{i,j}$ has radius $\rho_i = 2^{-i}\rho_0$ and centre $x_{i,j}$ such that $d(x_{i,j}, x) \leq 2^{-i} A d(x,z) < 2^{-i} Ar$.*
(c) *For all i, we have $m_i \leq Ar/\rho_0$.*
(d) *For large i, we have $m_i = 0$ and the balls $B_{i,0}$ are centred at x.*
(e) *The balls $B_{i,j}$ are ordered lexicographically, i.e. $B_{i,j}$ comes before $B_{i',j'}$ if and only if $i < i'$ or $i = i'$ and $j < j'$. If $B_{i,j}$ and $B_{i',j'}$ are two neighbours with respect to this ordering, then $B_{i,j} \cap B_{i',j'}$ is nonempty.*

Proof. Let γ be an A-uniform curve connecting z to x in Ω. Assume that γ is parameterized by arc length and that $\gamma(0) = z$ and $\gamma(L) = x$, where $L = \ell(\gamma) \leq Ad(x,z) < Ar$ is the length of γ. Choose the smallest $i_x \in \mathbf{N}$ such that both $\rho_{i_x} = 2^{-i_x}\rho_0 \leq \delta_\Omega(x)/8\sigma$ and $2^{-i_x} L \leq \delta_\Omega(x)/2$.

For each $i = 0, 1, \ldots, i_x - 1$ consider all $j \geq 0$ such that

$$t_{i,j} := (1 - 2^{-i})L + j\rho_i < (1 - 2^{-i-1})L.$$

Clearly, $j \leq L/2\rho_0 \leq Ar/2\rho_0$ for all such j. Similarly, consider all $j \geq 0$ such that

$$t_{i_x,j} := (1 - 2^{-i_x})L + j\rho_{i_x} < L.$$

Again, $j \leq L/\rho_0 \leq Ar/\rho_0$, thus proving (c).

Let $x_{i,j} = \gamma(t_{i,j})$. Then the balls $B_{i,j} = B(x_{i,j}, \rho_i)$ cover γ in the direction from z to x and neighbouring balls always have nonempty intersection. Note that $d(x_{i,j}, x) \leq L - t_{i,j} \leq 2^{-i}L$. We complete the chain by adding the balls $B_{i,0} = B(x, \rho_i)$ for $i > i_x$.

Now, it remains to show that this chain has the required properties. The properties (b)–(e) are clearly satisfied by the construction. To show that $4\sigma B \subset \Omega$ for all balls B in the chain, we shall distinguish several cases.

If the centre of B lies in the ball $B(z, \delta_\Omega(z)/2)$, then as $4\sigma\rho_i \leq 4\sigma\rho_0 \leq \delta_\Omega(z)/4A$, we have $4\sigma B \subset B(z, \delta_\Omega(z)) \subset \Omega$.

Assume therefore that the centre $y = x_{i,j}$ of B does not lie in $B(z, \delta_\Omega(z)/2)$, and consider the following three cases:

If $i = 0$, then as $\ell(\gamma_{x,y}) \geq L/2 \geq \ell(\gamma_{z,y}) \geq \delta_\Omega(z)/2$ and γ is an A-uniform curve,

$$4\sigma\rho_i \leq \frac{\delta_\Omega(z)}{2A} \leq \frac{\ell(\gamma_{z,y})}{A} \leq \delta_\Omega(y).$$

Thus, $4\sigma B \subset \Omega$.

If $1 \le i < i_x$, then as $t_{i,j} \ge L/2$ and $y \notin B(z, \delta_\Omega(z)/2)$,

$$\ell(\gamma_{z,y}) \ge L/2 \ge \ell(\gamma_{x,y}) \ge 2^{-i-1}L \ge 2^{-i-1}\ell(\gamma_{z,y}) \ge 2^{-i-2}\delta_\Omega(z).$$

Hence, as γ is an A-uniform curve and $\rho_0 \le \delta_\Omega(z)/16\sigma A$,

$$4\sigma\rho_i = 2^{2-i}\sigma\rho_0 \le \frac{2^{-i}\delta_\Omega(z)}{4A} \le \frac{\ell(\gamma_{x,y})}{A} \le \delta_\Omega(y).$$

Finally, if $i \ge i_x$, then by construction $y \in B(x, \delta_\Omega(x)/2)$, and as $4\sigma\rho_i \le 4\sigma\rho_{i_x} \le \delta_\Omega(x)/2$ we have $4\sigma B \subset \Omega$. $\square$

Our next theorem extends a recent result of Keith [21] to the metric space setting and to negative exponents. It will be used to show that uniform domains in metric spaces are extension domains for the Newtonian spaces.

Theorem 4.4. *Assume that μ supports a weak p-Poincaré inequality on X and let $\Omega \subset X$ be an A-uniform domain. Then there exists $\alpha_0 > 0$ such that for all $\alpha \ge -\alpha_0$, the measure $d\nu(y) := \delta_\Omega(y)^\alpha\, d\mu(y)$ is doubling on Ω and supports a weak p-Poincaré inequality on Ω with a dilation constant $3A$.*

Proof. By Lemma 4.2 and Theorem 2.8, ν is doubling on Ω. To prove the weak p-Poincaré inequality, let $x_0 \in \overline{\Omega}$, $0 < r \le \frac{1}{2}\operatorname{diam}\Omega$ and $B_0 = B(x_0, r)$. By Lemma 4.2, there exists $z \in \Omega \cap B_0$ such that $\delta_\Omega(z) \ge r/3A$. Note that $B_0 \subset B(z, 2r)$. Let σ be the dilation constant from the weak p-Poincaré inequality on X and let

$$\rho_0 = \frac{r}{48\sigma A^2} \le \frac{\delta_\Omega(z)}{16\sigma A}.$$

Let $u \in N^{1,p}(\Omega, \nu)$ and $g \in L^p(\Omega, \nu)$ be an upper gradient of u in Ω. Note that then $u \in N^{1,p}(B, \mu)$ for all $B \Subset \Omega$. Let $x \in B_0 \cap \Omega$ be a Lebesgue point of u with respect to μ (since μ is doubling, μ-almost every point in Ω is a Lebesgue point of u). Use Lemma 4.3 to find a chain $\mathcal{C}_x$ of balls connecting x and the ball $B_z = B(z, \rho_0) \subset \Omega$. For each $B \in \mathcal{C}_x$, we denote by B^* its immediate successor in the lexicographic ordering. As x is a Lebesgue point of u with respect to μ, we have

$$|u(x) - u_{B_z}| = \lim_{i\to\infty} |u_{B_{i,0}} - u_{B_z}| \le \sum_{B \in \mathcal{C}_x} |u_{B^*} - u_B|, \tag{4.1}$$

where $u_B = \fint_B u\, d\mu$. Note that $B^* \subset 3B$, and

$$|u_{B^*} - u_B| \le |u_{B^*} - u_{3B}| + |u_{3B} - u_B|.$$

As the radii of B and B^* differ by at most a factor 2 and $3\sigma B \Subset \Omega$, an application of the weak p-Poincaré inequality for μ on X yields

$$|u_{B^*} - u_{3B}| \le \fint_{B^*} |u - u_{3B}|\, d\mu \le C \fint_{3B} |u - u_{3B}|\, d\mu \le C \operatorname{diam} B \left(\fint_{3\sigma B} g^p\, d\mu \right)^{1/p}.$$

The difference $|u_{3B} - u_B|$ is estimated in the same way. Inserting these estimates into (4.1), we obtain for μ-a.e. $x \in B_0 \cap \Omega$,

$$|u(x) - u_{B_z}| \le C \sum_{B \in \mathcal{C}_x} \operatorname{diam} B \left(\fint_{3\sigma B} g^p\, d\mu \right)^{1/p}. \tag{4.2}$$

We now wish to estimate the measure of level sets of the function $x \mapsto |u(x) - u_{B_z}|$ in $B_0 \cap \Omega$. Assume that $|u(x) - u_{B_z}| \ge t$ and write $t = C_\varepsilon M t \sum_{i=0}^\infty 2^{-i\varepsilon}$, where

$\varepsilon \in (0,1)$ will be chosen later, and $M \le Ar/\rho_0 + 1 = 48\sigma A^3 + 1$ is the maximal number of balls in $\mathcal{C}_x$ with the same radius. Then

$$C_\varepsilon M t \sum_{i=0}^{\infty} 2^{-i\varepsilon} = t \le C \sum_{B \in \mathcal{C}_x} \operatorname{diam} B \biggl(\fint_{3\sigma B} g^p \, d\mu \biggr)^{1/p}.$$

Hence, there exists $B_x = B(x_{i,j}, \rho_i) \in \mathcal{C}_x$ such that

$$C_\varepsilon 2^{-i\varepsilon} t \le C\rho_i \biggl(\fint_{3\sigma B_x} g^p \, d\mu \biggr)^{1/p}.$$

We have $2^{-i} = \rho_i/\rho_0 = 48\sigma A^2 \rho_i / r$, and inserting this into the last inequality yields

$$t \le Cr \Bigl(\frac{\rho_i}{r} \Bigr)^{1-\varepsilon} \biggl(\fint_{3\sigma B_x} g^p \, d\mu \biggr)^{1/p},$$

where C depends only on ε, A and σ.

As $4\sigma B_x \subset \Omega$, we have $\delta_\Omega(y) \simeq \delta_\Omega(x_{i,j})$ for all $y \in 3\sigma B_x$ and hence

$$\fint_{3\sigma B_x} g^p \, d\mu \simeq \fint_{3\sigma B_x} g^p \, d\nu.$$

As $3\sigma B_x \subset 3AB_0$, a simple iteration of the doubling condition for ν shows that there exist $C > 0$ and $s > 0$, independent of B_x, such that

$$\frac{\nu(3\sigma B_x \cap \Omega)}{\nu(3AB_0 \cap \Omega)} \ge C \Bigl(\frac{3\sigma\rho_i}{3Ar} \Bigr)^s, \quad \text{i.e.} \quad \frac{\rho_i}{r} \le C \biggl(\frac{\nu(3\sigma B_x \cap \Omega)}{\nu(3AB_0 \cap \Omega)} \biggr)^{1/s}.$$

Hence

$$t \le Cr \biggl(\frac{\nu(3\sigma B_x \cap \Omega)}{\nu(3AB_0 \cap \Omega)} \biggr)^{(1-\varepsilon)/s} \biggl(\fint_{3\sigma B_x} g^p \, d\nu \biggr)^{1/p},$$

and choosing $\varepsilon \in (0,1)$ so that $\tau := 1 - (1-\varepsilon)p/s > 0$, we obtain

$$\nu(3\sigma B_x \cap \Omega)^\tau \le \frac{Cr^p}{t^p \nu(3AB_0 \cap \Omega)^{1-\tau}} \int_{3\sigma B_x} g^p \, d\nu. \tag{4.3}$$

Let $E_t = \{x \in B_0 \cap \Omega : |u(x) - u_{B_z}| \ge t\}$ and F_t be the set of all points in E_t which are Lebesgue points of u with respect to μ. By above, for every $x \in F_t$ there exists $B_x \in \mathcal{C}_x$ satisfying (4.3). Note also that by Lemma 4.3, we have $x \in B'_x := 96\sigma A^3 B_x$. A general covering lemma (e.g. Theorem 1.2 in Heinonen [18]) provides us with a pairwise disjoint collection $\{B'_{x_i}\}_{i=1}^\infty$ such that the union of all balls $5B'_{x_i}$ covers F_t. The doubling property of ν and (4.3) then yield

$$\begin{aligned}
\nu(E_t) = \nu(F_t) &\le \sum_{i=1}^\infty \nu(5B'_{x_i} \cap \Omega) \le C \sum_{i=1}^\infty \nu(3\sigma B_{x_i} \cap \Omega) \\
&\le \frac{Cr^{p/\tau}}{t^{p/\tau} \nu(3AB_0 \cap \Omega)^{1/\tau - 1}} \sum_{i=1}^\infty \biggl(\int_{3\sigma B_{x_i}} g^p \, d\nu \biggr)^{1/\tau} \\
&\le \frac{Cr^{p/\tau}}{t^{p/\tau} \nu(3AB_0 \cap \Omega)^{1/\tau - 1}} \biggl(\int_{3AB_0 \cap \Omega} g^p \, d\nu \biggr)^{1/\tau}.
\end{aligned}$$

Lemma 4.5 below, which is easily proved using the Cavalieri principle, now implies that

$$\fint_{B_0 \cap \Omega} |u - u_{B_z}| \, d\nu \le Cr \biggl(\fint_{3AB_0 \cap \Omega} g^p \, d\nu \biggr)^{1/p}$$

and a standard argument allows us to replace u_{B_z} by $\fint_{B_0 \cap \Omega} u \, d\nu$. $\square$

Lemma 4.5 (Lemma 4.22 of [18]). *Let $(Y, \tilde{\nu})$ be a measure space and f be a measurable function on Y such that for some $q_0 > 1$ and $C_0 > 0$ and all $t > 0$,*

$$\tilde{\nu}(\{y \in Y : |f(y)| > t\}) \le C_0 t^{-q_0}.$$

Then for each q with $1 \le q < q_0$,

$$\left(\int_Y |f|^q \, d\tilde{\nu}\right)^{1/q} \le \left(\frac{q_0}{q_0 - q}\right)^{1/q} C_0^{1/q_0} \tilde{\nu}(Y)^{1/q - 1/q_0}.$$

5. Extension domains for Newtonian functions

The following proposition is an easy consequence of a combination of several earlier results.

Proposition 5.1. *Let $1 < p < \infty$. If Ω satisfies the density condition* (1.1) *and supports a weak p-Poincaré inequality, then it is an $N^{1,p}$-extension domain.*

Proof. By Proposition 7.1 in Aikawa–Shanmugalingam [1], $\overline{\Omega}$ supports a weak p-Poincaré inequality for the class of functions in $N^{1,p}(\Omega)$ and $N^{1,p}(\Omega) = N^{1,p}(\overline{\Omega})$. As $p > 1$, by a result of Keith–Zhong [22] we know that $\overline{\Omega}$ supports a q-Poincaré inequality for some $1 \le q < p$.

Let $f \in N^{1,p}(\Omega) = N^{1,p}(\overline{\Omega})$. Theorem 4.9 in Shanmugalingam [30] implies that f belongs to the Hajłasz space $M^{1,p}(\overline{\Omega})$ and

$$\|f\|_{M^{1,p}(\overline{\Omega})} \le C \|f\|_{N^{1,p}(\Omega)}.$$

Theorem 1.4 in Hajłasz–Koskela–Tuominen [14] and Theorem 4.1 in Kinnunen–Latvala [23] then show that there exists a p-quasicontinuous function $Ef \in M^{1,p}(X)$ such that $Ef = f$ on $\overline{\Omega}$ and

$$\|Ef\|_{M^{1,p}(X)} \le C \|f\|_{M^{1,p}(\overline{\Omega})}.$$

Finally, by Theorem 4.8 in [30], $M^{1,p}(X)$ embeds continuously into $N^{1,p}(X)$ and hence

$$\|Ef\|_{N^{1,p}(X)} \le C \|Ef\|_{M^{1,p}(X)}. \qquad \square$$

Remark 5.2. If X is Ahlfors regular, then by Theorem 1.3 in Hajłasz–Koskela–Tuominen [14], the density condition (1.1) is also necessary for Ω to be an $N^{1,p}$-extension domain. At the same time, Example 2.5 in Koskela [24] shows that the domain Ω given by

$$\begin{aligned}
\Omega_+ &= \{x = (x_1, \dots, x_n) \in \mathbf{R}^n : 0 < x_j < 2, j = 1, \dots, n\}, \\
\Omega_- &= \{x = (x_1, \dots, x_n) \in \mathbf{R}^n : -2 < x_j < 0, j = 1, \dots, n\}, \\
\Omega &= \Omega_+ \cup \Omega_- \cup ((-2,2)^n \setminus [-1,1]^n)
\end{aligned}$$

is a $W^{1,p}$-extension domain for all $1 < p < n$, but it is neither uniform nor supports any weak p-Poincaré inequality.

The drawback of relying on extension results for Hajłasz spaces is that we need a better Poincaré inequality to know that $N^{1,p}(\overline{\Omega}) \subset M^{1,p}(\overline{\Omega})$ and thus we do not obtain the extension property in the case $p = 1$. To overcome this, we give a direct proof of the extension property in the case when Ω satisfies the corkscrew condition and supports a weak p-Poincaré inequality for $d\nu = \delta_{\overline{\Omega}}(x)^\alpha \, d\mu(x)$. The proof is a modification of the proof of Theorem 1 in Jones [20], and uses the density of Lipschitz functions in $N^{1,p}(\Omega)$ together with the Whitney decomposition of $X \setminus \overline{\Omega}$. We shall therefore first outline a Whitney type covering in X and state some of its properties. For the proofs see e.g. Björn–Björn–Shanmugalingam [2].

Theorem 5.3. *Let $F \subset X$ be a nonempty closed set, $R > 0$, and*

$$V = \{x \in X : 0 < \operatorname{dist}(x, F) \le 16R\}.$$

Then there exists a countable family of balls $\mathcal{W} = \{B_{i,j} = B(x_{i,j}, r_i) : i \in \mathbf{N}, j \in N_i\}$ with $N_i \subset \mathbf{N}$ such that
(a) $V \subset \bigcup_{B \in \mathcal{W}} B$,
(b) $r_i = 2^{-i}R$ *for all* i,
(c) $8r_i < \operatorname{dist}(x_{i,j}, F) \le 16r_i$ *for all* i *and* j,
(d) *The balls* $\left\{\frac{1}{2}B : B \in \mathcal{W}\right\}$ *are pairwise disjoint.*

Remark 5.4. By standard techniques, there exists a partition of unity

$$\Phi = \{\phi_{i,j} : i \in \mathbf{N}, j \in N_i\}$$

subordinate to the collection $\{2B : B \in \mathcal{W}\}$ and satisfying
(a) $\sum_{i,j} \phi_{i,j} = 1$ on V,
(b) $\operatorname{supp}(\phi_{i,j}) \subset 2B_{i,j}$ for all i and j,
(c) $0 \le \phi_{i,j} \le 1$ for all i and j,
(d) $\phi_{i,j}$ is C/r_i-Lipschitz for all i and j.

The following lemma gives us simple estimates on the overlap of the Whitney balls.

Lemma 5.5. *Let $0 < \lambda < 8$.*
(i) *If $\lambda B_{i,j} \cap \lambda B_{k,l} \neq \emptyset$, then $r_i < (16 + \lambda)r_k/(8 - \lambda)$.*
(ii) *For all $x \in \Omega$,*

$$\sum_{\mathcal{W} \ni B \ni x} \chi_{\lambda B}(x) \le M,$$

where M depends only on λ and the doubling constant of μ.

We are now ready for the main result of this section.

Theorem 5.6. *Let $1 \le p < \infty$ and $\Omega \subset X$ be a domain such that $\mu(\partial\Omega) = 0$ and Lipschitz functions are dense in $N^{1,p}(\Omega)$. Assume also that $\overline{\Omega}$ satisfies any of the conditions in Theorem* 2.8 *and that for some $\alpha > 0$ the measure $d\nu(x) = \delta_{\overline{\Omega}}(x)^\alpha \, d\mu(x)$ supports a weak p-Poincaré inequality on $\overline{\Omega}$. Then Ω is an $N^{1,p}$-extension domain, and moreover the following local estimates for the extension operator hold: For bounded Ω there exist $\lambda > 0$ and $C > 0$ such that for all balls B with radius at most* $\operatorname{diam}\Omega$ *and a centre in $\overline{\Omega}$,*

$$\|Ef\|_{L^p(B)} \le C\|f\|_{L^p(\lambda B \cap \Omega)} \quad \textit{and} \quad \|g_{Ef}\|_{L^p(B)} \le C\|g_f\|_{L^p(\lambda B \cap \Omega)}. \tag{5.1}$$

If Ω is unbounded and $R > 0$, then the estimates (5.1) *hold only for balls with radius at most R, and C depends on R.*

Remark 5.7. We are primarily interested in the case $p = 1$, but as the same proof works for all $p \ge 1$, we do not make that assumption in Theorem 5.6. If $1 < p < \infty$, then the weak p-Poincaré inequality for ν follows from the weak p-Poincaré inequality for μ and the local β-shell condition (2.2) for $\overline{\Omega}$. Indeed, by Theorem 1.0.1 in Keith–Zhong [22], μ supports a q-Poincaré inequality on $\overline{\Omega}$ for some $1 \le q < p$. Hence, if $B = B(x, r)$ with $x \in \overline{\Omega}$, then by the q-Poincaré inequality for μ and the Hölder inequality,

$$\begin{aligned}
\fint_{B\cap\Omega} |u - u_{B\cap\Omega}| \, d\mu &\le Cr \left(\fint_{\sigma B \cap \Omega} g(y)^q \delta_{\overline{\Omega}}(y)^{\alpha q/p} \delta_{\overline{\Omega}}(y)^{-\alpha q/p} \, d\mu(y) \right)^{1/q} \\
&\le Cr \left(\fint_{\sigma B \cap \Omega} g^p \, d\nu \right)^{1/p},
\end{aligned}$$

provided that

$$\left(\frac{\nu(\sigma B\cap\Omega)}{\mu(\sigma B\cap\Omega)}\right)^{1/p}\left(\fint_{\sigma B\cap\Omega}\delta_{\overline{\Omega}}(y)^{-\alpha q/(p-q)}\,d\mu(y)\right)^{1/q-1/p}\le C$$

with a constant C independent of B. For α sufficiently close to zero, this follows from Remark 2.9. We have thus obtained the two-weighted Poincaré inequality

$$\fint_{B\cap\Omega}|u-u_{B\cap\Omega}|\,d\mu\le Cr\left(\fint_{\sigma B\cap\Omega}g^p\,d\nu\right)^{1/p},$$

which implies the weak p-Poincaré inequality for ν on $\overline{\Omega}$, see Franchi–Hajłasz [9].

Remark 5.8. Note that by Theorem 4.1 in Shanmugalingam [30], the density of Lipschitz functions in $N^{1,p}(\Omega)$, $1\le p<\infty$, follows from the weak p-Poincaré inequality on Ω and the doubling property of μ on Ω, which in turn follows from Lemma 4.2 and Theorem 2.8. Together with Theorem 4.4, this observation implies the following proposition.

Proposition 5.9. *Let $1\le p<\infty$, and assume that μ supports a weak p-Poincaré inequality on X. Then every uniform domain in X is an $N^{1,p}$-extension domain.*

The following corollary of Theorem 5.6 provides an almost characterization of extension domains satisfying the local estimates (5.1).

Corollary 5.10. *Let $1<p<\infty$ and assume that μ supports a weak p-Poincaré inequality on X. Suppose that Ω is a bounded domain with $\mu(\partial\Omega)=0$ and that $\overline{\Omega}$ satisfies one of the conditions in Theorem* 2.8. *Then the following are equivalent*:

(i) *Ω is an $N^{1,p}$-extension domain and the local estimates* (5.1) *hold for all balls with centre in $\overline{\Omega}$ and radius at most* $\operatorname{diam}\Omega$.

(ii) *The measure μ supports a weak p-Poincaré inequality on Ω.*

Proof. The implication (ii)$\Rightarrow$(i) follows from Theorem 5.6 together with Remarks 5.7 and 5.8. Conversely, if Ω is an $N^{1,p}$-extension domain and (5.1) holds, then using $\mu(\partial\Omega)=0$, (1.1) and the weak p-Poincaré inequality on X, we have for all $f\in N^{1,p}(\Omega)$ and all balls B with centre in $\overline{\Omega}$ and radius at most $\operatorname{diam}\Omega$,

$$\begin{aligned}\fint_{B\cap\Omega}|f-f_{B\cap\Omega}|\,d\mu&\le 2\fint_{B\cap\overline{\Omega}}|f-(Ef)_B|\,d\mu\le C\fint_B|Ef-(Ef)_B|\,d\mu\\&\le Cr\left(\fint_{\sigma B}g_{Ef}^p\,d\mu\right)^{1/p}\le Cr\left(\fint_{\lambda\sigma B\cap\Omega}g_f^p\,d\mu\right)^{1/p}.\end{aligned}$$

As Ω is bounded, the weak p-Poincaré inequality for larger balls follows trivially from the weak p-Poincaré inequality for balls with radius $\operatorname{diam}\Omega$. $\square$

Proof of Theorem 5.6. Fix $0<R\le\operatorname{diam}\Omega$ and let $\mathcal{W}$ be the Whitney type covering of

$$V:=\{x\in X:0<\operatorname{dist}(x,\Omega)\le 16R\}$$

given by Theorem 5.3 with $F=\overline{\Omega}$. For each $B_{i,j}=B(x_{i,j},r_i)\in\mathcal{W}$, let $y_{i,j}$ be a point in $\overline{\Omega}$ closest to $x_{i,j}$ (there may be more than one choice for $y_{i,j}$), and let

$$U_{i,j}=B(y_{i,j},r_i)\cap\overline{\Omega}.$$

By assumption, Lipschitz functions are dense in $N^{1,p}(\Omega)$; hence it suffices to extend Lipschitz functions and control the Newtonian norm of the extension by the

Newtonian norm of the original function. Any $f \in \mathrm{Lip}(\Omega)$ can be extended to be a function in $\mathrm{Lip}(\overline{\Omega})$. We define a function $F : V \to \mathbf{R}$ by

$$F(x) = \sum_{i,j} f_{U_{i,j}}\, \phi_{i,j}(x),$$

where $f_{U_{i,j}} = \fint_{U_{i,j}} f\, d\nu$ and $\{\phi_{i,j}\}$ is the partition of unity subordinate to the collection $\{2B : B \in \mathcal{W}\}$ given by Remark 5.4.

The plan of the proof is as follows. We show that $F \in N^{1,p}(V)$ and estimate $\|F\|_{N^{1,p}(V)}$ in terms of $\|f\|_{N^{1,p}(\Omega)}$. As $\mu(\partial\Omega) = 0$, by Lemma 5.11 below it then only remains to show that the glueing of F and f is absolutely continuous across the boundary $\partial\Omega$ and in this instance we will show that the glueing of F and f is Lipschitz on $\overline{\Omega} \cup V$.

Let $\gamma \subset V$ be a curve connecting x and y. By splitting γ into parts if necessary, we can assume that $\gamma \subset B_{k,l} \in \mathcal{W}$. We then have

$$\begin{aligned} |F(x) - F(y)| &= \left| \sum_{i,j} (f_{U_{i,j}} - f_{U_{k,l}})(\phi_{i,j}(x) - \phi_{i,j}(y)) \right| \\ &\le \sum_{i,j} |f_{U_{i,j}} - f_{U_{k,l}}|\, |\phi_{i,j}(x) - \phi_{i,j}(y)| \\ &\le C\, d(x,y) \sum_{i,j:2B_{i,j}\cap B_{k,l}\neq\emptyset} \frac{|f_{U_{i,j}} - f_{U_{k,l}}|}{r_i}. \end{aligned} \tag{5.2}$$

Note that by Lemma 5.5, there are only M balls $B_{i,j} \in \mathcal{W}$ such that $2B_{i,j} \cap B_{k,l} \neq \emptyset$ and that $r_k/2 \le r_i \le 2r_k$ for all such balls. Let $B_{i,j}$ be one such ball. Then $U_{i,j} \subset B'_{k,l} \cap \overline{\Omega}$, where $B'_{k,l} = B(y_{k,l}, 55r_k)$, and the doubling property for ν (which follows from Lemma 2.5) implies

$$\begin{aligned} |f_{U_{i,j}} - f_{U_{k,l}}| &\le \fint_{U_{i,j}} |f - f_\nu|\, d\nu + \fint_{U_{k,l}} |f - f_\nu|\, d\nu \\ &\le C \fint_{B'_{k,l}} |f - f_\nu|\, d\nu, \end{aligned}$$

where $f_\nu = \fint_{B'_{k,l}} f\, d\nu$. (Note that $\nu(X \setminus \Omega) = 0$ and thus $\int_{B'_{k,l}} f\, d\nu = \int_{B'_{k,l}\cap\Omega} f\, d\nu$.)

Let g be an upper gradient of u in Ω. The weak p-Poincaré inequality for f with respect to ν on $\overline{\Omega}$ now yields

$$|f_{U_{i,j}} - f_{U_{k,l}}| \le C r_k \left(\fint_{\sigma B'_{k,l}} g^p\, d\nu \right)^{1/p}.$$

Inserting this into (5.2) shows that the function

$$G(x) := C \sum_{2B_{k,l}\ni x} \left(\fint_{\sigma B'_{k,l}} g^p\, d\nu \right)^{1/p}$$

is an upper gradient of F in V.

We now show that $G \in L^p(V)$. Note that for each $B_{i,j} \in \mathcal{W}$, there are only M balls $B_{k,l}$ such that $2B_{k,l} \cap B_{i,j} \neq \emptyset$. We therefore have

$$\begin{aligned} \int_V G^p\, d\mu &\le C \sum_{i,j} \mu(B_{i,j}) \Bigg(\sum_{k,l:2B_{k,l}\cap B_{i,j}\neq\emptyset} \left(\fint_{\sigma B'_{k,l}} g^p\, d\nu \right)^{1/p} \Bigg)^p. \\ &\le C \sum_{i,j} \mu(B_{i,j}) \sum_{k,l:2B_{k,l}\cap B_{i,j}\neq\emptyset} \fint_{\sigma B'_{k,l}} g^p\, d\nu. \end{aligned} \tag{5.3}$$

Since $r_i \le 2r_k$ for all such balls, we have by Theorem 2.8(iv) and Lemma 2.2,

$$\nu(\sigma B'_{k,l}) \ge \nu(U_{k,l}) \ge Cr_k^\alpha \mu(U_{k,l}) \ge Cr_k^\alpha \mu(B(y_{k,l}, r_k)) \simeq Cr_i^\alpha \mu(B_{i,j})$$

Inserting this into (5.3) yields

$$\begin{aligned}\int_V G^p \, d\mu &\le C \sum_{i,j} \frac{\mu(B_{i,j})}{r_i^\alpha \mu(B_{i,j})} \int_{CB_{i,j}} g^p \, d\nu \\ &= C \int_\Omega g(x)^p \sum_{i,j} \frac{\chi_{CB_{i,j}}(x) \delta_{\overline{\Omega}}(x)^\alpha}{r_i^\alpha} \, d\mu(x) \\ &\le C \int_\Omega g(x)^p \, d\mu(x),\end{aligned}$$

provided that for all $x \in \overline{\Omega}$,

$$\delta_{\overline{\Omega}}(x)^\alpha \sum_{i,j} \frac{\chi_{CB_{i,j}}(x)}{r_i^\alpha} \le C. \tag{5.4}$$

To prove (5.4), note that if $x \in CB_{i,j} \cap \overline{\Omega}$, then $d(x, x_{i,j}) \le Cr_i$ and hence $i \le \log_2(C/\delta_{\overline{\Omega}}(x))$. Furthermore, $B_{i,j} \subset B(x, Cr_i)$, and by the doubling property of μ, for each $x \in \overline{\Omega}$ and i there are only M balls $B_{i,j}$ such that $x \in CB_{i,j} \cap \Omega$. Thus, as $\alpha > 0$,

$$\delta_{\overline{\Omega}}(x)^\alpha \sum_{i,j} \frac{\chi_{CB_{i,j}}(x)}{r_i^\alpha} \le C\delta_{\overline{\Omega}}(x)^\alpha \sum_{0 \le i \le \log_2(C/\delta_{\overline{\Omega}}(x))} 2^{i\alpha} \le C,$$

which proves (5.4).

Next, we show that $F \in L^p(V)$. Recall that μ is doubling and that $\overline{\Omega}$ satisfies the corkscrew condition. Since each $B \in \mathcal{W}$ intersects at most M balls $2B_{i,j}$ and $\mu(U_{i,j}) \ge C\mu(B_{i,j}) \ge C\mu(B)$ for all such balls, we have by the Hölder inequality and Theorem 2.8(iv),

$$\begin{aligned}\int_V |F|^p \, d\mu &\le \sum_{B \in \mathcal{W}} \mu(B) \Bigg(\sum_{i,j: 2B_{i,j} \cap B \ne \emptyset} |f_{U_{i,j}}| \Bigg)^p \\ &\le C \sum_{B \in \mathcal{W}} \sum_{i,j: 2B_{i,j} \cap B \ne \emptyset} \frac{\mu(B)}{\nu(U_{i,j})} \int_{U_{i,j}} |f|^p \, d\nu \\ &\le C \sum_{B \in \mathcal{W}} \sum_{i,j: 2B_{i,j} \cap B \ne \emptyset} \frac{\mu(B)}{r_i^\alpha \mu(U_{i,j})} \int_{U_{i,j}} |f|^p \, d\nu \\ &\le C \sum_{i,j} r_i^{-\alpha} \int_{U_{i,j}} |f|^p \delta_{\overline{\Omega}}(x)^\alpha \, d\mu \\ &\le C \int_\Omega |f|^p \, d\mu,\end{aligned}$$

since by (5.4) for all $x \in \overline{\Omega}$ we have

$$\delta_{\overline{\Omega}}(x)^\alpha \sum_{i,j} \frac{\chi_{U_{i,j}}(x)}{r_i^\alpha} \le C.$$

Similar estimates show that there exist $\lambda > 0$ and $C > 0$ such that for all balls centred in $\overline{\Omega}$,

$$\int_{B \cap V} |F|^p \, d\mu \le C \int_{\lambda B \cap \Omega} |f|^p \, d\mu, \quad \text{and} \quad \int_{B \cap V} |G|^p \, d\mu \le C \int_{\lambda B \cap \Omega} |g|^p \, d\mu. \tag{5.5}$$

Finally, let $\eta(x) = \max\{0, \min\{1, 2 - \operatorname{dist}(x,\Omega)/R\}\}$ and define $Ef : X \to \mathbf{R}$ by

$$Ef(x) = \begin{cases} f(x), & \text{if } x \in \overline{\Omega}, \\ \eta(x)F(x), & \text{otherwise.} \end{cases}$$

Note that $G\eta + Fg_\eta$ is a p-weak upper gradient of Ef in $X \setminus \overline{\Omega}$ and hence for balls with radii at most R, the estimates (5.5) hold with F and G replaced by Ef and g_{Ef}. In view of Lemma 5.11 below we need now only to show that Ef is absolutely continuous across $\partial\Omega$. If $x \in B_{k,l} \subset V$ and $z \in \overline{\Omega}$, then

$$|F(x) - f(z)| = \biggl|\sum_{i,j} (f_{U_{i,j}} - f(z))\phi_{i,j}(x)\biggr| \le \sum_{i,j:2B_{i,j}\cap B_{k,l}\neq\emptyset} |f_{U_{i,j}} - f(z)|. \qquad (5.6)$$

By the construction of $\mathcal{W}$, if $2B_{i,j} \cap B_{k,l} \neq \emptyset$ then $r_i \le 2r_k \le \frac{1}{3}\operatorname{dist}(x,\overline{\Omega}) \le \frac{1}{3} d(x,z)$. Thus, for $y \in U_{i,j}$,

$$d(y,z) \le d(y,y_{i,j}) + d(y_{i,j},x_{i,j}) + d(x_{i,j},x) + d(x,z) \le Cr_i + d(x,z) \le Cd(x,z).$$

Hence, if $2B_{i,j} \cap B_{k,l} \neq \emptyset$,

$$|f_{U_{i,j}} - f(z)| \le \fint_{U_{i,j}} |f(y) - f(z)|\, d\nu(y) \le L \fint_{U_{i,j}} d(y,z)\, d\nu(y) \le CL\, d(x,z),$$

where L is the Lipschitz constant of f. Inserting this into (5.6), we get that $|F(x) - f(z)| \le CL\, d(x,z)$ whenever $x \in V$ and $z \in \overline{\Omega}$. It follows that Ef is Lipschitz continuous in X and hence $Ef \in N^{1,p}(X)$ with norm bounded by the $N^{1,p}$-norm of f. Since Lipschitz functions are dense in $N^{1,p}(\Omega)$, the extension operator $E : f \mapsto Ef$ extends continuously to the whole of $N^{1,p}(\Omega)$. $\square$

The following lemma is used in the proof of Theorem 5.6, but may also be of independent interest.

Lemma 5.11. *Let $\Omega \subset X$ be open and $F \subset \partial\Omega$ such that $\mu(F) = 0$. If u is a function defined on $\Omega \cup F$ such that $u \in N^{1,p}(\Omega)$ and is absolutely continuous on p-almost every curve in $\Omega \cup F$, then $u \in N^{1,p}(\Omega \cup F)$. Moreover, if g is a p-weak upper gradient of u in Ω, then the zero extension of g to F is a p-weak upper gradient of u in $\Omega \cup F$.*

Proof. Let $g \in L^p(\Omega)$ be a p-weak upper gradient of u in Ω. We will denote the zero extension of g to F also by g. Then $g \in L^p(\Omega \cup F)$. We shall show that g is a p-weak upper gradient of u in $\Omega \cup F$.

Let $\gamma : [0, \ell] \to \Omega \cup F$ be a curve parameterized by its arc length such that u is absolutely continuous on γ, the set $\gamma^{-1}(F) \subset [0, \ell]$ has 1-dimensional Lebesgue measure zero and g satisfies (3.1) on every subcurve of $\gamma \setminus F$. By our assumptions, p-almost every curve in $\Omega \cup F$ has these properties. The set $\gamma^{-1}(\Omega) \cap (0, \ell)$ is open and can be written as a union of at most countably many pairwise disjoint intervals (a_i, b_i), $i = 1, 2, \ldots$. For a fixed $n \ge 2$, let $U_n = \bigcup_{i=1}^n (a_i, b_i)$ and reorder the intervals so that $b_i \le a_{i+1}$ for all $i = 1, 2, \ldots, n$. Also let $\alpha_i = a_i + (b_i - a_i)/n^2$, $\beta_i = b_i - (b_i - a_i)/n^2$, $\beta_0 = 0$ and $\alpha_{n+1} = \ell$. As $\gamma|_{[\alpha_i, \beta_i]} \subset \Omega$, we have by the triangle inequality

$$|u(\gamma(0)) - u(\gamma(\ell))| \le \sum_{i=1}^n \int_{\gamma_i} g\, ds + \sum_{i=0}^n |u(\gamma(\beta_i)) - u(\gamma(\alpha_{i+1}))|,$$

where γ_i is the restriction of γ to the interval $[\alpha_i, \beta_i]$. Let $\varepsilon > 0$ and note that

$$\sum_{i=0}^{n} |\alpha_{i+1} - \beta_i| \le |(0,\ell) \setminus U_n| + \frac{2n\ell}{n^2} \to 0,$$

as $n \to \infty$. The absolute continuity of $u \circ \gamma$ then yields that for large n,

$$|u(\gamma(0)) - u(\gamma(\ell))| \le \int_\gamma g\, ds + \varepsilon$$

and letting $\varepsilon \to 0$ completes the proof. $\square$

The assumption of absolute continuity in Lemma 5.11 is needed as the following example shows.

Example 5.12. Let $C \subset [0,1]$ be the Cantor $\frac{1}{3}$-discontinuum, $F = (C \times [0,1])$ and let

$$\Omega = ((-1,2) \times (-1,2)) \setminus F \subset \mathbf{R}^2.$$

Let f be the Cantor staircase function on $[0,1]$, in particular, f is nondecreasing and continuous on $[0,1]$, constant on each interval in $[0,1] \setminus C$ and not absolutely continuous on $[0,1]$. Let η be a Lipschitz continuous function with compact support in $(0,1)$ and $\eta = 1$ on $\left[\frac{1}{4}, \frac{3}{4}\right]$. Then $u(x_1, x_2) := f(x_1)\eta(x_2) \in N^{1,p}(\Omega)$, but u has no extension in $N^{1,p}(\Omega \cup F)$.

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