Dirichlet forms, Poincaré inequalities, and the Sobolev spaces of Korevaar-Schoen[1]

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Abstract

We answer a question of Jost on the validity of Poincaré inequalities for metric space-valued functions in a Dirichlet domain. We also investigate the relationship between Dirichlet domains and the Sobolev-type spaces introduced by Korevaar and Schoen.

1 Introduction

In recent years the subject of analysis and function theory on singular (non-Riemannian) metric spaces has seen significant progress. Motivation for such an abstract formulation comes from applications to sub-Riemannian manifolds and Carnot-Carathéodory geometry, analysis on fractals, and the classification of boundaries of word hyperbolic groups. Some principal tools required in these theories are a notion of first-order Sobolev space and a suitably formulated Poincaré inequality for elements of such a space. In cases where this machinery can be defined, one can reasonably proceed to consider variational problems, partial differential equations (at least, those that admit variational formulations), potential theory and other considerations of classical analysis.

There are various approaches to the development of Sobolev-type spaces on metric spaces. First, there is a geometric approach via the notion of upper gradients which was investigated independently by Cheeger [C] and Shanmugalingam [Sh1], [Sh2], [KiSh]. This method yields a comprehensive first-order theory (by the results of [C]) which allows for natural applications of variational methods in potential theory and partial differential equations (cf. [KiSh]). Moreover, this method yields metric space analogs of the classical Sobolev spaces $W^{1,p}$ for all values of p between 1 and ∞. However, there is a class of metric spaces not covered by this theory. In fact, the types of spaces for which the results of [C], [Sh1], [Sh2] have nontrivial content are those which support "sufficiently rich" families of rectifiable curves. While this class includes a number of geometrically diverse examples (Carnot-Carathéodory spaces, boundaries of hyperbolic groups, and other exotic metric spaces), it nevertheless rules out possibilities such as classical self-similar fractals.

A second avenue to the development of first-order analysis on metric spaces lies in what we would like to call the probabilistic approach. Here the central object of interest is a Dirichlet form, which in turn yields a version of the Hilbert-type Sobolev space $W^{1,2}$. Starting from a Dirichlet form, one constructs a panoply of standard probabilistic machinery (heat hernels, Markov processes, etc.). The canonical reference here is Fukushima–Oshima–Takeda [FOT]. While this approach is currently limited to the Hilbert setting $W^{1,2}$ ($p = 2$), it has been shown to apply in more general metric spaces than those of [C] and [Sh1]. Barlow and Bass [BB] used probabilistic methods to construct Dirichlet forms on self-similar fractals and direct constructions were later carried out by Kigami [Ki1], [Ki2]. The literature on Dirichlet forms on self-similar sets is rapidly expanding and we cannot hope to provide a bibliography which does justice to the field here. Concurrently the abstract theory of Dirichlet forms on general metric spaces has seen substantial progress, see e.g. Mosco [M], Biroli–Mosco [BM1], [BM2], Sturm [St1], [St2], [St3] and the references therein. Here a natural question of interest is the regularity of minimizers for the canonical variational problem (i.e., harmonic functions). Such regularity is known to hold under the assumption of a Poincaré inequality expressed in terms of the Dirichlet form.[1]

After the real-valued case has been studied, it is natural to turn to the question of mappings between metric spaces. Sobolev maps from Euclidean or Riemannian domains into metric spaces have been considered by Korevaar–Schoen [KS], Reshetnyak [R] and Tyson [T]. The paper [HKST] formulated a general theory of Sobolev maps between metric spaces making use of the upper gradient definition from [HeK]. Dirichlet forms for maps between metric spaces have been considered by Jost [J]. Jost also considers the question of the regularity of minimizers for maps into NPC (non-positively curved) spaces under the assumption of a Poincaré-type inequality for metric space-valued Dirichlet functions.

In light of the fundamental importance of the Poincaré inequality in both of these contexts, one would like to know in what cases it can be verified. In particular, it is natural to ask whether satisfaction of the scalar-valued Poincaré inequality necessarily implies the satisfaction of the corresponding metric space-valued Poincaré inequality. (This would reduce the question to the scalar valued case, whis is presumably easier.) Jost poses this question explicitly in [J] for the case of Poincaré inequalities defined in terms of Dirichlet forms. The principal result of this paper (Theorem 3.6) answers Jost's question affirmatively. In fact, we considered precisely the same question in [HKST] for Poincaré inequalities defined in terms of upper gradients. We show here that the techniques from [HKST] can be adapted to the setting of Dirichlet forms. Thus maps between metric spaces with finite Dirichlet energy satisfy a

[1] We do not wish to imply that these are the only approaches to first-order Sobolev analysis on metric spaces. In particular, we would be remiss in not mentioning the pioneering work of Hajłasz [Ha]. Hajłasz's construction makes sense in a variety of contexts, however—in contrast with the upper gradient approach [C], [Sh1], [Sh2]—it fails to satisfy certain natural localization properties which make it somewhat more difficult to work with. In particular, the functions in Hajłasz's construction always satisfies a Poincaré inequality regardless of the metric space under consideration provided $p > 1$. It would be interesting to know what connections can be drawn between the Hajłasz-type Sobolev spaces and the domains of the Dirichlet forms on fractals considered in [BB] and [Ki1].

Poincaré inequality whenever all corresponding extended real-valued functions with finite Dirichlet energy satisfy a Poincaré inequality.

As has been mentioned, there are various approaches to the subject of Sobolev functions on and between metric spaces. In [HKST, §§5-6], we studied the relationship between some of these definitions. One can ask how the definition via Dirichlet forms fits into the picture. A canonical construction of a Dirichlet form between metric spaces was given by Korevaar and Schoen [KS]. It is reasonable to ask: in what situations is the Korevaar–Schoen form essentially unique (i.e., up to embedding of the relevant Sobolev spaces)? In the final section of this paper we consider this question. We identify suitable conditions on the underlying space and on the Dirichlet form which suffice to guarantee that it generates the same Sobolev space as the Korevaar-Schoen form.

This paper is designed as follows. The second section recapitulates the definition of Dirichlet energy forms, see also [J]. We answer Jost's question [J, p. 12] in the third section. In the fourth section we study the connection between the Sobolev spaces of Korevaar and Schoen [KS] and the domains of general Dirichlet forms.

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2 Definitions

We follow in spirit the approach of Jost in defining Dirichlet forms using energy forms; see [J, §2]. We begin by defining Dirichlet forms for (extended) real-valued functions on metric measure spaces and then use this to define Dirichlet forms for maps into general metric spaces. In this latter case we do not in general require that the form be bilinear; in fact, it is not at all clear how to polarize a given energy form if the target is not a Hilbert space. However, we do assume in general the existence of local energy measures (see condition 5 of Definition 2.5); these play an important role in Jost's proof of the regularity of minimizers.

Throughout this paper we make the standing assumption that $X = (X, d)$ is a metric space equipped with a nontrivial Borel regular measure μ. Recall that $L^2(X) = L^2(X,\mu)$ becomes a Hilbert space when endowed with the inner product $(u,v) := \int_X uv\, d\mu$.

Definition 2.1. We say that a functional $\mathcal{E}_{\mathbb{R}} : L^2(X,\mu) \to \mathbb{R}^+ \cup \{\infty\}$ is a *Dirichlet bilinear energy form* if the following four conditions are satisfied:

1. *Quadratic contraction property* - If $\mathcal{E}_{\mathbb{R}}(u) < \infty$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is L-Lipschitz with $\varphi(0) = 0$, then

$$\mathcal{E}_{\mathbb{R}}(\varphi \circ u) \leq L^2 \mathcal{E}_{\mathbb{R}}(u). \tag{2.2}$$

2. *Closedness* - If $\{u_n\}$ is a Cauchy sequence in $L^2(X)$ so that $\mathcal{E}_{\mathbb{R}}(u_n) < \infty$ for all n and $\mathcal{E}_{\mathbb{R}}(u_n - u_m) \to 0$ as $n, m \to \infty$, then $\mathcal{E}_{\mathbb{R}}(u_n - u) \to 0$ as $n \to \infty$. Here u denotes the L^2-limit of the Cauchy sequence $\{u_n\}$.

3. *Density condition* - The domain $D(\mathcal{E}_{\mathbb{R}})$ of the bilinear form, consisting of all functions u in $L^2(X)$ for which $\mathcal{E}_{\mathbb{R}}(u) < \infty$, is dense in $L^2(X)$ with respect to its norm and is dense in the space $C_0(X)$ of continuous functions with compact support with respect to the supremum norm.

4. For every pair of functions u, v in $L^2(X)$, we have the *Minkowski inequality*

$$\sqrt{\mathcal{E}_{\mathbb{R}}(u+v)} \leq \sqrt{\mathcal{E}_{\mathbb{R}}(u)} + \sqrt{\mathcal{E}_{\mathbb{R}}(v)}, \tag{2.3}$$

   and the *parallelogram rule*

$$\mathcal{E}_{\mathbb{R}}(u+v) + \mathcal{E}_{\mathbb{R}}(u-v) = 2\left(\mathcal{E}_{\mathbb{R}}(u) + \mathcal{E}_{\mathbb{R}}(v)\right). \tag{2.4}$$

Remarks (1) The restriction in condition 1 to basepoint preserving functions $\varphi : \mathbb{R} \to \mathbb{R}$ is necessary in the case when the measure of X is infinite (otherwise $\varphi \circ f$ will not even be in $L^2(X)$). It is natural to extend the domain of definition of $\mathcal{E}_{\mathbb{R}}$ to $D(\mathcal{E}_{\mathbb{R}}) \oplus \mathbb{R}$ by defining $\mathcal{E}_{\mathbb{R}}(u+k) = \mathcal{E}_{\mathbb{R}}(u)$ for $k \in \mathbb{R}$. In this case (2.2) holds for all L-Lipschitz functions $\varphi : \mathbb{R} \to \mathbb{R}$.

(2) By the Minkowski inequality (2.3) and the quadratic contraction property (2.2), $D(\mathcal{E}_{\mathbb{R}})$ is a vector space, and indeed is a normed space when equipped with the norm

$$\|u\|_{\mathcal{E}_{\mathbb{R}}} := \|u\|_{L^2(X)} + \sqrt{\mathcal{E}_{\mathbb{R}}(u)}.$$

Conditions 2 and 4 together imply that if $\{u_n\}$ is a sequence which is Cauchy in this norm with L^2-limit u, then $\mathcal{E}_{\mathbb{R}}(u) = \lim_{n\to\infty} \mathcal{E}_{\mathbb{R}}(u_n)$ is finite. Moreover, as a consequence of condition 2, the normed space $(D(\mathcal{E}_{\mathbb{R}}), \|\cdot\|_{\mathcal{E}_{\mathbb{R}}})$ is a Banach space.

(3) The usual polarization technique yields a bilinear form from an energy form satisfying conditions 1–4. To wit, set

$$E_{\mathbb{R}}(u,v) := \frac{1}{4}\left(\mathcal{E}_{\mathbb{R}}(u+v) - \mathcal{E}_{\mathbb{R}}(u-v)\right).$$

Condition 4 guarantees that $E_{\mathbb{R}}(\cdot,\cdot)$ is a bilinear form on $D(\mathcal{E}_{\mathbb{R}}) \times D(\mathcal{E}_{\mathbb{R}})$.[2]

(4) Using the fact that $L^2(X)$ is a Hilbert space, Beurling and Deny [BD] construct a unique measure-valued bilinear form associated with $\mathcal{E}_{\mathbb{R}}$:

$$\eta_{\mathbb{R}} : D(\mathcal{E}_{\mathbb{R}}) \times D(\mathcal{E}_{\mathbb{R}}) \to \mathcal{M}(X)$$

(where $\mathcal{M}(X)$ is the collection of all finite signed Radon measures on X) which satisfies $E_{\mathbb{R}}(u,v) = \int_X d\eta_{\mathbb{R}}(u,v)$. The measure $\eta_{\mathbb{R}}(u,u)$ associated with the Dirichlet function u plays the role of the (square of the norm of the) pointwise derivative of u in the general setting (cf. the discussion in the appendix of this paper).

For further properties of Dirichlet forms, we refer the reader to [J], [BD], [St1], [St2], [M] and the text [FOT].

Let now Y also denote a metric space (with metric d_Y), fix $y_0 \in Y$, and let $L^2(X;Y)$ be the collection of all maps $f : X \to Y$ for which the "norm" $\|f\|_{L^2} := (\int_X d_Y(f(x),y_0)^2\, d\mu(x))^{1/2}$ is finite. The distance function

$$d_{L^2}(f,g) := \left(\int_X d_Y(f(x),g(x))^2\, d\mu(x)\right)^{1/2}, \qquad f,g \in L^2(X;Y),$$

makes $L^2(X;Y)$ into a metric space. Moreover, $L^2(X;Y)$ is complete whenever Y is complete. However, in general $L^2(X;Y)$ is not a vector space.

Definition 2.5. Given a complete metric space Y, a functional $\mathcal{E} : L^2(X;Y) \to \mathbb{R}^+ \cup \{\infty\}$ is said to be a *Dirichlet energy form* if there is a corresponding bilinear Dirichlet energy form $\mathcal{E}_{\mathbb{R}} : L^2(X,\mu) \to \mathbb{R}^+ \cup \{\infty\}$ so that the following conditions 1–4 are satisfied. (As before, we denote by $D(\mathcal{E}) := \{f \in L^2(X;Y) : \mathcal{E}(f) < \infty\}$ the *Dirichlet domain* for $\mathcal{E}$.)

[2]Condition 4 is not explicitly stated in [J] as part of the definition of the energy form, however, it is both necessary and sufficient to yield a bilinear form via polarization.

1. Closedness - Let $\{f_n\}$ be a sequence in $D(\mathcal{E})$ which is Cauchy in $L^2(X;Y)$ and assume that, for every Lipschitz map $\varphi : Y \to \mathbb{R}$, the sequence $\{\varphi \circ f_n\}$ satisfies the hypotheses of Definition 2.1, condition 2 for the functional $\mathcal{E}_{\mathbb{R}}$. Then the L^2-limit f of the sequence $\{f_n\}$ is in $D(\mathcal{E})$.

2. If $\varphi \circ f \in D(\mathcal{E}_{\mathbb{R}})$ for every Lipschitz map $\varphi : Y \to \mathbb{R}$ with $\varphi(y_0) = 0$, then $f \in D(\mathcal{E})$.

3. Locality - For every open subset A of X the form $\mathcal{E}$ has a restriction to a form $\mathcal{E}_A$ on $L^2(A;Y)$ (i.e. $f \in D(\mathcal{E}) \Rightarrow f\big|_A \in D(\mathcal{E}_A)$) that also satisfies the preceding two conditions.

4. There is a Radon measure-valued form $\eta : D(\mathcal{E}) \to \mathcal{M}(X)$ which satisfies the locality condition $\mathcal{E}_A(f\big|_A) = \int_A d\eta(f)$ for open sets A in X and the contraction inequality

   $$d\eta_{\mathbb{R}}(\varphi \circ f, \varphi \circ f) \leq L^2 d\eta(f) \tag{2.6}$$

   for L-Lipschitz functions $\varphi : Y \to \mathbb{R}$. Here $\eta_{\mathbb{R}}$ is the energy measure associated with the form $\mathcal{E}_{\mathbb{R}}$ and inequality (2.6) is assumed to hold on the level of measures (this property is called *strong locality* in [J] and [M]).

Remarks (1) By applying (2.6) to the whole space X, we deduce the quadratic contraction property for $\mathcal{E}$: if $f \in D(\mathcal{E})$ and $\varphi : Y \to \mathbb{R}$ is an L-Lipschitz map with $\varphi(y_0) = 0$, then $\varphi \circ f \in D(\mathcal{E}_{\mathbb{R}})$ and

$$\mathcal{E}_{\mathbb{R}}(\varphi \circ f) \leq L^2 \mathcal{E}(f).$$

(2) In the case $Y = \mathbb{R}$, any form $\mathcal{E}$ satisfying Definition 2.1 also satisfies Definition 2.5. Indeed, conditions 1 and 2 are obvious. For the energy measure η of condition 4, we may choose the Beurling-Deny measure $\eta_{\mathbb{R}}$, and for the localized form $\mathcal{E}_A$ of condition 3, we may take $\mathcal{E}_A(f) = \int_A d\eta_{\mathbb{R}}(f, f)$.

(3) Note that while conditions 3 and 4 in Definition 2.5 are not explicitly stated in [J], they are used in Jost's proof of the regularity of minimizers. See page 15 of [J].

(4) In the metric-space valued case, the energy measure cannot be directly constructed by the method of [BD]. Therefore we assume this condition as part of the definition. In most examples, such measure-valued forms are easily seen to exist by the nature of the construction of the energy form $\mathcal{E}$.

Examples The classical example of a Dirichlet form is given by the gradient operator: $D(\mathcal{E}) = W^{1,2}(\mathbb{R}^n)$, and $\mathcal{E}(f) = \int_{\mathbb{R}^n} |\nabla f|^2$ for functions f in $W^{1,2}(\mathbb{R}^n)$. In this case the associated measure-valued form is the symmetric bilinear form $d\eta(u,v)(x) := \nabla u(x) \cdot \nabla v(x)\, dx$. A generalization of the classical Sobolev space $W^{1,2}(\mathbb{R}^n)$ for metric measure spaces X is the Newtonian space $N^{1,2}(X)$ (see [Sh1]), or more generally, by the Newtonian space $N^{1,2}(X;Y)$ of mappings from X to another metric space Y (see §3 of [HKST]).[3] Denoting by ρ_f the minimal weak upper gradient of a function $f \in N^{1,2}(X;Y)$, we observe that the energy integral

$$\mathcal{E}_1(f) = \int_X \rho_f^2\, d\mu, \qquad f \in N^{1,2}(X;Y), \tag{2.7}$$

defines a Dirichlet energy form in the sense of Definition 2.5.

Even in the case $Y = \mathbb{R}$, the energy form in (2.7) is in general not bilinear as the parallelogram rule typically fails: $\rho_{f+g}^2 + \rho_{f-g}^2 \neq 2(\rho_f^2 + \rho_g^2)$. However, in this case an equivalent bilinear Dirichlet form on $N^{1,2}(X)$ can be constructed using Cheeger's differentiation theory. Assuming that the measure on X is doubling and supports a $(1,2)$-Poincaré inequality (see [HeK] or [C] for the definition of the Poincaré inequality used here), Cheeger [C, Theorem 4.38] establishes a version of Rademacher's theorem for Lipschitz functions. To wit, he constructs a finite-dimensional vector bundle $\mathbf{F}$ over X (the *generalized cotangent bundle*) and a differential operator d taking $N^{1,2}(X)$ into the L^2-sections of $\mathbf{F}$. The pointwise norms $||\cdot||_x$ on the fibers of $\mathbf{F}$ is defined so that the identity $||du(x)||_x = \rho_u(x)$ holds almost everywhere. Thus the Dirichlet form of (2.7) can be rewritten

$$\mathcal{E}_1(u) = \int_X ||du||^2\, d\mu.$$

Since the fibers of $\mathbf{F}$ are uniformly finite-dimensional, they can be uniformly renormed with equivalent inner product norms $|||\cdot|||_x = \langle \cdot, \cdot \rangle_x^{1/2}$ so that for all $u \in N^{1,2}(X)$ we have $|||du|||_x \leq \rho_u(x)$ for μ-almost every $x \in X$ (compare the discussion on page 460 of [C]). Then

$$\mathcal{E}_2(u) = \int_X |||du|||^2\, d\mu, \qquad u \in N^{1,2}(X) \tag{2.8}$$

[3] In [C], Cheeger gives an alternate definition of a Sobolev type space on metric measure spaces. By a result of [Sh1], Cheeger's space is isometrically isomorphic with the Newtonian space studied in [Sh1] and [HKST].

defines a new Dirichlet form which is equivalent to $\mathcal{E}_1$ on $N^{1,2}(X)$. Indeed, $\mathcal{E}_2$ plays the role of the bilinear form $\mathcal{E}_{1,\mathbb{R}}$ related to $\mathcal{E}_1$ in Definition 2.5. Polarization yields the bilinear form

$$E_2(u,v) = \int_X \langle du, dv\rangle \, d\mu. \tag{2.9}$$

It is easy to verify that the forms $\mathcal{E}_1$ and $\mathcal{E}_2$ satisfy the conditions in Definitions 2.5 and 2.1 respectively.

In the real-valued case, bilinear Dirichlet forms on certain fractals and more general self-similar spaces have been constructed intrinsically by Barlow–Bass [BB] and Kigami [Ki1], [Ki2]. Note that these spaces typically do not satisfy the assumptions of Cheeger's theory. We do not know of any attempts to generalize the constructions of [BB] and [Ki1], [Ki2] to the case of metric space targets.

3 Independence of Poincaré inequalities on target spaces

In this section we use a technique developed in [HKST] to answer a question of Jost [J, p. 12] regarding Poincaré inequalities for Dirichlet forms.

The Poincaré inequality for a Dirichlet form $\mathcal{E}$ is typically expressed using the so-called *intrinsic metric* associated with $\mathcal{E}$. We briefly recall the definition of this concept. Given a real-valued Dirichlet form $\mathcal{E}_{\mathbb{R}}$ satisfying Definition 2.1 we say that a subspace Γ of $D(\mathcal{E}_{\mathbb{R}})\cap C_0(X)$ is a *μ-separating core* if it is dense in $C_0(X)$ with respect to the supremum norm, is dense in $D(\mathcal{E}_{\mathbb{R}})$ with respect to the norm $||\cdot||_{\mathcal{E}_{\mathbb{R}}}$, and has the following separation property: for every pair of distinct points x,y in X there is a function $\varphi\in\Gamma$ so that $\varphi(x)\neq\varphi(y)$ and $d\eta_{\mathbb{R}}(\varphi,\varphi)\leq d\mu$. Then the intrinsic metric d_E is defined as follows:

$$d_E(x,y) = \sup\{\varphi(x)-\varphi(y) \ : \ \varphi\in\Gamma, d\eta_{\mathbb{R}}(\varphi,\varphi)\leq d\mu\}, \qquad x,y\in X. \tag{3.1}$$

Following [J] and [BM1], we make the standing assumption that the topology induced by d_E coincides with the underlying topology on X and that the measure μ is doubling for balls in the new metric, i.e., there exists a constant $C\geq 1$ so that

$$\mu(2B_E)\leq C\mu(B_E)$$

for every ball B_E in X. Here and henceforth we denote by $B_E = B_E(x,r) := \{y \in X : d_E(x,y) < r\}$ a ball in the intrinsic metric d_E, and we write $\lambda B_E = B_E(x, \lambda r)$ for $\lambda > 0$.

Definition 3.2. Let (Y, d_Y) be a complete metric space and let $\mathcal{E}$ be a Dirichlet form acting on maps from X to Y. We say that X supports a *weak* $(2,2)$-*Poincaré inequality for the form* $\mathcal{E}$ *with respect to* Y (briefly, a *weak Poincaré inequality with respect to* Y), if there are constants $\lambda \geq 1$ and $C > 0$ so that the inequality

$$\inf_{a\in Y} \int_{B_E} d_Y(f(x), a)^2 \, d\mu(x) \leq C \operatorname{diam}_E(B_E)^2 \int_{\lambda B_E} d\eta(f), \tag{3.3}$$

holds for every map $f \in D(\mathcal{E})$ and every d_E-ball B_E in X. Here η denotes the energy measure associated with $\mathcal{E}$ from condition 4 of Definition 2.5. If $Y = \mathbb{R}$, then we say that X supports a *weak Poincaré inequality for* $\mathcal{E}_{\mathbb{R}}$. The strong form of the inequality corresponds to the choice $\lambda = 1$.

In place of (3.3) we could require instead the *a priori* stronger inequality

$$-\!\!\!\!\!\!\int_{B_E} \int_{B_E} d_Y(f(x), f(y))^2 \, d\mu(x) \, d\mu(y) \leq C \operatorname{diam}_E(B_E)^2 \int_{\lambda B_E} d\eta(f) \tag{3.4}$$

provided we are willing to change the constant C. Indeed, it is clear that (3.4) implies (3.3) and a simple application of the triangle inequality in Y shows that (3.3) implies (3.4) with C replaced by $4C$.

We now invoke the fact that every metric space can be isometrically embedded in a Banach space to obtain a third equivalent form of the Poincaré inequality. Recall that any metric space Y can be isometrically embedded in the Banach space $\ell^\infty(Y)$ of bounded real-valued functions on Y by mapping $y \in Y$ to the function $\varphi_y(z) = d_Y(y,z) - d_Y(y, y_0)$. Here as before y_0 denotes a fixed basepoint in Y.

Let $Y \subset V$ be an isometric embedding into some Banach space (e.g. $V = \ell^\infty(Y)$) and denote the norm in V by $\|\cdot\|$. We then have the following third equivalent formulation of the weak Poincaré inequality:

$$\int_{B_E} \|f(x) - f_{B_E}\|^2 \, d\mu(x) \leq C \operatorname{diam}_E(B_E)^2 \int_{\lambda B_E} d\eta(f), \tag{3.5}$$

where

$$f_{B_E} := \frac{1}{\mu(B_E)} \int_{B_E} f \, d\mu =: \fint_{B_E} f \, d\mu \in V$$

denotes the integral average of f over B_E, defined as a Bochner integral. See section 2 of [HKST] and the references therein for more on integration of Banach space-valued functions. We note here that for Borel sets A the mean value f_A satisfies the identity $\Lambda f_A = \frac{1}{\mu(A)} \int_A \Lambda f \, d\mu$ for every Λ in the dual space V^*. Again, it is easy to see that (3.5) is equivalent to (3.3) and (3.4), even though in general f_{B_E} need not be in the target space Y itself.

Let X be a metric space equipped with a non-trivial Borel measure μ, and let $\mathcal{E}$ be a Dirichlet form acting on L^2-maps from X to a collection $\mathcal{Y}$ of non-trivial metric spaces Y so that there is a common Dirichlet form $\mathcal{E}_{\mathbb{R}}$ satisfying conditions 1, 2, and 4 of Definition 2.5 for *each* Dirichlet domain corresponding to the metric spaces Y in $\mathcal{Y}$.

Theorem 3.6. *Let X and $\mathcal{E}$ be as above. Consider the following three conditions:*

(1) X supports a weak Poincaré inequality for the form $\mathcal{E}_{\mathbb{R}}$.

(2) X supports a weak Poincaré inequality for $\mathcal{E}$ with respect to some metric space Y in $\mathcal{Y}$.

(3) For every metric space Y in $\mathcal{Y}$, X supports a weak Poincaré inequality for $\mathcal{E}$ with respect to Y.

Then condition (1) implies conditions (2) and (3).

If in addition we assume that for each $Y \in \mathcal{Y}$ there is a map $\psi : [-1,1] \to Y$ with $\psi(0) = y_0$ and a constant $K \geq 1$ such that for every $s, t \in [-1,1]$,

$$|t - s| \leq K \, d_Y(\psi(t), \psi(s)) \leq K^2 |t - s|, \tag{3.7}$$

and the contraction inequality

$$d\eta(\psi \circ f) \leq K \, d\eta_{\mathbb{R}}(f, f) \tag{3.8}$$

holds for each $f \in D(\mathcal{E}_{\mathbb{R}})$, then all three conditions are equivalent.

We remind the reader that we assume the measure μ is a doubling measure with respect to the induced metric d_E. The fact that condition (1) implies condition (3) answers the question posed by Jost in [J]; in fact (1) and (3) are equivalent if $\mathcal{Y}$ contains the metric space $\mathbb{R}$. The existence of the map ψ in Equation (3.8) implies that Y has at least one rectifiable curve passing through y_0, and its existence is guaranteed if Y has at least one geodesic line segment passing through y_0. In this case, if $\alpha : [-\ell/2, \ell/2] \to Y$ is such a geodesic with $\alpha(0) = y_0$, the function ψ is given by $\psi(t) = \alpha(\ell t/2)$.

The proof of the above theorem follows closely that of Theorem 4.3 of [HKST] using the embedding of metric spaces into Banach spaces.

Proof. The implication (3) $\Rightarrow$ (2) is trivial. The implication (2) $\Rightarrow$ (1) follows from an application of (3.8), (3.7) together with the Poincaré inequality (3.4) to the truncated function $\psi \circ f_n : X \to Y$, followed by the dominated convergence theorem. Here, for each $n \in \mathbb{N}$, f_n is the truncated function defined by

$$f_n(x) = \frac{1}{n} \min\{n, \max\{-n, f(x)\}\}.$$

Observe that the condition $\psi(0) = y_0$ guarantees that $\psi \circ f_n \in L^2(X : Y)$ and the right hand side of Equation (3.7) ensures that $\psi \circ f_n \in D(\mathcal{E})$. It can be seen that $n|f_n| \leq |f|$ and $nf_n \to f$ in $L^2(X)$, and by Condition 4 of Definition 2.5 applied to $\mathcal{E}_{\mathbb{R}}$,

$$d\eta(\psi \circ f_n) \leq K d\eta_{\mathbb{R}}(f_n, f_n) \leq \frac{K}{n^2} d\eta_{\mathbb{R}}(f, f).$$

To complete the proof, we will show the implication (1) $\Rightarrow$ (3).

Let $Y \in \mathcal{Y}$ and V be a Banach space containing an isometric copy of Y, with the base point $y_0 \in Y$ mapped to the zero vector in V. Let $f \in D(\mathcal{E}) \cap L^2(X; Y) \subset L^2(X; V)$ and let $\Lambda \in V^*$ with $\|\Lambda\|_{V^*} \leq 1$. Then $\Lambda : V \to \mathbb{R}$ is an L-Lipschitz map, $L = \|\Lambda\|_{V^*}$, and we have the strong locality inequality (2.6). By quadratic contraction, $\Lambda \circ f$ is in $D(\mathcal{E}_{\mathbb{R}})$. Hence

$$\fint_{B_E} |\Lambda \circ f - (\Lambda \circ f)_{B_E}|^2 \, d\mu \leq C L^2 \operatorname{diam}_E(B_E)^2 \frac{\eta(f)(\lambda B_E)}{\mu(\lambda B_E)}$$

for every $\Lambda \in V^*$ and every d_E-ball $B_E \subset X$. If x, y are Lebesgue points of $\Lambda \circ f$, then letting $B_0 = B_E(x, 2d_E(x, y))$ and $B_i = B_E(x, 2^{-i} d_E(x, y))$, $B_{-i} =$

$B_E(y, 2^{-i}d_E(x,y))$ for $i \in \mathbb{N}$ and using the standard telescoping argument we see that

$$
\begin{aligned}
&|\Lambda \circ f(x) - \Lambda \circ f(y)| \\
&\qquad \leq \sum_{i\in\mathbb{Z}} |(\Lambda \circ f)_{B_i} - (\Lambda \circ f)_{B_{i+1}}| \\
&\qquad \leq CLd_E(x,y) \left(M_{2\lambda d_E(x,y)}\eta(f)(x)^{1/2} + M_{2\lambda d_E(x,y)}\eta(f)(y)^{1/2} \right),
\end{aligned} \tag{3.9}
$$

where

$$
M_R\eta(f)(x) = \sup_{0<r<R} \frac{\eta(f)(B_E(x,r))}{\mu(B_E(x,r))}. \tag{3.10}
$$

(Note that d_E-balls are open sets in the metric space (X,d); since μ is a Borel measure, such balls are μ-measurable, and so the Lebesgue differentiation theorem holds.)

By Proposition 3.12 below, it now suffices to show that for μ-almost every x, y in X we have the inequality

(3.11)

$$
\begin{aligned}
d_Y(f(x), f(y)) &= \|f(x) - f(y)\| \\
&\leq Cd_E(x,y) \left(M_{\lambda_1 d_E(x,y)}\eta(f)(x)^{1/2} + M_{\lambda_1 d_E(x,y)}\eta(f)(y)^{1/2} \right)
\end{aligned}
$$

for some $\lambda_1 \geq 1$ independent of x, y and f. This can be done as in [HKST, Theorem 4.3]; the proof will be included here for the sake of completeness.

For each $\Lambda \in V^*$ there is a set $Z_\Lambda \subset X$ of measure zero so that inequality (3.9) holds for the function $\Lambda \circ f$ and all pairs of points $x, y \in X \backslash Z_\Lambda$. Furthermore, f is essentially separably valued: there exists a set $Z_0 \subset X$ of measure zero so that $f(X \backslash Z_0)$ is separable. See, for example, section 2 of [HKST]. Hence the difference set $\{f(x) - f(y) \,:\, x, y \in X\backslash Z_0, x \neq y\}$ is also separable. Choose a countable dense subset $\{v_i\}_{i\in\mathbb{N}}$ of this difference set. By the Hahn-Banach theorem, there exist functionals $\Lambda_i \in V^*$ so that $\|\Lambda_i\|_{V^*} = 1$ and $\Lambda_i(v_i) = \|v_i\|$. We abbreviate $Z_i = Z_{\Lambda_i}$. Now the measure of $Z := Z_0 \cup \bigcup_i Z_i$ is zero, and for every pair of points x, y in $X \setminus Z$, inequality (3.9) holds for all of the functions $\Lambda_i \circ f$, $i \in \mathbb{N}$. Now for each pair x, y in $X \setminus Z$ we can approximate $\|f(x) - f(y)\|$ by a sub-sequence $\{\Lambda_{i_k} \circ f(x) - \Lambda_{i_k} \circ f(y)\}$, and inequality (3.11) follows. $\square$

The following proposition, used in the above proof, is similar to Proposition 4.6 in [HKST].

Proposition 3.12. *Let (X, d_E) be a metric space equipped with a doubling measure μ, let $V \supset Y$ be a Banach space, and let $f \in D(\mathcal{E}) \subset L^2(X; V)$. If*

$$\|f(x) - f(y)\| \le C d_E(x,y) \left(M_{\lambda_1 d_E(x,y)} \eta(f)(x)^{1/2} + M_{\lambda_1 d_E(x,y)} \eta(f)(y)^{1/2} \right) \tag{3.13}$$

for μ-almost every $x, y \in X$, then f satisfies the Poincaré inequality (3.3).

Proof. Let B_E be a d_E-ball and abbreviate $d_E := \mathrm{diam}_E(B_E)$. By (3.13)

$$\fint_{B_E}\fint_{B_E} \|f(x) - f(y)\| \, d\mu(y)\, d\mu(x) \le C d_E \fint_{B_E} M_{\lambda_1 d_E} \eta(f)^{1/2}\, d\mu.$$

An application of Lemma 3.21 below yields

$$\fint_{B_E}\fint_{B_E} \|f(x) - f(y)\| \, d\mu(y)\, d\mu(x) \le C d_E \sqrt{\frac{\eta(f)(2\lambda_1 B_E)}{\mu(B_E)}},$$

whence

$$\fint_{B_E} \|f(x) - f_{B_E}\| \, d\mu(x) \le 2C d_E \sqrt{\frac{\eta(f)(2\lambda_1 B_E)}{\mu(B_E)}}, \tag{3.14}$$

where we recall that $f_{B_E} = \fint_{B_E} f\, d\mu \in V$ is the integral average of f discussed above. This is the so-called (weak) $(1,2)$-Poincaré inequality (expressed in terms of Dirichlet forms) which implies the desired $(2,2)$-Poincaré inequality

$$\fint_{B_E} \|f(x) - f_{B_E}\|^2 \, d\mu(x) \le 2C' d_E^2 \frac{\eta(f)(4\lambda_1 B_E)}{\mu(B_E)} \tag{3.15}$$

by a standard argument, cf. Theorem 4.18 of [He]. For the sake of completeness, we recall this derivation. Fix $0 < \epsilon < 1$ (to be determined later), and, for $t > 0$, let $A_t = \{x \in B_E : \|f(x) - f_{B_E}\| > t\}$. For each $i \in \mathbb{N}$ let $B_i = \{z \in X : d_E(x,z) < 2^{-i} d_E\}$. Note that $B_i \subset 2B_E$. At each Lebesgue point $x \in A_t$ for the map $f : X \to Y \subset V$, we have

$$\begin{aligned} C(\epsilon)t \sum_{i\in\mathbb{N}} 2^{-i(1-\epsilon)} = t < \|f(x) - f_{B_E}\| &\le \sum_{i\in\mathbb{N}} \|f_{B_{i+1}} - f_{B_i}\| \\ &\le C \fint_{B_i} \|f(z) - f_{B_i}\|\, d\mu(z) \\ &\le C\, d_E \sum_{i\in\mathbb{N}} 2^{-i} \sqrt{\frac{\eta(f)(2\lambda_1 B_i)}{\mu(B_i)}}. \end{aligned}$$

by a standard telescoping argument and inequality (3.14). Hence there exists a positive integer i_x so that

$$C(\epsilon)t\, 2^{-i_x(1-\epsilon)} \le C d_E\, 2^{-i_x} \sqrt{\frac{\eta(f)(2\lambda_1 B_{i_x})}{\mu(B_{i_x})}},$$

equivalently,

$$\mu(B_{i_x}) \le \frac{C\, d_E^2}{t^2}\, 2^{-2i_x\epsilon} \eta(f)(2\lambda_1 B_{i_x}). \tag{3.16}$$

Since μ is doubling with respect to the metric d_E, there exists $Q > 0$ so that $\left(\frac{r}{d_E}\right)^Q \le C \frac{\mu(B_r)}{\mu(B_E)}$ for all balls B_r of radius $0 < r < d_E$, where C depends only on the doubling constant. Applying this inequality to each ball B_i yields

$$2^{-i_x Q} \le C \frac{\mu(B_{i_x})}{\mu(B_E)}. \tag{3.17}$$

We now assume that $\epsilon > 0$ is chosen so that $\epsilon < Q/2$. In this case, substituting (3.17) into (3.16) and solving for $\mu(B_{i_x})$ gives

$$\mu(B_{i_x}) \le C \left(\frac{d_E}{t}\right)^{2s} \frac{\eta(f)(2\lambda_1 B_{i_x})^s}{\mu(B_E)^{s-1}}$$

where we have abbreviated $s = Q/(Q - 2\epsilon)$.

Observe that the balls $5B_{i_x}$ constitute a cover of the set of all Lebesgue points x of f in A_t. By the basic covering theorem (see [HaK, Theorem 14.12] or Theorem 1.2 of [He]), we can extract a countable subcover $\{5B'_k\}_{k\in\mathbb{N}}$ so that the balls B'_k are pairwise disjoint. Since almost every point of A_t is a Lebesgue point,

$$\begin{aligned}\mu(A_t) \le \sum_{k\in\mathbb{N}} \mu(5B'_k) \le C\sum_{k\in\mathbb{N}} \mu(B'_k) &\le C\left(\frac{d_E}{t}\right)^{2s} \mu(B_E)^{1-s} \sum_{k\in\mathbb{N}} \eta(f)(2\lambda_1 B'_k)^s \\ &\le C\left(\frac{d_E}{t}\right)^{2s} \mu(B_E)^{1-s} \left(\sum_{k\in\mathbb{N}} \eta(f)(2\lambda_1 B'_k)\right)^s \\ &\le C\left(\frac{d_E}{t}\right)^{2s} \mu(B_E)^{1-s} \eta(f)(4\lambda_1 B_E)^s\end{aligned}$$

Now (3.15) follows by applying Lemma 3.23 to the function $g(x) = ||f(x) - f_{B_E}||$ with $C_0 = C^{1/s} d_E^2 \eta(f)(4\lambda_1 B_E)$. □

To complete this section, we recall some basic facts about the Hardy-Littlewood maximal operator in doubling metric measure spaces which were used in the above proof.

Lemma 3.18. *Let (X, d_E, μ) be a metric measure space and let η be a second Borel measure on X for which $0 < \eta(X) < \infty$. Then the restricted Hardy-Littlewood maximal function*

$$M_R\eta(x) = \sup_{0<r<R} \frac{\eta(B_E(x,r))}{\mu(B_E(x,r))} \tag{3.19}$$

satisfies the weak $(1,1)$-inequality: there exists a constant $C > 0$ so that

$$\mu(\{x \in B_E \ : \ M_R\eta(x) > \tau\}) \le \frac{C}{\tau}\eta(2B_E) \tag{3.20}$$

for all $\tau > 0$ and all d_E-balls B_E of radius greater than or equal to R.

Using Lemma 3.18, we can prove the following lemma as in [HaK, 14.11].

Lemma 3.21. *Under the assumptions of Lemma 3.18,*

$$\fint_{B_E} M_R\eta^{1/2}\, d\mu \le C\sqrt{\frac{\eta(2B_E)}{\mu(B_E)}}. \tag{3.22}$$

Proof of Lemma 3.18. The classical proof of the Hardy-Littlewood theorem is as follows. Let A denote the set of points x in B_E for which $M_R\eta(x) > \tau$. For each $x \in A$ there is a radius $r_x > 0$ so that $r_x \leq R$ and $\frac{\eta(B_E(x,r_x))}{\mu(B_E(x,r_x))} > \tau$. Covering the set A by these balls $B_E(x, r_x)$, $x \in A$, and applying again the basic covering theorem (see [HaK, Theorem 14.12] or Theorem 1.2 of [He]), we find a countable sub-cover $\{5B_i\}$ for which the balls B_i are pairwise disjoint. Since μ is doubling and $B_E(x, r_x) \subset 2B_E$,

$$\mu(A) \leq \sum_i \mu(5B_i) \leq C\sum_i \mu(B_i) \leq \frac{C}{\tau}\sum_i \eta(B_i) \leq \frac{C}{\tau}\eta\left(\cup_i B_i\right) \leq \frac{C}{\tau}\eta(2B_E).$$

The proof of Lemma 3.18 is complete. □

Proof of Lemma 3.21. Let $\beta > 0$. Using Cavalieri's principle and Lemma 3.18,

$$\begin{aligned}\int_{B_E} M_R\eta(x)^{1/2}\,d\mu(x) &= \int_0^\infty \mu(\{x \in B_E \,:\, M_R\eta(x) > \tau^2\})\,d\tau \\ &\leq \int_0^\beta \mu(B_E)\,d\tau + \int_\beta^\infty \frac{C\eta(2B_E)}{\tau^2}\,d\tau \\ &= \mu(B_E)\beta + \frac{C}{\beta}\eta(2B_E).\end{aligned}$$

Choosing $\beta = \sqrt{\frac{\eta(2B_E)}{\mu(B_E)}}$ yields (3.22). □

In a similar manner, we prove

Lemma 3.23. *Let g be a nonnegative function on B_E and, for $t > 0$, set $A_t = \{x \in B_E : g(x) > t\}$. If*

$$\mu(A_t) \leq C_0^s \frac{\mu(B_E)^{1-s}}{t^{2s}}$$

for some $s > 1$, then

$$\int_{B_E} g^2\,d\mu \leq \frac{2C_0}{(s-1)^{1/s}}. \tag{3.24}$$

Proof of Lemma 3.23. Fixing $\beta > 0$ and using Cavalieri's principle,

$$\begin{aligned}\int_{B_E} g^2\,d\mu = 2\int_0^\infty \mu(A_t)t\,dt &\leq 2\int_0^\beta \mu(B_E)t\,dt + 2C_0^s\mu(B_E)^{1-s}\int_\beta^\infty t^{1-2s}\,dt \\ &= \beta^2\mu(B_E) + \frac{C_0^s}{s-1}\mu(B_E)^{1-s}\beta^{2-2s}.\end{aligned}$$

Choosing $\beta = (s-1)^{-1/(2s)}\sqrt{\frac{C_0}{\mu(B_E)}}$ yields (3.24). $\square$

By means of Theorem 3.6 we strengthen the principal results of [J]. The assumption of a metric space-valued Poincaré inequality can be replaced with the corresponding real-valued inequality, which is in principle more easily verifiable. Recall that [J] proves the Harnack inequality and Hölder continuity for metric space-valued Dirichlet energy minimizers, under the additional (key) assumption that the target space has non-positive curvature. The proof depends on the crucial fact that if the target space Y has non-positive curvature, then for each $f : X \to Y$ which is a local minimizer (in the sense of [J]) on a domain and each $a \in Y$, the real-valued function $v_a(x) := d_Y(f(x), a)^2$ is a subsolution for the Dirichlet problem.

4 Dirichlet domains and the Sobolev spaces of Korevaar and Schoen

In this final section we will assume that the Dirichlet form $\mathcal{E}_{\mathbb{R}}$ satisfies the following condition in addition to conditions 1 through 4 of Definition 2.1.

5. We assume that the separating core Γ consists of all Lipschitz functions $\varphi : X \to \mathbb{R}$ with bounded support and that

$$C^2 \left(\operatorname{Lip}\varphi\right)^2\, d\mu \geq d\eta_{\mathbb{R}}(\varphi,\varphi) \geq \frac{1}{C^2}\left(\operatorname{Lip}\varphi\right)^2\, d\mu, \tag{4.1}$$

   for all such Lipschitz functions φ, where

$$\operatorname{Lip}\varphi(x) := \lim_{r\to 0}\sup_{y\in B(x,r)}\frac{|\varphi(x)-\varphi(y)|}{d(x,y)}$$

   denotes the pointwise Lipschitz constant of φ and the constant $C \geq 1$ is independent of x and φ.

We also assume henceforth that X is quasi-convex, that is, there is a constant $C_0 > 0$ so that every pair of points x, y in X can be joined by a rectifiable curve with length at most $C_0 d(x,y)$. For a relationship between Poincaré inequalities and quasi-convexity see [HaK, Proposition 4.4].

In general the computation of the metric d_E on X is very difficult, and it is in our interest to know when we can use the underlying metric of X instead of

d_E. The next lemma shows that by condition 5 we can replace the d_E-balls in the Poincaré inequality with the underlying metric balls and the corresponding d_E-diameter of these balls by the regular diameter.

Lemma 4.2. *If the Dirichlet form $\mathcal{E}_{\mathbb{R}}$ satisfies condition 5, then the metric d_E is bilipschitz equivalent to the underlying metric d.*

Proof. Let x and y be two points in X. Then we can construct a 1-Lipschitz map $\varphi : X \to \mathbb{R}$ so that $\varphi(x) = d(x, y)$, $\varphi(y) = 0$, and the support of φ lies in the closure of the ball $B(x, d(x, y))$. By the preceding assumptions, the modified function $C^{-1}\varphi$ is an admissible function for calculating $d_E(x, y)$, and therefore we see that $d_E(x, y) \geq C^{-1}d(x, y)$. Now suppose that $\varphi \in \Gamma$ and $d\eta_{\mathbb{R}}(\varphi, \varphi) \leq d\mu$. Then φ is Lipschitz with $\operatorname{Lip}\varphi(x) \leq C$, and hence the global Lipschitz constant of φ is bounded above by a constant $L > 0$ which depends only on C and the quasi-convexity constant C_0. Thus we have $|\varphi(x) - \varphi(y)| \leq Ld(x, y)$, and hence $d_E(x, y) \leq Ld(x, y)$. $\square$

By the above lemma, the assumption in [J] and [BM1] regarding the topology induced by d_E is automatically satisfied (see the discussion in Section 3) provided condition 5 is satisfied by $\mathcal{E}_{\mathbb{R}}$.

As stated in section 2, the Newtonian space $N^{1,2}(X)$ is an example of a Dirichlet domain. Another Sobolev type space defined on metric spaces was considered by Korevaar and Schoen [KS]. Sturm [St3] proved that the Korevaar-Schoen space $KS^{1,2}(X; Y)$ is a Dirichlet domain corresponding to a local Dirichlet form. Given the discussion in [St3], it is natural to ask whether $KS^{1,2}$ is the only Dirichlet domain (up to isomorphism) of functions on a metric space. In this section we explore the relationship between the Sobolev type spaces $KS^{1,2}(X; Y)$ defined by Korevaar and Schoen [KS] and the domain $D(\mathcal{E})$ of a general Dirichlet form

$$\mathcal{E} : L^2(X; Y) \to \mathbb{R}^+ \cup \{\infty\}.$$

As a consequence of Proposition 4.3 an extension of Sturm's result follows: the Dirichlet domain constructed from the space $KS^{1,2}(X; Y)$ cannot be extended as a Dirichlet domain to a larger space.

Recall that the space $KS^{1,2}(X; Y)$ is defined to be the collection of all functions $f \in L^2(X; Y)$ for which the approximating energies

$$e_\epsilon^2(x; f) := \fint_{B(x,\epsilon)} \frac{d_Y(f(x), f(y))^2}{\epsilon^2} \, d\mu_X(y)$$

converge to a finite energy

$$E_{KS}(f) := \sup_{\text{balls } B} \limsup_{\epsilon \to 0} \int_B e_\epsilon^2(x; f)\, d\mu_X(x) < \infty.$$

We show that, under some additional conditions on the Dirichlet energy form $\mathcal{E}$, the corresponding Dirichlet domain $D(\mathcal{E})$ embeds into the Korevaar-Schoen space $KS^{1,2}(X;Y)$ continuously.

Recall that condition 5 is assumed for the following two propositions. Consequently a measure is doubling with respect to one of the metrics d and d_E if and only if it is doubling with respect to the other metric.

Proposition 4.3. *Consider a metric measure space X supporting a doubling measure and a Poincaré inequality for the Dirichlet energy form $\mathcal{E}_\mathbb{R}$ related to the Dirichlet form $\mathcal{E} : L^2(X;Y) \to \mathbb{R}^+ \cup \{\infty\}$. We assume in addition that $\mathcal{E}_\mathbb{R}$ satisfies condition 5. Then $D(\mathcal{E})$ embeds Lipschitz continuously into $KS^{1,2}(X;Y)$.*

We follow the argument of [KMc, Section 4] in proving the above proposition.

Proof. We again use an isometric embedding $Y \subset V$ of Y into a Banach space V that maps y_0 to the zero vector. By Theorem 3.6 X supports a Poincaré inequality for $\mathcal{E}$ with respect to Y. Indeed, by the proof of Theorem 3.6 we have the following version of the Poincaré inequality:

$$\fint_B \|f(x) - f_B\|\, d\mu(x) \le C\,(\operatorname{diam}(B)) \left(\frac{\eta(f)(\lambda B)}{\mu(\lambda B)}\right)^{1/2}.$$

Let $f \in D(\mathcal{E}) \subset L^2(X;V)$ and $x, y \in X$ be two Lebesgue points for f. (By the doubling property of the measure on X such points constitute a set of full measure in X.) By the usual telescoping argument,

$$\|f(x) - f(y)\| \le \sum_{i\in\mathbb{Z}} \|f_{B_i} - f_{B_{i+1}}\| \le \sum_{i\in\mathbb{Z}} \fint_{B_i} \|f - f_{B_i}\|\, d\mu$$

where $|\cdot|$ denotes the norm in V and $B_0 = B(x, d(x,y))$, $B_i = 2^{-i}B_0$, and $B_{-i} = 2^{-i}B(y, d(x,y))$ for $i \in \mathbb{N}$. Now by the Poincaré inequality for $\mathcal{E}$,

$$\|f(x) - f(y)\| \le C \sum_{i\in\mathbb{Z}} 2^{-|i|} d(x,y) \left(\frac{\eta(f)(\lambda B_i)}{\mu(\lambda B_i)}\right)^{1/2}.$$

As in [HaK, Section 5] and [KMc, Theorem 4.1], we define

$$J_r\eta(f)(x) := \sum_{i=0}^{\infty} 2^{-i} r \left(\frac{\eta(f)(B(x, 2^{-i}\lambda r))}{\mu(B(x, 2^{-i}\lambda r))} \right)^{1/2}$$

as a generalization of the Riesz potential. Then we have

$$\|f(x) - f(y)\| \leq C\left(J_r\eta(f)(x) + J_r\eta(f)(y)\right)$$

whenever $r \geq d(x,y)$. Hence

$$\|f(x) - f(y)\|^2 \leq C\left(J_r\eta(f)(x)^2 + J_r\eta(f)(y)^2\right).$$

Fix $\epsilon > 0$ and let $y \in B(x, \epsilon)$. Choosing $r = 2\epsilon \geq d(x,y)$ we have

$$\frac{\|f(x) - f(y)\|^2}{\epsilon^2} \leq \frac{C}{\epsilon^2}\left(J_{2\epsilon}\eta(f)(x)^2 + J_{2\epsilon}\eta(f)(y)^2\right),$$

and therefore

$$e_\epsilon^2(x; f) \leq \frac{C}{\epsilon^2}\left(J_{2\epsilon}\eta(f)(x)^2 + \fint_{B(x,\epsilon)} J_{2\epsilon}\eta(f)(y)^2 \, d\mu_X(y) \right).$$

By an adaptation of the proof of [HaK, Theorem 5.3] we know that

$$\int_{B(x_0, r)} J_{2r}\eta(f)(z)^2 \, d\mu_X(z) \leq Cr^2 \eta(f)(B(x_0, 2\lambda r)) \tag{4.4}$$

for every $x_0 \in X$. Thus

$$e_\epsilon^2(x; f) \leq \frac{C}{\epsilon^2} J_{2\epsilon}\eta(f)(x)^2 + C\frac{\eta(f)(B(x, 2\lambda\epsilon))}{\mu(B(x, 2\lambda\epsilon))}$$

and hence (using (4.4) again)

$$\int_{B(x_0,\epsilon)} e_\epsilon^2(x; f) \, d\mu_X(x) \leq C\eta(f)(B(x_0, 2\lambda\epsilon)) + C\int_{B(x_0,\epsilon)} \frac{\eta(f)(B(x, 2\lambda\epsilon))}{\mu(B(x, 2\lambda\epsilon))} \, d\mu_X(x).$$

From this we can conclude that

$$\int_{B(x_0,\epsilon)} e_\epsilon^2(x;f)\,d\mu_X(x) \leq C\eta(f)(B(x_0,3\lambda\epsilon)),$$

and now the doubling property and a standard covering argument imply that

$$\int_X e_\epsilon^2(x;f)\,d\mu_X(x) \leq C\eta(f)(X) = C\mathcal{E}(f).$$

Thus $E_{KS}(f) \leq C\mathcal{E}(f)$ and we have the required embedding. $\square$

The next result provides us with a converse to the above proposition under some additional conditions on the form $\mathcal{E}$.

Definition 4.5. We say that a Dirichlet form $\mathcal{E} : D(\mathcal{E}) \to \mathbb{R}_+$ has the *strong quadratic contraction property* if

$$\mathcal{E}(f) \leq \sup_\varphi \mathcal{E}_{\mathbb{R}}(\varphi \circ f), \tag{4.6}$$

the supremum being taken over all 1-Lipschitz functions φ from Y into $\mathbb{R}$.

Note that in fact equality holds in (4.6) by Remark (1) on page 6. Observe also that inequality (4.6) is a strengthened form of condition 2 in Definition 2.5.

The example $N^{1,2}(X;Y)$ of a Dirichlet domain for metric space-valued maps satisfies the condition in Definition 4.5, see e.g. [HKST, Section 3].

Proposition 4.7. *Let X be a metric measure space which is proper (closed balls are compact) and let $\mathcal{E}$ be a Dirichlet form on maps from X to a complete metric space Y as before which satisfies conditions 1 through 5 as well as the strong quadratic contraction property (4.6). If in addition X supports a Poincaré inequality for the Korevaar-Schoen energy E_{KS}, then $KS^{1,2}(X;Y)$ embeds in $D(\mathcal{E})$.*

Proof. Let $f \in KS^{1,2}(X;Y)$ and $\varphi : Y \to \mathbb{R}$ be a 1-Lipschitz function. Then $\varphi \circ f$ belongs to $KS^{1,2}(X;\mathbb{R}) = KS^{1,2}(X)$ with $E_{KS}(\varphi \circ f) \leq E_{KS}(f)$.

Given $\epsilon > 0$ we can cover X by a countable number of balls $B_i = B(x_i,\epsilon)$ of uniformly bounded overlap (since the measure on X is doubling). Fix such an $\epsilon > 0$ and define a locally Lipschitz approximation f_ϵ as follows:

$$f_\epsilon(x) := \sum_{i\in\mathbb{N}} \varphi_i(x)\,(\varphi \circ f)_{B_i}$$

where the functions $\{\varphi_i\}$, $i \in \mathbb{N}$, constitute a C/ϵ-Lipschitz partition of unity subordinate to the cover $\{B_i\}$. By [KMc, Lemma 4.6] we have

$$\begin{aligned} |f_\epsilon(x) - f_\epsilon(y)| &\leq C d(x,y) \fint_{B(x,2\epsilon)} \left(e_{5\epsilon}^2(z;\varphi\circ f)\right)^{1/2} d\mu(z) \\ &\leq C d(x,y) \left(\fint_{B(x,2\epsilon)} e_{5\epsilon}^2(z;\varphi\circ f)\, d\mu(z) \right)^{1/2} \end{aligned}$$

and so

$$\operatorname{Lip} f_\epsilon(x) \leq C \left(\fint_{B(x,2\epsilon)} e_{5\epsilon}^2(z;\varphi\circ f)\, d\mu(z) \right)^{1/2}.$$

By the assumption (4.1) on the energy measure $\eta_{\mathbb{R}}$,

$$\begin{aligned} \int_B d\eta_{\mathbb{R}}(f_\epsilon, f_\epsilon)(x) &\leq C \int_B \fint_{B(x,2\epsilon)} e_{5\epsilon}^2(z;\varphi\circ f)\, d\mu(z)\, d\mu(x) \\ &\leq C \int_B e_{5\epsilon}^2(x;\varphi\circ f)\, d\mu(x) \end{aligned}$$

for all balls B of radius ϵ. By a standard covering argument the above inequality yields

$$\int_{B_0} d\eta_{\mathbb{R}}(f_\epsilon, f_\epsilon) \leq C \int_{B_0} e_{5\epsilon}^2(x;f)\, d\mu(x) \tag{4.8}$$

for arbitrary balls B_0, where $C > 0$ is independent of B_0.

Now suppose f is a Lipschitz function with compact support—that is, outside a compact set, f takes on the constant value $y_0 \in Y$. By the Poincaré inequality $E_{KS}(f) = 0$ implies that f is constant and hence $\mathcal{E}(f) = 0$. Therefore we may assume without loss of generality that $E_{KS}(f) > 0$. Since $d\eta_{\mathbb{R}}(f_\epsilon + a, f_\epsilon + a) = d\eta_{\mathbb{R}}(f_\epsilon, f_\epsilon)$ for every constant $a \in \mathbb{R}$, we may further assume that $\varphi(0) = 0$. We may choose a sufficiently large ball B_0 so that f and all of the functions f_ϵ, $0 < \epsilon < 1$, have support inside B_0. Then

$$\int_{B_0} d\eta_{\mathbb{R}}(f_\epsilon, f_\epsilon) = \int_X d\eta_{\mathbb{R}}(f_\epsilon, f_\epsilon)$$

by the strong locality of $\eta_{\mathbb{R}}$. Since $E_{KS}(f) > 0$, we have $\int_{B_0} e^2_{5\epsilon}(x;\varphi\circ f) \leq 2E_{KS}(f)$ for sufficiently small $\epsilon > 0$. Using these facts in (4.8) we deduce that $\mathcal{E}_{\mathbb{R}}(f_\epsilon) \leq CE_{KS}(f)$ for sufficiently small $\epsilon > 0$. Thus $\{f_\epsilon\}$ is a bounded sequence of functions in $(D(\mathcal{E}_{\mathbb{R}}), ||\cdot||_{\mathcal{E}_{\mathbb{R}}})$, where

$$\|u\|_{\mathcal{E}_{\mathbb{R}}} := \mathcal{E}_{\mathbb{R}}(u)^{1/2} + \|u\|_{L^2}, \qquad u \in D(\mathcal{E}_{\mathbb{R}}),$$

and $f_\epsilon \to \varphi\circ f$ almost everywhere by the proof of [HKST, Lemma 8.2]. Since $(D(\mathcal{E}_{\mathbb{R}}), ||\cdot||_{\mathcal{E}_{\mathbb{R}}})$ is a Hilbert space, we can use Mazur's lemma to conclude that $\varphi\circ f \in D(\mathcal{E}_{\mathbb{R}})$, and in this case we also see that $\mathcal{E}_{\mathbb{R}}(\varphi\circ f) \leq CE_{KS}(f)$. Finally, $f \in D(\mathcal{E})$ and $\mathcal{E}(f) \leq \sup_\varphi \mathcal{E}_{\mathbb{R}}(\varphi\circ f) \leq CE_{KS}(f)$ by (4.6) (if we do not have (4.6), then by condition 2 of Definition 2.5 we can still conclude that $f \in D(\mathcal{E})$, but we do not have control over the Dirichlet energy of f in terms of $E_{KS}(f)$).

As X supports a Poincaré inequality for the Korevaar-Schoen energy measure, Lipschitz functions of bounded support are dense in $KS^{1,2}(X;Y)$ (see [KS] or [KMc]). Hence all functions $f \in KS^{1,2}(X;Y)$ are in $D(\mathcal{E})$ with $\mathcal{E}(f) \leq CE_{KS}(f)$, and thus we have the embedding stated in the proposition. □

5 Appendix

If $\mathcal{E}$ is a Dirichlet form on real-valued functions, then to guarantee the existence of the corresponding measure-valued form η it suffices to know that

$$E(\varphi u, u) - \frac{1}{2}E(u^2,\varphi) \geq 0 \tag{5.1}$$

for every non-negative continuous function φ with bounded support. Under this assumption, the Riesz representation theorem applied to the positive definite linear operator $T_u(\varphi) = E(\varphi u, u) - \frac{1}{2}E(u^2,\varphi)$ yields a finite Radon measure $\eta(u,u)$ so that $T_u(\varphi) = \int_X \varphi\, d\eta(u,u)$. By covering X with a countable collection of balls with bounded overlap and considering a partition of unity ψ_k, $k \in \mathbb{N}$, we see that $E(u,u) = \sum_{k\in\mathbb{N}}(E(\psi_k u, u) - \frac{1}{2}E(u^2,\psi_k))$ is given by $E(u,u) = \eta(u,u)(X)$. (Here we use the fact that $E(v,w) = 0$ whenever w is constant.) A corresponding polarization yields the measure-valued bilinear form $\eta(u,v)$. Such polarization is bilinear because the form E is. In fact, we have the identity

$$2\int_X \varphi\, d\eta(u,v) = E(\varphi u, v) + E(u,\varphi v) - E(uv,\varphi). \tag{5.2}$$

The Dirichlet forms on fractals constructed in [BB] and [Ki1], [Ki2] as well as the Dirichlet forms obtained by using the Cheeger derivatives (as discussed in section 2) satisfy condition (5.1). It is not clear what its analogue for forms on general metric space- or Banach space-valued functions should be. One possible analogue would be to require that the Dirichlet energy form comes from a bilinear form E and that $E(\varphi u, u) - \frac{1}{2}E(|u|^2, \varphi) \geq 0$ for all non-negative continuous functions with bounded support, where $|u(x)|$ denotes the Banach space norm of the element $u(x)$.

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