Some Convergence Results for p-harmonic Functions on Metric Measure Spaces

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Abstract

In this note we consider a stability property of p-harmonic functions in general metric measure spaces. Despite the fact that there may not be a corresponding differential equation, it is shown here that a family of uniformly convergent p-harmonic functions converge to a p-harmonic function; that is, the collection of p-harmonic functions form a closed and locally compact subspace in a certain Hölder class. As a consequence, it is shown that a family of p-harmonic functions bounded in a Sobolev type space (called the Newtonian space) has a sequence that converges locally uniformly in the domain of harmonicity to a p-harmonic function. The main result of this paper is used to construct p-harmonic functions on unbounded domains.

1 Introduction

In this note we study some stability properties of p-harmonic functions on certain metric measure spaces. A p-harmonic function is a Sobolev-type class function that locally minimizes the p-energy. More specifically, a p-harmonic function u on a domain Ω satisfies the condition that whenever V is a relatively compact subdomain of Ω and v is a Sobolev-type function that takes on the value of 0 on $\Omega\setminus V$, then the p-energy of $u+v$ on V is not smaller than the p-energy of u on V. In the setting of Euclidean space $\mathbb{R}^n$, the Sobolev-type class is the classical Sobolev space $W^{1,p}(\mathbb{R}^n)$ and the p-energy of a Sobolev function f on a domain Ω is given by the integral $\int_\Omega |\nabla f(x)|^p\, dx$. In this setting, p-harmonic functions u satisfy the corresponding Euler-Lagrange equation

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0.$$

The setting considered in this note is that of a general metric space X, endowed with a metric d and a local doubling Borel regular measure μ; see Definition 2.9 for precise definitions. Observe that if the measure on X is doubling and X is proper, then it is a separable metric space. We also require the metric space X to be proper (that is, closed and bounded subsets of X are compact) and supporting a Poincaré type inequality; see Definition 2.8. A Sobolev-type space of functions on metric measure spaces has been extensively studied in [C], [HK], [He], and [Sh1], and some regularity properties of corresponding p-harmonic functions have been studied in [Bj], [KiSh], and [Sh2].

It is a well-known fact that a sequence of locally bounded p-harmonic functions on a domain in $\mathbb{R}^n$ has a locally uniformly convergent subsequence that converges to a p-harmonic function on that domain; see [HKM, Theorem 3.78] or [Ru, Theorem 14.6]. The aim of this paper is to extend this result to p-harmonic functions on abstract metric measure spaces. We thus unify the analogous results already proved for Euclidean spaces equipped with $\mathcal{A}_p$-weights and Riemannian manifolds (see [HKM], [LM1] and [LM2]) and extend the results to more general metric spaces including Carnot groups such as the Heisenberg groups (both with the Carnot-Carathéodory metric and with the intrinsic metric derived from the Lie group norm), regular connected graphs (see Section 3), and the more exotic non-Riemannian metric spaces of Bordon–Pajot (see [BP]) and Laakso (see [L]). The crucial tool used in the standard proof of the above classical fact is the Euler-Lagrange equation satisfied by the p-harmonic functions. In the general setting of metric measure spaces we do not have an analog of the Euler-Lagrange equation, and therefore the proof of the corresponding result on metric measure spaces uses variational techniques.

There are many notions of Sobolev spaces in metric spaces; see for example [Ha], [C], [He], [HK], [KM2], [Sh1], and [T]. In this paper we will follow the definition $N^{1,p}(X)$ of [Sh1]; see Section 2 for the definition of $N^{1,p}(X)$. If Ω is a bounded open set such that the interior of $X\setminus\Omega$ is non-empty, then it is known that for every $f \in N^{1,p}(X)$ there is a unique solution to the p-Dirichlet problem on Ω with boundary values f; see [Sh2].

The question studied here is a convergence problem about p-harmonic functions. Suppose $\{f_i\}$ is a Cauchy sequence of functions in $N^{1,p}(X)$, converging to a function f. Let u_i be the solution to the Dirichlet problem with boundary data f_i. We would like to know whether u_i converges to the corresponding Dirichlet solution with boundary data f, either almost everywhere or in the L^p-norm or better yet, in the $N^{1,p}(X)$-norm. In the Euclidean setting, this question is answered by the papers [LM1] and [LM2], using the setting of the differential equation. A combination of the initial part of the proof of [LM1, Theorem 1.4] together with [LM2, Lemma 2.7] yields that if $X = \mathbb{R}^n$ then $u_i \to u$ in $N^{1,p}(X)$, after noting that $u_i \to u$ weakly in $N^{1,p}(X)$ and that u is the p-harmonic function with boundary data f.

The proof in the papers [LM1] and [LM2] uses the Euler-Lagrange differential equation. We have no such differential equation in the general setting of metric measure spaces considered in this note. Therefore, the two main results here, Theorem 1.1 and Theorem 1.2, are weaker versions of their results.

Recall that a subset of the metric space X is a domain if it is a non-empty open connected set. See Definition 2.7 for the p-energy integral $I(f)$ of a Sobolev-type function f.

Theorem 1.1. *Let X be a proper metric measure space equipped with a local doubling measure supporting a weak local $(1,p)$-Poincaré inequality, and let Ω be a bounded domain in X with a boundary point $x_0 \in \partial\Omega$ so that the measure of $B(x_0,r)\backslash\Omega$ is positive for every $r > 0$. Consider a Cauchy sequence $\{f_i\}$ in $N^{1,p}(X)$ that converges to f, and suppose u_i is the solution to the Dirichlet problem on a bounded domain Ω with boundary data f_i. Then there is a convex combination sequence, say $\tilde{u}_n = \sum_{i=n}^{N(n)} \lambda_{n,i}u_i$; $\sum_{i=n}^{N(n)} \lambda_{n,i} = 1$, so that $\tilde{u}_n$ converges to u in $N^{1,p}(X)$ and u is the solution to the Dirichlet problem with boundary values f. Moreover, the p-energy integrals $I(u_i)$ converges to the energy $I(u)$. Furthermore, if X supports a weak $(1,q)$-Poincaré inequality for some q with $1 \le q < p$, then u_i converges to u locally uniformly in Ω.*

The assumption on the boundary of Ω is not as restrictive as it sounds. Indeed, if X is path-connected and $X\backslash\overline{\Omega}$ is non-empty, any point x_0 in the nonempty set $\partial\overline{\Omega}$ would satisfy the condition that the measure of $B(x_0,r)\backslash\Omega$ is positive for every $r > 0$. Also, if the measure of $X\backslash\Omega$ is positive and X is path connected, then again the ssumption on the boundary of Ω in the above theorem is satisfied.

In the next theorem there are no restrictions on the domain Ω. Theorem 1.2 is a generalization of [VK, Proposition 1].

Theorem 1.2. *Let X be a proper metric measure space equipped with a local doubling measure supporting a weak local $(1,p)$-Poincaré inequality. If u_i is a sequence of p-harmonic functions on a domain Ω such that $u_i \to u$ locally uniformly in Ω, and u is in $N^{1,p}_{loc}(\Omega)$, then u is a p-harmonic function.*

The restricton of Theorem 1.1 to bounded domains is essential. Should the boundedness condition be removed, we lose the knowledge that solutions to Dirichlet problems are unique. In fact, even in the Euclidean spaces we do not have uniqueness of solutions to the Dirichlet problem if the domain of the problem is unbounded. Indeed, a suitable truncation of the Green function is a solution to the Dirichlet problem on the domain whose complement is the compact set on which the Green function is modified; the same Dirichlet problem has a constant solution as well. By losing the uniqueness result for Dirichlet solutions we lose information about local uniform convergence of the *entire* sequence $\{u_i\}$ considered in Theorem 1.1 to the same solution u. Moreover, if Ω is unbounded, then the limit function u in Theorem 1.1 may not be in $N^{1,p}(X)$, in which case we are also unable to conclude that $\|g_{u_i}\|_{L^p(X)}$ converges to $\|g_u\|_{L^p(X)}$. Thus removing the condition that Ω is bounded results in a substantial weakening of Theorem 1.1; see Proposition 4.9.

Examples of spaces for which the results in this paper applies are the Euclidean spaces, the Carnot-Carathéodory spaces (see [GN], [FLW1], [FLW2], and [HK, Section 11] for more on these spaces), the Laakso spaces ([L]), the Bordon-Pajot spaces ([BoPa]), and Riemannian manifolds with non-negative curvature. Theorems 1.1 and 1.2 hold for general Riemannian manifolds as well, as Riemannian manifolds have locally doubling measure and support a local $(1,1)$-Poincaré inequality. As mentioned above, there is a strengthening of the above theorems in the setting of the Euclidean spaces; see [LM1] and [LM2]. In the setting of spaces for which the p-harmonic function satisfies the p-Laplacian equation (for example, Riemannian manifolds and the Carnot-Carathéodory spaces), the result [HKM, Theorem 3.78] yields Theorem 1.2. See also [VK, Proposition 1] for Theorem 1.2 in the setting of Carnot groups.

Classical harmonic analysis uses locally bounded classes of harmonic functions to construct functions that have an essential singularity at a point and are p-harmonic away from this singularity. Such functions are called p-Green functions; see [Ho]. The results in this paper indicate that that such an approach can be taken in a more abstract metric space setting to obtain singular p-harmonic functions; see [HoSh], [HoShT].

The paper [Sh2] shows that the Dirichlet problem is solvable if the domain is bounded. In the last section of this paper we will use Theorem 1.2 to construct solutions to the Dirichlet problem on unbounded domains.

This note is organized as follows. In the second section we briefly recall the necessary notations and definitions. In the third section we show that the theory of p-harmonic functions studied in this note is applicable to p-harmonic functions on certain (regular) graphs. More specifically, we show that a p-harmonic function as defined in [HS] on a regular graph can be extended in a natural way to the connected version of the graph to be p-harmonic, and that the corresponding connected graph is locally doubling and supports a Poincaré type inequality. The fourth section concentrates on proving the above mentioned convergence results, and the last section contains two applications of these results; a normal family characterization of p-harmonic functions, and existence of solutions to p-Dirichlet problems on general domains.

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2 Background

This section catalogs the definitions necessary for the rest of the paper.

Let X be a metric measure space equipped with a complete metric d and a Borel regular measure μ. It is assumed throughout this paper that the measure of every open set is positive, and that the measure of every bounded set is finite. Let $1 < p < \infty$ (most of the definitions in this section makes sense even for $p = 1$, but the proof of the results in this note are valid only when $p > 1$).

Paths γ in X are continuous maps $\gamma : I \to X$, where I is some interval in $\mathbb{R}$. The following definitions are from [HeK1], [HK], and [Sh1].

Definition 2.1. Let u be a real-valued function on X. A non-negative Borel-measurable function $\rho : X \to [0, \infty]$ is said to be an *upper gradient* of u if for all compact rectifiable paths γ the following inequality holds:

$$|u(x) - u(y)| \leq \int_\gamma \rho ds, \tag{1}$$

where x and y denote the endpoints of the path.

See [HeK1, Section 2.9], [He], [KM1], and [Sh1] for a discussion on upper gradients; [HeK1] uses the term *very weak gradients* for this concept.

Definition 2.2. Let Γ be a collection of paths in X. The *p-modulus* of the family Γ, denoted $\mathrm{Mod}_p\Gamma$, is defined to be the number

$$\inf_\rho \|\rho\|^p_{L^p},$$

where the infimum is taken over the set of all non-negative Borel-measurable functions $\rho : X \to [0, \infty]$ such that for all rectifiable paths γ in Γ the path integral $\int_\gamma \rho ds$ is not smaller than 1. Such functions ρ are said to be *admissible* for the family Γ.

A property relevant to paths is said to hold for *p-almost all paths* if the p-modulus of the family of curves for which the property does not hold is zero. See [R, Chapter II] for a treatment of moduli of curve families in Euclidean spaces, and [F] for the setting in general metric measure spaces.

Definition 2.3. Let u be an arbitrary real-valued function on X, and let ρ be a non-negative Borel function on X. If inequality (1) holds for p-almost all paths γ, then ρ is said to be a *p-weak upper gradient* of u.

An extended real-valued Borel function f on X is said to be *p-integrable* if $\int_X |f|^p d\mu < \infty$.

It is easy to see that if ρ_1 and ρ_2 are p-weak upper gradients of f_1 and f_2 respectively then $\rho_1 + \rho_2$ is a p-weak upper gradient of $f_1 + f_2$ and $|\alpha|\rho_1$ is a p-weak upper gradient of αf_1 for every $\alpha \in \mathbb{R}$. It is shown in [Sh1] that if

U is a measurable subset of X and a function f on X is constant on U, and moreover, ρ is a p-integrable p-weak upper gradient of f, then $\rho\chi_U$ is also a p-weak upper gradient of f. It is also shown in [Sh1] and [Sh2] that if u is a bounded Lipschitz function and f a function on X with a p-integrable p-weak upper gradient ρ then the function $L|f| + \|u\|_\infty \rho$ is a p-weak upper gradient of uf, where L is the Lipschitz constant of u and $\|u\|_\infty$ is a bound for u. Furthermore, if $\Phi : \mathbb{R} \to \mathbb{R}$ is a C^1-function and f is a real-valued function on X with a p-integrable p-weak upper gradient, then $\rho\, \Phi' \circ f$ is a p-weak upper gradient of $\Phi \circ f$; that is, the chain rule applies (see for example [C]).

A p-weak upper gradient g_u of u is said to be the *minimal p-weak upper gradient* of u if whenever ρ is a p-weak upper gradient of u the inequality $g_u(x) \le \rho(x)$ holds for μ-a.e. $x \in X$. Recall that we assume $1 < p < \infty$. If a function u has a p-integrable p-weak upper gradient, then it has a minimal p-weak upper gradient; see [C] or [Sh2]. This is a consequence of the uniform convexity of the Banach space $L^p(X)$ when $1 < p < \infty$.

Definition 2.4. Let the set $\tilde{N}^{1,p}(X, d, \mu)$ be the collection of all real-valued p-integrable functions u on X that have a p-integrable p-weak upper gradient.

If u is a function in $\tilde{N}^{1,p}$, let

$$\|u\|_{\tilde{N}^{1,p}} = \|u\|_{L^p} + \inf_\rho \|\rho\|_{L^p} = \|u\|_{L^p} + \|g_u\|_{L^p},$$

where the infimum is taken over all p-integrable p-weak upper gradients of u. If u and v are functions in $\tilde{N}^{1,p}$, let $u \sim v$ if $\|u - v\|_{\tilde{N}^{1,p}} = 0$. Note that $\sim$ is an equivalence relation, partitioning $\tilde{N}^{1,p}$ into equivalence classes.

Definition 2.5. The *Newtonian space* corresponding to the index p, $1 \le p < \infty$, is the normed space $\tilde{N}^{1,p}(X, d, \mu)/\sim$, with norm $\|u\|_{N^{1,p}} := \|u\|_{\tilde{N}^{1,p}}$, and is denoted $N^{1,p}(X)$. If Ω is a domain in X, the Newtonian space with zero boundary values in Ω, $N^{1,p}_0(\Omega)$, is the collections of all equivalence classes $[u]$ in $N^{1,p}(X)$ for which there is a representative function $u \in [u]$ so that $u|_{X\setminus\Omega} = 0$.

The local analog $N^{1,p}_{loc}(X)$ and $N^{1,p}_{0,loc}(\Omega)$ are obtained by requiring local p-integrability rather than global p-integrability on X for the functions. Here a function f is said to be locally p-integrable if $f|_B \in L^p(B)$ for every bounded measurable subset B of X. This definition is not the usual notion of local integrability. The standard definition of local integrability requires only that the function be in $L^p(K)$ for every compact subset K of X. The definition used here is stronger than the standard definition of local integrability, but is useful here. For example, in constructing the extremal function u for the p-capacity $\mathrm{cap}_p(E, F)$ where E and F are non-empty, non-intersecting compact sets, there is no reason to expect the extremal function u to be p-integrable over the whole space X unless the measure of X itself is finite (note that

$0 \leq u \leq 1$), but u is locally p-integrable in our sense. See also the proofs in Section 4. With this non-standard definition, $N^{1,p}_{loc}(X) = N^{1,p}(X)$ if X is bounded. Note that we do still require the upper gradients to be globally p-integrable. For additional uses of this definition of $N^{1,p}_{loc}(X)$ see [KSh].

It is shown in [Sh1] that the Newtonian spaces are Banach spaces. The paper [C] has another definition of Sobolev spaces, but [Sh1] shows that when $p > 1$ the definition of [C] is isometrically equivalent to the definition given above. Hence by the results in [C], under the conditions we assume on X in this paper, the Newtonian spaces are reflexive when $1 < p < \infty$; see also [T].

A more sensitive alternative to measure can be defined on subsets of X by using the Newtonian spaces. Measure is the natural gauge for defining $L^p(X)$ functions; two functions in $L^p(X)$ are in the same L^p class if and only if the set on which they disagree is of measure zero. A finer gauge is needed to explore $N^{1,p}(X)$ function classes. One such canonical gauge is p-capacity. Different definitions for a capacity of a set can be found in literature. For more information on p-capacity, see [AO], [HKM], [HeK2], [KM1], and [KSh].

Definition 2.6. The *p-capacity* of a set $E \subset X$ with respect to the space $N^{1,p}(X)$ is defined by

$$\mathrm{Cap}_p E = \inf_u \|u\|^p_{N^{1,p}}, \tag{2}$$

where the infimum is taken over all the equivalence classes u in $N^{1,p}(X)$ that have a representative function, also denoted u, whose restriction to E is bounded below by 1.

It is easy to see that sets of zero p-capacity have zero measure. In general, sets of zero measure need not have zero p-capacity. It is shown in [Sh1] that if u_1 and u_2 are two functions in $N^{1,p}(X)$ so that $\|u_1 - u_2\|_{L^p} = 0$, then $u_1 \sim u_2$ and hence both functions belong to the same equivalence class in $N^{1,p}(X)$. It is also easy to see in this case that the p-capacity of the set where u_1 and u_2 differ is zero. In fact, u_1 and u_2 belong to the same equivalence class in $N^{1,p}(X)$ if and only if they differ only on a set of p-capacity zero; see [Sh1]. A property relevant to points in X is said to hold for *p-capacity almost every point* (or, *p-q.e.*), if the set of all points in X for which the property fails to hold is of zero p-capacity.

Definition 2.7. Given a function $f \in N^{1,p}(X)$, the energy operator I_f on the space $N^{1,p}_0(\Omega)$ is defined by $I_f(v) = \int_X g^p_{v+f}$, where g_{v+f} is the minimal p-weak upper gradient of $v + f$. The function $u = f + v$; $v \in N^{1,p}_{0,loc}(\Omega)$, is said to be a *solution to the p-Dirichlet problem* in Ω with boundary data f if for all $w \in N^{1,p}_0(\Omega)$ we have $I(u) := I_u(0) \leq I_u(w)$. Note that as $(w+u)|_{X\setminus\Omega} = u$, we have $g_u = g_{w+u}$ almost everywhere in $X\setminus\Omega$. Hence effectively the minimization takes place for the energy integral over Ω. Moreover, by [Sh2, Theorem 4.8], we can restrict attention to functions $w \in N^{1,p}_0(\Omega)$ whose support is compactly contained in Ω. If Ω is bounded, then we see that such harmonic function u belongs to $N^{1,p}(X)$, and there is only one solution u; see [C] or [Sh2]. As mentioned earlier, the uniqueness of solutions on unbounded domains is not true, even if X supports global Poincaré inequality with a measure that is globally doubling.

Let Ω be a domain in X such that the set $X\backslash\Omega$ has a positive measure. A function u on Ω is said to be *p-harmonic* in Ω if $u \in N^{1,p}(V)$ for every compact subdomain V of Ω and for every function $w \in N_0^{1,p}(V) + u$ we have $\int_V g_u^p \leq \int_V g_w^p$. It was shown in [Sh2] that every solution to a p-Dirichlet problem in the class $N^{1,p}(X)$ is a p-harmonic function and that every p-harmonic function in $N^{1,p}(X)$ is a solution to a p-Dirichlet problem.

In light of the result [Sh2, Theorem 4.8] and by the above definition of p-harmonic functions, we have the following equivalent definition of a solution to the Dirichlet problem. A function $u \in N_{loc}^{1,p}(X)$ is said to be a solution to the Dirichlet problem on a domain $\Omega \subset X$ with boundary data $f \in N^{1,p}(X)$ if u is p-harmonic on Ω and $u - f \in N_{0,loc}^{1,p}(\Omega)$. See [Bj], [C], [KiSh], and [Sh2] for some properties of p-harmonic functions.

Definition 2.8. The space X supports a *weak local* $(1,p)$*-Poincaré inequality* if there exist constants $r_0 > 0$, $\tau \geq 1$, and $C > 0$, so that for all open balls B in X with radius smaller than r_0 and all pairs of measurable functions u and ρ defined on τB, whenever ρ is an upper gradient of u in τB and u is integrable on B the following inequality holds true:

$$\fint_B |u - u_B| \leq C diam(B)(\fint_{\tau B} \rho^p)^{1/p}, \tag{3}$$

where, if f is a measurable function on X, then

$$f_B := \frac{1}{\mu(B)} \int_B f =: \fint_B f.$$

The space X supports a *weak* $(1,p)$*-Poincaré inequality* if we can remove the restriction that the radius of the balls B in inequality (3) is less than r_0. The space X supports a $(1,p)$*-Poincaré inequality* if it supports a weak $(1,p)$-Poincaré inequality with $\tau = 1$.

For details about Poincaré inequalities, see [HeK1], [HeK2], [HK], and the references therein.

Definition 2.9. A measure μ on a metric measure space is said to be *locally doubling* if it is a Borel measure so that the measure of open sets is positive and the measure of bounded sets is finite and moreover there are constants $r_0 > 0$ and $C \geq 1$ so that for all points x in X and all r with $0 < r \leq r_0$,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

The measure is said to be *doubling* if it satisfies the inequality for the local doubling condition for all $r > 0$.

Remark 2.10. It is known that if X has a doubling measure supporting a weak $(1, p)$-Poincaré inequality and Ω is a bounded domain in X such that the interior of $X \backslash \Omega$ is nonempty, the p-Dirichlet problem is solvable on Ω, and such solutions are unique and support classical properties such as the Harnack inequality, comparison principle, and the maximum principle; see [Sh2] and [KiSh]. In order to apply the results from [Sh2], in particular the results about existence of the solution to the Dirichlet problem on bounded domains, we need a version of the Poincaré inequality on bounded domains. In particular, we need to know that if $u \in N^{1,p}_{0,\,loc}(\Omega)$ where Ω is a bounded domain, then

$$\int_\Omega |u|^p \leq C(\Omega, p) \int_\Omega g_u^p \tag{4}$$

where the constant $C(\Omega, p) > 0$ is independent of u. If X is proper, the measure on X is locally doubling, and the measure supports a local $(1, p)$-Poincaré inequality, then it can be shown that X supports the above inequality (4) whenever there is a point x_0 on the boundary of Ω so that the measure of the set $B(x_0, r_0/4) \backslash \Omega$ is positive. Indeed, in this case if Ω is a bounded set, then the closed set $\overline{\Omega}$ is compact, and hence can be covered by a finite number of balls of radius $r = r_0/4$, one of them being $B(x_0, r)$. Recall that Ω is a domain and hence is a connected set; hence its closure $\overline{\Omega}$ is also connected. Now a standard chaining argument using the fact that Ω is connected, together with the inequality given in [KiSh, Lemma 2.10] applied to the ball $B(x_0, r)$, yield the above inequality. However, to simplify the proofs and avoid cumbersome notations that do not provide additional geometric insight, we only assume global Poincaré inequality and global doubling property here. It is worthwhile noting that because of the reasons given above, the two main theorems of this paper are applicable even to the general setting of Riemannian manifolds, where we do not normally have globally doubling measures supporting a global Poincaré inequality.

3 p-harmonic functions on regular graphs

p-harmonic functions on graphs have been studied by Holopainen and Soardi in [HS]. In particular, it has been shown that if the graph is regular, that is the measure on the graph is a doubling measure (see inequality (5) that supports a $(1,p)$-Poincaré type inequality (see inequality 6), then p-harmonic functions on the graph satisfies a Harnack inequality and the mean value property. Thus it is of interest to study stability properties of p-harmonic functions on graphs. We will show in this section that every p-harmonic function on such a graph extends to a p-harmonic function on the connected version of the graph together with the natural (one-dimensional) measure, and that the natural measure on this connected graph satisfies a doubling property and a $(1,p)$-Poincaré inequality as defined in Section 2 above. Thus the results of this note also apply to p-harmonic functions on such graphs.

Let G be a graph with vertex set V and (neighbourhood) relation $\sim$. For $x, y \in V$, we say x and y are neighbours if $x \sim y$. The measure on the graph is the counting measure on the vertices of the graph. The metric on G is obtained as follows: If x, y are vertices, then the metric $d(x,y)$ is the smallest number of vertices $x = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = y$ needed so that $x_i \sim x_{i+1}$.

As considered in [HS], the measure on G is doubling if whenever r is a positive number and $x \in V$, we have

$$\#B(x,2r) \le c\#B(x,r) \tag{5}$$

with the constant c independent of x and r. Here, for subsets U of the graph, $\#U$ denotes the number of vertices in U, that is, the measure of U.

Again, following [HS], we say G supports a $(1,p)$-Poincaré inequality if for all functions $u : V \to \mathbb{R}$,

$$\frac{1}{\#B}\sum_{x\in B}|u(x)-u_B| \le c\ \mathrm{diam}(B)\Big(\frac{1}{\#B}\sum_{x\in B, y\sim x}|u(x)-u(y)|^p\Big)^{1/p}, \tag{6}$$

and $u : V \to \mathbb{R}$ is said to be p-harmonic in $U \subset V$ if

$$\Delta_p u(x) = \sum_{y\sim x}|u(y)-u(x)|^{p-2}(u(y)-u(x)) = 0 \tag{7}$$

for every $x \in U \cup \partial U$, or equivalently, for every function $w : V \to \mathbb{R}$ with finite support in U,

$$\sum_{x\in U, y\sim x}|u(y)-u(x)|^{p-2}(u(y)-u(x))(w(y)-w(x)) = 0. \tag{8}$$

A graph that has both the doubling property and the $(1,p)$-Poincaré inequality is said to be *regular*. The paper [HS] shows that if a graph is regular, then p-harmonic functions satisfy Harnack's inequality and hence such graphs have no positive non-constant p-harmonic functions. The proofs in that paper are calculation-intensive, and avoids the problems of choosing locally supported test functions in classical methods of proving Harnack inequality. The purpose of this section is to show that such graphs, which are discrete objects, can be equated with a corresponding continuous object having doubling property and $(1,p)$-Poincaré inequality and hence the machinery developed in the previous sections can be applied to obtain Harnack inequality and strong maximum principle.

To the graph G we associate a connected graph G_0 as follows. If $x \sim y$, then connect x and y by a closed interval of unit length. Associate a measure μ on this graph as follows: If U is a subset of this graph, then

$$\mu(U) = \sum_I \text{length}(I \cap U), \tag{9}$$

where the sum is over the intervals I such that $I \cap U \neq \emptyset$.

Lemma 3.1. *The measure μ is doubling.*

Proof. First note that by the doubling property (5), there are atmost C number of neighbours to each vertex. If x is a vertex, then we have the following two cases:

Case 1: If $r \leq 1$, then

$$r \leq \mu(B(x,r)) \leq cr.$$

Case 2: If $r > 1$, then

$$\#B(x,r)/2 \leq \mu(B(x,r)) \leq c\#B(x,r+1).$$

The situation of x not a vertex but $r \leq 1$ is also covered by case 1. The situation of x not a vertex but $r > 1$ is covered by case 2 after taking the closest vertex to x. Note that the closest vertex is of distance no more than $1/2$ from x.

Now, if $r \leq 1/2$, then

$$\mu(B(x,2r)) \leq 2cr \leq 2c\mu(B(x,r)).$$

If $r > 1/2$, then

$$\begin{aligned}\mu(B(x,2r)) &\leq c\#B(x,2r+1) \leq c^2\#B(x,r+1/2)\\ &\leq c^3\#B(x,r) \leq 2c^3\mu(B(x,r)).\end{aligned}$$

Hence the measure μ is doubling. $\square$

Lemma 3.2. *If G supports a $(1,p)$-Poincaré inequality in the sense of inequality (6), then the connected graph G_0 supports a $(1,p)$-Poincaré inequality in the usual sense.*

Proof. Just as in the proof of Lemma 3.1, we can assume the ball B is centered at a vertex x.

Case 1: The radius $r \leq 1$. Then $\mu(B(x,r)) = t\deg(x)$ is comparable to t. Note that if y is a neighbour of x, then $u(ty) - u(x) = \int_x^{ty} \nabla u$, where ty denotes a point on the unit interval connecting x to y a distance t away from the endpoint x. Therefore, as $t \leq 1$,

$$|u(ty) - u(x)| \leq \int_x^{ty} |\nabla u|.$$

Hence,

$$\fint_B |u - u(x)|\, d\mu = \frac{1}{\mu(B)} \sum_{y\sim x} \int_x^{ry} |\nabla u|\, dt \leq \frac{r}{\mu(B)} \int_B |\nabla u|\, d\mu.$$

Now by the Hölder inequality, we have a $(1,p)$-Poincaré inequality.

Case 2: The radius $r > 1$. Then by the proof of Lemma 3.1, $\mu(B)$ is comparable to $\#B$. By inequality (6),

$$\frac{1}{\#B} \sum_{x\in B} |u(x) - u_B| \leq cr\Big(\frac{1}{\#B} \sum_{x\in B, y\sim x} |u(x) - u(y)|^p\Big)^{1/p}.$$

We know from one-dimensional calculus that linear functions have the smallest p-energy integrals amongst all Sobolev functions with the same boundary values. Hence $\int_I |u(x) - u(y)|^p \leq \int_I |\nabla u|^p$ whenever I is an edge connecting x to y. Thus,

$$\frac{1}{\#B} \sum_{x\in B} |u(x) - u_B| \leq cr\Big(\frac{1}{\mu(B)} \int_B |\nabla u|^p\, d\mu\Big)^{1/p}.$$

Without loss of generality, we may assume that

$$u_B = \frac{1}{\#B} \sum_{x\in B} u(x) = 0.$$

Then

$$\frac{1}{\#B} \sum_{x\in B} |u(x)| \leq cr\Big(\fint_B |\nabla u|^p\, d\mu\Big)^{1/p}.$$

Let $x \sim y$ and $x, y \in B$. Then, along the line segment Γ_{xy} connecting x to y,

$$\int_{\Gamma_{xy}} |u(t)| dt \leq \frac{|u(x)| + |u(y)|}{2}.$$

This, together with the fact that the measure of B is comparable to $\#B$ yields the desired Poincaré inequality

$$\fint_B |u|\, d\mu \le cr \Big(\fint_B |\nabla u|^p \, d\mu \Big)^{1/p}.$$

$\square$

Let u_0 be a p-harmonic function on $U \subset V$ in the sense of inequality (8). We can associate a function, called u, on G_0, so that for every vertex x we have $u(x) = u_0(x)$, and on the unit intervals connecting x to y, where $x \sim y$, we have $u(t) = (u_0(y) - u_0(x))t + u_0(x)$. This is the piecewise linear extension of u_0 to all of G_0.

Lemma 3.3. *If U is connected in the sense that every vertex in U is connectable to every other vertex in U by a sequence of vertex neighbours in U, then the linear extension u of u_0 is p-harmonic on the domain U_0 associated with U: that is, for every function $w \in C_0^\infty(U_0)$,*

$$\int_{U_0} |\nabla u|^{p-2} \nabla u \cdot \nabla w = 0. \tag{10}$$

Proof. Suppose u_0 satisfies inequality (8), and let u be the corresponding piecewise linear extension to G_0. Let $w \in C_0^\infty(U_0)$. Then we can define w_1 on G_0 so that for every vertex x we have $w_1(x) = w(x)$ and w_1 is linear on each edge. Then $w_2 = w - w_1$ vanishes on the vertex set V, and on each edge w_2 is smooth and vanishes on the endpoints of the edge. Therefore, by inequality (8),

$$\begin{aligned} 0 &= \sum_{x \in U, y \sim x} |u(y) - u(x)|^{p-2}(u(y) - u(x))(w(y) - w(x)) \\ &= \int_{U_0} |\nabla u|^{p-2} \nabla u \cdot \nabla w_1. \end{aligned}$$

Since the linear extension u is p-harmonic on each edge I of G_0, we have

$$\int_I |\nabla u|^{p-2} \nabla u \cdot \nabla w_2 = 0.$$

This is because w_2 is smooth on I and vanishes at the endpoints of I. Therefore,

$$\sum_I \int_I |\nabla u|^{p-2} \nabla u \cdot \nabla w_2 = 0,$$

where the sum is over all edges I with one endpoint in U. Thus

$$\int_{U_0} |\nabla u|^{p-2} \nabla u \cdot \nabla w = 0,$$

and u is p-harmonic on U_0. $\square$

4 Proof of the Convergence Results

In this section we look at what happens to a sequence of p-harmonic functions if the corresponding boundary functions converge to a given function f. We also obtain a weak compactness result for the space of all p-harmonic functions on a given domain in X. The aim of this section is to prove Theorems 1.1 and 1.2.

In Theorem 1.1 the case of the limit function $f = 0$ is easy to tackle, and in fact we obtain a stronger convergence result; see Proposition 4.1 below.

As stated in the previous section, we assume that X is proper. Hence X is separable. Moreover, we assume that the measure on X is doubling and supports a weak $(1, p)$-Poincaré inequality.

Proposition 4.1. *Let $\{f_i\}$ be a sequence in $N^{1,p}(X)$ such that $f_i \to 0$ in the $N^{1,p}(X)$-norm. Suppose u_i is the Dirichlet solution for the boundary value problem of f_i on the domain Ω, where the measure of the complement of Ω is positive. Then $u_i \to 0$ in $N^{1,p}_{loc}(X)$.*

Proof. Since $f_i \to 0$ in $N^{1,p}(X)$, we have $g_{f_i} \to 0$ in the L^p-norm. Now as u_i is p-harmonic and agrees with f_i on $X \backslash \Omega$, we have $\|g_{u_i}\|_{L^p(X)} \leq \|g_{f_i}\|_{L^p(X)}$, and hence $g_{u_i} \to 0$ in $L^p(X)$.

By the Minkowski inequality,

$$\|u_i\|_{L^p(X)} \leq \|u_i - f_i\|_{L^p(X)} + \|f_i\|_{L^p(X)}.$$

Note that $u_i - f_i$ vanishes outside of Ω. By assumption, $\mu(X \backslash \Omega) > 0$. Hence by the Sobolev-Poincaré inequality (see [KiSh, Lemma 2.10] or Remark 2.10), for every sufficiently large ball B (for which the measure of $B \backslash \Omega$ is positive) we have the following inequality:

$$\|u_i - f_i\|_{L^p(B)} \leq C(B)\|g_{u_i - f_i}\|_{L^p(B)} \leq C(B)\left(\|g_{u_i}\|_{L^p} + \|g_{f_i}\|_{L^p}\right).$$

Hence

$$\|u_i\|_{L^p(B)} \leq C\|g_{u_i}\|_{L^p(X)} + C\|f_i\|_{N^{1,p}(X)},$$

and the right-hand side of the above inequality tends to zero. Hence

$$\|u_i\|_{N^{1,p}_{loc}(X)} = \|u_i\|_{L^p_{loc}} + \|g_{u_i}\|_{L^p_{loc}} \to 0.$$

Note that 0 is the Dirichlet solution to the boundary value problem with boundary data 0. $\square$

Now we would like to tackle the more general case; Theorem 1.1.

Recall that for functions $f \in N^{1,p}(X)$, $I(f)$ denotes the p-energy integral $\int_X g_f^p$; see Definition 2.7. Let $I_{min}(f) := \inf_v I(v + f)$, where the infimum is taken over all functions $v \in N^{1,p}_0(\Omega)$.

Lemma 4.2. *If $f_i \to f$ in $N^{1,p}(X)$, then*

$$\lim_i I_{min}(f_i) = I_{min}(f).$$

Proof. Note that as $f_i \to f$ in $N^{1,p}(X)$, we have $v + f_i \to v + f$ in $N^{1,p}(X)$. Hence, $I(v + f_i) \to I(v + f)$. Indeed,

$$|I(v + f_i)^{1/p} - I(v + f)^{1/p}| \le \|g_{f_i - f}\|_{L^p},$$

and hence for every positive number ϵ, for sufficiently large i (the largeness depending only on ϵ), we have the following inequality for every function $v \in N_0^{1,p}(\Omega)$:

$$|I(v + f_i)^{1/p} - I(v + f)^{1/p}| \le \epsilon.$$

Let $v \in N_0^{1,p}(\Omega)$. Then

$$I_{min}(f)^{1/p} \le I(v + f)^{1/p} \le \epsilon + I(v + f_i)^{1/p}.$$

Taking infimum over $v \in N_0^{1,p}(\Omega)$, we see that $I_{min}(f)^{1/p} \le \epsilon + I_{min}(f_i)^{1/p}$ for sufficiently large i. Now letting $i \to \infty$ and then $\epsilon \to 0$, we see that

$$I_{min}(f) \le \liminf_{i\to\infty} I_{min}(f_i). \tag{11}$$

On the other hand,

$$I(v + f)^{1/p} \ge I(v + f_i)^{1/p} - \epsilon \ge I_{min}(f_i)^{1/p} - \epsilon.$$

Taking infimum over $v \in N_0^{1,p}(\Omega)$, then $\limsup$ over i, and subsequently letting ϵ go to 0, we see that

$$I_{min}(f) \ge \limsup_{i\to\infty} I_{min}(f_i). \tag{12}$$

By inequalities (11) and (12), we see that $I_{min}(f) = \lim_{i\to\infty} I_{min}(f_i)$. $\square$

From the proof of [Sh1, Theorem 3.7] we obtain the following lemma.

Lemma 4.3. *If f_i is a sequence in $N^{1,p}(X)$ that converges in the norm of $N^{1,p}(X)$ to a function f, then there is a subsequence f_{i_k} so that for p-capacity almost every point $x \in X$ we have $\lim_k f_{i_k}(x) \to f(x)$.*

The following useful lemma is from [KSh].

Lemma 4.4 (KSh, Lemma 3.1). *Let $\{u_i\}_i$ be a sequence in $N^{1,p}_{loc}(X)$ so that the following two conditions hold: There is a positive constant C such that $\int_X g_{u_i}^p \le C$; and for every ball $B \subset X$ there is a constant C_B independent of i such that $\int_B u_i^p < C_B$. Then there is a convex combination sequence $\tilde{u}_n = \sum_{i=n}^{N(n)} \lambda_{n,i} u_i$ where $\sum_{i=n}^{N(n)} \lambda_{n,i} = 1$, and a function $u \in N^{1,p}_{loc}(X)$ so that for every ball $B \subset X$ we have $u_i \to u$ in $L^p(B)$ and $g_{u_i} \to g$ in $L^p(X)$ for some weak upper gradient g of u.*

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Note that the operator $I^{1/p}$ is sublinear:

$$I(h_1+h_2)^{1/p} \le I(h_1)^{1/p} + I(h_2)^{1/p},$$

and $I(\lambda h)^{1/p} = \lambda I(h)^{1/p}$. Since $f_i \to f$ in $N^{1,p}(X)$, we see that $\|g_{f_i-f}\|_{L^p} \to 0$ and that $\|f_i - f\|_{L^p} \to 0$. Therefore, by the minimizing property of u_i and by the fact that $u_i \in N^{1,p}(X)$ and hence $f_i - u_i \in N_0^{1,p}(\Omega)$,

$$\|g_{u_i}\|_{L^p} \le \|g_{f_i}\|_{L^p} \le \|g_{f_i-f}\|_{L^p} + \|g_f\|_{L^p} < C \tag{13}$$

for some postive finite constant C. Since $\mu(X\backslash\Omega) > 0$, by the Sobolev-Poincaré inequality (or inequality (4)) applied to sufficiently large balls B containing Ω (see [KiSh, Lemma 2.10] or Remark 2.10) we have the following inequality:

$$\begin{aligned}\|u_i\|_{L^p(B)} &\le \|u_i - f_i\|_{L^p(B)} + \|f_i - f\|_{L^p(X)} + \|f\|_{L^p(X)}\\ &\le C(B)\|g_{u_i-f_i}\|_{L^p(B)} + C \le C_B.\end{aligned} \tag{14}$$

Hence u_i is a bounded sequence in $N^{1,p}(B)$, and as $u_i = f_i$ on $X\backslash B$ we observe that u_i is a bounded sequence in $N^{1,p}(X)$. Hence by [KSh, Lemma 3.1] (see Lemma 4.4), there is a linear convex combination sequence $\tilde{u}_n = \sum_{i=n}^{N(n)} \lambda_{n,i} u_i$ of u_i and a function u in $N^{1,p}(X)$ so that $\tilde{u}_n$ converges to u in $L^p(X)$ and $\tilde{g}_n = \sum_{i=n}^{N(n)} \lambda_{n,i} g_{u_i}$ converges to a p-weak upper gradient g of u in $L^p(X)$. Note that we can choose the convex combination sequence $\tilde{u}_n$ so that $\tilde{g}_n \to g$ in $L^p(X)$ because of the inequality (13) and the fact that $L^p(X)$ is reflexive; $1 < p < \infty$. Therefore $u \in N^{1,p}(X)$.

Using the notation from Lemma 4.2, as $u_i \in N^{1,p}(X)$, we see that $I_{min}(f_i) = I(u_i)$. By the sublinearity mentioned above,

$$\left(\int_X \tilde{g}_n^p\right)^{1/p} \le \sum_{i=n}^{N(n)} \lambda_{n,i} I(u_i)^{1/p} = \sum_{i=n}^{N(n)} \lambda_{n,i} I_{min}(f_i)^{1/p}.$$

By Lemma 4.2, if n is sufficiently large, then $I_{min}(f_i) \le I_{min}(f) + \epsilon$. Hence,

$$\left(\int_X \tilde{g}_n^p\right)^{1/p} \le \sum_{i=n}^{N(n)} \lambda_{n,i}(I_{min}(f)+\epsilon)^{1/p} = (I_{min}(f)+\epsilon)^{1/p} \tag{15}$$

because $\sum_{i=n}^{N(n)} \lambda_{n,i} = 1$. Since $\tilde{g}_n$ converges to $g \ge g_u$ in the $L^p(X)$-norm, therefore $\liminf_n I(\tilde{u}_n) \ge I(u)$. Thus we see that

$$I(u) \le I_{min}(f) + \epsilon.$$

Observe that $u_i|_{X\backslash\Omega} = f_i|_{X\backslash\Omega}$, and we can choose a subsequence of f_i if necessary so that $f_i \to f$ in $N^{1,p}(X)$ and p-q.e. in X by Lemma 4.3. Hence

$\tilde{u}_n|_{X\setminus\Omega} = \sum_{i=n}^{N(n)} \lambda_{n,i} u_i|_{X\setminus\Omega}$ converges p-q.e. to $f|_{X\setminus\Omega}$. Hence $u \in N_0^{1,p}(\Omega) + f$, and therefore we have $I(u) \geq I_{min}(f)$. Hence $I(u) = I_{min}(f)$, and by the uniqueness of solution to the Dirichlet problem, u is the Dirichlet solution with boundary data f.

Next note that by the uniqueness again, every subsequence of the sequence u_i that converges weakly in $N^{1,p}(X)$ converges weakly to the same function u. Hence the entire sequence $\{u_i\}$ converges weakly to u, both in $N^{1,p}(X)$ and in L^p. By Lemma 4.2 and by the fact that as $u_i \in N^{1,p}(X)$ we have $I_{min}(f_i) = I(u_i)$, see that $\|g_{u_i}\|_{L^p} \to \|g_u\|_{L^p}$, and hence as Ω is bounded, u_i converges to u in the relaxed topology of $N^{1,p}(X)$; that is, $\|u_i\|_{N^{1,p}(X)} \to \|u\|_{N^{1,p}(X)}$ (this terminology was used by Cheeger in [C]).

By the above argument, u_i are uniformly bounded in $L^p(X)$. If we have a $(1,q)$-Poincaré inequality for some $q < p$, by [KiSh, Theorem 5.4] and [KiSh, Theorem 4.8] we have positive constants $C(B)$ and α, independent of i, so that

$$|u_i(x) - u_i(y)| \leq C(B)\|u_i\|_{L^p(B)} d(x,y)^\alpha \leq C(B) d(x,y)^\alpha$$

whenever $x, y \in \frac{1}{2}B$ and B is a ball contained in the domain Ω. Hence u_i are locally equibounded and equicontinuous on the domain Ω, and by the Arzela-Ascoli theorem, it follows that u_i converges to u locally uniformly in Ω (recall that X is separable). $\square$

Note that if the norm on $N^{1,p}(X)$ is uniformly convex, then by Lemma 4.2 and by [BaPr, Chapter 1, Theorem 2.9] we have that $u_i \to u$ in $N^{1,p}(X)$. Examples of metric measure spaces for which the corresponding Newtonian spaces have that property include the Euclidean spaces and the Carnot-Carathéodory spaces. The modified norm used by Cheeger in [C] to show reflexivity of Newtonian spaces is uniformly convex, but this modified norm is only known to be equivalent to the Newtonian norm, and hence yields the apropriate result only for a particular class of quasi-minimizers. If we denote the Cheeger derivative of a Lipschitz function as df, it would be interesting to know whether $|df| = g_f$ if X is a geodesic space. See [C] or [T] for the definition of df.

We would also like to know an analog of [LM1, Theorem 1.4]: if u_i is a bounded sequence of p-harmonic functions in $N^{1,p}(X)$ such that $u_i \to u$ p-q.e., then u is p-harmonic and $\min\{u_i, u\} \to u$ in the relaxed topology of $N^{1,p}(X)$. Unfortunately we have been unable to show the convergence. However, it can be shown that u is a p-harmonic function. This is the content of Theorem 1.2. We now prove this theorem.

The following definitions follow the Euclidean analogs of [HKM].

Definition 4.5. A function $u \in N^{1,p}(X)$ is a *p-supersolution* (or supersolution) to the energy minimizing problem in Ω if for every non-negative function $v \in N_0^{1,p}(\Omega)$ we have that $I(u) \leq I(u+v)$.

Definition 4.6. A function $u : X \to (-\infty, \infty]$ is *p-superharmonic* (or superharmonic) in Ω if it is not identically ∞ on Ω and whenever $w \in N^{1,p}(X)$ is a p-harmonic function on $\Omega_1 \subset\subset \Omega$ so that $w \le u$ on $\overline{\Omega_1} \setminus \Omega_2$ for some subdomain Ω_2 of Ω_1 it follows that $w \le u$ on Ω_2 as well.

The standard definition of superharmonic functions require that the function also be lower semicontinuous. However, the semicontinuity propery is not needed in the applications here, and hence is omitted. However, it can be shown that supersolutions are indeed lower semicontinuous by combining the results of [KiSh] with the proof of [HKM, Theorem 3.63]. Hence supersolutions would indeed be superharmonic in the standard definition of superharmonicity; see also the discussion at the end of this section.

Proof of Theorem 1.2. Note first that as u_i converges to u locally uniformly, u is continuous on the domain Ω.

Suppose such u is not superharmonic. Then there is a p-harmonic function w on a subdomain Ω_1 and a subdomain $\Omega_2 \subset \Omega_1$ so that $u \ge w$ on $\overline{\Omega_1} \setminus \Omega_2$ but it is not true that $u \ge w$ on Ω_2. Let

$$V_\epsilon := \{x \in \Omega_2 \ : \ \epsilon + u(x) < w(x)\}.$$

By assumption there is a positive number ϵ so that V_ϵ is non-empty. As w, u are continuous on Ω_2, the set V_ϵ is open. Since u_i converges locally uniformly to u, there is a ball $B \subset V_\epsilon$ so that $u_i(x) + \epsilon/2 \le w(x)$ for sufficiently large i. Therefore by the maximum principle, for sufficiently large i there are corresponding points x_i in $\Omega_1 \setminus \Omega_2$ so that $u_i(x_i) + \epsilon/2 \le w(x_i)$. Since the closure of Ω_1 is compactly contained in Ω, by the local uniform convergence of u_i to u again, we have $u(x_i) < w(x_i)$ for sufficiently large i, which is a contradiction of the choice of w. Hence u must be superharmonic. The same argument applied to the sequence $-u_i$ and the function $-u$ yields that $-u$ also is superharmonic. Hence u must be p-harmonic, by the following lemma. $\square$

Observe that in the following lemma, and hence in Theorem 1.2, it suffices to know that $u \in N^{1,p}(V)$ for every relatively compact open set $V \subset \Omega$. This guarantees the existence of the solution to the Dirichlet problems considered in the proof of Lemma 4.7.

Lemma 4.7. *Let X be a metric measure space equipped with a doubling measure supporting a weak $(1, p)$-Poincaré inequality, and let u be a superharmonic function on a domain Ω so that $-u$ is also a superharmonic function on Ω. Assume also that $u \in N^{1,p}(V)$ for every relatively compact open subset V of Ω. Then u is p-harmonic.*

Proof. Let B be a ball whose closure lies in Ω. Let w be the solution to the Dirichlet problem on B with boundary data u. Then on $\Omega\setminus B$ we have $u \geq w$. Hence by the definition of superharmonicity, $u \geq w$ on B.

Note that $-w$ is the solution to the Dirichlet problem on B with boundary data $-u$. Hence we also have $-u \geq -w$ on B. Hence $u = w$ on B, indicating that u is p-harmonic on B.

Instead of a ball, we can take any compactly contained subdomain of Ω in the above argument. Hence u is p-harmonic on compact subdomains of Ω. Now by definition (or by [Sh2, Theorem 4.8]), u is p-harmonic on Ω. $\square$

As a corollary to Theorem 1.2 we obtain the following weak analog of a result of Li and Martio; see [LM1].

Corollary 4.8. *Let Ω be a bounded domain in a proper metric space X that is equipped with a doubling measure supporting a weak $(1,p)$-Poincaré inequality, and let u_i be a sequence of functions that are p-harmonic in Ω and are bounded in $N^{1,p}_{loc}(X)$. Then there is a subsequence u_{i_k} converging to an $N^{1,p}_{loc}(X)$-function u locally uniformly in Ω, and furthermore, u is p-harmonic in Ω.*

Proof. Since u_i are bounded in $N^{1,p}_{loc}(X)$ and are p-harmonic in Ω, by the local Hölder continuity of p-harmonic functions (see [KiSh, Theorem 5.4]), on compact subsets of X the sequence is equicontinuous and equibounded. Since X is proper and locally doubling and hence is separable, we can apply the Arzela-Ascoli theorem to the sequence u_i to extract a subsequence, also denoted u_i, that converges locally uniformly in Ω to a function u. Therefore, by [KSh, Lemma 3.1] (see Lemma 4.4), the function $u \in N^{1,p}_{loc}(X)$. Hence by Theorem 1.2, we have the required result. $\square$

Note that as both u and u_i are superharmonic (p-harmonic functions are superharmonic by the maximum principle), so is $\min\{u, u_i\}$ superharmonic, and by the Lebesgue dominated convergence theorem, $\min\{u, u_i\}$ converges to u in the L^p-norm. In going from the proof of Theorem 1.1 to Theorem 1.2, we seem to have lost information about convergence of the corresponding weak upper gradients. We therefore were unable to obtain even the relaxed topology convergence of the subsequence of u_i.

By arguments similar to above, it is easy to see that a supersolution must be a superharmonic function. Indeed, if u is a supersolution on Ω and w is a p-harmonic function on $\Omega_1 \subset\subset \Omega$ so that $u \geq w$ on $\overline{\Omega}\setminus\Omega_1$, then letting $v := \max\{u, w\}$, we see that $v = u$ on $\overline{\Omega}\setminus\Omega_1$. Now by the fact that u is a supersolution we see that

$$\|g_v\|_{L^p(\Omega_1)} \geq \|g_u\|_{L^p(\Omega_1)}.$$

But

$$g_v = g_w \chi_{\{u<w\}} + g_u \chi_{\{u\geq w\}},$$

and hence we see that

$$\|g_w\|_{L^p(\{u<w\})} \geq \|g_u\|_{L^p(\{u<w\})}.$$

Here the function χ_A denotes the characteristic function of the set A. However, $v = w$ p-q.e. on the boundary of the set $\{u < w\}$. Hence by the energy minimizing property of the harmonic function w, we see that

$$\|g_w\|_{L^p(\{u<w\})} = \|g_u\|_{L^p(\{u<w\})}.$$

By the uniqueness of the Dirichlet solutions, it now follows that $u = w$ p-q.e. on this set. Hence on Ω_1 we have $u \geq w$ p-q.e., and so u is superharmonic.

It is not however clear to the author whether superharmonic functions are supersolutions if they are locally bounded, which is the analogue of [HKM, Corollary 7.20]. Note also that a function that is superharmonic or a supersolution on Ω is superharmonic or is a supersolution on every subdomain of Ω. It again is not clear that a locally superharmonic function or a local supersolution is a global superharmonic function or a global supersolution. It seems that we lost the local to global results in considering general metric measure spaces without partial differential equations. Another example of such loss is in the definition of p-harmonic functions. If the p-harmonic function is obtained via a differential equation in divergence form, then it is easy to see that a locally p-harmonic function in a domain is globally p-harmonic. That is, if every point in the domain has a neighbourhood in which the function is p-harmonic, then the function is p-harmonic in the entire domain. it is not clear that this property holds in this generality.

As mentioned in the introduction, if we do not know that the domain Ω is bounded, then Theorem 1.1 is substantially weakened. However, because of Theorem 1.2, we have the following weaker version of Theorem 1.1 for unbounded domains.

Proposition 4.9. *Let X be a proper metric measure space equipped with a locally doubling measure supporting a weak local $(1, p)$-Poincaré inequality, and let Ω be a domain in X with a boundary point $x_0 \in \partial\Omega$ so that the measure of $B(x_0, r)\backslash\Omega$ is positive for every $r > 0$. Consider a Cauchy sequence $\{f_i\}$ in $N^{1,p}(X)$ that converges in $N^{1,p}(X)$ to a function f, and suppose u_i is a solution to the Dirichlet problem on Ω with boundary data f_i. Moreover, assume that $u_i \in N^{1,p}(X)$. Then there is a subsequence of the sequence $\{u_i\}$ converging locally uniformly to a solution u of the Dirichlet problem on Ω with boundary data f.*

Proof. By the hypothesis about the boundary of Ω, we can apply the Sobolev-Poincaré inequality (or inequality (2.10) to the functions $u_i - f_i$. Also by hypothesis $u_i \in N^{1,p}(X)$. Hence we have the uniform local L^p-bound (14) for the functions u_i. Now by the results [KiSh, Theorem 5.4] and [KiSh, Theorem 4.8] we see that u_i is a locally equibounded and equicontinuous sequence. Now the Arzela-Ascoli theorem yields a subsequence that converges locally uniformly to a function u. Moreover, as f_i converges in $N^{1,p}(X)$ to f, we see that $u|_{X\setminus\Omega} = f|_{X\setminus\Omega}$. By Theorem 1.2 we also see that u is p-harmonic on Ω. Moreover, as each u_i satisfies the inequalities (13) and (14), by [KSh, Lemma 3.1] (see Lemma 4.4) and inequality (15) we see that $g_u \in L^p(X)$. Hence $u \in N^{1,p}_{loc}(X)$. Thus we can conclude that u is a solution to the Dirichlet problem on Ω with boundary data f. □

5 Two Applications

In this section we look at two applications of the results of Section 4.

The first application states that a family of p-harmonic functions in a path-connected domain Ω in X is a normal family if it is uniformly bounded at a point in the domain. Recall that a family of functions is said to be normal if for every sequence of functions from that family there is a subsequence that converges to an element in that family. Harnack inequality, together with the facts that the family of p-harmonic functions is bounded at a point in the domain and that every other point in the domain can be connected to this point by a compact path lying in the domain, indicates that the family is locally uniformly bounded on that domain. Another application of the Harnack inequality (see [KiSh]) shows that the family is locally equicontinuous. Thus for any sequence of functions in the family the Arzela-Ascoli theorem yields a function that is a local uniform limit of a subsequence, and this limit function is in $N^{1,p}_{loc}(\Omega)$ because of the Cacciopoli type inequality [KiSh, (3,8), Proposition 3.9] and by [KSh, Lemma 3.1]. Hence this limit function is p-harmonic on the domain by Theorem 1.2. Thus we have the following proposition.

Proposition 5.1. *Let Ω be a path-connected subdomain of a metric space X equipped with a doubling measure supporting a weak $(1, q)$-Poincaré inequality for some q for which $1 \le q < p$, and $\mathcal{F}$ be a collection of non-negative functions that are p-harmonic on Ω. If there is a constant $C > 0$ and a point $y \in \Omega$ so that for every function $u \in \mathcal{F}$ we have $0 \le u(y) \le C$, then for every sequence in the family $\mathcal{F}$ there is a subsequence that converges locally uniformly to a function that is p-harmonic on Ω.*

If the family $\mathcal{F}$ consists of p-harmonic functions that are locally uniformly bounded in Ω, then the above proposition holds true for this family even if Ω is not path-connected. The path-connectedness assumption was needed only to ensure that from the hypothesis that the functions in the family are uniformly bounded at one point we can conclude that the functions have to be locally uniformly bounded on Ω.

Another application of the results in Section 4 is in solving the p-Dirichlet problem on unbounded domains. We know from the results of [Sh2] that if Ω is a bounded domain in X so that the interior of $X \setminus \Omega$ is non-empty, then for every boundary data $f \in N^{1,p}(X)$ there is a corresponding solution to the p-Dirichlet problem in Ω.

Suppose Ω is an unbounded domain in X so that $C = X \setminus \Omega$ is compact and has positive measure. Let $\{G_i\}_i$ be an exhaustion of X by bounded domains so that $X \setminus G_i$ has non-empty interior and $C \subset G_1$, $\overline{G}_i \subset G_{i+1}$. Denote by f a function in $N^{1,p}(X)$, and let u_i be the solution to the p-Dirichlet problem on $\Omega \cap G_i = G_i \setminus C$ with boundary data f. Since the complement of Ω is compact, without loss of generality we may assume that f has support in G_1. By inequality (14) and by [KiSh, Remark 4.11(1)], the sequence $\{u_j\}_j$ is locally uniformly bounded in Ω, and hence by the above proposition we can find a locally uniformly convergent subsequence converging to a function u which is p-harmonic on $\Omega \cap G_i$, and hence is p-harmonic on Ω. Also $u|_C = f$. Note that $\int_X g_{u_j}^p \le \int_X g_f^p < \infty$. Hence as $f \in N^{1,p}(X)$, we see by the fact that $L^p(X)$ is reflexive and by inequality (15) (together with [KSh, Lemma 3.1] or Lemma 4.4) that $g_u \in L^p(X)$. Hence by inequality (14) $u \in N^{1,p}_{loc}(X)$. By the definition of solutions to the Dirichlet problem, such a function u is a solution to the Dirichlet problem with boundary values f on Ω. Now we have the following result.

Lemma 5.2. *If X is a path-connected proper metric measure space equipped with a doubling measure supporting a weak $(1,p)$-Poincaré inequality, is unbounded, and if Ω is a domain in X with $X \setminus \Omega$ compact in X so that the measure of $X \setminus \Omega$ is positive, then for every function $f \in N^{1,p}(X)$ there is a solution to the p-Dirichlet problem on Ω with boundary data f.*

Using the above lemma, it is now possible to prove an existence result for the p-Dirichlet problem on some general domains.

Proposition 5.3. *If Ω is a domain in an unbounded, proper, path-connected metric measure space X equipped with a doubling measure supporting a weak $(1,p)$-Poincaré inequality, and the measure of $X \setminus \Omega$ is positive, then for every function $f \in N^{1,p}(X)$ there is a solution to the p-Dirichlet problem on Ω with boundary data f.*

Another version of this proposition has now been proven in [BBS], where the results of Section 4 above have been used to construct solutions to a Dirichlet problem on bounded domains with more general continuous functions defined solely on the boundary of the domain as boundary data, thus eliminating the artificial constraint of membership in the class $N^{1,p}(X)$ for the boundary data functions as considered in [Sh2].

Proof. If $X\backslash\Omega$ is bounded, then by the above lemma we have the result. So without loss of generality, assume that $C = X\backslash\Omega$ is unbounded. Let $x_0 \in C$, and let $C_n = C \cap \overline{B}(x_0, n)$. Then C_n is compact, and for sufficiently large n C_n has positive measure. By above lemma, we have a function u_n that is p-harmonic on $X\backslash C_n$ so that $u_n|_{C_n} = f|_{C_n}$. Since f is in $N^{1,p}(X)$, by inequality (14) and [KiSh, Remark 4.11] the function u_n is locally bounded independently of n. Also, the minimal upper gradients of the functions u_n are bounded in $L^p(X)$. Applying Theorem 1.2 to the sequence $\{u_n\}$ on the domain Ω, we obtain a function u that is p-harmonic on Ω and has f as its boundary data. ☐

For example, the above proposition implies that if E and F are two non-empty, disjoint, compact subsets of Ω, then there is a p-harmonic function u in $X\backslash(E \cup F)$ so that $u|_E = 1$ and $u|_F = 0$; that is, $\mathrm{cap}_p(E, F) = \mathrm{Mod}_p(E, F) = \int_X g_u^p$. See [HeK1], [He], [KM2], or [KSh] for the definition of $\mathrm{Mod}_p(E, F)$.

Remark 5.4. By a capacitary analog of the Sobolev-Poincaré inequality for Newtonian functions (see [Bj] or [BMS]), the requirement of positivity of the measure of $X\backslash\Omega$ in the results of this paper can be replaced with the weaker requirement of positivity of the p-capacity of $X\backslash\Omega$. In this case the relevant constants depend also on the p-capacity of $X\backslash\Omega$.

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