BOUNDARY HARNACK PRINCIPLE FOR p-HARMONIC FUNCTIONS IN SMOOTH EUCLIDEAN DOMAINS

HIROAKI AIKAWA, TERO KILPELÄINEN, NAGESWARI SHANMUGALINGAM, AND XIAO ZHONG

Abstract. We establish a scale-invariant version of the boundary Harnack principle for p-harmonic functions in Euclidean $C^{1,1}$-domains and obtain estimates for the decay rates of positive p-harmonic functions vanishing on a segment of the boundary in terms of the distance to the boundary. We use these estimates to study the behavior of conformal Martin kernel functions and positive p-superharmonic functions near the boundary of the domain.

Contents

1. Boundary Harnack principle for $C^{1,1}$-domains in $\mathbb{R}^n$

By a *boundary Harnack principle* we mean the property that two positive harmonic functions in a domain vanishing on a portion of the boundary decay at the same speed toward a smaller portion of the boundary. More specifically, let Ω be a Euclidean domain. Suppose a compact set K and an open set V satisfy $K \subset V$, $K \cap \Omega \neq \emptyset$ and $V \cap \partial\Omega \neq \emptyset$. We say that the boundary Harnack principle holds if for all u and v are two positive harmonic functions on Ω continuously vanishing at every regular boundary point of $V \cap \partial\Omega$ and bounded near every irregular boundary point of $V \cap \partial\Omega$ we have

$$\frac{u(x)/u(y)}{v(x)/v(y)} \le A \quad \text{for all } x, y \in K \cap \Omega$$

with some constant $A \ge 1$ depending only on Ω, K and V.

J. T. Kemper [18] first proposed the boundary Harnack principle for a Lipschitz domain. After Kemper's pioneering work, the boundary Harnack principle was proved for a Lipschitz domain by Ancona [4], Dahlberg [13] and Wu [26] independently. Since then, the boundary

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Harnack principle has garnered a lot of attention, and many papers have been devoted to the study of the boundary Harnack principle not only for Euclidean domains but also for manifolds. Among them, Caffarelli-Fabes-Mortola-Salsa [11], Jerison-Kenig [17] and Bass-Burdzy-Bañuelos [9, 8] gave significant extensions for the boundary Harnack principle for Euclidean domains. The boundary Harnack principle is now well-known for Lipschitz domains, non-tangentially accessible (NTA) domains, and more generally for Hölder domains. In [1], the first named author showed that a uniform domain satisfies a local version of the boundary Harnack principle which is stronger than the above boundary Harnack principle. It has applications to the study of the Martin boundary. In particular, this version of the boundary Harnack principle shows that the Martin boundary and the Euclidean boundary of a uniform domain are homeomorphic, a fact well-known for a Lipschitz domain and an NTA domain.

Replacing positive harmonic functions by positive solutions to non-linear partial differential equations, we can easily formulate a boundary Harnack principle for non-linear partial differential equations. Nevertheless, such a boundary Harnack principle was almost unknown, whereas the interior Harnack principle for non-linear partial differential equations is well established. The classical methods such as the box argument due to Bass-Burdzy [9] heavily rely on the linearity of harmonic measure and do not seem to extend to non-linear partial differential equations.

The main aim of this paper is to establish a local version of the boundary Harnack principle for p-harmonic functions (with $1 < p < \infty$) on $C^{1,1}$-domains in $\mathbb{R}^n$. Let $B(x, r)$ and $S(x, r)$ be the open ball and the sphere with center x and radius r, respectively.

Definition 1.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say that Ω is a $C^{1,1}$-domain if there exist positive constants r_0 and A such that to each point $\xi \in \partial\Omega$ there corresponds a coordinate system (x', x_n) with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$, and a C^1 function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $|\nabla\varphi(x') - \nabla\varphi(y')| \le A|x' - y'|$ and $\Omega \cap B(\xi, r_0) = \{(x', x_n) : x_n > \varphi(x')\} \cap B(\xi, r_0)$.

Let $1 < p < \infty$. We say that u is p-harmonic in Ω if $u \in W^{1,p}_{loc}(\Omega)$ is continuous in Ω and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ in Ω in the weak sense; that is, whenever Ω' is a relatively compact subdomain of Ω and $\varphi \in W^{1,p}_0(\Omega')$, we have

$$\int_{\Omega'} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx = 0.$$

The main theorem of this note is the following boundary Harnack principle for p-harmonic functions.

Theorem 1.2 (Boundary Harnack Principle). *Let Ω be a bounded $C^{1,1}$-domain and let $1 < p < \infty$. Then there exist constants $A_1 > 1$, $A_2 > 1$ and $r_1 > 0$ with the following property: Let $0 < r < r_1$ and $\xi \in \partial\Omega$. If u and v are positive p-harmonic functions on $\Omega \cap B(\xi, A_1 r)$ vanishing on $\partial\Omega \cap B(\xi, A_1 r)$, then*

$$\frac{u(x)/u(y)}{v(x)/v(y)} \le A_2 \quad \text{for } x, y \in \Omega \cap B(\xi, r).$$

Remark 1.3. Recently, in [10] Bennewitz and Lewis established a boundary Harnack principle for p-harmonic functions (with $1 < p < \infty$) in planar domains whose boundary is a quasicircle. While in $\mathbb{R}^2$ the results in [10] are far more general than ours, it seems that their method does not extend to higher dimensions; see Lemma 2.16 of [10]. Furthermore, their method does not provide the decay estimate for the p-harmonic functions in terms of distance $\delta_\Omega(\cdot)$ to the boundary; we will provide such a decay estimate in the last section of this note.

In the study of the (linear) Laplace operator Δ, it is known that the validity of Carleson estimates together with the boundary Harnack principle on a given domain Ω implies that the Martin boundary and the Euclidean boundary of the domain Ω are homeomorphic. This is accomplished by demonstrating that Martin kernel functions corresponding to the Laplace operator are extremal in the following sense: if u is a non-negative harmonic function on the domain vanishing on $\partial\Omega \setminus \{\xi\}$ for some $\xi \in \partial\Omega$, then there is a constant $\lambda \geq 0$ such that $u = \lambda M_\xi$ for some Martin kernel function M_ξ with singularity at ξ. In the non-linear setting of the n-Laplacian Δ_n, Aikawa and Shanmugalingam showed the Carleson estimates for a uniform domain; see [3]. However, the above boundary Harnack principle is not sufficient to give the homeomorphism between the conformal Martin boundary and the Euclidean boundary. We still do have a result analogous to the minimality property of conformal Martin kernel functions. Such a version of minimality property is studied in Section 5, see Theorem 5.3.

We shall use the following notation. Recall that $B(x,r)$ (resp. $S(x,r)$) denotes the open ball (resp. sphere) with center x and radius r. We write $[x,y]$ for the line segment connecting x and y. Throughout the paper, Ω stands for a bounded domain in $\mathbb{R}^n$ and $\delta_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$. By the symbol A we denote an absolute positive constant whose value is unimportant and may change even in the same line. If necessary, we use $A_0, A_1, \ldots,$ to specify them. We shall say that two positive quantities f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $A \geq 1$ such that $A^{-1} f_1 \leq f_2 \leq A f_1$. The constant A will be called the constant of comparison. Hence the boundary Harnack principle implies that $\dfrac{u/u(y)}{v/v(y)} \approx 1$.

This note is organized as follows. Section 2 will explore a geometric characterization of the $C^{1,1}$-property of Euclidean domains Ω. Theorem 1.2 will be proved in the following section. The final two sections will be devoted to some applications of the boundary Harnack principle to the study of the conformal Martin boundary of a $C^{1,1}$-domain. In Section 4 we explicitly compute the conformal Martin kernel functions for balls in $\mathbb{R}^n$, and then in Section 5 we study the singular behavior of conformal Martin kernel functions for general $C^{1,1}$-domains in $\mathbb{R}^n$ using the boundary Harnack principle. The final section explores further properties of p-superharmonic functions related to boundary behavior.

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2. $C^{1,1}$-DOMAINS AND BALL CONDITIONS

While the definition of $C^{1,1}$-domain makes sense in the Euclidean setting, in more general setting of metric measure spaces it is desirable to have a more geometric characterization. We demonstrate in this section that a bounded domain is a $C^{1,1}$-domain if and only if it satisfies the following ball condition. It is worth noting that the results and techniques discussed in this paper are valid for other metric measure spaces such as the Heisenberg groups and polarizable Carnot groups (see [7] for more on polarizable Carnot groups), provided we concentrate on domains that satisfy the ball condition. Such a condition makes sense in the polarizable Carnot groups, and at least in the setting of Heisenberg groups most balls satisfy this condition. Indeed, all balls in the Heisenberg groups satisfy this condition at all boundary points that are not in the singular set (see [12]); since the singular set has Hausdorff dimension 1, all the results studied in this note applies in a relative neighbourhood of the non-singular boundary points of the balls.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say that Ω satisfies the *interior ball condition* (with radius r_2) if there exists $r_2 > 0$ satisfying the following condition: For every $\xi \in \partial\Omega$ there exists a point $\xi^i \in \Omega$ such that $B(\xi^i, r_2) \subset \Omega$ and $\xi \in S(\xi^i, r_2)$. We say that Ω satisfies the *exterior ball condition* (with radius r_2) if there exists $r_2 > 0$ satisfying the following condition: For every $\xi \in \partial\Omega$ there exists a point $\xi^e \in \mathbb{R}^n \setminus \Omega$ such that $B(\xi^e, r_2) \subset \mathbb{R}^n \setminus \Omega$ and $\xi \in S(\xi^e, r_2)$. We say that Ω satisfies the *ball condition* (with radius r_2) if Ω satisfies both the interior and the exterior ball conditions (with radius r_2).

The following lemma is a probably well-known folklore, but for the convenience of the reader we provide the proof here.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then Ω is a $C^{1,1}$-domain if and only if Ω satisfies the ball condition.*

Proof. First suppose that Ω is a $C^{1,1}$-domain. Let $\xi \in \partial\Omega$ be an arbitrary boundary point. Without loss of generality we may assume that $\xi = 0$. By a suitable choice of coordinate system we have

$$\Omega \cap B(0, r_0) = \{(x', x_n) : x_n > \varphi(x')\} \cap B(0, r_0),$$

where $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\varphi(0) = 0$ and $|\nabla\varphi(x') - \nabla\varphi(X')| \le A|x' - X'|$. Without loss of generality (by a rotation of the coordinate system if necessary) we may assume that $\nabla\varphi(0) = 0$, whence we conclude that $|\nabla\varphi(x')| \le A|x'|$ for all $x' \in \mathbb{R}^{n-1}$. The Taylor expansion gives $\varphi(x') = \nabla\varphi(\theta x') \cdot x'$ for some $0 < \theta < 1$, so that

$$|\varphi(x')| \le |\nabla\varphi(\theta x')||x'| \le A\theta|x'|^2 \le A|x'|^2. \tag{1}$$

Let $\vec{e}_n = (0, \dots, 0, 1)$. Observe that $S(r\vec{e}_n, r)$ touches the hyperplane $\{x_n = 0\}$ at the origin and the lower hemisphere is represented as

$$x_n = \psi_r(x') = r - \sqrt{r^2 - |x'|^2} = r\left(1 - \sqrt{1 - \left(\frac{|x'|}{r}\right)^2}\right).$$

Since $1 - t \le \sqrt{1 - t} \le 1 - t/2$ for $0 \le t \le 1$, it follows that

$$\frac{|x'|^2}{2r} \le \psi_r(x') \le \frac{|x'|^2}{r}. \tag{2}$$

In view of (1) and (2), we obtain that if $r > 0$ is sufficiently small, then

$$-\psi_r(x') \le \varphi(x') \le \psi_r(x') \quad \text{for } |x'| < r.$$

The first (resp. second) inequality implies $B(-r\vec{e}_n, r) \subset \mathbb{R}^n \setminus \Omega$ (resp. $B(r\vec{e}_n, r) \subset \Omega$). Thus the ball condition holds.

Conversely, suppose that Ω satisfies the ball condition with radius r_2. Let $\xi \in \partial\Omega$ be an arbitrary boundary point. Without loss of generality we may assume that $\xi = 0$. Moreover, we may assume that $B(r_2\vec{e}_n, r_2) \subset \Omega$ and $B(-r_2\vec{e}_n, r_2) \subset \mathbb{R}^n \setminus \Omega$ with $\vec{e}_n = (0, \dots, 0, 1)$. Let $L^+ = \{(x', r_2) : |x'| < r_2\}$ and $L^- = \{(x', -r_2) : |x'| < r_2\}$. Then $L^+ \subset B(r_2\vec{e}_n, r_2) \subset \Omega$ and $L^- \subset B(-r_2\vec{e}_n, r_2) \subset \mathbb{R}^n \setminus \Omega$. Hence, there exists a point $(x', \varphi(x')) \in \partial\Omega$ on the line segment connecting $(x', r_2) \in L^+$ and $(x', -r_2) \in L^-$ with $|x'| < r_2$. The ball condition implies that such a point is unique. Thus the function $\varphi(x')$ is defined for $x' \in \mathbb{R}^{n-1}$ with $|x'| < r_2$. See Figure 1. Now we claim that φ is a $C^{1,1}$-function for $|x'| < r_2/2$. By (2) and

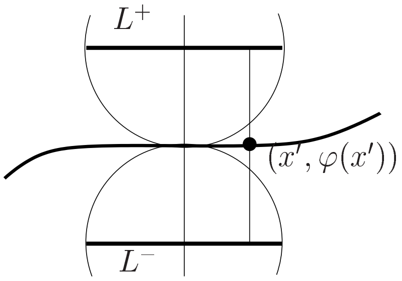

FIGURE 1.

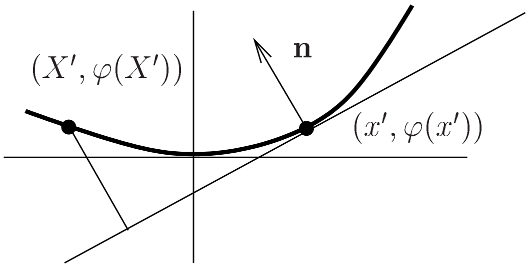

FIGURE 2.

the ball condition

$$-\frac{|x'|^2}{r_2} \le \varphi(x') \le \frac{|x'|^2}{r_2} \quad \text{for } |x'| < r_2. \tag{3}$$

This shows that $\nabla\varphi(0) = 0$. Moreover, the ball condition at $(x', \varphi(x'))$ shows that $\nabla\varphi(x')$ exists for $|x'| < r_2$ and it is bounded. The ball condition at $(x', \varphi(x'))$ also implies that

$$|\{(X', \varphi(X')) - (x', \varphi(x'))\} \cdot \mathbf{n}| \le A|(X', \varphi(X')) - (x', \varphi(x')|^2,$$

where $\mathbf{n}$ is the unit normal vector at $(x', \varphi(x'))$ given by $\mathbf{n} = (-\nabla\varphi(x'), 1)/\sqrt{1 + |\nabla\varphi(x')|^2}$. See Figure 2. Let $h' = -|x'||\nabla\varphi(x')|^{-1}\nabla\varphi(x')$ and $X' = x' + h'$ with $|x'| < r_2/2$. Then the above inequality becomes

$$\left| \frac{|x'||\nabla\varphi(x')| + \varphi(x' + h') - \varphi(x')}{\sqrt{1 + |\nabla\varphi(x')|^2}} \right| \le A(|h'|^2 + |\varphi(x' + h') - \varphi(x')|^2).$$

Since $|\nabla\varphi(x')|$ is bounded, it follows from $|h'| = |x'|$, $|x' + h'| \le 2|x'|$ and (3) that $|\nabla\varphi(x')| \le A|x'|$. Since $\nabla\varphi(0) = 0$, this shows that φ is a $C^{1,1}$-function at 0. $\square$

3. PROOF OF THEOREM 1.2

We shall prove Theorem 1.2 by comparing positive p-harmonic functions and fundamental p-harmonic functions. Let

$$\phi_p(x) = \begin{cases} \log \dfrac{1}{|x|} & \text{for } p = n, \\ \\ |x|^{(p-n)/(p-1)} & \text{for } p \ne n. \end{cases}$$

It is easy to see that $\phi_p(x)$ is a p-harmonic function. Translations and dilations of ϕ_p are p-harmonic functions as well. For $r > 0$ we define $G_r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ by

$$G_r(x,z) = \begin{cases} \log \dfrac{r}{|x-z|} & \text{if } p = n, \\[2ex] \left(\dfrac{|x-z|}{r}\right)^{(p-n)/(p-1)} - 1 & \text{if } 1 < p < n, \\[2ex] 1 - \left(\dfrac{|x-z|}{r}\right)^{(p-n)/(p-1)} & \text{if } p > n. \end{cases} \tag{4}$$

It should be noted that the restriction of $G_r(\cdot, z)$ to $B(z,r)$ is a constant multiple of a p-singular function for the ball $B(z,r)$ with singularity at the center z, see Section 4 for the definition of p-singular functions. This is the *only* p-singular function on this ball with singularity at z; see [19] and [15].

Throughout this section, let Ω be a bounded $C^{1,1}$-domain. We have observed in Lemma 2.2 that Ω satisfies the ball condition with radius r_2, i.e. for each $\xi \in \partial\Omega$ there exist $\xi^i \in \Omega$ and $\xi^e \in \mathbb{R}^n \setminus \Omega$ such that $B(\xi^i, r_2) \subset \Omega$, $B(\xi^e, r_2) \subset \mathbb{R}^n \setminus \Omega$, $\xi \in S(\xi^i, r_2) \cap S(\xi^e, r_2) \cap \partial\Omega$.

Lemma 3.1 (Lower Estimate)**.** *There exist constants $A_3 \geq 1$ and $r_3 > 0$ with the following property: Let $\xi \in \partial\Omega$ and $0 < r < r_3$. If u is a positive p-harmonic function on $\Omega \cap B(\xi, 6r)$, then*

$$A_3 \frac{u(x)}{u(\xi_r)} \geq \frac{\delta_\Omega(x)}{r} \quad \text{for } x \in \Omega \cap B(\xi, r),$$

where $\xi_r \in [\xi, \xi^i]$ is such that $\delta_\Omega(\xi_r) = |\xi_r - \xi| = r$.

A weak version of this lemma holds true for positive p-superharmonic functions; see Proposition 6.1.

Proof. Let $x \in \Omega \cap B(\xi, r)$ with $0 < r < r_2/2$. Then there is $\eta \in \partial\Omega$ such that $\delta_\Omega(x) = |x - \eta|$. By the interior ball condition at η we find a point η^i such that $B(\eta^i, r_2) \subset \Omega$ and $\eta \in S(\eta^i, r_2)$. Take $\eta^i_{2r} \in [\eta, \eta^i]$ with $\delta_\Omega(\eta^i_{2r}) = |\eta^i_{2r} - \eta| = 2r$. Observe that $B(\eta^i_{2r}, 2r) \subset \Omega \cap B(\xi, 6r)$ as $B(\eta^i_{2r}, 2r) \subset B(\eta^i, r_2) \subset \Omega$ and $|\eta^i_{2r} - \xi| \leq |\eta^i_{2r} - \eta| + |\eta - \xi| < 4r$. Since the unit normal for the $C^{1,1}$-domain Ω varies Lipschitz continuously, we infer that $\operatorname{dist}([\xi_r, \eta^i_{2r}], \partial\Omega) \geq Ar$ provided r is sufficiently small. Hence the Harnack principle yields

$$u(\xi_r) \approx u(\eta^i_{2r}) \approx u(y) \quad \text{for } y \in S(\eta^i_{2r}, r),$$

since $|\eta^i_{2r} - \xi_r| \leq |\eta^i_{2r} - \eta| + |\eta - \xi| + |\xi - \xi_r| < 5r$. See Figure 3. We compare $u/u(\xi_r)$ and

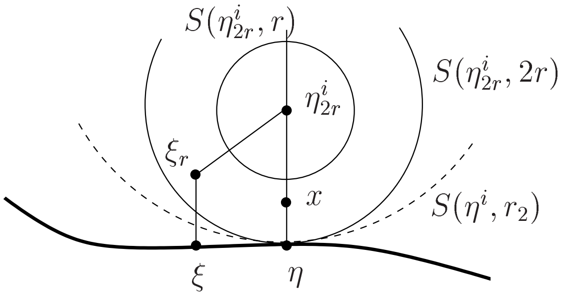

FIGURE 3.

$G_{2r}(\cdot,\eta^i_{2r})$ in the annulus $B(\eta^i_{2r},2r)\setminus B(\eta^i_{2r},r)$. Since $G_{2r}(y,\eta^i_{2r})=0$ when $y\in S(\eta^i_r,2r)$, and

$$G_{2r}(y,\eta^i_{2r})=\begin{cases}\log 2 & \text{if } p=n,\\ 2^{(n-p)/(p-1)}-1 & \text{if } 1<p<n,\\ 1-2^{(n-p)/(p-1)} & \text{if } p>n\end{cases}$$

when $y\in S(\eta^i_r,r)$, it follows from the comparison principle that

$$A_3\,\frac{u(y)}{u(\xi_r)}\geq G_{2r}(y,\eta^i_{2r})\quad \text{for } y\in B(\eta^i_{2r},2r)\setminus B(\eta^i_{2r},r).$$

Observe that $x\in[\eta,\eta^i_{2r}]\cap B(\eta^i_{2r},2r)\setminus B(\eta^i_{2r},r)$. Since

$$G_{2r}(x,\eta^i_{2r})=\begin{cases}\log\dfrac{2r}{2r-\delta_\Omega(x)}\approx\dfrac{\delta_\Omega(x)}{r} & \text{if } p=n,\\ \left(\dfrac{2r-\delta_\Omega(x)}{2r}\right)^{(p-n)/(p-1)}-1\approx\dfrac{\delta_\Omega(x)}{r} & \text{if } 1<p<n,\\ 1-\left(\dfrac{2r-\delta_\Omega(x)}{2r}\right)^{(p-n)/(p-1)}\approx\dfrac{\delta_\Omega(x)}{r} & \text{if } p>n,\end{cases}$$

letting $y=x$ we obtain

$$A_3\,\frac{u(x)}{u(\xi_r)}\geq\frac{\delta_\Omega(x)}{r}.$$

□

Definition 3.2. We say that the *Carleson estimate* holds on Ω if there exist constants $A_4>1$ and $r_4>0$ with the following property: Suppose $\xi\in\partial\Omega$, $0<r<r_4$ and u is a positive p-harmonic function in $\Omega\cap B(\xi,A_4r)$ vanishing on $\partial\Omega\cap B(\xi,A_4r)$. If $y\in\Omega\cap B(\xi,r)$ and $\delta_\Omega(y)\geq\varepsilon r$ with $0<\varepsilon<1$, then

$$u(x)\leq C_\varepsilon u(y)\quad\text{for } x\in\Omega\cap B(\xi,r),$$

where $C_\varepsilon>1$ depends only on Ω, A_4, r_4 and ε.

In [3] it was shown that the Carleson estimate holds for a uniform domain. Obviously, a $C^{1,1}$-domain is a uniform domain, and hence the Carleson estimate is available for the domain Ω under consideration in this section.

Lemma 3.3 (Upper Estimate)**.** *There exist constants $A_5\geq 1$, $A_6>1$ and $r_5>0$ with the following property: Let $\xi\in\partial\Omega$ and $0<r<r_5$. If u is a positive p-harmonic function on $\Omega\cap B(\xi,A_6r)$ vanishing on $\partial\Omega\cap B(\xi,A_6r)$, then*

$$\frac{u(x)}{u(\xi_r)}\leq A_5\frac{\delta_\Omega(x)}{r}\quad \textit{for } x\in\Omega\cap B(\xi,r),$$

where $\xi_r\in[\xi,\xi^i]$ is such that $\delta_\Omega(\xi_r)=|\xi_r-\xi|=r$.

Proof. Let $A_6=7A_4$ and $r_5=\min\{r_2/2,r_4/7\}$ with A_4 and r_4 from the Carleson estimate condition. Let $x\in\Omega\cap B(\xi,r)$ with $0<r<r_5$. Then there is $\eta\in\partial\Omega$ such that $\delta_\Omega(x)=|x-\eta|$. By the exterior ball condition at η we find a point η^e such that $B(\eta^e,r_2)\subset\mathbb{R}^n\setminus\Omega$ and $\eta\in S(\eta^e,r_2)$. Take $\eta^e_{2r}\in[\eta,\eta^e]$ such that $|\eta^e_{2r}-\eta|=2r$. Then $B(\eta^e_{2r},2r)\subset B(\eta^e,r_2)\subset\mathbb{R}^n\setminus\Omega$.

It is easy to see that $\eta^e_{2r} \in B(\xi, 4r)$ and that $x \in B(\eta^e_{2r}, 3r) \subset B(\xi, 7r)$. By the Carleson estimate, $u \leq Au(\xi_r)$ on $\Omega \cap B(\xi, 7r)$, and in particular,

$$u \leq Au(\xi_r) \quad \text{on } \Omega \cap S(\eta^e_{2r}, 3r). \tag{5}$$

Here, $A = C_\varepsilon$ with $\varepsilon = 1/7$ from the definition of the Carleson estimate; see Figure 4.

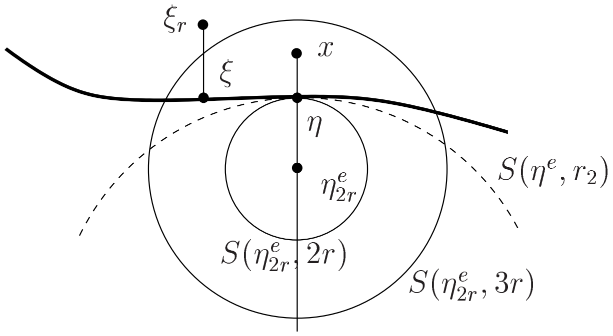

FIGURE 4.

Note that when $y \in S(\eta^e_{2r}, 3r)$,

$$-G_{2r}(y, \eta^e_{2r}) = \begin{cases} -\phi_n\left(\dfrac{y - \eta^e_{2r}}{2r}\right) = \log(3/2) > 0 & \text{if } p = n, \\[2ex] 1 - \phi_p\left(\dfrac{y - \eta^e_{2r}}{2r}\right) = 1 - (3/2)^{(p-n)/(p-1)} > 0 & \text{if } 1 < p < n, \\[2ex] \phi_p\left(\dfrac{y - \eta^e_{2r}}{2r}\right) - 1 = (3/2)^{(p-n)/(p-1)} - 1 > 0 & \text{if } p > n. \end{cases}$$

Since $-G(\cdot, \eta^e_{2r})$ is a positive p-harmonic function on $\mathbb{R}^n \setminus B(\eta^e_{2r}, 2r)$, it follows from (5) and the comparison principle that

$$\frac{u(y)}{u(\xi_r)} \leq -AG_{2r}(y, \eta^e_{2r}) \quad \text{for } y \in \Omega \cap B(\eta^e_{2r}, 3r).$$

The point x lies on the extended line connecting η and η^e_{2r} and $|x - \eta| = \delta_\Omega(x) < r$, and hence $x \in B(\eta^e_{2r}, 3r)$. Letting $y = x$, we obtain

$$\frac{u(x)}{u(\xi_r)} \leq A_5 \frac{\delta_\Omega(x)}{r},$$

since

$$-G_{2r}(x, \eta^e_{2r}) = \begin{cases} \log \dfrac{2r + \delta_\Omega(x)}{2r} \approx \dfrac{\delta_\Omega(x)}{r} & \text{if } p = n, \\[2ex] 1 - \left(\dfrac{2r + \delta_\Omega(x)}{2r}\right)^{(p-n)/(p-1)} \approx \dfrac{\delta_\Omega(x)}{r} & \text{if } 1 < p < n, \\[2ex] \left(\dfrac{2r + \delta_\Omega(x)}{2r}\right)^{(p-n)/(p-1)} - 1 \approx \dfrac{\delta_\Omega(x)}{r} & \text{if } p > n. \end{cases}$$

$\square$

Now we are ready to prove the main theorem of this note.

Proof of Theorem 1.2. Let $r_1 = \min\{r_3, r_5\}$ and let $A_1 = A_6$ from Lemma 3.3. Let $0 < r < r_1$, $\xi \in \partial\Omega$ and ξ_r be as in Lemmas 3.1 and 3.3. Suppose u and v are positive p-harmonic functions on $\Omega \cap B(\xi, A_1 r)$ vanishing on $\partial\Omega \cap B(\xi, A_1 r)$. Then Lemmas 3.1 and 3.3 give

$$\frac{u(x)}{u(\xi_r)} \leq A\frac{\delta_\Omega(x)}{r} \leq A\frac{v(x)}{v(\xi_r)} \quad \text{for } x \in \Omega \cap B(\xi, r).$$

Replacing u and v, letting $x = y$ and multiplying the resultant inequalities, we obtain

$$\frac{u(x)/u(y)}{v(x)/v(y)} \leq A_5 A_3 \quad \text{for } x, y \in \Omega \cap B(\xi, r).$$

Thus the theorem is proved with $A_2 = A_5 A_3$. □

The requirement of Ω being of class $C^{1,1}$ leaves out a wide class of domains such as Lipschitz domains on which the boundary Harnack principle for p-harmonic functions might still hold. We will now show that Lemmas 3.1 and 3.3 can be used to prove the boundary Harnack principle for a wider class of Euclidean domains such as conical domains.

Definition 3.4. Given a set $E \subset \mathbb{R}^n$, we say that E is a *uniformly spherically porous set* if E is a closed set and there are constants $A_7 > 1$ and $r_6 > 0$ such that whenever $\eta \in E$ and $0 < r < r_6$ there exists $r/A_7 < \rho < r$ such that $B(\eta, \rho) \setminus B(\eta, \rho/A_7)$ does not intersect E.

In other words, uniformly spherically porous sets are closed sets with uniformly spaced annular gaps. Examples of porous sets include uniform Cantor sets and finite sets.

Definition 3.5. given a set $E \subset \partial\Omega$ we say that Ω is of class $C^{1,1}_{loc}$ *uniformly in* $\partial\Omega \setminus E$ if there is a constant $L > 0$ such that $\partial\Omega \setminus E = \bigcup_{k\in\mathbb{N}} \Gamma_k$ with each Γ_k a relatively open subset of $\partial\Omega$ that is the graph of a $C^{1,1}$-function (in local coordinates) φ_k with $\nabla\varphi_k$ an L-Lipschitz map, and $\{\Gamma_k\}_k$ forms a locally uniform cover of $\partial\Omega \setminus E$ in the sense that there is a $A_8 \geq 1$ so that whenever $\eta \in \partial\Omega \setminus E$ with $\operatorname{dist}(\eta, E) < r_0$ there exists $k \in \mathbb{N}$ such that $B(\eta, \operatorname{dist}(\eta, E)/A_8) \cap \partial\Omega \subset \Gamma_k$.

Examples of domains satisfying the above definition include conical domains and domains whose boundary contains a Cantor set of corners.

Proposition 3.6. *Let Ω be a uniform domain in $\mathbb{R}^n$ and $E \subset \partial\Omega$ such that E is a uniformly spherically porous set, and Ω is of class $C^{1,1}_{loc}$ uniformly in $\partial\Omega \setminus E$. Let $\xi \in \partial\Omega$. Then there is a constant $A \geq 1$ such that if $0 < 6A_6 r < \min\{r_0, r_6\}$ and u, v are both p-harmonic in $B(\xi, 6A_6 r) \cap \Omega$ and vanish on $B(\xi, 6A_6 r) \cap \partial\Omega$, and $\xi_r \in \Omega \cap S(\xi, r)$ with $\delta_\Omega(\xi_r) \approx r$, then*

$$\frac{1}{A} \leq \frac{u(x)/u(\xi_r)}{v(x)/v(\xi_r)} \leq A \quad \textit{whenever } x \in B(\xi, r) \cap \Omega.$$

Proof. Note that $\partial\Omega \setminus E$ satisfies the ball condition in the following sense. If $\xi \in \partial\Omega \setminus E$, then Ω satisfies the interior and exterior ball conditions at ξ with $r_2 = \min\{r_0, \operatorname{dist}(\xi, E)\}/(LA_8)$.

Let $\xi \in \partial\Omega$. If $B(\xi, r)$ does not intersect E, then we can directly apply Theorem 1.2 to obtain the boundary Harnack principle. If $B(\xi, r)$ does intersect E, then without loss of generality we may assume that $\xi \in E$. The idea of the proof in this case is to use Lemmas 3.1 and 3.3 to obtain control over the decay of the functions $u/u(\xi_r)$ and $v/v(\xi_r)$ on $B(\eta, r/A_8) \cap \Omega$ for each $\eta \in S(\xi, r/A_7) \cap \partial\Omega$, and then use the Harnack inequality to show that these two p-harmonic functions are comparable on $S(\xi, r) \cap \Omega$. Now the comparison theorem for p-harmonic functions, together with the fact that both u and v vanish on $B(\xi, r) \cap \partial\Omega$, tells us these two functions are comparable on $S(\xi, r) \cap \overline{\Omega}$.

In this situation we do not get nice decay estimates as in Lemma 3.1 and Lemma 3.3, but that is not essential for the boundary Harnack principle. □

4. Conformal Martin boundary of a ball in $\mathbb{R}^n$.

Singular functions for the p-Laplacian operator on domains in smooth manifolds were first constructed by Holopainen in his thesis [15] for $1 < p \le n$. Using his constructions, an analog of Martin boundary was proposed in [16]. The aim of Section 5 is to further the study of conformal Martin boundaries by obtaining growth estimates of the conformal Martin kernel functions of $C^{1,1}$-domains near their singularities and prove a generalization of extremal properties of Martin kernel functions. To do so, we need to know the explicit form of the conformal Martin kernels of a ball in $\mathbb{R}^n$; this is the aim of this section.

Definition 4.1. A function $G^p_\Omega : \Omega \times \Omega \to \mathbb{R}$ is said to be a p-singular function on Ω if G^p_Ω satisfies the following conditions for each $y \in \Omega$:

(i) $G^p_\Omega(\cdot, y)$ is non-negative and p-harmonic on $\Omega \setminus \{y\}$.

(ii) Let $g_y(x) = G^p_\Omega(x, y)$ for $x \in \Omega$ and let $g_y(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$. Then $g_y \in W^{1,p}(\mathbb{R}^n \setminus B(y, r))$ whenever $r > 0$.

(iii) Whenever φ is a compactly supported Lipschitz function on Ω,

$$\int_\Omega |\nabla_x G^p_\Omega(x, y)|^{p-2} \nabla_x G^p_\Omega(x, y) \cdot \nabla_x \varphi(x)\, dx = \varphi(y).$$

The construction in [15] demonstrates that whenever the p-capacity of the complement of Ω is positive, such a p-singular function always exists and that it is unique if $p = n$. In the rest of section we restrict ourselves to the case $p = n$ and write simply G_Ω for G^n_Ω. Following [16], we give the definition of conformal Martin kernels and the conformal Martin boundary of such a domain as follows.

Definition 4.2. Let $x_0 \in \Omega$ be fixed. A sequence $\xi = (x_k)_k$ from Ω is said to be a fundamental sequence if it has no accumulation point in Ω and in addition the sequence of n-singular functions

$$\frac{G_\Omega(\cdot, x_k)}{G_\Omega(x_0, x_k)}$$

converges locally uniformly in Ω to an n-harmonic function, denoted M_ξ. Such a limit function is called a *conformal Martin kernel function.* The collection of conformal Martin kernel functions is called the *conformal Martin boundary* of Ω. The conformal Martin boundary is equipped with the local uniform topology: A sequence of kernel functions $(M_{\xi_k})_k$ is said to converge to a kernel function M_ξ if the sequence of kernel functions converge locally uniformly in Ω to M_ξ.

It can be easily seen that the notion of fundamental sequence is independent of the reference point x_0; indeed, two conformal Martin boundaries obtained using two reference points are homeomorphic - see the discussion in [16] or [3]. It can be seen with the aid of Harnack's inequality that the conformal Martin boundary is a compactification of Ω in this topology. It is a consequence of Theorem 6.1 of [3] that if Ω is a John domain (in particular, if it is a uniform domain), then every fundamental sequence must converge to a boundary point of Ω; hence, every conformal Martin kernel function M_ξ is associated with a unique boundary point $\xi \in \partial\Omega$ (however, it is still open whether in a uniform domain more than one conformal Martin kernel function can be associated with a given $\xi \in \partial\Omega$). Here and throughout the rest of the paper, by an abuse of notation we will denote every conformal Martin kernel function whose associated fundamental sequence converges to $\xi \in \partial\Omega$ by M_ξ, even though ξ is not itself a fundamental sequence.

While the above notions make sense even in certain metric measure spaces (see [16] and [3]), even in the elementary setting of Euclidean spaces not much is known about n-singular functions and conformal Martin boundaries. An open problem in this setting is whether the n-singular functions are symmetric if the domain in sufficiently smooth. In this section we explicitly determine the conformal Martin boundary of the ball $B(0,R)$. It can be seen that the n-singular functions and conformal Martin boundaries are invariant under conformal mappings between two domains in $\mathbb{R}^n$. We will use this fact strongly in the following discussion of this section. Recall that conformal mappings between two domains in $\mathbb{R}^n$ are Möbius maps when $n \geq 3$. It is interesting to note that the n-singular functions on balls are symmetric because of the high symmetry of the Euclidean balls.

Lemma 4.3. *The n-singular function (or conformal Green function) for the ball $B(0,R)$ with singularity at y is given by*

$$G_{B(0,R)}(x,y) = \begin{cases} G_R(x,0) = \log \dfrac{R}{|x|} & \text{if } y = 0, \\[2ex] \log \dfrac{|x-y'||y|}{|x-y|R} & \text{if } y \neq 0, \end{cases}$$

where $y' = \dfrac{R^2}{|y|^2}y$.

Proof. If $y = 0$, then it is easy to see that $G_{B(0,R)}(\cdot, 0)$ is nonnegative and continuous on $\overline{B}(0,R) \setminus \{0\}$, and

$$\begin{aligned} \Delta_n G_{B(0,R)}(\cdot,0) &= 0 \quad \text{on } B(0,R) \setminus \{0\}, \\ G_{B(0,R)}(\cdot,0) &= 0 \quad \text{on } S(0,R). \end{aligned}$$

So, the lemma follows in this case from the uniqueness result of [15].

Now let us consider the case $y \neq 0$. Let $\vec{e}_1 = (1,0,\ldots,0)$. By dilation we may assume that $R = 1$ without loss of generality. Moreover, by rotation we may assume that $y = a\,\vec{e}_1$ for some $0 < a < 1$, and $x = (b_1, b_2, 0, \ldots, 0)$ with $b_1^2 + b_2^2 < 1$. Then $y' = (1/a)\vec{e}_1$. Consider the inversion with respect to the sphere $S(y',1)$:

$$TX = y' + \frac{X - y'}{|X - y'|^2} \quad \text{for } X \in \mathbb{R}^n.$$

Then we infer from the symmetry that the sphere $S(0,1)$ is mapped by T to the sphere with diameter $[T\vec{e}_1, T(-\vec{e}_1)]$. Moreover, we observe from an elementary calculation that $T\alpha$ is the midpoint of $[T\vec{e}_1, T(-\vec{e}_1)]$, i.e., the center of the mapped sphere.

Let us calculate $\dfrac{|Tx - Ty|}{|T\vec{e}_1 - Ty|}$ to evaluate the singular function $G_{B(0,R)}(x,y)$. We restrict ourselves to the x_1x_2-plane and identify the plane with the complex plane $\mathbb{C}$ by $z = x_1 + ix_2$.

We identify x with b_1+ib_2 and y with the real number a, and retain the same notation. The restriction of the inversion T to the the x_1x_2-plane can be identified by

$$Tz=\frac{1}{a}+\frac{z-1/a}{|z-1/a|^2}\quad \text{for } z\in\mathbb{C}.$$

The composition of T and the inversion with respect to the real axis becomes the following Möbius transform:

$$Uz=\overline{Tz}=\frac{1}{a}+\frac{\overline{z}-1/a}{|z-1/a|^2}=\frac{1}{a}+\frac{a}{az-1}.$$

A direct computation shows that

$$\frac{|Tx-Ty|}{|T\vec{e}_1-Ty|}=\frac{|Ux-Ua|}{|U1-Ua|}=\frac{\left|\dfrac{a}{ax-1}-\dfrac{a}{a^2-1}\right|}{\left|\dfrac{a}{a-1}-\dfrac{a}{a^2-1}\right|}=\frac{|x-a|}{a|x-1/a|}.$$

Hence, the first case, together with the fact that Ty is the midpoint of the diameter $[T\vec{e}_1,T(-\vec{e}_1)]$ of the ball $TB(0,1)$, and the invariance of the conformal Green function yields

$$G_{B(0,R)}(x,y)=\log\frac{a|x-1/a|}{|x-a|}=\log\frac{|y||x-y'|}{|x-y|}.$$

Thus the lemma follows. □

Proposition 4.4. *The conformal Martin kernel function for the ball $B(0,R)$ with singularity at $\xi\in S(0,R)$ and the fixed reference point $x_0=0$ is given by*

$$M_\xi(x)=\frac{R^2-|x|^2}{|x-\xi|^2},$$

and the conformal Martin boundary of $B(0,R)$ is homeomorphic to the sphere $S(0,R)$.

Contrast this with the classical Martin kernel (corresponding to the case $p=2$) of $B(0,R)$:

$$M_{\text{classical},\xi}(x)=\frac{R^2-|x|^2}{|x-\xi|^n},$$

see Example 8.1.9 of [6].

Proof. Let us calculate the limit

$$M_\xi(x)=\lim_{y\to\xi}\frac{G_{B(0,R)}(x,y)}{G_{B(0,R)}(0,y)}$$

for $x\in B(0,R)$ and $\xi\in S(0,R)$. Let $y\in B(0,R)$ and let $r=|x|$, $\rho=|y|$, $\theta=\angle x0y$, $\alpha=\angle x0\xi$. By the law of cosines we have

$$G_{B(0,R)}(x,y)=\frac{1}{2}\log\left(\frac{|x-y'|^2|y|^2}{|x-y|^2R^2}\right)=\frac{1}{2}\log\frac{r^2\rho^2+R^4-2r\rho R^2\cos\theta}{R^2(r^2+\rho^2-2r\rho\cos\theta)}.$$

Hence

$$M_\xi(x)=\frac{1}{2}\lim_{\rho\to R,\ \theta\to\alpha}\frac{\log\dfrac{r^2\rho^2+R^4-2r\rho R^2\cos\theta}{R^2(r^2+\rho^2-2r\rho\cos\theta)}}{\log\dfrac{R}{\rho}}.$$

Since $\log(1+t) = t + O(t^2)$ as $t \to 0$, it follows that

$$\begin{aligned}
M_\xi(x) &= \frac{1}{2} \lim_{\substack{\rho\to R\\ \theta\to\alpha}} \frac{\dfrac{r^2\rho^2 + R^4 - 2r\rho R^2\cos\theta}{R^2(r^2+\rho^2-2r\rho\cos\theta)} - 1}{\dfrac{R}{\rho} - 1} \\
&= \frac{1}{2} \lim_{\substack{\rho\to R\\ \theta\to\alpha}} \frac{\rho(r^2\rho^2 + R^4 - 2r\rho R^2\cos\theta - R^2(r^2+\rho^2-2r\rho\cos\theta))}{(R-\rho)R^2(r^2+\rho-2r\rho\cos\theta)} \\
&= \frac{1}{2} \lim_{\substack{\rho\to R\\ \theta\to\alpha}} \frac{\rho(R^2-r^2)(R^2-\rho^2)}{(R-\rho)R^2(r^2+\rho-2r\rho\cos\theta)} \\
&= \frac{1}{2}\frac{R(R^2-r^2)2R}{R^2(r^2+R-2rR\cos\alpha)} = \frac{R^2-r^2}{r^2+R-2rR\cos\alpha} = \frac{R^2-|x|^2}{|x-\xi|^2}.
\end{aligned}$$

Therefore the conformal Martin kernel M_ξ is independent of the way y approaches ξ; thus the conformal Martin boundary of $B(0,R)$ is homeomorphic to $S(0,R)$ and the kernel function corresponding to $\xi \in S(0,R)$ is given as above. □

5. Consequences to the conformal Martin kernels of $C^{1,1}$ Euclidean domains

Theorem 6.1 of [3] shows that if Ω is a John domain and M_ξ is any conformal Martin kernel function for the domain associated with $\xi \in \partial\Omega$, then M_ξ is bounded in a neighborhood of every point in $\partial\Omega \setminus \{\xi\}$, and that M_ξ tends to infinity along all non-tangential approaches from Ω to ξ. In this section we obtain blow-up estimates of M_ξ near ξ and prove a weak version of extremal property of Martin kernel functions for Euclidean $C^{1,1}$-domains.

Using Lemmas 3.1 and 3.3, we have the following improvement of Theorem 6.1 of [3].

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a $C^{1,1}$-domain. If $\chi \in \partial\Omega$ and M_χ is a conformal Martin kernel function associated with χ, then there exists $\alpha \ge 1$ such that*

(i) *If $\xi \in \partial\Omega \setminus \{\chi\}$ and $0 < r < |\xi-\chi|/(2A_1)$, then*
$$\frac{M_\chi(\xi_r)}{A_2 r}\delta_\Omega(x) \le M_\chi(x) \le A_2\frac{M_\chi(\xi_r)}{r}\,\delta_\Omega(x) \quad \text{for } x \in B(\xi, r),$$
where $\xi_r \in \Omega \cap B(\xi, 2r)$ such that $\delta_\Omega(\xi_r) = r$.

(ii) *If $0 < r < r_1$, then for $x \in \Omega \cap B(\chi, r)$,*
$$M_\chi(x) \approx \frac{\delta_\Omega(x)}{|x-\chi|}\, M_\chi(\xi_r) \le A_2\,\frac{\delta_\Omega(x)}{|x-\chi|^\alpha},$$
where $\xi_r \in B(\chi, 2r) \cap \Omega$ such that $\delta_\Omega(\xi_r) = r$.

Now using the above lemma and the explicit computation of the conformal Martin kernel functions for balls in $\mathbb{R}^n$, we obtain the following improvement.

Since Ω is a $C^{1,1}$-domain and hence satisfies both the interior and the exterior ball conditions, there exists $r_2 > 0$ such that if B is a ball in $\mathbb{R}^n$ with radius $r < r_2$ and satisfies the internal ball condition for Ω with $\partial\Omega \cap \partial B = \{\chi\}$, and we obtain a ball B^* satisfying the exterior ball condition for Ω at χ by reflecting B about the $(n-1)$-dimensional hyperplane that is the tangent plane to $\partial\Omega$ at χ. If P is the center of B, then $Q = P^*$ is the center of B^*. Given a point $x \in B$ we denote the point in B^* obtained via reflecting x about the tangential hyperplane by x^*. Let I and I^* denote the linear maps that map B and B^* respectively to $B(0,r)$ such that $I(\chi) = I^*(\chi)$. Then I and I^* are compositions of translation and rotation maps with $I^*(x) = I(x^*)$, and map P and Q to the origin 0. Let M_{r,χ_0} denote the conformal Martin kernel function for $B(0,r)$ with singularity at χ_0; that is,

$$M_{r,\chi_0}(x) = \frac{r^2 - |x|^2}{|x-\chi_0|^2}.$$

Let T denote the inversion of $\mathbb{R}^n$ about the sphere ∂B^*;

$$T(z) = r^2\,\frac{x-Q}{|x-Q|^2} + Q.$$

Theorem 5.2. *Let Ω be a $C^{1,1}$-domain in $\mathbb{R}^n$, and let $\chi \in \partial\Omega$. If M^*_χ is a conformal Martin kernel function for Ω associated with a fundamental sequence contained in a conical subregion of Ω in $\Omega \cap B = B$ converging to χ, then there is a constant $C > 0$ such that for all $x \in B$,*

$$\frac{1}{C}\,M_{r,I(\chi)}(I(x)) \le M^*_\chi(x) \le C\,M_{r,I^*(\chi)}(I^*(T(x))). \tag{6}$$

In particular, one can take $\alpha = 2$ in Lemma 5.1, and moreover, if x lies in a conical subregion $\Gamma(\chi,\tau)$ of Ω with the vertex of the cone at χ, then

$$M^*_\chi(x) \approx \frac{1}{\delta_\Omega(x)}.$$

Here by a conical subregion we mean a subregion of points whose distance to the boundary $\partial\Omega$ is comparable to the distance to χ. These are sets of the form

$$\Gamma(\xi,\tau) = \{x \in \Omega \,:\, \delta_\Omega(x) \ge \tau\,|x-\xi| \text{ and } |x-\xi| < \tau\}$$

for $\xi \in \partial\Omega$ and $0 < \tau < 1$. The number τ is the *cone width* of $\Gamma(\xi,\tau)$. Note that $\Gamma(\chi,\tau) \cap B(\chi,\rho) \subset B$ if $\rho > 0$ is sufficiently small; that is, $\Gamma(\chi,\tau) \cap B(\chi,\rho)$ is a conical subregion of B with vertex at χ.

Proof. Let $G_\Omega : \Omega\times\Omega \to \mathbb{R}\cup\{\infty\}$ denote the n-singular function for the domain Ω, and g_r, g^*_r denote the respective n-singular functions for B and B^*. Note that $(x,y) \mapsto g^*_r(T(x),T(y))$ is the n-singular function for the domain $\mathbb{R}^n \setminus \overline{B^*}$ and that $g_r(x,y) = G_{B(0,r)}(I(x),I(y))$ and $g^*_r(x,y) = G_{B(0,r)}(I^*(x),I^*(y))$. Here $G_{B(0,r)}$ denotes the n-singular function for the ball $B(0,r)$ given by Lemma 4.3.

Let $(x_n)_n$ be a fundamental sequence from $\Gamma(\chi,\tau)$ associated with M^*_χ. Then by the comparison theorem, for $x \in B$ we have

$$\frac{g_r(x,x_n)}{g^*_r(T(P),T(x_n))} \le \frac{G_\Omega(x,x_n)}{G_\Omega(P,x_n)} \le \frac{g^*_r(T(x),T(x_n))}{g_r(P,x_n)}.$$

Thus we obtain

$$\frac{G_\Omega(x,x_n)}{G_\Omega(P,x_n)} \le \frac{g_r(T(x)^*,T(x_n)^*)}{g_r(P,x_n)} = \frac{g_r(T(x)^*,T(x_n)^*)}{g_r(P,T(x_n)^*)}\,\frac{g_r(P,T(x_n)^*)}{g_r(P,x_n)} \le C\,\frac{g_r(T(x)^*,T(x_n)^*)}{g_r(P,T(x_n)^*)},$$

whence we get the desired right hand side of (6). Here we used the fact that as the sequence $(x_n)_n$ is in the conical subdomain $\Gamma(\xi,\tau)$ and as for $x_n \in \Gamma(\xi,\tau)$ we have $T(x_n)^* \in \Gamma(\xi,\tau')$ for some $\tau' > 0$ that depends only on τ, we have

$$\frac{g_r(P,T(x_n)^*)}{g_r(P,x_n)} \le C.$$

The constant C depends on the cone width τ. Note that by the results in the previous section, as $\lim_n x_n = \chi$, the limit $\lim_{n\to\infty} \dfrac{g_r(T(x)^*, T(x_n)^*)}{g_r(P, T(x_n)^*)}$ exists and equals the conformal Martin kernel function for B associated with the boundary point χ, evaluated at the point $T(x)^*$; but this is equal to $M_{r,I^*(\chi)}(I^*(T(x)))$. Similarly,

$$\frac{G_\Omega(x,x_n)}{G_\Omega(P,x_n)} \geq \frac{g_r(x,x_n)}{g_r^*(T(P),T(x_n))} = \frac{g_r(x,x_n)}{g_r(P,x_n)}\,\frac{g_r(P,x_n)}{g_r(T(P)^*,x_n)}\,\frac{g_r(T(P)^*,x_n)}{g_r(T(P)^*,T(x_n)^*)} \geq \frac{1}{C}\,\frac{g_r(x,x_n)}{g_r(P,x_n)},$$

from whence we obtain the left hand side inequality of (6). □

It is well-known that if M is a classical Martin kernel for a "nice" Euclidean domain, then it is extremal in the sense that whenever u is a non-negative harmonic function on that domain with $u \leq M$, then there is a real number λ such that $u = \lambda M$; see for example Theorem 6.1 of [4], Chapter 8 of [6], or the discussion in [22] and [5]. Such minimal Martin boundary points for John domains were studied in [2]. For more general domains it is not true that all Martin kernels are minimal, see for example [5]. Existence of minimal conformal Martin kernel functions is not known in the non-linear setting, even under the strong assumptions that the domain is smooth. However, using the above results we have a weak version as follows.

Theorem 5.3. *Let Ω be a $C^{1,1}$-domain in $\mathbb{R}^n$, and let $\chi \in \partial\Omega$. If u is a positive n-harmonic function on Ω such that u vanishes continuously on all of $\partial\Omega \setminus \{\chi\}$, and u is bounded in $\Omega \setminus B(\chi, r)$ for all $r > 0$, then u tends to infinity at χ at a rate no slower than the function M^*_χ studied in Theorem 5.2; that is,*

$$\liminf_{\Gamma(\chi,\tau)\ni x\to\chi} \delta_\Omega(x)\, u(x) > 0.$$

Moreover, there is a constant $\lambda > 0$, that depends on u, such that $\frac{1}{\lambda} M^*_\chi \leq u \leq \lambda M^*_\chi$.

Here $\Gamma(\chi,\tau)$ is a conical subregion of Ω with vertex at χ. Moreover, we have the improvement of the estimate from Lemma 5.1.

Proof. To see this, note that if

$$\liminf_{\Gamma(\chi,\tau)\ni x\to\chi} \delta_\Omega(x)\, u(x) = 0,$$

then by the above theorem,

$$\liminf_{\Gamma(\chi,\tau)\ni x\to\chi} \frac{u(x)}{M^*_\chi(x)} = 0,$$

and hence we can find a monotone decreasing sequence of radii $(\rho_n)_n$ with $\lim_n \rho_n = 0$ and a corresponding sequence of points $(x_n)_n$ from $\Gamma(\chi,\tau) \subset \Omega$ such that $n\, u(x_n) \leq M^*_\chi(x_n)$ and $\delta_\Omega(x_n) \approx d(x_n,\chi) = \rho_n$. Given $\eta \in \partial\Omega \cap S(\chi,\rho_n)$, by Theorem 1.2 together with Harnack's inequality and the fact that we can cover $S(\chi,\rho_n)$ by a bounded number of balls of radii $\rho_n/(20A_1)$ with the bound independent of ρ_n, we know that whenever $x \in \Omega \cap B(\eta, \rho_n/(10A_1))$,

$$\frac{n\, u(x)}{n\, u(x_n)} \leq A_2 \frac{M^*_\chi(x)}{M^*_\chi(x_n)}.$$

As $n\, u(x_n) \leq M^*_\chi(x_n)$, we see that whenever $x \in (\Omega \cap S(\chi,\rho_n)) \cup (\partial\Omega \setminus B(\chi,\rho_n))$,

$$n\, u(x) \leq A_2\, M^*_\chi(x).$$

Hence by the comparison theorem we see that $n\,u \le A_2\,M^*_\chi$ on $\Omega \setminus B(\chi,\rho_1)$ for every $n \in \mathbb{N}$; thus $u \equiv 0$ (note that both u and M^*_χ are bounded on $\Omega \setminus B(\chi,\rho_1)$, and hence we can use the comparison theorem and compare these two n-harmonic functions on this set).

Now reversing the roles of u and M^*_χ in the above argument shows that $u \approx M^*_\chi$ in a conical neighborhood $\Gamma(\chi,\tau)$ of χ. Combining this with Theorem 1.2 will show that $u \approx M^*_\chi$ on all of Ω (provided Ω is bounded), with the comparison constant depending on u. □

6. On p-superharmonic functions

While Theorem 5.3 tells us how non-negative n-harmonic functions that vanish on the boundary of the domain except for a single boundary point behave, it does not tell us how general non-negative n-harmonic functions behave near the boundary, nor does it tell us how p-harmonic functions behave near the boundary when $1 < p < n$. The following proposition partially addresses this shortcoming. Analogous results on Euclidean domains for the case $p = 2$ (that is, for potential theory of the Laplacian operator) can be found in [25], Theorem 7'-16 from page 226 of [23], [24], [21], and [20].

Proposition 6.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the interior ball condition, $1 < p \le n$, and let u be a nonnegative p-superharmonic function on Ω. If there exists a point $\eta \in \partial\Omega$ such that*

$$\liminf_{\Omega \ni y \to \eta} \frac{u(y)}{\delta_\Omega(y)} = 0,$$

then $u \equiv 0$ in Ω.

The proof of this proposition uses some of the ideas from the proof of Lemma 3.1.

Proof. Suppose u is a non-negative p-superharmonic function on Ω such that $u \not\equiv 0$. Then $u > 0$ in Ω by the strong maximum principle (see Theorem 6.5 in [14]). Let

$$m_0 := \inf\{u(x) \,:\, x \in \Omega \text{ and } \delta_\Omega(x) \ge r_2/5\},$$

where $r_2 > 0$ is as in Definition 2.1. Since $u > 0$ on Ω and u is lower semicontinuous on Ω, it is true that $m_0 > 0$. Moreover, as the set of points where u is not finite is a zero p-capacity set, we see that m_0 is finite as well.

Let $x \in \Omega$ such that $\delta_\Omega(x) < r_2/4$, and fix $\eta \in \partial\Omega$ such that $\delta_\Omega(x) = |x - \eta|$. Using the interior ball condition, for $r = r_2/2$ we fix $\eta^r_i \in \Omega$ such that $\delta_\Omega(\eta^r_i) = |\eta^r_i - \eta| = r$, $x \in [\eta^i_r, \eta]$, and $B(\eta^i_r, r) \subset \Omega$. We let $G_r(\cdot, x_0)$ denote the p-singular function on $B(\eta^i_r, r)$ with singularity at η^i_r; see (4). By the comparison theorem (see Theorem 7.6 of [14]),

$$\frac{u(y)}{m_0} \ge A G_r(y, \eta^i_r) \quad \text{whenever } y \in B(\eta^i_r, r) \setminus B(\eta^i_r, r/2),$$

with the constant A depending solely on n, p, and r. In $B(\eta^i_r, r) \setminus B(\eta^i_r, r/2)$ we have $G_r(y, \eta^i_r) \approx \delta_\Omega(y)/r$ with the comparison constant also dependent solely on n, p, and r (see (4)). It therefore follows that if $y \in B(\eta^i_r, r) \setminus B(\eta^i_r, r/2)$, then

$$\frac{u(y)}{m_0} \ge C\frac{\delta_\Omega(y)}{r},$$

and hence, as $x \in B(\eta^i_r, r) \setminus B(\eta^i_r, r/2)$,

$$\frac{u(x)}{\delta_\Omega(x)} \ge C\frac{m_0}{r} > 0.$$

Observe that C depends solely on the constants p, n, and r; specifically, it is *independent* of x. It is also clear that the constant m_0 is independent of x (but depends on r and the semi-global bound of u). Therefore,

$$\liminf_{\Omega \ni x \to \partial\Omega} \frac{u(x)}{\delta_\Omega(x)} \geq C\frac{m_0}{r} > 0,$$

as any sequence in Ω that tends to $\partial\Omega$ eventually satisfies $\delta_\Omega(x) < r_2/4$. □

Using this proposition and the Harnack inequality for p-harmonic functions, and using the fact that $C^{1,1}$-domains are uniform domains, we obtain the following decay estimate for positive p-harmonic functions.

Corollary 6.2. *If u is a positive p-harmonic function on a bounded $C^{1,1}$-domain $\Omega \subset \mathbb{R}^n$ then there exist constants $\beta > 0$ and $C_u > 0$ such that whenever $\eta \in \partial\Omega$ and $x \in B(\eta, r_2)$, we have*

$$C_u^{-1}\,\delta_\Omega(x) \leq u(x) \leq \; C_u\,\delta_\Omega(x)^{-\beta}$$

for any $x \in \Omega$, where $C_u > 0$ depends on u.

We do not know the exact value of the constant β in the above corollary for general p-harmonic functions when $1 < p < \infty$. However, when $p = n$ we are able to prove in the following theorem that we can set $\beta = 1$. This shows that the conformal Martin kernel functions exhibit the extremal behavior near the boundary among the class of all positive n-harmonic functions. The proof follows from the explicit form of the n-singular functions in the ball, as formulated in Lemma 4.3. Unfortunately, such explicit p-singular functions, with singularities not at the center of the ball, are not available for general values of p. In the linear case $p = 2$, the extremal behavior of classical (Euclidean) Martin kernel functions was proved in Serrin [25] (using the integral representation for harmonic functions).

Theorem 6.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the interior ball condition. Then corresponding to every positive n-harmonic function u there is a constant $C_u > 0$ such that*

$$C_u^{-1}\,\delta_\Omega(x) \leq u(x) \leq \; C_u\,\delta_\Omega(x)^{-1}$$

for all $x \in \Omega$.

Proof. Let u be a positive n-harmonic function in Ω. In the light of the above proposition, it suffices to prove the second inequality in the theorem. We will compare u with the n-singular functions from Lemma 4.3. Fix $x \in \Omega$. We may assume that $\delta_\Omega(x) < r_2/5$, where r_2 is the constant associated with the interior ball condition of Ω, for otherwise, the second inequality in the theorem is true by taking $C_u = M_0 \operatorname{diam}(\Omega)^{-1}$, where $M_0 = \sup\{u(x) : x \in \Omega \text{ and } \delta_\Omega(x) \geq r_2/5\}$. By the Harnack principle, M_0 is comparable to m_0 in the proof of Proposition 6.1. Now let $\eta \in \partial\Omega$ such that $\delta_\Omega(x) = |x - \eta|$ and $B(\eta^i, r_2) \subset \Omega$ be the ball touching $\partial\Omega$ at η such that $x \in [\eta^i, \eta]$. For simplicity, by a translation we assume that $\eta^i = 0$. We note that then $\delta_\Omega(x) = r_2 - |x|$. On one hand, by Lemma 4.3 the n-singular function in the ball $B(0, r_2)$ with singularity at x is given by

$$G_{B(0,r_2)}(y, x) = \log \frac{|y - x'||x|}{|y - x|r_2},$$

where $x' = \frac{r_2^2}{|x|^2}x$. Then an easy computation gives the following estimate for all $y \in \partial B(x, \delta_\Omega(x)/2)$:

$$G_{B(0,r_2)}(y,x) \le \log \frac{(|y-x|+|x-x'|)|x|}{|y-x|r_2} = \log \frac{3|x|+2r_2}{r_2} \le \log 5.$$

On the other hand, by the Harnack principle, we have that for all $y \in B(x, \delta_\Omega(x)/2)$,

$$u(y) \ge Cu(x),$$

where the constant C depends solely on n. Now, since u and $G_{B(0,r_2)}(\cdot, x)$ are two n-harmonic functions in $B(0,r_2) \setminus B(x, \delta_\Omega(x)/2)$, the comparison principle tells us that

$$u(y) \ge \frac{Cu(x)}{\log 5} G_{B(0,r_2)}(y,x)$$

for all $y \in B(0,r_2) \setminus B(x, \delta_\Omega(x)/2)$. Taking $y = 0$, we have

$$u(0) \ge \frac{cu(x)}{\log 5} G_{B(0,r_2)}(0,x) \ge \frac{Cu(x)}{\log 5} \log \frac{r_2}{|x|},$$

that is,

$$u(x) \le Cu(0)\delta_\Omega(x)^{-1} \le CM_0\delta_\Omega(x)^{-1},$$

which proves the second inequality in the theorem. □

A natural problem is to characterize the domains Ω for which the corresponding conformal Martin kernel functions exhibit the following extremal behavior: Whenever u is a positive n-harmonic function on Ω, for all conformal Martin kernel functions M on Ω,

$$0 < \liminf_{\Omega \ni x \to \partial\Omega} \frac{u(x)}{M(x)}$$

and if M has a singularity at a point $\chi \in \partial\Omega$, then

$$\limsup_{\Gamma(\chi,\tau)\cap\Omega \ni x \to \chi} \frac{u(x)}{M(x)} < \infty.$$

References

[1] H. Aikawa, *Boundary Harnack principle and Martin boundary for a uniform domain*, J. Math. Soc. Japan **53** (2001), no. 1, 119–145.

[2] H. Aikawa, K. Hirata, and T. Lundh, *Martin boundary points of John domains and unions of convex sets*, J. Math. Soc. Japan (2006).

[3] H. Aikawa and N. Shanmugalingam, *Carleson-type estimates for p-harmonic functions and the conformal Martin boundary of John domains in metric measure spaces*, Michigan Math. J. **53** (2005), no. 1, 165–188.

[4] A. Ancona, *Principe de Harnack à la frontière et théorème de Fatou pour un opérateur elliptique dans un domaine lipschitzien*, Ann. Inst. Fourier (Grenoble) **28** (1978), no. 4, 169–213, x.

[5] ———, *Une propriété de la compactification de Martin d'un domaine euclidien*, Ann. Inst. Fourier (Grenoble) **29** (1979), no. 4, ix, 71–90.

[6] D. H. Armitage and S. J. Gardiner, *Classical potential theory*, Springer Monographs in Mathematics, Springer-Verlag London Ltd., London, 2001.

[7] Z. M. Balogh and J. T. Tyson, *Polar coordinates in Carnot groups*, Math. Z. **241** (2002), no. 4, 697–730.

[8] R. Bañuelos, R. F. Bass, and K. Burdzy, *Hölder domains and the boundary Harnack principle*, Duke Math. J. **64** (1991), no. 1, 195–200.

[9] R. F. Bass and K. Burdzy, *A boundary Harnack principle in twisted Hölder domains*, Ann. of Math. (2) **134** (1991), no. 2, 253–276.

[10] B. Bennewitz and J. Lewis, *On the dimension of p-harmonic measure*, Ann. Acad. Sci. Fenn. Math. **30** (2005), no. 2, 459–505.
[11] L. Caffarelli, E. Fabes, S. Mortola, and S. Salsa, *Boundary behavior of nonnegative solutions of elliptic operators in divergence form*, Indiana Univ. Math. J. **30** (1981), no. 4, 621–640.
[12] L. Capogna and P. Tang, *Uniform domains and quasiconformal mappings on the Heisenberg group*, Manuscripta Math. **86** (1995), 267–281.
[13] B. E. J. Dahlberg, *Estimates of harmonic measure*, Arch. Rational Mech. Anal. **65** (1977), no. 3, 275–288.
[14] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1993, Oxford Science Publications.
[15] I. Holopainen, *Nonlinear potential theory and quasiregular mappings on Riemannian manifolds*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes (1990), no. 74, 45.
[16] I. Holopainen, N. Shanmugalingam, and J. T. Tyson, *On the conformal Martin boundary of domains in metric spaces*, Papers on analysis, Rep. Univ. Jyväskylä Dep. Math. Stat., vol. 83, Univ. Jyväskylä, Jyväskylä, 2001, pp. 147–168.
[17] D. S. Jerison and C. E. Kenig, *Boundary behavior of harmonic functions in nontangentially accessible domains*, Adv. in Math. **46** (1982), no. 1, 80–147.
[18] J. T. Kemper, *A boundary Harnack principle for Lipschitz domains and the principle of positive singularities*, Comm. Pure Appl. Math. **25** (1972), 247–255.
[19] S. Kichenassamy and L. Veron, *Singular solutions of the p-Laplace equation*, Math. Ann. **275** (1986), 599–615.
[20] Ü. Kuran, *On positive superharmonic functions in a-admissible domains*, J. London Math. Soc. (2) **29** (1984), no. 2, 269–275.
[21] Ü. Kuran and J. L. Schiff, *A uniqueness theorem for nonnegative superharmonic functions in planar domains*, J. Math. Anal. Appl. **93** (1983), no. 1, 195–205.
[22] R. S. Martin, *Minimal positive harmonic functions*, Trans. Amer. Math. Soc. **49** (1941), 137–172.
[23] L. Naïm, *Sur le rôle de la frontière de R. S. Martin dans la théorie du potentiel*, Ann. Inst. Fourier, Grenoble **7** (1957), 183–281.
[24] J. L. Schiff, *A uniqueness theorem for superharmonic functions in $\mathbf{R}^n$*, Proc. Amer. Math. Soc. **84** (1982), no. 3, 362–364.
[25] J. Serrin, *On the Harnack inequality for linear elliptic equations*, J. Analyse Math. **4** (1955/56), 292–308.
[26] J. M. G. Wu, *Comparisons of kernel functions, boundary Harnack principle and relative Fatou theorem on Lipschitz domains*, Ann. Inst. Fourier (Grenoble) **28** (1978), no. 4, 147–167, vi.

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