Math. Nachr. **279,** No. 9–10, 1083 – 1098 (2006) / **DOI** 10.1002/mana.200510411

Equivalence of AMLE, strong AMLE, and comparison with cones in metric measure spaces

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Received 25 July 2005, accepted 12 January 2006
Published online 7 June 2006

Key words p-harmonic function, infinity-harmonic, absolutely minimizing Lipschitz extensions, comparison with cones, length space
MSC (2000) Primary: 31C35; Secondary: 31C45, 30C65

In this paper, we study the relationship between p-harmonic functions and absolutely minimizing Lipschitz extensions in the setting of a metric measure space (X, d, μ). In particular, we show that limits of p-harmonic functions (as $p \to \infty$) are necessarily the ∞-energy minimizers among the class of all Lipschitz functions with the same boundary data. Our research is motivated by the observation that while the p-harmonic functions in general depend on the underlying measure μ, in many cases their asymptotic limit as $p \to \infty$ turns out have a characterization that is independent of the measure.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f : \partial\Omega \to \mathbb{R}$ be a given Lipschitz continuous function. A well-known theorem due to Bhattacharya, DiBenedetto and Manfredi [5], suggested earlier in the work of Aronsson [2], states that the sequence (u_p) of the unique p-harmonic extensions of f to Ω, that is, $u_p \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ satisfying $u_p = f$ on $\partial\Omega$ with

$$\int_\Omega |\nabla u_p|^p \, dx \;\le\; \int_\Omega |\nabla v|^p \, dx \quad \text{for all } v \text{ such that } u - v \in W_0^{1,p}(\Omega)\,,$$

converges as $p \to \infty$ to a function $u_\infty \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ that satisfies

$$\operatorname*{ess\,sup}_{x\in V} |\nabla u_\infty(x)| \;\le\; \operatorname*{ess\,sup}_{x\in V} |\nabla v(x)| \tag{1.1}$$

whenever $V \subset \Omega$ is open and $v \in W^{1,\infty}(V)$ is such that $u_\infty = v$ on ∂V. Such functions are necessarily ∞-energy minimizers among the class of all Lipschitz functions with the same boundary data. In the literature, functions that satisfy (1.1) are usually called *absolutely minimizing Lipschitz extensions* (AMLEs for short). The name refers to the fact that a function satisfies (1.1) if and only if it is an optimal Lipschitz extension of f to Ω in the sense that

$$\operatorname{Lip}(u_\infty, V) \;\le\; \operatorname{Lip}(v, V) \quad \text{whenever } V \subset \Omega \text{ and } u_\infty = v \text{ on } \partial V\,. \tag{1.2}$$

The equivalence of (1.1) and (1.2), which is not at all trivial (see [3]), shows, in particular, that while the definition of p-harmonic functions clearly depends on the measure used in integration (above the n-dimensional Lebesgue measure), the limit function u_∞ can be characterized without this measure. This observation raises many natural questions. What happens if we replace the Lebesgue measure by another measure μ in the definition of

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p-harmonic functions? Do the p-harmonic extensions (if there are any) then converge to some function as $p \to \infty$; and if so, is the limit function the same as in the case of the Lebesgue measure? Or more generally, if we are given two measures μ_1 and μ_2, what conditions ensure that the asymptotic limits of their associated p-harmonic extensions coincide?

The objective of this paper is to investigate these questions in the setting of a metric measure space (X, d, μ). Under suitable assumptions on the space, there is a relatively well-developed theory of both p-harmonic extensions and AMLEs in this generality (see [23], [15], [21] and [7]), and thus there is no need to restrict the attention to the special case of $\mathbb{R}^n$. The abstract setting makes it easier to identify the properties of the space and measure relevant to our study, and also gives more flexibility when we need to construct counterexamples. As a first step, we will show that, under certain natural assumptions, the p-harmonic functions converge, as $p \to \infty$, to a limit function that satisfies a metric space version of (1.1). In general, the limit of p-harmonic functions is not even Lipschitz continuous, nor does its Lipschitz continuity guarantee that it would satisfy the condition (1.2). In order to establish that a limit function for which (1.1) holds also satisfies (1.2), we need to assume that the space (X, d, μ) has a "weak Fubini property"; this is used to show, roughly speaking, that sets of measure zero can be neglected when computing the Lipschitz constant of a function, a fact that is not true in general. The proof of the equivalence of (1.1) and (1.2) is rather involved, and as in [3], it is done with aid of an auxiliary concept called "comparison with cones" introduced in [9]. Let us also mention that while, in viewing (1.1), it might seem that the asymptotic limits of the p-harmonic extensions associated to the measures μ_1 and μ_2 coincide if the measures are mutually absolutely continuous, this is not always the case; we give a counterexample in Section 3.

The paper is organized as follows. In Section 2, we state the metric space versions of conditions (1.1) and (1.2), and give some remarks concerning their relationship. Note that in order to generalize (1.1), we have to find an appropriate substitute for the modulus of the gradient $|\nabla u|$. It turns out that the local Lipschitz constant will do quite well for this purpose since in the case of Lipschitz functions it coincides with the minimal p-weak upper gradient that appears in the definition the p-harmonic functions in the metric setting. Section 3 contains a proof for the convergence of p-harmonic functions to a function satisfying (1.1) in the case when (X, d, μ) is a complete length space supporting an appropriate Poincaré inequality and the measure μ is doubling. The comparison with cones property (which is also defined in Section 2 below) is shown in Section 4 to be equivalent to (1.2) in any length space. A much harder task is to show that (1.1) is also equivalent to comparison with cones. This is done in Section 5 under the key assumption of the weak Fubini property.

There is nowadays a vast literature on AMLEs and associated problems in $\mathbb{R}^n$, especially on the closely related topic of the infinity Laplace equation. We refer the reader to the survey [3], which contains an extensive list of references on the subject. The closely related issue of the dependence of the asymptotic limit of p-harmonic functions on the metric d has been considered e.g. in [3], [24], and [4].

2 Definitions

We assume that (X, d, μ) is a metric measure space such that the measure μ is Borel regular, nonempty open sets have positive measure and bounded sets have finite measure.

Given a set $A \subset X$ and a Lipschitz function f on A, we define the global Lipschitz constant to be the number

$$\operatorname{Lip}(f, A) := \sup_{x,y \in A,\, y \neq x} \frac{|f(x) - f(y)|}{d(x, y)},$$

and the local Lipschitz constant to be the function $\operatorname{Lip} f$ defined on A by

$$\operatorname{Lip} f(x) := \lim_{r \to 0^+} \sup_{y \in B(x,r) \setminus \{x\}} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Let $\Omega \subset X$ be a bounded domain and let $f : \overline{\Omega} \to \mathbb{R}$ be a Lipschitz function. Using the global and local Lipschitz constants, we define the metric space versions of conditions (1.1) and (1.2) of the introduction.

Definition 2.1 A function $u \in C(\overline{\Omega})$ is said to be an *absolutely minimizing Lipschitz extension* of f in Ω, abbreviated $\mathrm{AMLE}_f(\Omega)$, if $u = f$ on $\partial\Omega$ and for all subdomains $U \subset \Omega$,

$$\operatorname{Lip}(u, U) = \operatorname{Lip}(u, \partial U).$$

Notice that this definition is equivalent to (1.2) in $\mathbb{R}^n$ because

$$\operatorname{Lip}(v,U) \;=\; \operatorname{Lip}\left(v,\overline{U}\right) \;\geq\; \operatorname{Lip}(v,\partial U)$$

for any Lipschitz function. Furthermore, the McShane–Whitney extensions Λ_u, Υ_u of u from ∂U to U satisfy $\operatorname{Lip}(\Lambda_u, U) = \operatorname{Lip}(\Upsilon_u, U) = \operatorname{Lip}(u, \partial U)$.

Definition 2.2 A function $u \in \operatorname{Lip}(\overline{\Omega})$ is said to be an *strongly absolutely minimizing Lipschitz extension* of f in Ω, abbreviated st-AMLE$_f(\Omega)$, if $u = f$ on $\partial\Omega$ and in addition, for all subdomains $U \subset \Omega$ and for all functions $v \in \operatorname{Lip}\left(\overline{U}\right)$ with $u = v$ on ∂U the following inequality holds true:

$$\mu\text{-}\operatorname*{ess\,sup}_{x\in U} \operatorname{Lip} u(x) \;\leq\; \mu\text{-}\operatorname*{ess\,sup}_{x\in U} \operatorname{Lip} v(x)\,.$$

If $X = \mathbb{R}^n$, equipped with the usual Euclidean distance, and if μ is the Lebesgue measure, then $\operatorname{Lip} u(x) = |\nabla u(x)|$ at every point of differentiability. Thus in this special case,

$$\mu\text{-}\operatorname*{ess\,sup}_{x\in U} \operatorname{Lip} u(x) \;=\; \mu\text{-}\operatorname*{ess\,sup}_{x\in U} |\nabla u(x)|$$

for all Lipschitz functions by the Rademacher theorem, and hence Definition 2.2 is a natural generalization of (1.1). The descriptor "strongly" is reminiscent of the Euclidean case [3], where the inclusion st-AMLE$_f(\Omega) \subset$ AMLE$_f(\Omega)$ is relatively easy to establish, while the reverse inclusion AMLE$_f(\Omega) \subset$ st-AMLE$_f(\Omega)$ is more difficult. Note, however, that the inclusion st-AMLE$_f(\Omega) \subset$ AMLE$_f(\Omega)$ does not always hold in setting of metric spaces, see Example 5.3.

We will prove, under suitable conditions, that the classes AMLE$_f(\Omega)$ and st-AMLE$_f(\Omega)$ coincide by using an intermediate concept of comparison with cones, which we define next. To this end, given $a, b \in \mathbb{R}$ and $x_0 \in X$, let us denote by C_{a,b,x_0} the "cone function" on X defined by

$$C_{a,b,x_0}(x) \;:=\; b \,+\, a\, d(x, x_0)\,.$$

The motivation for considering the cone functions comes from the fact that the McShane–Whitney extensions Λ_u, Υ_u of u from ∂U to U are given by

$$\Upsilon_u(x) \;=\; \inf\{C_{a,b,y}(x) \,:\, y \in \partial U,\ b = u(y),\ \text{and } a = \operatorname{Lip}(u, \partial U)\}$$

and

$$\Lambda_u(x) \;=\; \sup\{C_{a,b,y}(x) \,:\, y \in \partial U,\ b = u(y),\ \text{and } a = -\operatorname{Lip}(u, \partial U)\}\,.$$

Definition 2.3 A function $u \in C(\overline{\Omega})$ is said to satisfy the property of *comparison with cones* in Ω, abbreviated $CC(\Omega)$, if the following two conditions hold:

1. For all subdomains $U \subset \Omega$ and for all $a \geq 0$, all $b \in \mathbb{R}$, and all $z_0 \in X \setminus U$, we have $u \leq C_{a,b,z_0}$ on U whenever $u \leq C_{a,b,z_0}$ on ∂U.

2. For all subdomains $U \subset \Omega$ and for all $a \geq 0$, all $b \in \mathbb{R}$, and all $z_0 \in X \setminus U$, we have $u \geq C_{-a,b,z_0}$ on U whenever $u \geq C_{-a,b,z_0}$ on ∂U.

The concept of comparison with cones was originally introduced in [9], where it was used to study the regularity properties of AMLEs in $\mathbb{R}^n$. See also [3] and the references therein. Its adaptation to the metric space setting is quite straightforward, and only the sign restriction $a \geq 0$ needs some thought, cf. [7]. Indeed, with some additional assumptions on the metric space one can get rid of the requirement $a \geq 0$ in the above definition. For example, if X has the property that for all nonempty bounded open subsets U of X with nonempty boundary and for all points $x_0 \in X \setminus U$ there exists a point $x_1 \in \partial U$ and a point $z \in U$ so that $d(x_0, z) + d(z, x_1) = d(z_0, x_1)$, then we can remove the restriction $a \geq 0$ in the above definition. To see this, suppose that u satisfies the comparison with cones property in a domain Ω, and suppose that $U \subset \Omega$ is a subdomain such that $u \leq C_{a,b,z_0}$ on ∂U for some $a < 0$ and $z_0 \in X \setminus U$. Note that if $W = \{x \in U \,:\, u(x) > C_{a,b,z_0}(x)\}$ is nonempty, then $u = C_{a,b,z_0}$ on ∂W. By the above assumption, we find $x_1 \in \partial W$ and a point $z \in W$ so that $d(z_0, z) + d(z, x_1) = d(z_0, x_1)$. It can be verified via the triangle inequality that $C_{-a,b+ad(z_0,x_1),x_1} \geq C_{a,b,z_0}$. Since $-a > 0$ and $u \leq C_{-a,b+ad(z_0,x_1),z_0}$

on ∂W, we see that $u \le C_{-a,b+ad(z_0,x_1),x_1}$ on W. However, as $d(z_0,z)+d(z,x_1) = d(z_0,x_1)$ and hence $u(z) > C_{a,b,z_0}(z) = C_{-a,b+ad(z_0,x_1),x_1}(z)$, we have a contradiction; thus, W is empty. Metric spaces whose every geodesic line segment is extendable to a bi-infinite geodesic line have the above property.

Propositions 4.1, 5.5 and 5.8 together demonstrate that in a proper length space that has the weak Fubini property, a function is a strongly absolutely minimizing Lipschitz extension if and only if it is an absolutely minimizing Lipschitz extension. If the measure in addition is doubling and supports a Poincaré inequality (see below), then Theorem 3.1 demonstrates the existence of such extensions. To prove the existence of st-AMLE$_f(\Omega)$ functions, we use p-harmonic functions associated with the measure μ as follows.

Given an open set $U \subset X$ and a function $f : U \to \mathbb{R}$, we say that a nonnegative Borel measurable function ρ on U is an *upper gradient* of f if for all compact rectifiable curves γ in U the following inequality

$$|f(x) - f(y)| \le \int_\gamma \rho\, ds \tag{2.1}$$

is satisfied, where x and y denote the two endpoints of γ. If either of $|f(x)|, |f(y)|$ is infinite, the right-hand side of the above inequality is also required to be infinite. It can be shown that if f is a Lipschitz function on U, then $\operatorname{Lip} f$ is an upper gradient of f.

For now let us fix an index p with $1 < p < \infty$. We say that a family of non-constant compact rectifiable curves in U is of zero p-modulus if there is a nonnegative Borel measurable function g on U such that $g \in L^p(U)$ and for all curves γ in this family the path integral $\int_\gamma g\, ds$ is infinite. If the collection of non-constant compact rectifiable curves for which the inequality (2.1) fails is a zero p-modulus family of curves, then the function ρ is said to be a *p-weak upper gradient* of f. The uniform convexity of $L^p(U)$, together with the fact that the collection of all p-weak upper gradients of f in $L^p(U)$ forms a closed convex subset of $L^p(U)$, implies that if this convex subset is nonempty then there is a p-weak upper gradient ρ_f of f in $L^p(U)$, uniquely determined up to sets of μ-measure zero, so that $\|\rho_f\|_{L^p(U)} \le \|\rho\|_{L^p(U)}$ for all p-weak upper gradients ρ of f. Furthermore, this *minimal p-weak upper gradient* has the property that $\rho_f \le \rho$ μ-a.e. in U whenever $\rho \in L^p(U)$ is a p-weak upper gradient of f.

A metric measure space is said to support a $(1,p)$-Poincaré inequality if there exist constants $C > 0$ and $\tau \ge 1$ such that for all functions $f : X \to \mathbb{R}$, for all p-weak upper gradients $\rho \in L^p_{\mathrm{loc}}(X)$ of f, and for all balls $B \subset X$,

$$\inf_{c\in\mathbb{R}} \fint_B |u-c|\, d\mu \le C \operatorname{rad}(B) \left(\fint_{\tau B} \rho^p\, d\mu \right)^{1/p}.$$

For more on upper gradients and Poincaré inequality see [14], [22], [20], [8], and the references therein.

The collection of all functions $f \in L^p(X)$ that have a p-weak upper gradient $\rho \in L^p(X)$ is called the Newton–Sobolev class $N^{1,p}(X)$; see [22] for more on this class. Under the above conditions it is known that functions in the Newton–Sobolev class satisfy versions of the Sobolev embedding theorems; see for example [11] or [22]. Given a set $E \subset X$ and $1 \le p < \infty$, the p-capacity of E is the number

$$\operatorname{Cap}_p(E) := \inf_u \left(\|u\|^p_{L^p(X)} + \inf_\rho \|\rho\|^p_{L^p(X)} \right),$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$ with $u \ge 1$ on E, and over all p-weak upper gradients ρ of u. For more on the definition and properties of p-capacity, we refer the reader to [18], [19], [17], and [23].

We say that the measure μ is *doubling* if there exists a constant $C \ge 1$ such that whenever $x \in X$ and $r > 0$, we have

$$\mu(B(x,2r)) \le C\, \mu(B(x,r)).$$

It was shown in [8] that whenever the measure μ is doubling and supports a $(1,p)$-Poincaré inequality, $\operatorname{Lip} f$ is the minimal p-weak upper gradient of any locally Lipschitz function f.

Throughout the rest of this note, let $\Omega \subset X$ be a bounded domain with $\operatorname{Cap}_p(X \setminus \Omega) > 0$ for sufficiently large $p \ge 1$; it should be noted here that $\operatorname{Cap}_p(X \setminus \Omega) > 0$ for sufficiently large $p \ge 1$ if and only if $\operatorname{Cap}_q(X \setminus \Omega) > 0$ for some finite $q \ge 1$. Furthermore, if $\mu(X \setminus \Omega) > 0$ then necessarily $\operatorname{Cap}_p(X \setminus \Omega) > 0$ for all finite $p \ge 1$.

Given a bounded domain $\Omega \subset X$ such that $X \setminus \Omega$ has positive p-capacity and a function $f : X \to \mathbb{R}$ such that $f \in L^p(X)$ and f has a p-weak upper gradient in $L^p(X)$, we say that a function $u : \overline{\Omega} \to \mathbb{R}$ is a *p-harmonic function on Ω with boundary data f* if the following two conditions hold:

1. the zero extension of $u - f$ is in $N^{1,p}(X)$,
2. whenever $v : \Omega \to \mathbb{R}$ also has the property that the zero extension of $v - f$ is in the class $N^{1,p}(X)$, then

$$\int_\Omega \rho_u^p \, d\mu \;\le\; \int_\Omega \rho_v^p \, d\mu \, .$$

We will show that if μ is doubling and supports a $(1, p_0)$-Poincaré inequality for some $1 \le p_0 < \infty$, then the p-harmonic extensions of a Lipschitz boundary function f converge to a strongly absolutely minimizing Lipschitz extension of f as $p \to \infty$. Moreover, if (X, d, μ) is a proper length space satisfying a weak Fubini property (see Section 5), then u is a strongly absolutely minimizing Lipschitz extension of f if and only if u is an $\mathrm{AMLE}_f(\Omega)$. In particular, in that class of metric measure spaces, the properties of such limits of p-harmonic extension are independent of μ. In general metric spaces, the class of p-harmonic functions changes as the underlying measure changes; in particular, the notion of minimal p-weak upper gradient changes with both the measure on X and the index p, as illustrated in the example below. However, it is a deep theorem of Cheeger that if the metric space is a length space (a property that is independent of the measure μ imposed on X), then amongst the class of all doubling Borel regular measures μ with respect to which the metric space supports a $(1, p_0)$-Poincaré inequality, the minimal p-weak upper gradient of a locally Lipschitz function f on X (the Lipschitz property of a function is again merely a property of the metric and does not depend on the measure μ) is the local Lipschitz constant $\mathrm{Lip}\, f$ whenever $p \ge p_0$; hence the functions obtained as limits of p-harmonic functions (with respect to such measures μ) exhibit properties that are independent of the measure μ. The first example below illustrates this property. The second example below demonstrates that when the metric measure space supports no Poincaré inequality the results obtained in this paper fail.

Example 2.4 As in the book [13], we may consider the *weighted p-Laplacian* equation

$$-\mathrm{div}\big(w(x)\, |\nabla u(x)|^{p-2}\, \nabla u(x)\big) \;=\; 0 \tag{2.2}$$

on a domain in $\mathbb{R}^n$. Solutions to (2.2) are p-harmonic on the domain in the metric measure space obtained by looking at the Euclidean space $\mathbb{R}^n$ endowed with the Euclidean metric, but with the measure μ given by

$$d\mu(x) \;=\; w(x)\, d\omega_n(x)\, ,$$

where ω_n is the canonical Lebesgue measure on $\mathbb{R}^n$. Should w be a p-admissible weight in the sense of [13] (see Section 1.1 of [13]), then we always have solutions to (2.2). In order to consider explicit solutions, let us consider more explicit p-admissible weights. Given $\beta > 0$, consider the weight function w given by

$$w(x) \;=\; |x|^{2\beta}\, .$$

From the discussion preceding Theorem 1.8 of [13], it is clear that this is a p-admissible weight for all $p > 1$. Let $\Omega = B(0,1) \setminus \{0\}$ be the punctured unit ball in $\mathbb{R}^n$ centered at the origin, and for $p > n$ we consider the boundary data for Ω given by $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = \max\{0,\, 1 - \mathrm{dist}(x, S(0,1))\}$ (where $S(0,1)$ is the unit sphere in $\mathbb{R}^n$ centered at the origin 0). It can be seen by the use of Hölder's inequality together with a polar coordinate integration that the set $\{0\}$ is a set of positive p-capacity when $p > n + 2\beta$. A basic calculation shows that the solution to (2.2) in Ω with the Lipschitz boundary data f is given by

$$u_p(x) \;=\; |x|^\alpha \quad \text{where} \quad \alpha \;=\; \frac{p - 2\beta - n}{p - 1}\, .$$

Note that u_p depends on the measure μ via the exponent β, but as $p \to \infty$ we have $u_p \to u_\infty$, where

$$u_\infty(x) \;=\; |x|$$

is clearly independent of μ.

Example 2.5 Let X be the metric space obtained by imposing the Euclidean metric on the set

$$X := \{z \in \mathbb{C} : |\mathrm{Arg}(z)| \leq \pi/4 \text{ or } |\mathrm{Arg}(-z)| \leq \pi/4\}.$$

We may consider two measures μ_1 and μ_2 on X as follows. Let μ_1 denote the standard two-dimensional Lebesgue measure on X, and let μ_2 be given by $d\mu_2(z) = e^{-1/|z|^2} d\mu_1(z)$. Then the collection of all non-constant curves passing through the origin has positive p-modulus with respect to the measure μ_1 whenever $p > 2$, but has zero p-modulus for all p with respect to the measure μ_2. Observe that these two measures are absolutely continuous with respect to each other. However, for $p > 2$ the metric measure space (X, d, μ_1) is a doubling measure space supporting a $(1, p)$-Poincaré inequality, whereas the metric measure space (X, d, μ_2) never supports a Poincaré inequality. If we consider the domain $\Omega \subset X$ given by $\Omega := \{z \in X : |z| < 1\}$, the boundary of Ω consists of two disjoint circular arcs separated by a distance $\sqrt{2}$. If we consider the boundary function obtained by setting the function f to take on the value of 1 on one of the two arcs and 0 on the other arc, the p-harmonic extension u_p to Ω with respect the measure μ_2 is given by $u_p(z) = 1$ if z lies in the quarter-disc whose boundary is the arc on which the data is 1, and $u_p(z) = 0$ otherwise. The limit of these u_p functions, as $p \to \infty$, yields the same function which is not even locally Lipschitz continuous. On the other hand, the p-harmonic extensions v_p obtained with respect to the measure μ_1 yield a Lipschitz function as a limit as $p \to \infty$; this limit function is necessarily of the class st-AMLE$_f(\Omega)$, though the limit of the functions u_p is never a member of this class.

3 Existence of st-AMLE$_f(\Omega)$

In this section we will prove the existence of st-AMLE$_f(\Omega)$ under the following additional assumptions. We will assume that X is a complete length space (that is, the distance between each pair of points $x, y \in X$ is given by $d(x, y) = \inf_\gamma \ell(\gamma)$, where $\ell(\gamma)$ denotes the length of the curve γ and the infimum is taken over all compact rectifiable curves γ in X with end-points x and y) and that the measure μ is doubling. We will also assume that (X, d, μ) supports a $(1, p_0)$-Poincaré inequality for some $1 \leq p_0 < \infty$. Under the assumption of the doubling property of the measure μ it is known that there exists $Q > 0$ such that whenever $x \in X$, $0 < r < R$, and $y \in B(x, R)$,

$$\left(\frac{r}{R}\right)^Q \mu(B(x, R)) \leq C\, \mu(B(y, r)).$$

If $p > Q$, then functions f in the Newton–Sobolev class $N^{1,p}(X)$ satisfy the following inequality for all pairs of points $x, y \in X$:

$$|f(x) - f(y)| \leq C\left(\sum_{j\in\mathbb{N}} 2^{-j(1-Q/p)}\right)\|\rho_f\|_{L^p(X)}\, d(x, y)^{1-Q/p}. \tag{3.1}$$

The important point here is that the constant C is independent of x, y, and f, and is also independent of p (but it depends on the doubling constant and the $(1, p_0)$-Poincaré inequality constant, with $p_0 \leq p$). Here $\rho_f \in L^p(X)$ denotes the minimal p-weak upper gradient of f. It should be mentioned that the results from the paper [8] show that if f is a Lipschitz function, then $\rho_f = \mathrm{Lip}\, f$ μ-almost everywhere (here we use the fact that as a complete doubling length space X is a geodesic space).

It was shown in [23] that under the hypotheses considered in this section, for every function $f \in N^{1,p}(X)$ there is a function $u \in N^{1,p}(X)$ such that $u = f$ on $X \setminus \Omega$ and whenever $v \in N^{1,p}(X)$ is another function such that $v = f$ on $X \setminus \Omega$,

$$\int_\Omega \rho_u^p\, d\mu \leq \int_\Omega \rho_v^p\, d\mu.$$

Such p-energy minimizing functions are called *p-harmonic extensions* of f to Ω. It was also shown in [23] that such p-harmonic functions satisfy the comparison property: If f and h are two functions from the class $N^{1,p}(X)$ such that $f \geq h$ on $\partial\Omega$, then their p-harmonic extensions u_f and u_h satisfy the inequality $u_f \geq u_h$ on Ω.

Theorem 3.1 *Suppose that (X, d, μ) is a complete length space. Under the assumptions that the measure μ is doubling and supports a $(1, p_0)$-Poincaré inequality, for every Lipschitz function f on $\partial\Omega$ there exists a st-AMLE$_f(\Omega)$-extension.*

Proof. Without loss of generality we may assume that f is a Lipschitz function on X with bounded support (by extending f to all of X by a McShane extension and then damping down f by a Lipschitz function which is identically 1 on a bounded neighbourhood of $\overline{\Omega}$ and vanishes outside a larger bounded neighbourhood). For each $p > \max\{p_0, Q\}$ let u_p denote the p-harmonic extension of f to Ω as above. Then by inequality (3.1) and by the fact that

$$\|\rho_{u_p}\|^p_{L^p(X)} \;=\; \int_{X\setminus\Omega} \rho_f^p \, d\mu + \int_\Omega \rho_{u_p}^p \, d\mu \;\leq\; \int_{X\setminus\Omega} \rho_f^p \, d\mu + \int_\Omega \rho_f^p \, d\mu \;=\; \int_X (\operatorname{Lip} f)^p \, d\mu \,,$$

we see that the family $\{u_p : p \geq q\}$ is an equibounded and equicontinuous family on X for every $q > \max\{p_0, Q\}$, and as X is complete, by the Arzela–Ascoli theorem it is a normal family, yielding a subsequence $(u_{p_k})_k$ that converges locally uniformly in X to a function that is $1 - Q/q$-Hölder continuous. By a Cantor diagonalization argument we can extract a subsequence, also denoted $(u_{p_k})_k$, so that the limit function u_∞ is Lipschitz continuous and $\lim_k p_k = \infty$.

Let $v \in N^{1,p}(X)$ be another continuous function on X such that $v = f$ on $\partial\Omega$ and locally Lipschitz continuous on Ω. Since u_{p_k} is the p_k-harmonic extension of f to Ω, we have

$$\int_\Omega \rho_{u_{p_k}}^{p_k} \, d\mu \;\leq\; \int_\Omega \rho_v^{p_k} \, d\mu \,,$$

and hence

$$\left(\fint_\Omega \rho_{u_{p_k}}^{p_k} \, d\mu \right)^{1/p_k} \;\leq\; \left(\fint_\Omega \rho_v^{p_k} \, d\mu \right)^{1/p_k} \;=\; \left(\fint_\Omega \operatorname{Lip} v^{p_k} \, d\mu \right)^{1/p_k} \;\leq\; \mu\text{-}\operatorname*{ess\,sup}_{x\in\Omega} \operatorname{Lip} v(x) \,.$$

Thus, by Hölder inequality, whenever $k \geq k_0$,

$$\left(\fint_\Omega \rho_{u_{p_k}}^{p_{k_0}} \, d\mu \right)^{1/p_{k_0}} \;\leq\; \mu\text{-}\operatorname*{ess\,sup}_{x\in\Omega} \operatorname{Lip} v(x) \,.$$

Now arguing as in the proof of Lemma 3.1 of [17], we may conclude that $u_\infty \in N^{1,p}(X)$ with a p_{k_0}-weak upper gradient ρ satisfying the inequality

$$\left(\fint_\Omega \rho^{p_{k_0}} \, d\mu \right)^{1/p_{k_0}} \;\leq\; \mu\text{-}\operatorname*{ess\,sup}_{x\in\Omega} \operatorname{Lip} v(x) \,,$$

and hence as $\operatorname{Lip} u_\infty$ is the minimal p_{k_0}-weak upper gradient of u_∞, we see that

$$\left(\fint_\Omega \operatorname{Lip} u_\infty^{p_{k_0}} \, d\mu \right)^{1/p_{k_0}} \;\leq\; \mu\text{-}\operatorname*{ess\,sup}_{x\in\Omega} \operatorname{Lip} v(x) \,.$$

Next letting $k_0 \to \infty$, we see that

$$\mu\text{-}\operatorname*{ess\,sup}_{x\in\Omega} \operatorname{Lip} u_\infty(x) \;\leq\; \mu\text{-}\operatorname*{ess\,sup}_{x\in\Omega} \operatorname{Lip} v(x) \,. \tag{3.2}$$

It is also clear that $u_\infty = f$ on $X \setminus \Omega$, and in particular, $u_\infty = f$ on $\partial\Omega$.

It now only remains to prove that for all subdomains $U \subset \Omega$ and for all functions $v \in \mathrm{Lip}\left(\overline{U}\right)$ with $u_\infty = v$ on ∂U the following inequality holds true:

$$\mu\text{-}\operatorname*{ess\,sup}_{x\in U} \mathrm{Lip}\, u_\infty(x) \;\le\; \mu\text{-}\operatorname*{ess\,sup}_{x\in U} \mathrm{Lip}\, v(x)\,.$$

To prove this, fix a subdomain $U \subset \Omega$, and for each p_k we find the p_k-harmonic extension v_{p_k} of u_∞ to U, and as before obtain a locally uniform convergence of these functions to a function v_∞ such that whenever $v \in \mathrm{Lip}\left(\overline{U}\right)$ with $v = v_\infty = u_\infty$ on ∂U, as in inequality (3.2) we get

$$\mu\text{-}\operatorname*{ess\,sup}_{x\in U} \mathrm{Lip}\, v_\infty(x) \;\le\; \mu\text{-}\operatorname*{ess\,sup}_{x\in U} \mathrm{Lip}\, v(x)\,.$$

Hence it suffices to show that $v_\infty = u_\infty$. To do so, it is important to note that the sequence $(p_k)_k$ was a subsequence of the sequence used to construct u_∞.

Since $u_{p_k} \to u_\infty$ uniformly on $\overline{U} \subset \overline{\Omega}$ (which is a compact set as X is a complete doubling space and hence is proper), we see that for all $\epsilon > 0$ there is a positive integer k_ϵ such that $\|u_{p_k} - u_\infty\|_{L^\infty(\partial U)} \le \epsilon$ whenever $k \ge k_\epsilon$; that is, $u_\infty - \epsilon \le u_{p_k} \le u_\infty + \epsilon$ on ∂U. Hence by the comparison theorem, we have $v_{p_k} - \epsilon \le u_{p_k} \le v_{p_k} + \epsilon$ on U whenever $k \ge k_\epsilon$. Letting $k \to \infty$ yields

$$v_\infty - \epsilon \;\le\; u_\infty \;\le\; v_\infty + \epsilon$$

on U. Letting $\epsilon \to 0$ now yields the desired result, completing the proof of the theorem. □

If X is not a length space, we will have to replace the condition

$$\mu\text{-}\operatorname*{ess\,sup}_{x\in U} \mathrm{Lip}\, u(x) \;\le\; \mu\text{-}\operatorname*{ess\,sup}_{x\in U} \mathrm{Lip}\, v(x)$$

with

$$\mu\text{-}\operatorname*{ess\,sup}_{x\in U} \rho_u(x) \;\le\; \mu\text{-}\operatorname*{ess\,sup}_{x\in U} \rho_v(x)$$

in the definition of st-AMLE$_f(\Omega)$ in order for the above proof to work. Note that by the results of [8], we have $\rho_u \approx \mathrm{Lip}\, u$ if u is a local Lipschitz function.

As Example 2.5 demonstrates, without the additional assumptions of the doubling property and the support of $(1, p)$-Poincaré inequality, the limit of p-harmonic functions, as $p \to \infty$, may not yield a function of the class st-AMLE$_f(\Omega)$ (even though the measure μ_2 considered in that example is mutually absolutely continuous with a "nice" measure μ_1). The limit function obtained in that example was not even locally Lipschitz; for an example where the limit function is also Lipschitz but fails to be of class st-AMLE$_f(\Omega)$, see the example discussed in Example 5.3 below.

The existence of AMLE$_f(\Omega)$ can be obtained in any length space (without any assumptions on the measure μ) by using a variant of the classical Perron's method. See [21], [15] and [16] for details.

4 Equivalence of $CC(\Omega)$ and AMLE$_u(\Omega)$

Proposition 4.1 *Suppose X is a length space. Then a function u is of class $CC(\Omega)$ if and only if it is of class AMLE$_u(\Omega)$.*

Proof. First suppose u is of class $CC(\Omega)$, and fix $U \subset \Omega$. Let $x, y \in U$. We will first show that $\mathrm{Lip}(u, \partial(U \setminus \{x\})) = \mathrm{Lip}(u, \partial U)$. To do so, fix $z_0 \in \partial U$, and let $a = \mathrm{Lip}(u, \partial U)$, $b = u(z_0)$. Then $a \ge 0$, and as u is a-Lipschitz on ∂U, we see that for all $y \in \partial U$ we have $|u(y) - u(z_0)| \le a\, d(y, z_0)$; that is, $C_{-a,b,z_0} \le u \le C_{a,b,z_0}$ on ∂U. Hence as u is of class $CC(\Omega)$, we see that $C_{-a,b,z_0} \le u \le C_{a,b,z_0}$ on U. In particular, as $x \in U$,

$$u(z_0) - \mathrm{Lip}(u, \partial U)\, d(x, z_0) \;\le\; u(x) \;\le\; u(z_0) + \mathrm{Lip}(u, \partial U)\, d(x, z_0)\,.$$

Since $z_0 \in \partial U$ was arbitrary, we have that for all $z_0 \in \partial U$,

$$|u(x) - u(z_0)| \ \leq \ \mathrm{Lip}(u, \partial U)\, d(x, z_0)\,,$$

that is, u is $\mathrm{Lip}(u, \partial U)$-Lipschitz on $\partial U \cup \{x\} = \partial(U \setminus \{x\})$; hence the equality $\mathrm{Lip}(u, \partial(U \setminus \{x\})) = \mathrm{Lip}(u, \partial U)$ follows. Repeating this process for the set $U \setminus \{x\}$ with respect to the point y, we also see that

$$\mathrm{Lip}(u, \partial(U \setminus \{x, y\})) \ = \ \mathrm{Lip}(u, \partial U)\,,$$

that is, $|u(x) - u(y)| \leq \mathrm{Lip}(u, \partial U)\, d(x, y)$. Since $x, y \in U$ were arbitrary, we obtain $\mathrm{Lip}(u, U) = \mathrm{Lip}(u, \partial U)$, in other words, u is of class $\mathrm{AMLE}_u(\Omega)$.

To complete the proof, we now show that functions of class $\mathrm{AMLE}_u(\Omega)$ are also of class $CC(\Omega)$. We prove this by contradiction. Suppose u is a function of class $\mathrm{AMLE}_u(\Omega)$ but not of class $CC(\Omega)$; therefore, there exists a subdomain $U \subset \Omega$ and $a \geq 0$, $b \in \mathbb{R}$, and a point $z_0 \in X \setminus U$, such that either

1. $u \leq C_{a,b,z_0}$ on ∂U but it is not true that $u \leq C_{a,b,z_0}$ on U,

or

2. $u \geq C_{-a,b,z_0}$ on ∂U but it is not true that $u \geq C_{-a,b,z_0}$ on U.

Since $-u$ is of class $CC(\Omega)$ whenever u is also of class $CC(\Omega)$, without loss of generality we may assume that the first case above occurs; that is, the set

$$W \ := \ \{x \in U \ : \ u(x) > C_{a,b,z_0}(x)\}$$

is nonempty. Since the two functions u and C_{a,b,z_0} are continuous, W is a nonempty open subset of U with $u = C_{a,b,z_0}$ on ∂W. We fix a point $x \in W$. Since $u = C_{a,b,z_0}$ on ∂W, we see that $\mathrm{Lip}(u, \partial W) \leq a$. As u is of class $\mathrm{AMLE}_u(\Omega)$, we therefore have $\mathrm{Lip}(u, W) \leq a$ (we may replace W with a connected component of W containing x here). On the other hand, as X is a length space, for every $\epsilon > 0$ we can find a curve γ_ϵ in X joining x and z_0 such that it's length $\ell(\gamma_\epsilon) \leq (1 + \epsilon) d(x, z_0)$. Since $x \in W$ and $z_0 \notin W$, the curve γ_ϵ must cross ∂W; let y_ϵ be such a point. Let $\gamma_{\epsilon,1}$ and $\gamma_{\epsilon,2}$ denote the two subcurves of γ_ϵ joining x to y_ϵ and joining y_ϵ to z_0 respectively. Then

$$\ell(\gamma_\epsilon) \ \geq \ \ell(\gamma_{\epsilon,1}) + \ell(\gamma_{\epsilon,2}) \ \geq \ d(x, y_\epsilon) + d(y_\epsilon, z_0)\,.$$

By the definition of W, as $x \in W$, we see that there is a positive real number $\delta > 0$ such that

$$u(x) \ \geq \ C_{a,b,z_0}(x) + \delta \ = \ b + a\, d(x, z_0) + \delta \ \geq \ b + a\big[(1 + \epsilon)^{-1} \ell(\gamma_\epsilon)\big] + \delta\,.$$

We choose $0 < \epsilon < 1$ so that $(1 + \epsilon)^{-1} \geq 1 - \epsilon > 0$. Thus,

$$\begin{aligned} u(x) \ &\geq \ b + a(1 - \epsilon)[d(x, y_\epsilon) + d(y_\epsilon, z_0)] + \delta \\ &\geq \ b + a\, d(y_\epsilon, z_0) + a(1 - \epsilon) d(x, y_\epsilon) - a\epsilon d(y_\epsilon, z_0) + \delta\,. \end{aligned}$$

Since $y_\epsilon \in \partial W$ and hence $u(y_\epsilon) = C_{a,b,z_0}(y_\epsilon) = b + a\, d(y_\epsilon, z_0)$, we have

$$\frac{|u(x) - u(y_\epsilon)|}{d(x, y_\epsilon)} \ \geq \ a(1 - \epsilon) + \frac{\delta - a\epsilon d(y_\epsilon, z_0)}{d(x, y_\epsilon)}\,.$$

Since W is a bounded open set, we see that $0 < d(x, y_\epsilon) \leq D := \mathrm{diam}\big(\overline{W}\big) < \infty$, and $d(y_\epsilon, z_0) \leq A := \max\big\{d(w, z_0) \ : \ w \in \overline{W}\big\} < \infty$. Therefore we may choose ϵ small enough so that $\delta - a\epsilon A > \delta/2$, to obtain

$$\frac{|u(x) - u(y_\epsilon)|}{d(x, y_\epsilon)} \ \geq \ a(1 - \epsilon) + \frac{\delta}{2D}\,,$$

that is,

$$\mathrm{Lip}(u, W) \ = \ \mathrm{Lip}\left(u, \overline{W}\right) \ \geq \ a(1 - \epsilon) + \frac{\delta}{2D}\,.$$

Letting $\epsilon \to 0$ yields $\mathrm{Lip}(u, W) \geq a + \frac{\delta}{2D} > a$, a contradiction. Hence it is necessary that u be of class $CC(\Omega)$ as well, thus completing the proof. □

5 Equivalence of $CC(\Omega)$ and st-AMLE$_u(\Omega)$

In this section again we assume that (X, d, μ) is a length space. It turns out that this assumption is insufficient to prove that the classes $CC(\Omega)$ and st-AMLE$_u(\Omega)$, and consequently, the classes AMLE$_u(\Omega)$ and st-AMLE$_u(\Omega)$, are equivalent. We therefore need the following assumption in addition to the others above.

If Γ is a family of curves in X and $1 \le p < \infty$, the *p-modulus* of Γ is the number

$$\mathrm{Mod}_p(\Gamma) := \inf_\rho \|\rho\|^p_{L^p(X)},$$

where the infimum is taken over all nonnegative Borel measurable functions ρ on X such that for each curve $\gamma \in \Gamma$ the path integral $\int_\gamma \rho\, ds \ge 1$. Ahlfors and Beurling first gave the concept of moduli of curve families in the setting of planar domains in [1] (they termed this concept extremal length), and this concept was further developed and axiomatized to a more general setting by Fuglede in [10]. It was shown in [20] that a family Γ is of zero p-modulus if and only if there is a nonnegative Borel measurable function ρ on X with $\rho \in L^p(X)$ such that for each $\gamma \in \Gamma$, the integral $\int_\gamma \rho\, ds$ is infinite. It is also easy to see that the empty family has zero p-modulus and that whenever $E \subset X$ is of zero μ-measure, the collection Γ^+_E of all curves γ in X for which $\mathcal{H}^1(|\gamma| \cap E) > 0$ is of zero p-modulus.

Definition 5.1 We say that X has a *weak Fubini property* if there exist $1 \le p < \infty$ and two positive constants C and τ_0 such that whenever $0 < \tau \le \tau_0$ and B_1 and B_2 are two balls in X with $\mathrm{dist}\big(\overline{B}_1, \overline{B}_2\big) > \tau \max\{\mathrm{diam}(B_1), \mathrm{diam}(B_2)\}$, then $\mathrm{Mod}_p\Gamma(B_1, B_2, \tau) > 0$. Here $\Gamma(B_1, B_2, \tau)$ denotes the collection of all compact rectifiable curves γ in X joining B_1 and B_2 such that $\ell(\gamma) \le \mathrm{dist}\big(\overline{B}_1, \overline{B}_2\big) + C\tau$.

The following key lemma demonstrates the importance of the above property.

Lemma 5.2 *Let X be a length space that has the weak Fubini property and let W be a nonempty open subset of X. If $u \in \mathrm{Lip}(W)$, then μ-$\mathrm{ess\,sup}_{x\in W} \mathrm{Lip}\, u(x) = \sup_{x\in W} \mathrm{Lip}\, u(x)$.*

Example 5.3 For general metric measure spaces the conclusion of this lemma does not hold. Indeed, one can obtain a counterexample by pasting a line segment to two disjoint closed triangular regions in $\mathbb{R}^2$ and using the length metric and the restriction of the two-dimensional Lebesgue measure to this set. A non-constant function that is constant on the two triangular regions but changes in a Lipschitz manner along the line segment will fail to satisfy the above lemma. However, we do not know whether it is possible that the conclusion of the above lemma holds true always if we assume that the measure of nonempty open sets are positive. Note that in the above example functions that are constant on the closed triangular regions are p-harmonic for all $1 < p < \infty$, and hence can yield as a limit (as $p \to \infty$) a function that is of class st-AMLE$_f(\Omega)$, but not of class AMLE$_f(\Omega)$.

Proof of Lemma 5.2. Let $a = \mu$-$\mathrm{ess\,sup}_{x\in W} \mathrm{Lip}\, u(x)$. Clearly $a \le \sup_{x\in W} \mathrm{Lip}\, u(x)$. Let $E = \{y \in W : \mathrm{Lip}\, u(y) > a\}$. It suffices to show that E is empty. Note that $\mu(E) = 0$. Thus the family Γ^+_E of all curves γ in X for which $\mathcal{H}^1(|\gamma| \cap E) > 0$ is of zero p-modulus. Hence $\mathrm{Mod}_p\big(\Gamma(B_1, B_2, \tau) \setminus \Gamma^+_E\big) > 0$ whenever balls B_1, B_2 and a number τ satisfy the definition of weak Fubini property. It suffices to show that for every $x \in W$ there is a neighborhood of x in which u is a-Lipschitz continuous. Since W is open, we may choose $r > 0$ for which $B(x, 10Cr) \subset W$, where C is from the definition of the weak Fubini property. Let $y, z \in B(x, r)$, and for $0 < 2\tau < \min\{1, \tau_0, d(y, z)\}$, choose $B_1 = B(y, \tau/2)$ and $B_2 = B(z, \tau/2)$. Then $\overline{B}_1 \cup \overline{B}_2 \subset W$, and

$$\mathrm{dist}\big(\overline{B}_1, \overline{B}_2\big) \ge d(y, z) - \tau > 2\tau - \tau = \tau,$$

with $\max\{\mathrm{diam}(B_1), \mathrm{diam}(B_2)\} \le \tau \le 1$. Therefore as X satisfies the weak Fubini property, we have $\mathrm{Mod}_p\big(\Gamma(B_1, B_2, \tau) \setminus \Gamma^+_E\big) > 0$, in particular, $\Gamma(B_1, B_2, \tau) \setminus \Gamma^+_E \ne \emptyset$. Let γ_τ be a curve from this family, and let y_τ and z_τ be the endpoints of γ_τ from B_1 and B_2 respectively. Then as $\mathcal{H}^1(|\gamma_\tau| \cap E) = 0$, we see that

$$|u(y_\tau) - u(z_\tau)| \le \int_{\gamma_\tau} \mathrm{Lip}\, u\, ds \le \int_{\gamma_\tau} a\, ds = a\, \ell(\gamma_\tau) \le a\, [d(y_\tau, z_\tau) + C\tau].$$

As $\tau \to 0$ we see that $u(y_\tau) \to u(y)$, $u(z_\tau) \to u(z)$, and $d(y_\tau, z_\tau) + C\tau \to d(y, z)$. Hence, $|u(y) - u(z)| \le a\, d(y, z)$. Since $y, z \in B(x, r)$ were arbitrary, we see that u is a-Lipschitz on $B(x, r)$, and in particular, $\mathrm{Lip}\, u(x) \le a$, that is, $x \notin E$. Thus E is empty, completing the proof of the lemma. □

It should be noted here that the weak Fubini property does not by itself imply a $(1, p)$-Poincaré inequality. Indeed, by joining countably many Euclidean balls of small diameter by tubes of fixed length whose diameter gets narrower, we obtain a metric measure space (with length metric generated from the underlying Euclidean metric and the natural Lebesgue measure) that has the weak Fubini property for any $1 \le p < \infty$ (precisely because we can run a Fubini type decomposition of volume integrals on this space), but has no Poincaré inequality. On the other hand, even a $(1, 1)$-Poincaré inequality is not sufficient to guarantee the weak Fubini property, as demonstrated by the n-dimensional unit sphere, equipped with the Euclidean metric and the $(n-1)$-dimensional Hausdorff measure; the obstacle here is that under the Euclidean metric the space is not a length space. In general, the Poincaré inequalities only guarantee quasiconvexity. However, it would be interesting to know whether if the space is a length space and nonempty open sets have positive measure, then the space has the weak Fubini property or not. Examples of spaces exhibiting the weak Fubini property include Euclidean domains, Riemannian manifolds, Carnot groups, and the metric spaces constructed by Bourdon and Pajot in [6]; the proof of this fact essentially follows from the fact that in these spaces the measure admits a Fubini type decomposition.

Lemma 5.4 *Let X be a length space and $W \neq X$ be a nonempty open subset of X. If $u \in \operatorname{Lip}(W)$, then*

$$\operatorname{Lip}(u, W) \le \max\left\{\operatorname{Lip}(u, \partial W), \sup_{x\in W} \operatorname{Lip} u(x)\right\}.$$

P r o o f. Let $x, y \in W$. Since X is a length space, for every positive integer n we can find a curve γ_n in X joining x and y such that $\ell(\gamma_n) \le d(x, y) + \frac{1}{n}$. If γ_n lies in W for sufficiently large n, then as $\operatorname{Lip} u$ is an upper gradient of u (see for example [12]), we have

$$|u(x) - u(y)| \le \int_{\gamma_n} \operatorname{Lip} u \, ds \le \left(\sup_{x\in W} \operatorname{Lip} u(x)\right)\ell(\gamma_n),$$

and letting $n \to \infty$ we see that

$$|u(x) - u(y)| \le \left(\sup_{x\in W} \operatorname{Lip} u(x)\right) d(x, y). \tag{5.1}$$

If for all sufficiently large values of n we have γ_n leaving W, then for each such n let z_n and w_n denote the first time γ_n leaves W and the last time γ_n enters W, by breaking γ_n up into three pieces, $\gamma_{n,1}$, $\gamma_{n,2}$, and $\gamma_{n,3}$, where $\gamma_{n,1}$ joins x and z_n and lies in W except for the endpoint z_n, $\gamma_{n,3}$ joins w_n and y and lies in W except for the endpoint w_n, and the subcurve $\gamma_{n,2}$ joins the two points $z_n, w_n \in \partial W$. Therefore, as before we see that

$$\begin{aligned} |u(x) - u(y)| &\le |u(x) - u(z_n)| + |u(z_n) - u(w_n)| + |u(w_n) - u(y)| \\ &\le \left(\sup_{x\in W} \operatorname{Lip} u(x)\right)[\ell(\gamma_{n,1}) + \ell(\gamma_{n,3})] + \operatorname{Lip}(u, \partial W)\, d(z_n, w_n) \\ &\le \max\left\{\operatorname{Lip}(u, \partial W), \sup_{x\in W} \operatorname{Lip} u(x)\right\}[\ell(\gamma_{n,1}) + \ell(\gamma_{n,3}) + \ell(\gamma_{n,2})] \\ &\le \max\left\{\operatorname{Lip}(u, \partial W), \sup_{x\in W} \operatorname{Lip} u(x)\right\}[d(x, y) + (1/n)]. \end{aligned}$$

Letting $n \to \infty$ yields the desired inequality. □

Proposition 5.5 *If X satisfies the weak Fubini property, then every function of the class st-AMLE$_u(\Omega)$ is of class $CC(\Omega)$.*

P r o o f. Suppose u is of class st-AMLE$_u(\Omega)$ but not of class $CC(\Omega)$. Then as in the proof of Proposition 4.1, we obtain $a \ge 0$, $b \in \mathbb{R}$, an open set $W \subset \Omega$, and a point $z_0 \in X \setminus W$ such that $u = C_{a,b,z_0}$ on ∂W but $u > C_{a,b,z_0}$ on W. Since u is of class st-AMLE$_u(\Omega)$, we see that

$$\mu\text{-}\operatorname*{ess\,sup}_{x\in W} \operatorname{Lip} u(x) \le \mu\text{-}\operatorname*{ess\,sup}_{x\in W} \operatorname{Lip} C_{a,b,z_0}(x) \le a.$$

Thus, by Lemma 5.2, we have $\sup_{x\in W} \operatorname{Lip} u(x) \le a$. In particular, by Lemma 5.4, we see that $\operatorname{Lip}(u, W) \le \max\{\operatorname{Lip}(C_{a,b,z_0}, \partial W), a\} \le a$. But then, for all $x \in W$ and all $y \in \partial W$ we have $|u(x) - u(y)| \le a\, d(x, y)$, and therefore,

$$u(x) \;\le\; u(y) + a\, d(x,y) \;=\; C_{a,b,z_0}(y) + a\, d(x,y) \;=\; b + a\,[d(y,z_0) + d(x,y)]\,.$$

Since $y \in \partial W$ was arbitrary, we see that

$$u(x) \;\le\; b + a \inf_{y\in\partial W} [d(y,z_0) + d(x,y)]\,.$$

Now as in the proof of Proposition 4.1, for every $\epsilon > 0$ we choose a curve γ_ϵ joining x and z_0 in the length space X such that $\ell(\gamma_\epsilon) \le (1+\epsilon)\, d(x, z_0)$, and let $y_\epsilon \in \partial W$ be a point at which this curve intersects ∂W. Thus,

$$u(x) \;\le\; b + a[d(y_\epsilon, z_0) + d(x, y_\epsilon)] \;\le\; b + a\,\ell(\gamma_\epsilon) \;\le\; b + a(1+\epsilon) d(x, z_0)\,.$$

Letting $\epsilon \to 0$, we see that $u(x) \le b + a\, d(x, z_0) = C_{a,b,z_0}(x)$, that is, $x \notin W$, a contradiction. Therefore it must be true that u is of class $CC(\Omega)$. ☐

The proof of the converse implication, that every function of the class $CC(\Omega)$ is of class st-AMLE$_u(\Omega)$, is slightly more complicated and requires some preparatory work. Let us first introduce some notation. For $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$, we define

$$S_u^+(x,r) \;:=\; \sup_{\{z\,:\,d(z,x)=r\}} \left(\frac{u(z)-u(x)}{r}\right), \quad S_u^-(x,r) \;:=\; \inf_{\{z\,:\,d(z,x)=r\}} \left(\frac{u(z)-u(x)}{r}\right)$$

and, if the limits exist,

$$S_u^+(x) \;:=\; \lim_{r\to 0^+} S_u^+(x,r)\,, \quad S_u^-(x) \;:=\; \lim_{r\to 0^+} S_u^-(x,r)\,.$$

In the setting of general metric spaces, it is not always true that if $0 < r < \operatorname{dist}(x, \partial\Omega)$ then $B(x, r) \subset \Omega$. However, it should be noted that if the metric space is a length space then for all $x \in \Omega$ we have $\operatorname{dist}(x, X \setminus \Omega) = \operatorname{dist}(x, \partial\Omega)$ and hence $B(x, r) \subset \Omega$.

If u is of class $CC(\Omega)$, then $S_u^+(x, r)$ is nonnegative and nondecreasing in r, and $S_u^-(x, r)$ is nonpositive and nonincreasing. In particular, in that case $S_u^+(x)$ and $S_u^-(x)$ are well-defined. To prove these facts, we first notice that since $u \le C_{0,M,x_0}$ on $\partial B(x, r)$ for any $x_0 \in X \setminus B(x, r)$ and $M = \sup\{u(z) :\ d(x, z) = r\}$, we have

$$u(x) \;\le\; \sup_{\{z\,:\,d(z,x)=r\}} u(z)\,, \quad \text{that is}\,, \quad \sup_{\{z\,:\,d(z,x)=r\}} \left(\frac{u(z)-u(x)}{r}\right) \;\ge\; 0\,.$$

Thus $S_u^+(x, r) \ge 0$. The monotonicity follows by comparing u to the cone $C_{S_u^+(x,r),u(x),x}$ in the punctured ball $B(x, r) \setminus \{x\}$; this yields

$$u(z) \;\le\; u(x) + S_u^+(x,r)\, d(x,z) \quad \text{for all} \quad z \in B(x,r)\,.$$

Hence

$$\frac{u(z)-u(x)}{s} \;\le\; S_u^+(x,r) \;\text{ for all } z \text{ such that } d(x,z) = s\,,\; 0 < s < r\,,$$

and we obtain $S_u^+(x, s) \le S_u^+(x, r)$. The claims concerning $S_u^-(x, r)$ follow by noticing that u is of class $CC(\Omega)$ if and only if $-u$ is of class $CC(\Omega)$ and that $S_u^-(x, r) = -S_{-u}^+(x, r)$.

Lemma 5.6 *If u is of class $CC(\Omega)$, then $S_u^+(x) = -S_u^-(x) = \operatorname{Lip} u(x)$ for every $x \in \Omega$.*

P r o o f. First, by definition,

$$\begin{aligned}\operatorname{Lip} u(x) \;&=\; \lim_{r\to 0^+} \sup_{y\in B(x,r)\setminus\{x\}} \frac{|u(x)-u(y)|}{d(x,y)} \\ &\ge \limsup_{r\to 0^+} \sup_{d(x,y)=r} \left(\frac{u(y)-u(x)}{r}\right) \;=\; \lim_{r\to 0^+} S_u^+(x,r) \;=\; S_u^+(x)\,,\end{aligned}$$

whence $\operatorname{Lip} u(x) \geq S_u^+(x)$. For the converse, we fix $x \in \Omega$ and for $0 < r < \frac{1}{4}\operatorname{dist}(x, \partial\Omega)$ we consider a point $y \in \Omega$ for which $d(x, y) = r$. For $n \in \mathbb{N}$, let γ_n be a path, parametrized by arc length, joining x to y such that $\ell(\gamma_n) < d(x, y) + \frac{1}{n}$. Let I_n denote the interval which is the domain of the map γ_n, and consider the function $g_n : I_n \to \mathbb{R}$ given by $g_n(t) = u(\gamma_n(t))$. Then

$$\begin{aligned}
\frac{g_n(t+h) - g_n(t)}{h} &= \frac{u(\gamma_n(t+h)) - u(\gamma_n(t))}{h} \\
&= \frac{u(\gamma_n(t+h)) - u(\gamma_n(t))}{d(\gamma_n(t+h), \gamma_n(t))} \frac{d(\gamma_n(t+h), \gamma_n(t))}{h} \\
&\leq S_u^+(\gamma_n(t), d(\gamma_n(t+h), \gamma_n(t))) \frac{d(\gamma_n(t+h), \gamma_n(t))}{h} \\
&\leq S_u^+(\gamma_n(t), d(\gamma_n(t+h), \gamma_n(t))) \,,
\end{aligned}$$

where we used the fact that $0 < d(\gamma_n(t+h), \gamma_n(t))/h \leq 1$. Therefore, we see that $g_n'(t) \leq S_u^+(\gamma_n(t))$ whenever $g_n'(t)$ exists. Observe that as u is a Lipschitz function, so is g_n; therefore, for almost every $t \in I_n$ we see that $g_n'(t)$ exists and that

$$u(y) - u(x) = \int_{I_n} g_n'(t)\, dt \leq \int_{I_n} S_u^+(\gamma_n(t))\, dt \leq \left(\sup_{z \in \gamma_n(I_n)} S_u^+(z) \right) \ell(\gamma_n) \leq \left(\sup_{z \in \gamma_n(I_n)} S_u^+(z) \right) \left[d(x, y) + \frac{1}{n} \right].$$

After letting $n \to \infty$ we therefore have

$$\operatorname{Lip} u(x) = \lim_{r \to 0^+} \sup_{y \in B(x,r) \setminus \{x\}} \frac{|u(x) - u(y)|}{d(x, y)} \leq \limsup_{r \to 0^+} \left(\sup_{z \in B(x,2r)} S_u^+(z) \right).$$

Next we recall that $s \mapsto S_u^+(z, s)$ is nondecreasing and notice that $z \mapsto S_u^+(z, s)$ is continuous (because u itself is even Lipschitz continuous). Thus

$$\limsup_{r \to 0^+} \left(\sup_{z \in B(x,2r)} S_u^+(z) \right) \leq \limsup_{r \to 0^+} \left(\sup_{z \in B(x,2r)} S_u^+(z, s) \right) = S_u^+(x, s)$$

for any $s > 0$ small enough. This shows that $\operatorname{Lip} u(x) \leq S_u^+(x, s)$ for all s sufficiently small and consequently $\operatorname{Lip} u(x) \leq S_u^+(x)$ by the definition of $S_u^+(x)$.

We have thus far showed that $\operatorname{Lip} u(x) = S_u^+(x)$ for every $x \in \Omega$. Since u is of class $CC(\Omega)$ if and only if $-u$ is of class $CC(\Omega)$, and $S_u^-(x) = -S_{-u}^+(x)$, we have

$$S_u^-(x) = -S_{-u}^+(x) = -\operatorname{Lip}(-u)(x) = -\operatorname{Lip} u(x)\,,$$

which completes the proof. □

Lemma 5.7 *Let u be of class $CC(\Omega)$, $x_0 \in \Omega$, and $0 < r < \operatorname{dist}(x_0, \partial\Omega)$. If $x_1 \in \Omega$ is such that $d(x_0, x_1) = r$ and $u(x_1) = \sup_{d(x_0,z)=r} u(z)$, then*

$$S_u^+(x_0, r) \leq S_u^+(x_1, s) \quad \textit{for all} \quad 0 < s < \operatorname{dist}(x_0, \partial\Omega) - r\,.$$

Proof. Let γ_n be a path joining x_1 to x_0, parametrized by arc length, such that $\ell(\gamma_n) < r + \frac{1}{n}$. Since $u \leq C_{S_u^+(x_0,r), u(x_0), x_0}$ on the boundary of the punctured ball $B(x_0, r) \setminus \{x_0\}$, we have

$$u(x) \leq u(x_0) + S_u^+(x_0, r) d(x_0, x) \quad \text{for all} \quad x \in B(x_0, r)$$

by the assumption that u is of class $CC(\Omega)$. In particular,

$$\begin{aligned}
u(\gamma_n(t)) &\leq u(x_0) + S_u^+(x_0, r)\, d(x_0, \gamma_n(t)) \\
&\leq u(x_0) + t \left(\frac{u(x_1) - u(x_0)}{r} \right) \\
&= u(x_1) + (r - t) \left(\frac{u(x_0) - u(x_1)}{r} \right) \\
&= u(x_1) + (r - t) \big(-S_u^+(x_0, r) \big)
\end{aligned}$$

for all $t < r$ for which $\gamma_n(t) \in B(x_0, r)$. Here we used the fact that γ_n is parametrized by arc length and the assumption $S_u^+(x_0, r) = \frac{u(x_1)-u(x_0)}{r}$. For sufficiently large n we see that the curve γ_n lies entirely in Ω. For such n we look at all $t < r$ for which $\gamma_n(t)$ is in $B(x_0, r) \cap B(x_1, \operatorname{dist}(x_1, X \setminus \Omega))$; for such t we have

$$\begin{aligned} S_u^-(x_1, d(x_1, \gamma_n(t))) &= \inf_{d(x_1,z)=d(x_1,\gamma_n(t))} \left(\frac{u(z) - u(x_1)}{d(x_1, \gamma_n(t))} \right) \\ &\le \frac{u(\gamma_n(t)) - u(x_1)}{d(x_1, \gamma_n(t))} \\ &\le \frac{r - t}{d(x_1, \gamma_n(t))} \left(-S_u^+(x_0, r) \right) \\ &\le \frac{r - t}{r + \frac{1}{n} - t} \left(-S_u^+(x_0, r) \right) \end{aligned}$$

for all such t's. Here we used the facts that $d(x_1, \gamma_n(t)) \le r + \frac{1}{n} - t$ and that $-S_u^+(x_0, r) \le 0$. Note that for each fixed $t < r$ that is sufficiently close to r, $\gamma_n(t)$ is in $B(x_0, r) \cap B(x_1, \operatorname{dist}(x_1, X \setminus \Omega))$ for all sufficiently large n. We fix such t for now. Because u is Lipschitz continuous, $S_u^-(x_1, r - t) = \lim_{n\to\infty} S_u^-(x_1, r + \frac{1}{n} - t))$. Hence, as $d(x_1, \gamma_n(t)) \le r + \frac{1}{n} - t$ and $s \mapsto S_u^-(x_1, s)$ is nonincreasing, we finally obtain, by letting $n \to \infty$, that

$$S_u^-(x_1, r - t) = \lim_{n\to\infty} S_u^-(x_1, r + \tfrac{1}{n} - t)) \le \limsup_{n\to\infty} \left(S_u^-(x_1, d(x_1, \gamma_n(t))) \right) \le \left(-S_u^+(x_0, r) \right)$$

for each such fixed $t < r$. Letting $t \to r^-$ gives $S_u^-(x_1) \le -S_u^+(x_0, r)$, which by Lemma 5.6 yields

$$S_u^+(x_1) = -S_u^-(x_1) \ge S_u^+(x_0, r) .$$

Since $s \mapsto S_u^+(x_1, s)$ is nondecreasing, we have $S_u^+(x_1, s) \ge S_u^+(x_0, r)$ for all $0 < s < \operatorname{dist}(x_0, \partial\Omega) - r$, as desired. □

Recall that a metric space is said to be proper if every closed and bounded subset of that space is compact.

Proposition 5.8 *Let (X, d) be a proper length space satisfying the weak Fubini property. Then every function u of the class $CC(\Omega)$ is of class st-AMLE$_u(\Omega)$.*

Proof. We argue by contradiction and assume that a Lipschitz function u is of the class $CC(\Omega)$, but is not of class st-AMLE$_u(\Omega)$. This means that there exist an open set $V \subset \Omega$, a Lipschitz function v and $x_0 \in V$ such that $u = v$ on ∂V and

$$\operatorname{Lip} u(x_0) > \sup_{x\in V} \operatorname{Lip} v(x) \ge 0 .$$

Here we used Lemma 5.2. Define the points $x_1, x_2, \ldots$ inductively so that

$$d(x_j, x_{j+1}) = \min\left\{1, \frac{1}{2} \operatorname{dist}(x_j, \partial V)\right\}$$

and

$$S_u^+(x_j, d(x_j, x_{j+1})) = \frac{u(x_{j+1}) - u(x_j)}{d(x_j, x_{j+1})}, \quad j = 0, 1, 2, \ldots ;$$

such points exist because the “spheres” $\{z : d(x_j, z) = r\}$ are all nonempty (a consequence of the fact that X is a length space) and compact (a consequence of the assumption that X is proper). By Lemma 5.7,

$$S_u^+(x_{j+1}, d(x_{j+1}, x_{j+2})) \ge S_u^+(x_j, d(x_j, x_{j+1})) ,$$

and thus

$$\begin{aligned} u(x_{j+1}) - u(x_j) &= S_u^+(x_j, d(x_j, x_{j+1}))\, d(x_j, x_{j+1}) \\ &\ge S_u^+(x_0)\, d(x_j, x_{j+1}) \\ &= \operatorname{Lip} u(x_0)\, d(x_j, x_{j+1}) . \end{aligned}$$

Summing up these inequalities gives

$$u(x_m) - u(x_0) = \sum_{j=0}^{m-1} \big(u(x_{j+1}) - u(x_j)\big) \;\geq\; \mathrm{Lip}\, u(x_0) \left(\sum_{j=0}^{m-1} d\big(x_j, x_{j+1}\big) \right) \quad \text{for any} \quad m \in \mathbb{N}\,.$$

Since $\mathrm{Lip}\, u(x_0) > 0$ and $u(x_m) - u(x_0) \leq 2 \sup_{x\in V} u(x) < \infty$, the sequence $\big(x_j\big)$ is a Cauchy sequence, and thus it converges to a point $x_\infty \in \overline{V}$. In fact, as $d\big(x_j, x_{j+1}\big) = \min\big\{1, \frac{1}{2}\,\mathrm{dist}\big(x_j, \partial V\big)\big\}$, we must have $x_\infty \in \partial V$. Moreover, by the continuity of u, we have

$$u(x_\infty) - u(x_0) \;\geq\; \mathrm{Lip}\, u(x_0) \left(\sum_{j=0}^{\infty} d\big(x_j, x_{j+1}\big) \right). \tag{5.2}$$

Next we set $y_0 = x_0$ and choose the points $y_1, y_2, \ldots$ inductively so that

$$d\big(y_j, y_{j+1}\big) \;=\; \min\left\{1,\; \frac{1}{2}\,\mathrm{dist}\big(y_j, \partial V\big)\right\}$$

and

$$S_u^-\big(y_j, d\big(y_j, y_{j+1}\big)\big) \;=\; \frac{u\big(y_{j+1}\big) - u\big(y_j\big)}{d\big(y_j, y_{j+1}\big)}\,, \quad j \;=\; 0\,,1\,,2\,,\ldots\,.$$

As above, using Lemmas 5.6 and 5.7 we find $y_\infty \in \partial V$ such that

$$u(y_\infty) - u(x_0) \;\leq\; -\,\mathrm{Lip}\, u(x_0) \left(\sum_{j=0}^{\infty} d\big(y_j, y_{j+1}\big) \right). \tag{5.3}$$

By combining (5.2) and (5.3) we obtain

$$u(x_\infty) - u(y_\infty) \;\geq\; \mathrm{Lip}\, u(x_0) \left(\sum_{j=0}^{\infty} d\big(x_j, x_{j+1}\big) + \sum_{j=0}^{\infty} d\big(y_j, y_{j+1}\big) \right).$$

On the other hand, as in the proof of estimate (5.1) in Lemma 5.4,

$$v\big(x_{j+1}\big) - v\big(x_j\big) \;\leq\; \left(\sup_{x\in V} \mathrm{Lip}\, v(x)\right) d\big(x_j, x_{j+1}\big)$$

and

$$v\big(y_{j+1}\big) - v\big(y_j\big) \;\geq\; \left(\sup_{x\in V} \mathrm{Lip}\, v(x)\right) d\big(y_j, y_{j+1}\big)\,,$$

for all j, and therefore

$$v(x_\infty) - v(y_\infty) \;\leq\; \left(\sup_{x\in V} \mathrm{Lip}\, v(x)\right) \left(\sum_{j=0}^{\infty} d\big(x_j, x_{j+1}\big) + \sum_{j=0}^{\infty} d\big(y_j, y_{j+1}\big) \right).$$

Since $\mathrm{Lip}\, u(x_0) > \sup_{x\in V} \mathrm{Lip}\, v(x)$, this implies that $v(x_\infty) - v(y_\infty) < u(x_\infty) - u(y_\infty)$, which is impossible because $x_\infty, y_\infty \in \partial V$ and $u = v$ on ∂V. □

Acknowledgements We would like to thank Tom Bieske for his comments and suggestions regarding this work. This paper was partially written while the second named author was visiting University of Jyväskylä during Spring 2005. The second named author was partially supported by NSF grant DMS-0355027.

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