

# REGULARITY OF SETS WITH QUASIMINIMAL BOUNDARY SURFACES IN METRIC SPACES

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ABSTRACT. This paper studies regularity of perimeter quasiminimizing sets in metric measure spaces with a doubling measure and a Poincaré inequality. The main result shows that the measure theoretic boundary of a quasiminimizing set coincides with the topological boundary. We also show that such a set has finite Minkowski content and apply the regularity theory study rectifiability issues related to quasiminimal sets in strong  $A_\infty$ -weighted Euclidean case.

## 1. INTRODUCTION

It is now a well-known fact that Euclidean sets with (locally) minimal surfaces have smooth boundary apart from a set of co-dimension 2. This result is due to De Giorgi, see [DG1] and [DG2]. The analogous result for Euclidean quasiminimal surfaces is due to David and Semmes [DS1], who showed that bounded sets with quasiminimal boundary surfaces are uniformly rectifiable and are locally John domains.

The paper [DS1] considered a double obstacle problem in constructing quasiminimal surfaces in Euclidean spaces; A similar problem was considered by Caffarelli and de la Llave in [CL], where the setting is  $C^2$  Riemannian manifolds. In [CL, Theorem 1.1] it is shown that given an Euclidean hyperplane (and the manifold is obtained by a perturbation of the Euclidean metric in a  $C^2$ -fashion) there is a quasiminimal surface in the Riemannian metric that lies close to the hyperplane. In [KKST2] a double obstacle problem similar to the one considered by [DS1] was studied in the setting of doubling metric measure spaces supporting a  $(1, 1)$ -Poincaré inequality. It is therefore natural to ask what type of regularity properties do the minimizing sets have away from the boundaries of the obstacles.

In this paper we study the regularity properties of quasiminimal sets or, more precisely, quasiminimal boundary surfaces in the setting of metric measure spaces with a doubling measure that supports a  $(1, 1)$ -Poincaré inequality. We will show, by modifying De Giorgi's technique using a part of the argument of David and Semmes, that such a set

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is porous and satisfies a measure density property. In particular, this implies that the measure theoretic boundary of a quasiminimizing set coincides with the topological boundary. We also show that such a set has finite Minkowski content. In the metric setting the classical definition of rectifiability may not be as widely applicable. For instance, in the setting of Heisenberg groups there are sets of finite perimeter that are not rectifiable [Mag]. Hence the finiteness of the Minkowski content is the best one can hope for in this generality. Since the problem studied in [CL] is a minimization problem and comes with an associated PDE, the techniques used there are essentially of PDE. The problem studied in [DS1] is a quasiminimization problem, and hence we find some of the methods used in this paper to be more easily adaptable to the general metric measure space setting.

In the last two sections of this paper we apply the regularity theory developed in the first part of the paper to study rectifiability issues related to quasiminimal sets in strong  $A_\infty$ -weighted Euclidean setting. Observe that when equipped with a strong  $A_\infty$ -weight, the Euclidean space with Euclidean metric need not satisfy a 1-Poincaré inequality. However, there is a natural metric induced by the strong  $A_\infty$ -weight, and we show in Section 6 that the Euclidean space equipped with this natural metric and weighted measure satisfy a 1-Poincaré inequality. Hence we are able to use the theory developed in the first part to study rectifiability issues of the boundary of quasiminimal sets in this modified Euclidean space. We consider this application in Section 7 of this paper. It is known that every strong  $A_\infty$ -weight is not comparable to the Jacobian of a Euclidean quasiconformal mapping; it is therefore not possible to use (unweighted) Euclidean results about regularity of sets with quasiminimal surfaces to study rectifiability issues of boundaries of such sets in the strong  $A_\infty$ -weighted setting. We were able to apply the theory developed in the general metric setting in the first five sections of this paper to successfully address rectifiability issues in this weighted Euclidean setting.

For related results about isoperimetric sets in the Carnot group setting we refer the interested reader to [LR], where they show that isoperimetric sets (which are necessarily a special class of sets of quasiminimal boundary surfaces) are Ahlfors regular (which also now follows from Corollary 5.3) and are porous. Regularity for Euclidean quasiminimizers that are asymptotically minimizers was studied by Rigot [R], where it was shown that if the asymptotic minimality condition is sufficiently controlled, then the quasiminimal surface is Hölder smooth in big pieces. We point out that our results about the sets with quasiminimal boundary surfaces apply to every boundary point of the set (of course, a modification of such a set on a measure zero subset would still maintain quasiminimality while destroying the regularity at some boundary point; to avoid this trivial modification we ensure, without

loss of generality, that each point  $x$  of the boundary of the set  $E$  satisfies  $\mu(B(x, r) \cap E) > 0$  and  $\mu(B(x, r) \setminus E) > 0$  for all  $r > 0$ ), and hence our results are weaker than Hölder regularity of the boundary, but are applicable to each boundary point. Hence it might well be that the studies related to rectifiability and weak tangents of locally minimal surfaces in the Carnot group setting would be more approachable using the regularity properties studied in this paper. It is a result of Lu and Wheeden [LW] that Carnot groups are doubling and satisfy a 1-Poincaré inequality, and hence the results of this paper apply in the setting of Carnot groups (and indeed in more general Carnot-Carathéodory spaces, which satisfy local versions of these conditions). A nice survey about Poincaré inequalities and isoperimetric inequalities in the setting of Carnot groups can also be found in [J] and [Hei].

It was shown in [AKL] that a subset  $E$ , of a Carnot group, with locally finite perimeter has vertical weak tangents for  $\|D\chi_E\|$ -almost every point. Combining this with our results (in particular, the consequence that every boundary point of such a set is in the measure-theoretic boundary), we see that  $\mathcal{H}^{Q-1}$ -a.e. boundary point of a set of quasiminimal boundary surface in a Carnot group has a vertical weak tangent ( $Q$  is the homogeneous dimension of the group). The method of [AKL] uses the group structure; it would be interesting to know whether such results hold for other Carnot-Carathéodory spaces such as the Grushin spaces. Note that existence of weak tangents is weaker than rectifiability. For a different notion of rectifiability in the Carnot group setting see [Mag, Section 3].

## 2. PRELIMINARIES

A Borel regular outer measure is doubling if there is a constant  $C > 0$  such that for every ball  $B \subset X$  we have  $0 < \mu(B) < \infty$  with  $\mu(2B) \leq C \mu(B)$ . For such a measure  $\mu$ , there is a lower mass bound exponent  $Q > 0$ ; that is, whenever  $x \in X$ ,  $0 < r \leq R$ , and  $y \in B(x, R)$ , we have

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq \frac{1}{C} \left(\frac{r}{R}\right)^Q.$$

Given a function  $f$  and a non-negative Borel measurable function  $g$  on  $X$ , we say that  $g$  is an upper gradient of  $f$  if whenever  $\gamma$  is a rectifiable curve in  $X$  (that is, a curve with finite length), we have

$$|f(y) - f(x)| \leq \int_{\gamma} g \, ds, \quad (2.1)$$

where  $x$  and  $y$  denote end points of  $\gamma$ . Here the above inequality should be interpreted to mean that  $\int_{\gamma} g \, ds = \infty$  whenever at least one of  $|f(x)|$  and  $|f(y)|$  is infinite; see for example [HeiK]. The collection of all upper gradients, together, play the role of the modulus of the weak derivative

of a Sobolev function in the metric setting. We consider the norm

$$\|f\|_{N^{1,1}(X)} := \|f\|_{L^1(X)} + \inf_g \|g\|_{L^1(X)}$$

with the infimum taken over all upper gradients  $g$  of  $f$ . The Newton-Sobolev space considered in this paper is the space

$$N^{1,1}(X) = \{f : \|f\|_{N^{1,1}(X)} < \infty\} / \sim,$$

where the equivalence relation  $\sim$  is given by  $f \sim h$  if and only if

$$\|f - h\|_{N^{1,1}(X)} = 0.$$

We say that  $X$  supports a weak  $(1, 1)$ -Poincaré inequality if there are constants  $C > 0$  and  $\lambda \geq 1$  such that whenever  $f$  is a function on  $X$  with upper gradient  $g$  and  $B$  is a ball in  $X$ , we have

$$\int_B |f - f_B| d\mu \leq C \text{rad}(B) \int_{\lambda B} g d\mu.$$

A function  $f$  on  $X$  is said to be of bounded variation, and denoted  $f \in BV(X)$ , if  $f \in L^1(X)$  and there is a sequence  $\{f_n\}_n$  of functions from  $N^{1,1}(X)$  such that  $f_n \rightarrow f$  in  $L^1(X)$  and  $\limsup_n \|f_n\|_{N^{1,1}(X)} < \infty$ . The  $BV$  norm of such a function  $f$  is given by

$$\|f\|_{BV(X)} := \inf_{\{f_n\}_n} \liminf_{n \rightarrow \infty} \|f_n\|_{N^{1,1}(X)},$$

where the infimum is taken over all such convergent sequences. The  $BV$  energy norm of  $f$  is given by

$$\|Df\|(X) := \inf_{\{f_n\}_n} \liminf_{n \rightarrow \infty} \left[ \|f_n\|_{N^{1,1}(X)} - \|f_n\|_{L^1(X)} \right].$$

We say that a Borel set  $E \subset X$  is of *finite perimeter* if  $\chi_E \in BV(X)$ . The *perimeter measure* of the set  $E$  is  $P(E, X) := \|D\chi_E\|(X)$ . See [Mi2] and [A] for more on  $BV$  functions and sets of finite perimeter in the metric setting. We point out here that in the Euclidean case with Lebesgue measure the above notion coincides with the classical definition of  $BV$  functions; see for example [EG]. It was shown by Miranda in [Mi2] that if  $\mathcal{U}$  is the collection of all open subsets of  $X$ , then the map  $\mathcal{U} \ni O \mapsto \|Df\|(O)$  extends to a Radon measure on  $X$ . The *coarea formula*

$$\|Df\|(A) = \int_{-\infty}^{\infty} P(\{x \in X : f(x) > t\}, A) dt$$

was also proven in [Mi2].

In this paper, we assume that  $\mu$  is a doubling Borel measure with lower mass bound exponent  $Q > 1$  and that  $X$  is complete and supports a  $(1, 1)$ -Poincaré inequality. Note that we can increase the value of  $Q$  as we like, and so assuming  $Q > 1$  is not a serious restriction, and is assumed merely for book-keeping. We point out here that if  $X$  supports

a weak  $(1, 1)$ -Poincaré inequality, then whenever  $f \in BV(X)$  and  $B$  is a ball in  $X$ , we have

$$\int_B |f - f_B| d\mu \leq C \operatorname{rad}(B) \|Df\|(\lambda B).$$

When considering the function  $f = \chi_E$  for set  $E \subset X$ , the above inequality implies the *relative isoperimetric inequality*

$$\min\{\mu(B \cap E), \mu(B \setminus E)\} \leq C \operatorname{rad}(B) P(E, \lambda B).$$

In this paper  $C$  will denote constants whose precise values are not needed, and so the value of  $C$  might differ even within the same line.

It is well known that the Poincaré inequality implies a Sobolev-Poincaré inequality if the measure is doubling. Indeed, by [HaKo] we have

$$\left( \int_B |u - u_B|^t d\mu \right)^{1/t} \leq C \operatorname{rad}(B) \int_{\lambda B} g_u d\mu.$$

with  $t = Q/(Q - 1)$  for all  $u \in N^{1,1}(X)$ . By the definition of the  $BV$  class, lower semicontinuity of the  $BV$  norm, and the Lebesgue dominated convergence theorem, we obtain the Sobolev inequality

$$\left( \int_B |u - u_B|^t d\mu \right)^{1/t} \leq C \operatorname{rad}(B) \frac{\|Du\|(2\lambda B)}{\mu(2\lambda B)}$$

for all  $u \in BV(X)$ .

**Lemma 2.2.** *Let  $u \in BV(X)$  and  $A = \{x \in B : |u(x)| > 0\}$ . If  $\mu(A) \leq \gamma\mu(B)$  for some  $0 < \gamma < 1$ , then*

$$\left( \int_B |u|^t d\mu \right)^{1/t} \leq \frac{C}{1 - \gamma^{1-1/t}} \operatorname{rad}(B) \frac{\|Du\|(2\lambda B)}{\mu(2\lambda B)}$$

with  $t = Q/(Q - 1)$ .

*Proof.* By Minkowski's inequality and the above-mentioned Sobolev inequality,

$$\begin{aligned} \left( \int_B |u|^t d\mu \right)^{1/t} &\leq \left( \int_B |u - u_B|^t d\mu \right)^{1/t} + |u_B| \\ &\leq C \operatorname{rad}(B) \frac{\|Du\|(2\lambda B)}{\mu(2\lambda B)} + |u_B|. \end{aligned} \tag{2.3}$$

By the assumption on  $u$  and Hölder's inequality,

$$\begin{aligned}
|u_B| &\leq \int_B |u| d\mu = \frac{1}{\mu(B)} \int_A |u| d\mu \\
&\leq \frac{1}{\mu(B)} \left( \int_A |u|^t d\mu \right)^{1/t} \mu(A)^{1-1/t} \\
&= \frac{1}{\mu(B)} \left( \int_B |u|^t d\mu \right)^{1/t} \mu(A)^{1-1/t} \\
&= \left( \frac{\mu(A)}{\mu(B)} \right)^{1-1/t} \left( \int_B |u|^t d\mu \right)^{1/t} \\
&\leq \gamma^{1-1/t} \left( \int_B |u|^t d\mu \right)^{1/t}.
\end{aligned}$$

So by (2.3),

$$(1 - \gamma^{1-1/t}) \left( \int_B |u|^t d\mu \right)^{1/t} \leq C \operatorname{rad}(B) \frac{\|Du\|(2\lambda B)}{\mu(2\lambda B)},$$

from which the lemma follows.  $\square$

**Corollary 2.4.** *If  $u \in BV(X)$  such that  $u = 0$  in  $X \setminus B$  and  $X \setminus 2B$  is non-empty, then*

$$\left( \int_B |u|^t d\mu \right)^{1/t} \leq C \operatorname{rad}(B) \frac{\|Du\|(\overline{B})}{\mu(B)}.$$

*Proof.* Since  $X \setminus 2B$  is non-empty, and because by the Poincaré inequality  $X$  is path-connected, it follows that there is a point  $y \in 2B \setminus B$  such that  $d(y, x) = 3r/2$  where  $B = B(x, r)$ . Therefore by the doubling property of the measure  $\mu$ , we have

$$\mu(2B \setminus B) \geq \mu(B)/C \geq \mu(2B)/C^2$$

for some constant  $C > 1$ . Because  $A = \{z \in 2B : |u(z)| > 0\}$  is a subset of  $B$ , it follows that

$$\frac{\mu(A)}{\mu(2B)} \leq \frac{\mu(B)}{\mu(2B)} = \frac{\mu(2B) - \mu(2B \setminus B)}{\mu(2B)} \leq 1 - C^{-2} < 1.$$

We can take  $\gamma = 1 - C^{-2}$  in Lemma 2.2 to obtain the desired inequality.  $\square$

### 3. QUASIMINIMIZING SURFACES AND QUASIMINIMIZERS

**Definition 3.1.** Let  $E \subset X$  be a Borel set of finite perimeter and  $\Omega \subset X$  be an open set. We say that  $E$  is a  $K$ -quasiminimal set, or has

a  $K$ -quasiminimal boundary surface, in  $\Omega$  if for all open  $U \Subset \Omega$  and for all Borel sets  $F, G \Subset U$ ,

$$P(E, U) \leq K P((E \cup F) \setminus G, U).$$

We say that a function  $u \in BV(\Omega)$  is a  $K$ -quasiminimizer if for all  $\varphi \in BV(\Omega)$  with support in  $U \Subset \Omega$ ,

$$\|Du\|(U) \leq K \|D(u + \varphi)\|(U).$$

**Lemma 3.2.** *If  $E$  is a  $K$ -quasiminimal set in  $\Omega$ , then  $u = \chi_E|_{\Omega}$  is a  $K$ -quasiminimizer in  $\Omega$ .*

*Proof.* Since  $E$  is of finite perimeter, it follows that  $u \in BV(\Omega)$ . Also, if  $\varphi \in BV(\Omega)$  with support in  $U \Subset \Omega$ , then for  $0 < t < 1$ , when  $x \in X \setminus U$  we have  $(u + \varphi)(x) > t$  if and only if  $x \in E$ , and consequently

$$P(\{u + \varphi > t\}, U) \geq K^{-1} P(E, U),$$

and so by the coarea formula,

$$\begin{aligned} \|Du\|(U) &= P(E, U) = \int_0^1 P(E, U) dt \\ &\leq K \int_0^1 P(\{u + \varphi > t\}, U) dt \\ &\leq K \int_{\mathbb{R}} P(\{u + \varphi > t\}, U) dt = K \|D(u + \varphi)\|(U), \end{aligned}$$

which shows that  $u$  is a  $K$ -quasiminimizer.  $\square$

#### 4. DENSITY

The main result of this section is Theorem 4.2, where we prove a uniform measure density estimate for quasiminimal sets. To prove the main result, we need the following lemma. For a proof of this lemma, we refer to [Gia, Lemma 5.1].

**Lemma 4.1.** *Let  $R > 0$  and  $f : (0, R] \rightarrow [0, 1)$  be a bounded function. Suppose that there exist some  $\alpha > 0$ ,  $0 \leq \theta < 1$ , and  $\gamma \geq 0$  such that for all  $0 < \rho < r \leq R < \infty$  we have*

$$f(\rho) \leq \gamma(r - \rho)^{-\alpha} + \theta f(r).$$

*Then there is a constant  $c = c(\alpha, \theta)$  so that for all  $0 < \rho < r \leq R$ ,*

$$f(\rho) \leq c\gamma(r - \rho)^{-\alpha}.$$

The next result implies that every boundary point of a set of quasiminimal surface belongs to the measure theoretic boundary.

**Theorem 4.2.** *If  $E$  is a quasiminimal set in  $\Omega$ , then by modifying  $E$  on a set of measure zero if necessary, there exists  $\gamma_0 > 0$  such that for all  $z \in \Omega \cap \partial E$ ,*

$$\frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} \geq \gamma_0 \quad \text{and} \quad \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} \geq \gamma_0$$

whenever  $0 < r < \text{diam}(X)/3$  such that  $B(x, 2r) \subset \Omega$ . The density constant  $\gamma_0$  depends solely on the doubling constant, the constants associated with the Poincaré inequality, and the quasiminimality constant  $K$ .

*Proof.* We can modify  $E$  on a set of measure zero so that  $\mu(B(x, r) \cap E) > 0$  for all  $x \in E$  and  $r > 0$ , and  $\mu(B(x, r) \setminus E) > 0$  for all  $x \in X \setminus E$  and  $r > 0$ . This is done by removing points  $x \in E$  for which there is a positive number  $r_x$  such that  $\mu(B(x, r_x) \cap E) = 0$  (and in doing so, note that we remove the ball  $B(x, r_x)$  from  $E$  as well since all points in this ball also satisfy this condition) and adding into  $E$  points  $y$  for which there is a positive number  $r_y$  such that  $\mu(B(y, r_y) \setminus E) = 0$  (and in doing so, note that we include the ball  $B(y, r_y)$  back into  $E$ ). By Lebesgue differentiation theorem, such a modification is done only on a set of  $\mu$ -measure zero. This implies that for all  $x \in \partial E$  and  $r > 0$ , we have

$$\mu(B(x, r) \cap E) > 0 \quad \text{and} \quad \mu(B(x, r) \setminus E) > 0.$$

By the relative isoperimetric inequality, we conclude that

$$P(E, B(x, r)) > 0.$$

Let  $u = \chi_E$ , and for  $z \in \Omega$  let  $R > 0$  such that  $B(z, 2R) \subset \Omega$  with  $0 < R < \text{diam}(X)/3$ . For  $0 < r < R$  let  $\eta$  be a  $C/(R-r)$ -Lipschitz continuous function such that  $\eta = 1$  on  $B(z, r)$  and  $\eta = 0$  on  $X \setminus B(z, R)$ , with  $0 \leq \eta \leq 1$  on  $X$ . Set

$$v = u - \eta u = (1 - \eta)u.$$

Then  $v = u$  on  $X \setminus B(z, R)$ , and so by the quasiminimality property of  $u$  and the product rule

$$\begin{aligned} \|Du\|(B(z, r)) &\leq \|Du\|(B(z, R)) \leq K\|Dv\|(B(z, R)) \\ &\leq K \left( \|Du\|(B(z, R) \setminus B(z, r)) + \frac{C}{R-r} \int_{B(z, R)} u \, d\mu \right). \end{aligned}$$

Observe that  $\eta$  is a bounded Lipschitz function and so the product rule is valid. By setting  $\theta = K/(K+1) < 1$ , we see that

$$\|Du\|(B(z, r)) \leq \theta \|Du\|(B(z, R)) + \frac{C}{R-r} \int_{B(z, R)} u \, d\mu.$$



Hence by Lemma 4.1, there is a constant  $C > 0$ , which is independent of  $z, R$  and  $E$ , such that

$$\|Du\|(B(z, r)) \leq \frac{C}{R-r} \int_{B(z, R)} u d\mu = \frac{C}{R-r} \mu(E \cap B(z, R)).$$

For  $r = \frac{3}{4}R$ , from the above we get

$$\|Du\|(B(z, \frac{3}{4}R)) \leq \frac{2C}{R} \mu(B(z, R) \cap E). \quad (4.3)$$

Let  $\nu$  be a  $2C/R$ -Lipschitz function such that  $0 \leq \nu \leq 1$  on  $X$ ,  $\nu = 1$  on  $B(z, \frac{1}{2}R)$ , and  $\nu = 0$  on  $X \setminus B(z, \frac{3}{4}R)$ . Setting  $\varphi = \nu u$ , the product rule implies that

$$\|D\varphi\|(B(z, \frac{3}{4}R)) \leq \|Du\|(B(z, \frac{3}{4}R)) + \frac{2C}{R} \mu(E \cap B(z, R)).$$

So by (4.3), we arrive at

$$\|D\varphi\|(B(z, \frac{3}{4}R)) \leq \frac{2C}{R} \mu(E \cap B(z, R)). \quad (4.4)$$

Notice that  $\varphi^t = \varphi = \chi_E$  in  $B(x, R/2)$  and therefore by Corollary 2.4 and (4.4), we obtain

$$\begin{aligned} \left( \frac{\mu(B(z, \frac{R}{2}) \cap E)}{\mu(B(z, \frac{R}{2}))} \right)^{1-1/Q} &= \left( \int_{B(z, \frac{R}{2})} \varphi^t d\mu \right)^{1/t} \\ &\leq CR \frac{\|D\varphi\|(B(z, \frac{3}{4}R))}{\mu(B(z, \frac{3}{4}R))} \\ &\leq C \frac{\mu(B(z, R) \cap E)}{\mu(B(z, R))}. \end{aligned} \quad (4.5)$$

Applying the above argument also to  $X \setminus E$ , we see that

$$\left( \frac{\mu(B(z, \frac{R}{2}) \setminus E)}{\mu(B(z, \frac{R}{2}))} \right)^{1-1/Q} \leq C \frac{\mu(B(z, R) \setminus E)}{\mu(B(z, R))}. \quad (4.6)$$

Up to now we have been using an adaptation of a part of the De Giorgi machinery. To complete the proof we adapt the proof of [DS1, Lemma 3.30]. Recall that by our assumption, if  $x \in \Omega \cap (E \cup \partial E)$  and  $r > 0$  then  $\mu(B(x, r) \cap E) > 0$ . For  $x \in \Omega \cap (E \cup \partial E)$  and  $z \in B(x, \frac{R}{4})$ , by the doubling property of  $\mu$ , we have

$$\frac{\mu(B(z, \frac{R}{2}) \cap E)}{\mu(B(z, \frac{R}{2}))} \leq C_d \frac{\mu(B(x, R) \cap E)}{\mu(B(x, R))}. \quad (4.7)$$

Let  $\gamma_0 = 1/(C^Q C_d) > 0$ , where  $C$  is as in (4.5). Suppose that

$$\frac{\mu(B(x, R) \cap E)}{\mu(B(x, R))} = \gamma < \gamma_0.$$

For positive integers  $j$  we set  $B_j = B(z, R/2^j)$ . Then by a repeated application of (4.5), with  $t = Q/(Q-1) > 1$ , we obtain

$$\begin{aligned} \frac{\mu(B_j \cap E)}{\mu(B_j)} &\leq \left( C \frac{\mu(B_{j-1} \cap E)}{\mu(B_{j-1})} \right)^{Q/(Q-1)} \\ &\leq C^{Q/(Q-1)} \left( C \frac{\mu(B_{j-2} \cap E)}{\mu(B_{j-2})} \right)^{(Q/(Q-1))^2} \\ &\leq C^{t+t^2+\dots+t^{j-1}} \left( \frac{\mu(B_1 \cap E)}{\mu(B_1)} \right)^{t^{j-1}} \\ &\leq C^{Q t^{j-1}} (C_d \gamma)^{t^{j-1}} = (C^Q C_d \gamma)^{t^{j-1}}, \end{aligned}$$

where we also used (4.7). Since  $C^Q C_d \gamma < 1$ , it follows that for all  $z \in B(x, R/4)$ ,

$$\liminf_{r \rightarrow 0} \frac{\mu(B(z, r) \cap E)}{\mu(B(z, r))} = 0,$$

and the Lebesgue differentiation theorem now implies that  $\mu(B(x, R/4) \cap E) = 0$ , resulting in a contradiction. Consequently, we have

$$\frac{\mu(B(x, R) \cap E)}{\mu(B(x, R))} \geq \gamma_0.$$

A similar argument for  $X \setminus E$  also gives

$$\frac{\mu(B(x, R) \setminus E)}{\mu(B(x, R))} \geq \gamma_0.$$

This completes the proof.  $\square$

## 5. POROSITY

By a result of David and Semmes [DS1], sets with quasiminimal surfaces in the complement of two disjoint cubes in the Euclidean space are uniform domains whose complements are also uniform (and indeed, are isoperimetric sets). Whether quasiminimal surfaces must enclose uniform domains is still open in the general metric setting, but now that we know such sets have each boundary point as a point of density for both the set and its complement, we next show that these sets are uniformly locally porous. For us, the porosity is a reasonable weakening of the uniform domain condition.

By Theorem 4.2, without loss of generality we may assume that every point  $x \in \Omega \cap \partial E$  has the property that

$$\frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} \geq \gamma_0 \quad \text{and} \quad \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} \geq \gamma_0$$

whenever  $0 < r < \text{diam}(X)/3$  such that  $B(x, 2r) \subset \Omega$ ,

**Lemma 5.1.** *Let  $E$  be a quasiminimal set in  $\Omega$  and  $x \in \Omega \cap \partial E$ . Then there exist  $r_0$  and  $C > 0$  such that*

$$C^{-1} \frac{\mu(B(x, r))}{r} \leq P(E, B(x, r)) \leq C \frac{\mu(B(x, r))}{r},$$

whenever  $0 < r < r_0$  such that  $B(x, 2r) \subset \Omega$ . The constant  $C$  is independent of  $x$  and  $r$ .

*Proof.* The inequality on the left-hand side follows immediately from the density property of both  $E$  and  $X \setminus E$  together with the relative isoperimetric inequality, so it suffices to prove the inequality on the right-hand side.

By the results in [KKST1, Lemma 6.2], we have that for all  $r > 0$  there exists  $r < \rho < 2r$  (indeed, a positive 1-dimensional measure amount of them) such that

$$P(B(x, \rho)) \approx \frac{\mu(B(x, \rho))}{\rho}$$

and we can also choose such  $\rho$  so that  $P(E, S(x, \rho)) = 0$ , where

$$S(x, \rho) = \{z \in X : d(z, x) = \rho\}$$

is the sphere centered at  $x$  with radius  $\rho$ . Fix  $\epsilon > 0$ . Then  $B(x, r) \subset \overline{B(x, \rho)} \subset B(x, \rho + \epsilon)$ , and so by the quasiminimizer property of  $E$  we have

$$\begin{aligned} P(E, B(x, r)) &\leq P(E, B(x, \rho + \epsilon)) \leq K P(E \cup B(x, \rho), B(x, \rho + \epsilon)) \\ &\leq K [P(B(x, \rho), B(x, \rho + \epsilon)) + P(E, B(x, \rho + \epsilon) \setminus B(x, \rho - \epsilon))] \\ &= K [P(B(x, \rho)) + P(E, B(x, \rho + \epsilon) \setminus B(x, \rho - \epsilon))]. \end{aligned}$$

Since  $P(E, S(x, \rho)) = 0$ , we have that

$$\lim_{\epsilon \rightarrow 0} P(E, B(x, \rho + \epsilon) \setminus B(x, \rho - \epsilon)) = 0.$$

It follows from the choice of  $\rho$  and the doubling property of  $\mu$  that

$$P(E, B(x, r)) \leq K P(B(x, \rho)) \approx K \frac{\mu(B(x, \rho))}{\rho} \approx CK \frac{\mu(B(x, r))}{r}. \quad \square$$

**Theorem 5.2.** *If  $E$  is a quasiminimal set in  $\Omega$ , then  $E$  and  $X \setminus E$  are locally porous in  $\Omega$ ; that is, for every  $x \in \Omega \cap \partial E$  there exists  $r_x > 0$  and  $C \geq 1$  such that whenever  $0 < r < r_x$ , there are points  $y \in B(x, r)$  and  $z \in B(x, r)$  such that*

$$B(y, r/C) \subset E \cap \Omega \quad \text{and} \quad B(z, r/C) \subset X \setminus E.$$

The constant  $C$  is independent of  $x, r$ . Furthermore,  $r_x$  depends on  $x$  only so far as to have  $B(x, 10r_x) \subset \Omega$ .

*Proof.* Fix  $x \in \Omega \cap \partial E$ . For  $r > 0$  such that  $B(x, 4r) \subset \Omega$ , let  $0 < \rho \leq r$  such that for all  $y \in B(x, r) \cap E$  the ball  $B(y, \rho)$  intersects  $X \setminus E$ . Note that  $\rho = r$  would satisfy this requirement. If there is some  $\rho$  with

$r/20 < \rho < r/10$  such that the above condition fails, then there is some  $y \in B(x, r) \cap E$  such that  $B(y, r/20) \subset E$ , and the porosity requirement is satisfied at the scale  $r$ . If not, we can choose  $\rho < r/10$  so that for every  $y \in B(x, r) \cap E$ , the set  $B(y, \rho) \setminus E$  is non-empty. In this case, we can cover  $B(x, r) \cap E$  by a family of balls  $\{B(y_i, 10\rho)\}_i$ , such that the collection  $\{B(y_i, 2\rho)\}_i$  is pairwise disjoint. Then by the doubling property of  $\mu$  together with the density property of the previous section,

$$\begin{aligned} \gamma_0 \mu(B(x, r)) &\leq \mu(B(x, r) \cap E) \\ &\leq \sum_i \mu(B(y_i, 10\rho)) \leq C \sum_i \mu(B(y_i, \rho)). \end{aligned}$$

Note that by the density results of the previous section,

$$\mu(B(y_i, 2\rho) \cap E) \geq C \mu(B(y_i, 2\rho))$$

and

$$\mu(B(y_i, 2\rho) \setminus E) \geq C \mu(B(y_i, 2\rho)).$$

Hence by the relative isoperimetric inequality,

$$P(E, B(y_i, 2\rho)) \geq \frac{1}{C} \frac{\mu(B(y_i, \rho))}{\rho}.$$

By the pairwise disjointness property of the above family of balls and the relative isoperimetric inequality combined with the density property of the previous section and the choice of  $\rho$ , we have

$$\begin{aligned} P(E, B(x, 2r)) &\geq \sum_i P(E, B(y_i, 2\rho)) \\ &\geq \sum_i \frac{1}{C} \frac{\mu(B(y_i, \rho))}{\rho} \geq \frac{1}{C} \frac{\mu(B(x, r))}{\rho}. \end{aligned}$$

By Lemma 5.1, we now have

$$\frac{1}{C} \frac{\mu(B(x, r))}{\rho} \leq C \frac{\mu(B(x, r))}{r},$$

and consequently  $\rho \geq r/C$ . This means that there is a point  $y \in B(x, r) \cap E$  such that  $B(y, r/2C) \subset E$ , thus proving the porosity of  $E$ . Similar argument with  $X \setminus E$ , which also is a quasiminimal set since  $E$  is a quasiminimal set, gives the porosity of  $X \setminus E$  in  $\Omega$ .  $\square$

The following corollary is a consequence of the porosity property proved above. Note that in the Euclidean setting, if a set satisfies the conclusion of the following corollary, then it is uniformly rectifiable; see for example the discussion in [DS1]. Indeed, David and Semmes use this fact together with the notion of tangent hyperplanes to prove that  $E$  then has to be locally a John domain.

As a consequence of the following corollary together with the results from [LT, Theorem 4.1], the Assouad dimension of  $\partial E$  is at most  $Q - 1$ ,

and by [LT, Theorem 4.2], the Assouad dimension of  $\partial E$  is  $Q - 1$  if the measure  $\mu$  is Ahlfors  $Q$ -regular, that is,

$$\mu(B) \approx \text{rad}(B)^Q.$$

In [LT] the supremum of all such possible  $\alpha$  is called the Aikawa co-dimension of  $\partial E$ . We also refer to [LT] for the definition of the Minkowski content of codimension  $\alpha$ . We denote  $\delta_E(x) = \text{dist}(x, X \setminus E)$ .

**Corollary 5.3.** *If  $E$  a quasiminimal set in domain  $\Omega$ , then, then  $\Omega \cap \partial E$  has finite Minkowski content of codimension  $\alpha$  for  $0 < \alpha < 1$ , and*

$$\int_{B(x_0, r) \cap E} \frac{1}{\delta_E(y)^\alpha} d\mu(y) \leq C \frac{\mu(B(x_0, r))}{r^\alpha}$$

for all  $x_0 \in \partial E$  and  $r > 0$  such that  $B(x_0, 10\lambda r) \subset \Omega$ . Furthermore, if  $\alpha \geq 1$  then

$$\int_{B(x_0, r) \cap E} \frac{1}{\delta_E(y)^\alpha} d\mu(y) = \infty.$$

*Proof.* By the Cavalieri principle, we see that

$$\begin{aligned} & \int_{B(x_0, r) \cap E} \frac{1}{\delta_E(y)^\alpha} d\mu(y) \\ &= \int_0^\infty \mu(\{y \in B(x_0, r) \cap E : \delta_E(y)^{-\alpha} > t\}) dt \\ &\approx \int_0^\infty \mu(\{y \in B(x_0, r) \cap E : \delta_E(y) < s\}) \frac{ds}{s^{1+\alpha}} \\ &\approx \int_0^r \mu(\{y \in B(x_0, r) \cap E : \delta_E(y) < s\}) \frac{ds}{s^{1+\alpha}} \\ &\quad + \int_r^\infty \frac{\mu(E \cap B(x_0, r))}{s^{1+\alpha}} ds \\ &\approx \int_0^r \frac{\mu(E_s^+ \cap B(x_0, r))}{s^{1+\alpha}} ds + C \frac{\mu(B(x_0, r))}{r^\alpha}. \end{aligned}$$

Here

$$E_s^+ = \bigcup_{x \in \partial E} B(x, s) \cap E.$$

To compute the measure of  $E_s^+ \cap B(x_0, r)$ , we can cover  $E_s^+ \cap B(x_0, r)$  by countably many balls  $5\lambda B_j$  with radius  $5\lambda s$ , such that  $\lambda B_j$  are pairwise disjoint. We also ensure that  $5\lambda B_j$  has its center located in  $B(x_0, r) \cap \partial E$ . Now we have by the relative isoperimetric inequality and the porosity of  $E$  and  $\Omega \setminus E$  that

$$\mu(B_j) \leq C s P(E, \lambda B_j).$$

Thus by the doubling property of  $\mu$  we conclude that

$$\begin{aligned} \mu(E_s^+ \cap B(x_0, r)) &\leq \sum_j \mu(5\lambda B_j) \leq C \sum_j \mu(B_j) \\ &\leq Cs \sum_j P(E, \lambda B_j) \leq Cs P(E, B(x_0, 2\lambda r)). \end{aligned}$$

By Lemma 5.1, we know that

$$P(E, B(x_0, 2\lambda r)) \approx \frac{\mu(B(x_0, r))}{r}.$$

Hence we can conclude that

$$\int_0^r \frac{\mu(E_s^+ \cap B(x_0, r))}{s^{1+\alpha}} ds \leq C \frac{\mu(B(x_0, r))}{r} \int_0^r \frac{ds}{s^\alpha},$$

and so

$$\int_{B(x_0, r) \cap E} \frac{1}{\delta_E(y)^\alpha} d\mu(y) \leq C \frac{\mu(B(x_0, r))}{r^\alpha}.$$

To see the second part of the claim, we can use the fact that the balls  $\lambda B_j$  are pairwise disjoint, together with Lemma 5.1 from which we get that  $\mu(B_j) \approx s P(E, \lambda B_j)$ , to obtain

$$\mu(E_s^+ \cap B(x_0, r)) \geq \frac{1}{C} \sum_{\{j: 5\lambda B_j \subset B(x_0, r)\}} \mu(B_j) \geq \frac{s}{C} P(E, B(x_0, \frac{r}{5\lambda})),$$

from which we see that when  $\alpha \geq 1$ ,

$$\int_{B(x_0, r) \cap E} \frac{1}{\delta_E(y)^\alpha} d\mu(y) \geq \frac{1}{C} P(E, B(x_0, \frac{1}{5\lambda} r)) \int_0^r \frac{ds}{s^\alpha} = \infty.$$

□

*Remark 5.4.* The above proof also indicates that

$$C^{-1} s P(E, B(x_0, \frac{1}{5\lambda} r)) \leq \mu(E_s^+ \cap B(x_0, r)) \leq C s P(E, B(x_0, 2\lambda r))$$

whenever  $x_0 \in \partial E$  and  $B(x_0, 10\lambda r) \subset \Omega$ .

If  $E$  is a quasiminimal set, then by the density property of the previous section, we have  $\overline{E} = \text{int}(\overline{E})$ , and so we can replace  $E$  with its interior  $\text{int}(E)$ . So we may assume that  $E$  is open. From the density property again, we can see that connected components of  $E$  are also quasiminimal sets in  $\Omega$ . So throughout the rest of the paper we will assume that in addition  $E$  is a connected open set, that is, a domain, of (uniformly locally) quasiminimal surface.

We say that a domain  $E$  is a  $BV_I$ -extension domain (in the sense of [BuMa] and [BaMo]) if there are constants  $C \geq 1$  and  $\delta > 0$  such that whenever  $u \in BV(E)$  such that the diameter of the support of  $u$  is smaller than  $\delta$ , then there is a function  $Tu \in BV(X)$  such that

$$\|DTu\|(X) \leq C \|Du\|(E)$$

and  $Tu = u$  on  $E$ .

**Lemma 5.5.** *If  $E$  is a domain of (uniformly) locally quasiminimal surface, then it is a  $BV_l$ -extension domain.*

*Proof.* Let the uniform quasiminimality constant be denoted  $K \geq 1$ . Let  $F \subset E$  be a set of finite perimeter in  $E$  with  $\text{diam}(F) < \delta$ . By the criterion of  $BV_l$ -extension domains found in [BuMa] and [BaMo], to verify that  $E$  is a  $BV_l$ -extension domain it suffices to prove that we can extend every such  $F$  to a set  $\widehat{F}$  such that

$$P(\widehat{F}, X) \leq (1 + K)P(F, E).$$

Indeed, via approximation of  $BV$  functions by Lipschitz functions and by the coarea formula, it suffices to prove the above claim for relatively closed (in  $E$ ) and bounded subsets  $F$  of  $E$ . We will show that we can choose  $\widehat{F} = F$ .

To see this, note that

$$P(F, X) = P(F, E) + P(F, \partial E),$$

and so we need to show that

$$P(F, \partial E) \leq K P(F, E).$$

Fix  $\varepsilon > 0$ . Let  $\Omega$  be a relatively compact open subset of  $X$  such that  $F \Subset \Omega$ . We can take  $\Omega$  sufficiently small so that  $P(E, \Omega \setminus \overline{F}) < \varepsilon$ .

Then by the local quasiminimality property of  $E$ , we see that

$$P(E, \Omega) \leq KP(E \setminus F, \Omega). \quad (5.6)$$

Since

$$P(E, \Omega) = P(E, \Omega \setminus \overline{F}) + P(E, \overline{F}) = P(E, \Omega \setminus \overline{F}) + P(F, \partial E)$$

and

$$P(E \setminus F, \Omega) = P(E, \Omega \setminus \overline{F}) + P(E \setminus F, \overline{F}) = P(E, \Omega \setminus \overline{F}) + P(F, E),$$

the estimate (5.6) implies that

$$P(F, \partial E) \leq KP(F, E) + (K - 1)\varepsilon.$$

By letting  $\varepsilon \rightarrow 0$ , we obtain

$$P(F, \partial E) \leq KP(F, E). \quad \square$$

Recall that a domain  $E$  is a local John domain if there exist constants  $C, \delta > 0$  such that whenever  $x_0 \in \partial E$  and  $0 < r < \delta$ , for all points  $x \in B(x_0, r) \cap E$  there is a point  $y \in E \cap B(x_0, Cr)$  with  $\delta_E(y) \geq r/C$  and a curve  $\gamma \subset E$ , called a John curve, connecting  $x$  to  $y$  satisfying

$$\ell(\gamma_{x,z}) \leq C \delta_E(z)$$

for all  $z \in \gamma$ ; here  $\gamma_{x,z}$  denotes a subcurve of  $\gamma$  with end points  $x$  and  $z$ . We conclude this section with the following open question: if  $E$  is a domain of locally quasiminimal surface, then is it true that  $E$  is a local John domain? In the Euclidean setting this question was answered in the affirmative by David and Semmes [DS1]. The crucial

part of the proof of [DS1] is to show that the boundary of a set of quasiminimal surface lies locally close to a hyperplane; in the setting of metric measure spaces one does not have such a structure, and the challenge is to construct an alternative approach.

## 6. SUPPORT OF POINCARÉ INEQUALITY IN $(\mathbb{R}^n, d, \mu)$ .

Let  $\omega$  be a strong  $A_\infty$ -weight on  $\mathbb{R}^n$ . Then we have that with the measure  $\mu$  on  $\mathbb{R}^n$  defined by the density condition

$$d\mu(x) = \omega(x) d\mathcal{L}^n(x),$$

there is a metric  $d$  on  $\mathbb{R}^n$  and a constant  $C \geq 1$  such that whenever  $x, y \in \mathbb{R}^n$  and  $B_{x,y}$  is the smallest Euclidean ball in  $\mathbb{R}^n$  containing  $x$  and  $y$  (that is,  $B_{x,y} = B(\frac{x+y}{2}, \frac{|x-y|}{2})$ ),

$$\frac{1}{C}\mu(B_{x,y})^{1/n} \leq d(x, y) \leq C\mu(B_{x,y})^{1/n}.$$

Since strong  $A_\infty$ -weights are  $A_\infty$ -weights,  $\omega$  is a Muckenhoupt  $A_p$ -weight for some  $p$ . It follows that  $\mu$  is a doubling measure with respect to the Euclidean metric. Hence, we have a constant  $C \geq 1$  such that whenever  $x, y \in \mathbb{R}^n$ ,

$$\frac{1}{C}\mu(B(x, |x-y|)) \leq d(x, y)^n \leq C\mu(B(x, |x-y|)). \quad (6.1)$$

For the definition and properties of strong  $A_\infty$ -weights, we refer to [DS2].

The metric space we consider here is  $(\mathbb{R}^n, d, \mu)$ . Balls in this metric are denoted with the superscript  $d$  in order to distinguish them from the Euclidean balls. So

$$B^d(x, r) = \{y \in \mathbb{R}^n : d(x, y) < r\},$$

while

$$B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}.$$

We note that the topology generated by the metric  $d$  is the same one as the Euclidean topology.

In this section we show that when the measure  $\mu$  on  $\mathbb{R}^n$  is given by a strong  $A_\infty$ -weight, then the space  $(\mathbb{R}^n, d, \mu)$  is an Ahlfors regular space supporting a  $(1, 1)$ -Poincaré inequality. Note that not all strong  $A_\infty$ -weights are  $A_1$ -weights, and so in general the space  $(\mathbb{R}^n, |\cdot|, \mu)$  need not support a  $(1, 1)$ -Poincaré inequality. The next result states that  $(\mathbb{R}^n, d, \mu)$  is Ahlfors  $n$ -regular.

**Lemma 6.2.** *There is a constant  $C \geq 1$  such that whenever  $x \in \mathbb{R}^n$  and  $r > 0$ , we have*

$$\frac{1}{C}r^n \leq \mu(B^d(x, r)) \leq Cr^n.$$



*Proof.* Let  $y \in \partial B^d(x, r)$  such that

$$|y - x| = \sup\{|x - z| : z \in \partial B^d(x, r)\}.$$

Note that as  $\overline{B^d}(x, r)$  is compact, such  $y$  exists. Then  $B^d(x, r) \subset B(x, |x - y|)$ , and so by (6.1), we have

$$\mu(B^d(x, r)) \leq \mu(B(x, |x - y|)) \leq Cd(x, y)^n = Cr^n.$$

Next, let  $z \in \partial B^d(x, r)$  such that

$$|x - z| = \inf\{|x - z| : z \in \partial B^d(x, r)\}.$$

Then  $B(x, |x - z|) \subset B^d(x, r)$ , and so again by (6.1) we have

$$\mu(B^d(x, r)) \geq \mu(B(x, |x - z|)) \geq \frac{1}{C}d(x, z)^n = \frac{1}{C}r^n,$$

completing the proof.  $\square$

**Lemma 6.3.** *There is a Borel set  $F \subset \mathbb{R}^n$  with  $|F| = 0$  such that whenever  $\gamma$  is a curve in  $\mathbb{R}^n$  which is rectifiable with respect to the Euclidean metric, it is rectifiable with respect to the metric  $d$  if  $\int_\gamma \infty \chi_F + \omega^{1/n} ds$  is finite. In this case we also have that the length of  $\gamma$  with respect to the metric  $d$ , denoted  $\ell_d(\gamma)$ , satisfies*

$$\ell_d(\gamma) \approx \int_\gamma \omega^{1/n} ds.$$

*Proof.* Fix  $x \in \mathbb{R}^n$ . Then for  $y \in \mathbb{R}^n$ , by (6.1) we have

$$\begin{aligned} \frac{d(x, y)}{|x - y|} &\approx \frac{\mu(B(x, |x - y|))^{1/n}}{|x - y|} \\ &\approx \frac{1}{|B(x, |x - y|)|^{1/n}} \left( \int_{B(x, |x - y|)} \omega(z) dz \right)^{1/n} \\ &\approx \left( \frac{1}{|B(x, |x - y|)|} \int_{B(x, |x - y|)} \omega(z) dz \right)^{1/n}. \end{aligned}$$

Denote

$$\underline{\rho}(x) = \liminf_{y \rightarrow x} \frac{d(x, y)}{|x - y|} \quad \text{and} \quad \bar{\rho}(x) = \limsup_{y \rightarrow x} \frac{d(x, y)}{|x - y|}.$$

Since  $\omega \in L^1_{\text{loc}}(\mathbb{R}^n)$  (the integrals being taken with respect to the Lebesgue measure), we see by Lebesgue differentiation theorem that for almost every  $x \in \mathbb{R}^n$ ,

$$\begin{aligned} \omega(x)^{1/n} &= \lim_{y \rightarrow x} \left( \frac{1}{|B(x, |x - y|)|} \int_{B(x, |x - y|)} \omega(z) dz \right)^{1/n} \\ &\leq C \underline{\rho}(x) \leq C \bar{\rho}(x) \\ &\leq C^2 \lim_{y \rightarrow x} \left( \frac{1}{|B(x, |x - y|)|} \int_{B(x, |x - y|)} \omega(z) dz \right)^{1/n} = C^2 \omega(x)^{1/n}. \end{aligned}$$

Let  $F$  be the set of all non-Lebesgue points of  $\omega$ ; then  $\mu(F) = |F| = 0$ . Let  $\gamma$  be an Euclidean rectifiable curve with  $\int_{\gamma} \infty \chi_F + \omega^{1/n} ds < \infty$ . Then  $\mathcal{H}^1(\gamma^{-1}(\gamma \cap F)) = 0$ , and in addition we have

$$\int_{\gamma} \underline{\rho} ds \leq \ell_d(\gamma) \leq \int_{\gamma} \bar{\rho} ds,$$

where  $\ell_d(\gamma)$  is the length of  $\gamma$  in the metric  $d$ . It follows that

$$\int_{\gamma} \omega^{1/n} ds \leq C \ell_d(\gamma) \leq C^2 \int_{\gamma} \omega^{1/n} ds. \quad \square$$

**Lemma 6.4.** *If  $u$  is Lipschitz continuous with respect to the metric  $d$ , then  $u \in W_{\text{loc}}^{1,n}(\mathbb{R}^n)$ .*

*Proof.* Let  $h \in \mathbb{R}$ , and let  $e_j$ ,  $j = 1, \dots, n$  denote the standard orthonormal basis for  $\mathbb{R}^n$ . Since  $u$  is Lipschitz with respect to the metric  $d$ , we see by (6.1) that

$$|u(x + he_j) - u(x)| \leq Cd(x, x + he_j) \leq C\mu(B(x, |h|))^{1/n}.$$

Thus, for  $a \in \mathbb{R}^n$ ,  $R > 0$  and  $|h| \leq R$ , by Fubini's theorem we see that

$$\begin{aligned} \int_{B(a,R)} |u(x + he_j) - u(x)|^n dx &\leq C \int_{B(a,R)} \mu(B(x, |h|)) dx \\ &\leq C \int_{B(a,R)} \int_{B(a,2R)} \chi_{B(x,|h|)}(y) d\mu(y) dx \\ &= C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{B(a,R)}(x) \chi_{B(x,|h|)}(y) d\mu(y) dx \\ &= C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{B(a,R)}(x) \chi_{B(y,|h|)}(x) dx d\mu(y) \\ &\leq C|h|^n \int_{B(a,2R)} d\mu(y) = C|h|^n \mu(B(a, 2R)). \end{aligned}$$

So whenever  $\Omega \Subset \mathbb{R}^n$  is an open set and  $\Omega' \Subset \Omega$ , we can cover  $\Omega'$  by a countable collection  $\{B_i\}$  of balls of radius  $R = \text{dist}(\Omega', \mathbb{R}^n \setminus \Omega)/2$  centered at points in  $\Omega'$  and with bounded overlap of the balls  $\{2B_i\}$ , to obtain that

$$\begin{aligned} \int_{\Omega'} \frac{|u(x + he_j) - u(x)|^n}{|h|^n} dx &\leq \sum_i \int_{B_i} \frac{|u(x + he_j) - u(x)|^n}{|h|^n} dx \\ &\leq C \sum_i \mu(2B_i) \leq C\mu(\Omega) < \infty. \end{aligned}$$

So by [GT, Lemma 7.24], we see that  $u \in W_{\text{loc}}^{1,n}(\mathbb{R}^n)$ .  $\square$

By Lemma 6.4 we know that if  $u$  is Lipschitz continuous with respect to the metric  $d$ , then it has an Euclidean weak derivative  $\nabla u$ , and that  $|\nabla u|$  is a minimal 1-weak upper gradient (in the Euclidean metric) of  $u$ .

In order to prove that  $(\mathbb{R}^n, d, \mu)$  supports a  $(1, 1)$ -Poincaré inequality, it suffices to prove the inequality for Lipschitz functions with respect to  $d$  and their *continuous* upper gradients in the metric  $d$ ; see for example [Ke]. Let  $u$  be a Lipschitz function with continuous upper gradient  $g$  in  $(\mathbb{R}^n, d, \mu)$ .

**Lemma 6.5.** *We have a constant  $C > 0$ , that is independent of  $g$  and  $u$ , such that*

$$|\nabla u(x)| \leq C\omega(x)^{1/n}g(x)$$

for almost every  $x \in \mathbb{R}^n$ .

*Proof.* As in the proof of Lemma 6.4, we let  $e_j$ ,  $j = 1, \dots, n$ , denote the canonical orthonormal basis of  $\mathbb{R}^n$ . For each  $j = 1, \dots, n$ , we consider the collection  $\Gamma_j$  of all line segments parallel to the direction of  $e_j$ . From [V, Section 7.2, page 21], we know that whenever  $\Gamma \subset \Gamma_j$  satisfies  $\text{Mod}_1(\Gamma) = 0$ , we have

$$\left| \bigcup_{\gamma \in \Gamma} \bigcup_{z \in \gamma} z \right| = 0.$$

Let  $\Gamma_a$  denote the collection of all compact line segments in  $\Gamma_j$  for which  $\int_{\gamma} \infty \chi_F + \omega^{1/n} ds = \infty$  (with  $F$  as in Lemma 6.3). Since by Hölder's inequality we know that  $\omega^{1/n} \in L^1_{\text{loc}}(\mathbb{R}^n)$ , it follows that  $\text{Mod}_1(\Gamma_a) = 0$ ; see for example [KoMc].

Since  $u$  is Lipschitz continuous with respect to  $d$  and the topology induced by  $d$  and the Euclidean topology coincide, we see that  $u$  is continuous in the Euclidean space  $\mathbb{R}^n$ . By Lemma 6.4 we also know that  $u \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ . It follows from the discussion in [V] that  $u$  is absolutely continuous on  $\text{Mod}_1$ -almost every compact Euclidean rectifiable curve in  $\mathbb{R}^n$ . Let  $\Gamma_b$  denote the collection of all line segments  $\gamma$  in  $\Gamma_j$  along which  $(u, |\nabla u|)$  does not support the upper gradient inequality (2.1); that is, there is some sub-segment  $\beta$  of  $\gamma$  for which the inequality (2.1) fails. Since  $|\nabla u|$  is a 1-weak upper gradient of  $u$ , it follows that  $\text{Mod}_1(\Gamma_b) = 0$ . Furthermore, let  $\Gamma_c$  denote the collection of all segments  $\gamma \in \Gamma_j$  for which  $\int_{\gamma} |\nabla u| ds$  is infinite.

Because  $g$  is an upper gradient of  $u$  in the metric  $d$ , by Lemma 6.3 we know that whenever  $\gamma \in \Gamma_j \setminus \Gamma_a$ , for all sub-segments  $\beta$  of  $\gamma$  we have

$$|u(x_{\beta}) - u(y_{\beta})| \leq C \int_{\beta} \omega^{1/n} g ds.$$

Here  $x_{\beta}$  and  $y_{\beta}$  denote the two end points of  $\beta$ . It follows that if  $\gamma \notin \Gamma_b \cup \Gamma_c$  as well, then for  $\mathcal{H}^1$ -almost every point  $x \in \beta$ ,

$$|\partial_j u(x)| \leq C\omega^{1/n}(x)g(x).$$

Note that  $\text{Mod}_p(\Gamma_a \cup \Gamma_b \cup \Gamma_c) = 0$ . Hence by the use of [V] again, we see that for almost every  $x \in \mathbb{R}^n$  we have

$$|\partial_j u(x)| \leq C\omega^{1/n}(x)g(x).$$

Now the conclusion follows by summing up over  $j = 1, \dots, n$ .  $\square$

We next compare Euclidean balls with balls in the metric  $d$ .

**Lemma 6.6.** *There is a constant  $C > 0$  such that whenever  $x \in \mathbb{R}^n$  and  $r > 0$ , there exist positive numbers  $\lambda_x^r$  and  $\tau_x^r$  such that*

$$B(x, \lambda_x^r r) \subset B^d(x, r) \subset B(x, C\lambda_x^r r) \quad (6.7)$$

and

$$B^d(x, \tau_x^r r) \subset B(x, r) \subset B^d(x, C\tau_x^r r). \quad (6.8)$$

*Proof.* Since  $\mu$  is a doubling measure on the Euclidean space  $\mathbb{R}^n$  and  $\mathbb{R}^n$  is uniformly perfect, there is a constant  $Q_1 > 0$  such that whenever  $0 < r < R$ ,

$$\frac{\mu(B(x, r))}{\mu(B(x, R))} \leq C \left( \frac{r}{R} \right)^{Q_1}. \quad (6.9)$$

Let  $z_x^r, y_x^r \in \partial B^d(x, r)$  such that for  $y \in \partial B^d(x, r)$  we have  $|x - z_x^r| \leq |x - y| \leq |x - y_x^r|$ . Then set

$$\lambda_x^r = \frac{|x - z_x^r|}{r}.$$

We have

$$B(x, |x - z_x^r|) = B(x, \lambda_x^r r) \subset B^d(x, r) \subset B(x, |x - y_x^r|).$$

Because of the upper mass bound (6.9), by the twice-repeated use of (6.1),

$$\frac{1}{C} \leq \frac{\mu(B(x, |x - z_x^r|))}{\mu(B(x, |x - y_x^r|))} \leq C \left( \frac{|x - z_x^r|}{|x - y_x^r|} \right)^{Q_1},$$

and so it follows that  $|x - y_x^r| \leq C|x - z_x^r|$ , whence we obtain that  $B(x, |x - y_x^r|) \subset B(x, C|x - z_x^r|)$ , and this proves (6.7).

To prove (6.8), we consider  $w_x^r \in \partial B(x, r)$  such that  $d(x, w_x^r) \leq d(x, y)$  whenever  $y \in \partial B(x, r)$ , and set

$$\tau_x^r = \frac{d(x, w_x^r)}{r}.$$

As in the previous argument, we consider also  $a_x^r \in \partial B(x, r)$  such that  $d(x, a_x^r) \geq d(x, y)$  for all  $y \in \partial B(x, r)$ , and obtain by the use of Lemma 6.2 that

$$\frac{\mu(B^d(x, d(x, w_x^r)))}{\mu(B^d(x, d(x, a_x^r)))} \approx \left( \frac{d(x, w_x^r)}{d(x, a_x^r)} \right)^n,$$

and from (6.1) we also see that

$$\mu(B^d(x, d(x, w_x^r))) \approx d(x, w_x^r)^n \approx \mu(B(x, |x - w_x^r|)) = \mu(B(x, r)).$$

A similar argument as above also shows that

$$\mu(B^d(x, d(x, a_x^r))) \approx \mu(B(x, r)).$$

It follows that

$$\left(\frac{d(x, w_x^r)}{d(x, a_x^r)}\right)^n \geq \frac{1}{C},$$

that is,  $d(x, a_x^r) \leq C d(x, w_x^r)$ . From this (6.8) follows.  $\square$

Now we are ready to prove the main result of this section.

**Proposition 6.10.** *The metric measure space  $(\mathbb{R}^n, d, \mu)$  is an Ahlfors  $n$ -regular space supporting a  $(1, 1)$ -Poincaré inequality.*

*Proof.* As pointed out by [Ke], it suffices to prove the inequality for functions  $u$  that are Lipschitz continuous on  $(\mathbb{R}^n, d)$  with continuous upper gradient  $g$ . By [DS2, Inequality (1.10)], we know that when  $x \in \mathbb{R}^n$  and  $r > 0$ ,

$$\begin{aligned} & \int_{B(x, C\lambda_x^r r)} \int_{B(x, C\lambda_x^r r)} |u(x) - u(y)| d\mu(x) d\mu(y) \\ & \leq C\mu(B(x, C\lambda_x^r r))^{1/n} \int_{B(x, 2C\lambda_x^r r)} \omega(x)^{-1/n} |\nabla u(x)| d\mu(x). \end{aligned}$$

By the doubling property of  $\mu$  on the Euclidean  $\mathbb{R}^n$ , Lemma 6.5, and Lemma 6.2, we see that

$$\begin{aligned} & \int_{B(x, C\lambda_x^r r)} \int_{B(x, C\lambda_x^r r)} |u(x) - u(y)| d\mu(x) d\mu(y) \\ & \leq C\mu(B(x, C\lambda_x^r r))^{1/n} \int_{B(x, 2C\lambda_x^r r)} g(x) d\mu(x) \\ & \leq C\mu(B(x, \lambda_x^r r))^{1/n} \int_{B(x, 2C\lambda_x^r r)} g(x) d\mu(x) \\ & \leq C\mu(B^d(x, r))^{1/n} \int_{B(x, 2C\lambda_x^r r)} g(x) d\mu(x) \\ & \leq Cr \int_{B(x, 2C\lambda_x^r r)} g(x) d\mu(x). \end{aligned}$$

Hence,

$$\begin{aligned}
\inf_{c \in \mathbb{R}} \int_{B^d(x,r)} |u - c| d\mu &\leq \int_{B^d(x,r)} |u - u_{B(x, C\lambda_x^r r)}| d\mu \\
&\leq \int_{B(x, C\lambda_x^r r)} |u - u_{B(x, C\lambda_x^r r)}| d\mu \\
&\leq C \int_{B(x, C\lambda_x^r r)} |u - u_{B(x, C\lambda_x^r r)}| d\mu \\
&\leq C \int_{B(x, C\lambda_x^r r)} \int_{B(x, C\lambda_x^r r)} |u(x) - u(y)| d\mu(x) d\mu(y) \\
&\leq Cr \int_{B(x, 2C\lambda_x^r r)} g(x) d\mu(x).
\end{aligned}$$

By Lemma 6.6 we have

$$B(x, 2C\lambda_x^r r) \subset B^d(x, 2C^2\tau_x^{2C\lambda_x^r r} \lambda_x^r r).$$

Note that  $B^d(x, \tau_x^{2C\lambda_x^r r} 2C\lambda_x^r r)$  is the largest metric ball centered at  $x$  that fits inside the Euclidean ball  $B(x, 2C\lambda_x^r r)$ . Let  $\rho > 0$  such that  $B^d(x, \rho)$  is the largest metric ball centered at  $x$  and contained in the Euclidean ball  $B(x, \lambda_x^r r)$ , and let  $y_1 \in \partial B^d(x, \rho) \cap \partial B(x, \lambda_x^r r)$ , and correspondingly let  $y_2 \in \partial B^d(x, \tau_x^{2C\lambda_x^r r} 2C\lambda_x^r r) \cap \partial B(x, 2C\lambda_x^r r)$ . Then by (6.1),

$$\rho = d(x, y_1) \approx \mu(B(x, \lambda_x^r r))^{1/n}$$

and again by (6.1),

$$\tau_x^{2C\lambda_x^r r} 2C\lambda_x^r r = d(x, y_2) \approx \mu(B(x, 2C\lambda_x^r r))^{1/n}.$$

By the doubling property of  $\mu$  in the Euclidean space  $\mathbb{R}^n$ , we see that

$$\mu(B(x, \lambda_x^r r)) \approx \mu(B(x, 2C\lambda_x^r r)).$$

It follows that  $\rho \approx \tau_x^{2C\lambda_x^r r} 2C\lambda_x^r r$ . On the other hand, since  $B^d(x, \rho)$  is the largest metric ball centered at  $x$  and fitting inside the Euclidean ball  $B(x, \lambda_x^r r)$ , and by the construction of  $\lambda_x^r$  from Lemma 6.6 we know that  $B(x, \lambda_x^r r) \subset B^d(x, r)$ , we can conclude that  $\rho \leq r$ . Hence

$$\tau_x^{2C\lambda_x^r r} 2C\lambda_x^r r \leq C\rho \leq C_2r.$$

Thus we have

$$B(x, 2C\lambda_x^r r) \subset B^d(x, 2C^2\tau_x^{2C\lambda_x^r r} \lambda_x^r r) \subset B^d(x, C_2r),$$

from which we conclude that

$$\begin{aligned}
\inf_{c \in \mathbb{R}} \int_{B^d(x,r)} |u - c| d\mu &\leq Cr \int_{B(x, 2C\lambda_x^r r)} g(x) d\mu(x) \\
&\leq Cr \int_{B^d(x, C_2r)} g d\mu,
\end{aligned}$$

which is equivalent to the  $(1, 1)$ -Poincaré inequality on  $(\mathbb{R}^n, d, \mu)$  because the constants  $C, C_2$  are independent of  $x, r, u, g$ .  $\square$

## 7. RECTIFIABILITY OF QUASIMINIMAL SURFACES IN EUCLIDEAN SPACES WITH STRONG $A_\infty$ WEIGHTS

In this section we apply the results from the earlier part of this paper (from the first three sections) to the setting of weighted Euclidean spaces where the weight  $\omega$  is a strong  $A_\infty$ -weight.

If  $\omega$  is an  $A_1$ -weight, then the space  $(\mathbb{R}^n, |\cdot|, \mu)$  is a doubling metric measure space supporting a  $(1, 1)$ -Poincaré inequality. However, not all strong  $A_\infty$ -weights are  $A_1$ -weights, but as shown in the previous section, the metric measure space  $(\mathbb{R}^n, d, \mu)$  is also an Ahlfors  $n$ -regular space supporting a  $(1, 1)$ -Poincaré inequality. In this section we will prove that a set  $E \subset \mathbb{R}^n$  that has a locally quasiminimal boundary surface in  $(\mathbb{R}^n, d, \mu)$  will have a rectifiable boundary. Here of course, the notion of rectifiability is in terms of the Euclidean metric. As in [Mat, page 204, Definition 15.3], we say that a set  $A \subset \mathbb{R}^n$  is  $m$ -rectifiable if there is a countable collection  $\{f_i\}$  of Euclidean Lipschitz maps  $f_i : \mathbb{R}^m \rightarrow \mathbb{R}^n$  such that  $\mathcal{H}_{Euc}^m(A \setminus \bigcup_i f_i(\mathbb{R}^m)) = 0$ . In this section, we consider the issue of whether the boundary  $\partial E$  of the set with quasiminimal boundary surface in  $(\mathbb{R}^n, d, \mu)$  is  $(n - 1)$ -rectifiable in the above sense (with respect to the Euclidean metric). We also recall that a set  $K \subset \mathbb{R}^n$  is purely  $m$ -unrectifiable if whenever  $A \subset \mathbb{R}^n$  is  $m$ -rectifiable, we have  $\mathcal{H}_{Euc}^m(K \cap A) = 0$ .

Let  $E \subset \Omega \subset \mathbb{R}^n$  be a set of finite perimeter with locally quasiminimal boundary surface with respect to the metric  $d$  and measure  $\mu$ . Then the results obtained in the previous sections of this note apply to  $E$ . So we may assume that  $E = \text{int}(\overline{E})$ .

**Lemma 7.1.** *Let  $\Lambda > 0$  and  $A_\Lambda \subset \partial E$  be such that for all  $x \in A_\Lambda$  we have*

$$\limsup_{r \rightarrow 0^+} \frac{\mu(B(x, r))}{r^n} \geq \Lambda.$$

*Then*

$$\mathcal{H}_{Euc}^{n-1}(A_\Lambda) \leq \frac{C}{\Lambda^{(n-1)/n}} P(E, \Omega).$$

Note that in the above limes supremum condition, if we replace  $B(x, r)$  with  $B^d(x, r)$ , then by Lemma 6.2 we have  $A_\Lambda$  to be empty whenever  $\Lambda > C$ .

*Proof.* Fix  $\delta > 0$ ; then by the condition imposed on  $A_\Lambda$ , for every  $x \in A_\Lambda$  we can find  $0 < r_x < \delta/5$  such that

$$\mu(B(x, r_x)) \geq (\Lambda - \delta)r_x^n.$$

Note also that if  $x \in A_\Lambda$  and  $r > 0$  such that

$$\mu(B(x, r)) \geq (\Lambda - \delta)r^n,$$

then for any  $y \in \partial B(x, r)$  we have that

$$(\Lambda - \delta)r^n \leq \mu(B(x, r)) = \mu(B(x, |x - y|)) \leq Cd(x, y)^n,$$

and so we have  $d(x, y) \geq C^{-1/n}(\Lambda - \delta)^{1/n}r$ . It follows that

$$B^d(x, \frac{(\Lambda - \delta)^{1/n}}{C^{1/n}}r) \subset B(x, r).$$

The family  $B^d(x, \frac{(\Lambda - \delta)^{1/n}}{5C^{1/n}}r_x)$ ,  $x \in A_\Lambda$ , forms a cover of  $A_\Lambda$ , and hence we can find a pairwise disjoint countable subfamily

$$\{B^d(x_i, \frac{(\Lambda - \delta)^{1/n}}{5C^{1/n}}r_i)\}_i$$

such that

$$A_\Lambda \subset \bigcup_i B^d(x_i, \frac{(\Lambda - \delta)^{1/n}}{C^{1/n}}r_i).$$

Hence by Lemma 6.2 and the fact that the balls  $\{B(x_i, r_i)\}_i$  therefore also form a cover of  $A_\Lambda$  by Euclidean balls,

$$\begin{aligned} \mathcal{H}_{Euc, \delta}^{n-1}(A_\Lambda) &\leq \sum_i (r_i)^{n-1} = \sum_i \frac{r_i^n}{r_i} \\ &\leq \frac{C}{\Lambda - \delta} \sum_i \frac{\mu(B^d(x_i, \frac{(\Lambda - \delta)^{1/n}}{C^{1/n}}r_i))}{r_i} \\ &\leq \frac{C}{\Lambda - \delta} \sum_i \frac{\mu(B^d(x_i, \frac{(\Lambda - \delta)^{1/n}}{5C^{1/n}}r_i))}{r_i} \\ &\leq \frac{C}{(\Lambda - \delta)^{1 - \frac{1}{n}}} \sum_i \frac{\mu(B^d(x_i, \frac{(\Lambda - \delta)^{1/n}}{5C^{1/n}}r_i))}{\frac{(\Lambda - \delta)^{1/n}}{5C^{1/n}}r_i}. \end{aligned}$$

By Lemma 5.1, we now have

$$\mathcal{H}_{Euc, \delta}^{n-1}(A_\Lambda) \leq \frac{C}{(\Lambda - \delta)^{1 - \frac{1}{n}}} \sum_i P\left(E, B^d(x_i, \frac{(\Lambda + \delta)^{1/n}}{5C^{1/n}}r_i)\right).$$

Since the family  $\{B^d(x_i, \frac{(\Lambda + \delta)^{1/n}}{5C^{1/n}}r_i)\}_i$  is pairwise disjoint, we see that

$$\mathcal{H}_{Euc, \delta}^{n-1}(A_\Lambda) \leq \frac{C}{(\Lambda - \delta)^{1 - \frac{1}{n}}} P(E, \Omega).$$

Letting  $\delta \rightarrow 0$  completes the proof.  $\square$

**Lemma 7.2.** *For  $\mathcal{H}_{Euc}^{n-1}$ -almost every  $x \in \partial E$  we have*

$$\limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^n} < \infty.$$

*Proof.* By Lemma 7.1, we know that

$$\mathcal{H}_{Euc}^{n-1}(A_\Lambda) \leq C \Lambda^{-(n-1)/n} P(E, \Omega).$$



Since the set of all points  $x \in \partial E$  for which

$$\limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^n} = \infty$$

is the set  $\bigcap_{\Lambda > 0} A_\Lambda$ , we see that the claim of the lemma holds true.  $\square$

We set  $F_0$  to be the collection of all points  $x \in \partial E$  for which

$$\limsup_{r \rightarrow 0^+} \frac{\mu(B(x, r))}{r^n} = 0.$$

Let  $Z = \bigcap_{\Lambda > 0} A_\Lambda$ ; from the above discussion we know that  $\mathcal{H}_{Euc}^{n-1}(Z)$  is zero.

**Lemma 7.3.** *We have that  $F_\infty = \bigcup_{k \in \mathbb{N}} (A_{1/k} \setminus Z)$  is  $\sigma$ -finite with respect to the measure  $\mathcal{H}_{Euc}^{n-1}$ , and*

$$\partial E = Z \cup F_0 \cup F_\infty.$$

*Proof.* By Lemma 7.1, we know that  $\mathcal{H}_{Euc}^{n-1}(A_{1/k}) < \infty$ . Thus we see that  $F_\infty$  is  $\sigma$ -finite with respect to the measure  $\mathcal{H}_{Euc}^{n-1}$ .  $\square$

**Lemma 7.4.** *Either  $\mathcal{H}(F_0) = 0$  or  $\mathcal{H}_{Euc}^{n-1}(F_0) = \infty$ . Furthermore, with*

$$K_\epsilon = \left\{ x \in \partial E : \limsup_{r \rightarrow 0^+} \frac{\mu(B(x, r))}{r^n} < \epsilon \right\},$$

*we have*

$$\mathcal{H}(K_\epsilon) \leq C \epsilon^{(n-1)/n} \mathcal{H}_{Euc}^{n-1}(K_\epsilon).$$

Note that here  $\mathcal{H}$  is the codimension 1 Hausdorff measure with respect to the metric  $d$  and measure  $\mu$ , while  $\mathcal{H}_{Euc}^{n-1}$  is the  $(n-1)$ -dimensional Hausdorff measure with respect to the Lebesgue measure and Euclidean metric.

*Proof.* Suppose  $\mathcal{H}(F_0) > 0$ . We will show that then  $\mathcal{H}_{Euc}^{n-1}(F_0) = \infty$ . To this end, fix  $\epsilon > 0$ . For each  $x \in F_0$  there is a positive number  $\delta_x$  such that whenever  $0 < r < \delta_x$  we have  $\mu(B(x, r)) \leq 2\epsilon r^n$ . For each  $j \in \mathbb{N}$  let

$$F_j = \{x \in F_0 : \delta_x \geq 1/j\}.$$

Note that for large  $j$  we have  $F_j$  non-empty since  $F_0 = \bigcup_j F_j$  is non-empty. Furthermore, because  $\mathcal{H}(F_0) > 0$  we have that  $\mathcal{H}(F_j) > 0$  for sufficiently large  $j$ . Let  $0 < \delta < 1/j$ , and for each  $x \in F_j$ , whenever  $r < \delta$ , we have that  $\mu(B(x, r)) \leq 2\epsilon r^n$ . For  $y \in \partial B(x, r)$ , we see by (6.1) that

$$2\epsilon r^n \geq \mu(B(x, r)) = \mu(B(x, |x-y|)) \geq \frac{1}{C} d(x, y)^n.$$

It follows that  $d(x, y) \leq C\epsilon^{1/n} r$  whenever  $y \in \partial B(x, r)$ . Therefore  $B(x, r) \subset B^d(x, C\epsilon^{1/n} r)$ . Now we choose a countable cover of  $F_j$  by

balls  $B(x_i, r_i)$  with  $r_i < \delta$  and

$$\mathcal{H}_{Euc}^{n-1}(F_j) + \delta \geq \sum_i r_i^{n-1}.$$

But then by Lemma 6.2 we have

$$\begin{aligned} \mathcal{H}_{Euc}^{n-1}(F_j) + \delta &\geq \frac{1}{C\epsilon} \sum_i \frac{\mu(B^d(x_i, C\epsilon^{1/n}r_i))}{r_i} \\ &\geq \frac{1}{C\epsilon^{1-\frac{1}{n}}} \sum_i \frac{\mu(B^d(x_i, C\epsilon^{1/n}r_i))}{C\epsilon^{1/n}r_i} \\ &\geq \frac{1}{C\epsilon^{1-\frac{1}{n}}} \mathcal{H}_{C\epsilon^{1/n}\delta}(F_j). \end{aligned}$$

Now letting  $\delta \rightarrow 0$ , we can conclude that

$$\mathcal{H}_{Euc}^{n-1}(F_0) \geq \mathcal{H}_{Euc}^{n-1}(F_j) \geq \frac{1}{C\epsilon^{1-\frac{1}{n}}} \mathcal{H}(F_j).$$

Because  $E$  is of finite perimeter and  $E$  satisfies the density conditions discussed in the previous sections, we know that  $\mathcal{H}(F_0) < \infty$ . Also, if  $j_1 > j_2$  then  $F_{j_1} \supset F_{j_2}$ . Therefore we have  $\mathcal{H}(F_0) = \lim_j \mathcal{H}(F_j)$ . Therefore

$$\mathcal{H}_{Euc}^{n-1}(F_0) \geq \frac{1}{C\epsilon^{(n-1)/n}} \mathcal{H}(F_0),$$

and now the desired conclusion that  $\mathcal{H}_{Euc}^{n-1}(F_0) = \infty$  follows by taking  $\epsilon \rightarrow 0$ . The proof of the second claim of the lemma follows from an argument similar to the first part of the proof above.  $\square$

Recall that

$$F_0 := \left\{ x \in \partial E : \limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^n} = 0 \right\}.$$

**Theorem 7.5.** *With*

$$D_\infty := \left\{ x \in \partial E : \limsup_{r \rightarrow 0} \frac{\mathcal{H}_{Euc}^{n-1}(\partial E \cap B(x, r))}{r^{n-1}} = \infty \right\},$$

*the set  $\partial E \setminus (F_0 \cup D_\infty)$  is  $(n-1)$ -rectifiable.*

*Proof.* Suppose not. Then combining Lemma 7.1 with [Mat, page 205, Theorem 15.6], we know that there is a purely  $(n-1)$ -unrectifiable set  $K \subset \partial E \setminus (F_0 \cup D_\infty)$  with  $\mathcal{H}_{Euc}^{n-1}(K) > 0$ . Since  $K \cap D_\infty$  is empty, it follows that  $\mathcal{H}_{Euc}^{n-1}(K \cap B)$  is finite for balls  $B$  centered at points in  $K$  and with sufficiently small radii. Thus there is a density point  $x_0$  of  $K$  for the measure  $\mathcal{H}_{Euc}^{n-1}$ . By Lemma 7.2 we can assume without loss of generality that

$$Q := \limsup_{r \rightarrow 0} \frac{\mu(B(x_0, r))}{r^n} < \infty.$$

Since  $x_0 \notin F_0$ ,

$$\infty > Q = \limsup_{r \rightarrow 0} \frac{\mu(B(x_0, r))}{r^n} > 0.$$

Furthermore, because  $x_0 \notin D_\infty$ , we have

$$M = \limsup_{r \rightarrow 0} \frac{\mathcal{H}_{Euc}^{n-1}(\partial E \cap B(x_0, r))}{r^{n-1}} < \infty.$$

Let  $\epsilon$  be some small number to be determined later; by the choice of  $x_0$ , for sufficiently small  $r_0 > 0$ , for all  $0 < r < r_0$  by the definition of  $M$  above we have

$$\frac{\mathcal{H}_{Euc}^{n-1}((\partial E \setminus K) \cap B(x_0, r))}{\mathcal{H}_{Euc}^{n-1}(\partial E \cap B(x_0, r))} < \frac{\epsilon}{2M}. \quad (7.6)$$

Since  $x_0 \notin D_\infty$ , for sufficiently small  $r$  we also have

$$\mathcal{H}_{Euc}^{n-1}(\partial E \cap B(x_0, r)) \leq (M + \epsilon)r^{n-1}.$$

Therefore, by (7.6), for sufficiently small  $r > 0$ ,

$$\mathcal{H}_{Euc}^{n-1}((\partial E \setminus K) \cap B(x_0, r)) < \epsilon r^{n-1}. \quad (7.7)$$

We can find a small positive number  $r > 0$  that satisfies the above requirements and in addition satisfies

$$\mu(B(x_0, r)) \geq \frac{Q}{2}r^n.$$

By inequality (6.1),

$$B^d(x_0, C^{-1}\mu(B(x_0, r))^{\frac{1}{n}}) \subset B(x_0, r) \subset B^d(x_0, C\mu(B(x_0, r))^{\frac{1}{n}}).$$

So in particular, by the definition of  $Q$  and the choice of  $r$ , we have that  $B^d(x_0, cQ^{\frac{1}{n}}r) \subset B(x_0, r)$  for  $c = C^{-1}2^{-1/n}$ .

Now by Theorem 5.2, we can find  $y_0 \in E$  and  $y_1 \in E^c$  such that

$$B^d(y_0, c_1Q^{\frac{1}{n}}r) \subset E \cap B^d(x_0, cQ^{\frac{1}{n}}r) \subset B(x_0, r)$$

and

$$B^d(y_1, c_1Q^{\frac{1}{n}}r) \subset B^d(x_0, cQ^{\frac{1}{n}}r) \setminus E \subset B(x_0, r)$$

For  $i = 0, 1$  let

$$\gamma_i = \inf \left\{ h > 0 : B^d(y_i, c_1Q^{\frac{1}{n}}r) \subset B(y_i, h) \right\},$$

that is,  $B(y_i, \gamma_i)$  is the smallest Euclidean ball containing the metric ball  $B^d(y_i, c_1Q^{\frac{1}{n}}r)$ . Note that by Lemma 6.2,

$$\mu(B(y_i, \gamma_i)) \geq \mu(B^d(y_i, c_1Q^{\frac{1}{n}}r)) \geq cQr^n.$$

Because  $B^d(y_i, c_1Q^{\frac{1}{n}}r) \subset B(x_0, r)$ , it follows that  $B^d(y_i, c_1Q^{\frac{1}{n}}r) \subset B(y_i, 2r)$ , and so  $\gamma_i \leq 2r$ .

Since  $B(x_0, r) \subset B(y_i, 2r)$  we have by the doubling property of  $\mu$ , with respect to the Euclidean metric, that  $\mu(B(y_i, 2r)) \leq CQr^n$ .

As observed above,  $\gamma_i \leq 2r$ , and so (6.9) applies here to give

$$\frac{cQr^n}{CQr^n} \leq \frac{\mu(B(y_i, \tilde{\gamma}_i))}{\mu(B(y_i, 2r))} \leq C \left( \frac{\gamma_i}{2r} \right)^{Q_1}.$$

Thus  $\gamma_i \geq cr$  and so by Lemma 6.6 we have that

$$B(y_i, cr) \subset B^d(y_i, c_1 Q^{\frac{1}{n}} r).$$

As  $K$  is purely unrectifiable, by the Besicovich-Federer Projection theorem ([Mat, Theorem 18.1(2)]), for  $i = 0, 1$  there must exist points  $\tilde{y}_i \in B(y_i, cr/4)$  such that for  $v = \frac{\tilde{y}_1 - \tilde{y}_2}{|\tilde{y}_1 - \tilde{y}_2|}$  we have

$$\mathcal{H}_{Euc}^{n-1}(P_{v^\perp}(K)) = 0.$$

Here  $P_{v^\perp}$  is the projection to the  $(n-1)$ -dimensional hyperplane orthogonal to the vector  $v$ .

Let  $\xi_1 = v$ ; then we can find unit vectors  $\xi_2, \xi_3, \dots, \xi_n$  such that  $\{\xi_1, \xi_2, \dots, \xi_n\}$  forms an orthonormal basis for the vector space  $\mathbb{R}^n$ . For any  $z \in \mathbb{R}^n$  and  $\beta > 0$  let  $Q_\beta(z)$  denote the cube whose faces are normal to the vectors  $\{\xi_1, \xi_2, \dots, \xi_n\}$ , with Euclidean side length  $\beta$ , and center located at  $z$ . Note

$$Q_{\frac{cr}{8n^{1/2}}}(\tilde{y}_0) \subset E \quad \text{and} \quad Q_{\frac{cr}{8n^{1/2}}}(\tilde{y}_1) \subset \mathbb{R}^n \setminus E.$$

Consider the following cross-section of the cube  $Q_{\frac{cr}{8n^{1/2}}}(\tilde{y}_0)$ ,

$$\Pi := Q_{\frac{cr}{8n^{1/2}}}(\tilde{y}_0) \cap (\langle v \rangle^\perp + \tilde{y}_0),$$

where  $\langle v \rangle$  is the one-dimensional vector subspace of  $\mathbb{R}^n$  spanned by the vector  $v$ , and  $\langle v \rangle^\perp$  is the  $(n-1)$ -dimensional hyperplane orthogonal to the vector  $v$ . For each  $z \in \Pi$  let

$$e_z = P_{v^\perp}^{-1}(\{P_{v^\perp}(z)\}) \cap Q_{\frac{cr}{8n^{1/2}}}(\tilde{y}_1) \cap (\langle v \rangle^\perp + \tilde{y}_1),$$

that is,  $e_z$  is the point in the region

$$Q_{\frac{cr}{8n^{1/2}}}(\tilde{y}_1) \cap (\langle v \rangle^\perp + \tilde{y}_1) = \Pi + \tilde{y}_1 - \tilde{y}_0$$

corresponding to  $z \in \Pi$  such that  $z - e_z = \tilde{y}_0 - \tilde{y}_1$ .

Let

$$\Pi' = \{z \in \Pi : P_{v^\perp}^{-1}(P_{v^\perp}(z)) \cap K = \emptyset\},$$

that is,  $\Pi'$  is the collection of all points  $z \in \Pi$  such that the line segment  $[z, e_z]$  connecting  $z$  to  $e_z$  does *not* intersect the purely unrectifiable set  $K$ . By the choice of  $\tilde{y}_i$ ,  $i = 0, 1$ , we know that  $\mathcal{H}_{Euc}^{n-1}(\Pi \setminus \Pi') = 0$ . However, for any  $z \in \Pi'$  we know that  $z \in E$  and  $e_z \in E^c$  so we must have the line segment  $[z, e_z]$  intersect  $\partial E \setminus K$ . So for each  $z \in \Pi'$  we can

pick a point  $b_z \in (\partial E \setminus K) \cap [z, e_z]$ . On the other hand, as orthogonal projections do not increase the measure  $\mathcal{H}_{Euc}^{n-1}$ ,

$$\begin{aligned} \mathcal{H}_{Euc}^{n-1}(\partial E \setminus K) &\geq \mathcal{H}_{Euc}^{n-1}\left(\bigcup_{z \in \Pi'} b_z\right) \geq \mathcal{H}_{Euc}^{n-1}\left(P_{v^\perp}\left(\bigcup_{z \in \Pi'} b_z\right)\right) \\ &= \mathcal{H}_{Euc}^{n-1}(\Pi') = \left(\frac{c r}{8\sqrt{n}}\right)^{n-1} = \left(\frac{c}{8\sqrt{n}}\right)^{n-1} r^{n-1}. \end{aligned}$$

This contradicts (7.7) when we choose  $0 < \epsilon < \left(\frac{c}{8\sqrt{n}}\right)^{n-1}$ .

□

**Corollary 7.8.** *Suppose that there is a positive number  $\alpha$  such that  $\omega(x) \geq \alpha$  for  $\mathcal{L}^n$ -almost every  $x$  in a neighborhood of  $\bar{E}$ . Then  $\partial E$  has  $\mathcal{H}_{Euc}^{n-1}$  finite measure and is Euclidean  $(n-1)$ -rectifiable.*

*Proof.* By Lemma 7.3, we have  $\partial E = Z \cup F_\infty \cup F_0$ , with  $\mathcal{H}_{Euc}^{n-1}(Z) = 0$  and  $\mathcal{H}_{Euc}^{n-1}$  being  $\sigma$ -finite on  $F_\infty$ . It follows from the assumption  $\omega \geq \alpha$  almost everywhere that for all  $x \in \partial E$  we have

$$\liminf_{r \rightarrow 0^+} \frac{\mu(B(x, r))}{r^n} \geq C_n \alpha > 0,$$

that is,  $F_0$  is empty.

Now we look at  $F_\infty = \bigcup_{k \in \mathbb{N}} A_{1/k} \setminus Z$ . Because of the assumption that  $\omega \geq \alpha$  almost everywhere, we know that  $F_\infty = A_{1/k_0} \setminus Z$  where  $k_0 \in \mathbb{N}$  large enough so that  $1/k_0 < C_n \alpha$ . So by Lemma 7.1 we have that

$$H^{n-1}(\partial E) = \mathcal{H}_{Euc}^{n-1}(F_\infty) < \infty.$$

Thus an application of [Mat, Theorem 6.2] gives  $\mathcal{H}_{Euc}^{n-1}(D_\infty) = 0$ , and so  $\partial E$  is rectifiable by Theorem 7.5. □

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