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QUASICONTINUITY OF NEWTON–SOBOLEV FUNCTIONS AND DENSITY OF LIPSCHITZ FUNCTIONS ON METRIC SPACES

ANDERS BJÖRN, JANA BJÖRN, AND NAGESWARI SHANMUGALINGAM

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Abstract. We show that on complete doubling metric measure spaces X supporting a Poincaré inequality, *all* Newton–Sobolev functions u are quasicontinuous, i.e. that for every $\varepsilon > 0$ there is an open set $U \subset X$ such that $C_p(U) < \varepsilon$ and the restriction of u to $X \setminus U$ is continuous. This implies that the capacity is an outer capacity.

1. Introduction

Over the last decade there has been a lot of development in the theory of Sobolev type spaces on metric measure spaces, see e.g. Hajłasz [7], Cheeger [5], Shanmugalingam [19] and the references therein. Newtonian (Newton–Sobolev) spaces $N^{1,p}(X)$ on metric measure spaces X (see Section 2) were introduced and studied in [19]. The results in [19] demonstrate that if the measure on X is doubling and X supports a p-Poincaré inequality, $1 \le p < \infty$, then every function in $N^{1,p}(X)$ has a representative which is quasicontinuous, i.e. continuous on complements of open sets with arbitrarily small capacity. The main result of this paper is that a simple condition guarantees that *all* Newtonian functions (all representatives) are quasicontinuous, i.e. we prove the following theorem.

Theorem 1.1. *Let X be proper, $\Omega \subset X$ open and assume that continuous functions are dense in $N^{1,p}(X)$. Then every $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is quasicontinuous in Ω.*

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Recall that X is *proper* if all closed bounded subsets of X are compact. Note that if X is complete, the measure on X is doubling, and X supports a p-Poincaré inequality, then the assumptions of Theorem 1.1 are satisfied; see Theorem 4.1 of [19]. Our results, however, apply to a wider class of spaces, see e.g. Example 5.2.

In Remark 4.4 of [19] it was claimed that the proof of Theorem 4.1 therein showed that every $u \in N^{1,p}(X)$ is quasicontinuous whenever X is doubling and supports a p-Poincaré inequality. Unfortunately, this claimed fact does not follow from the proof given in [19]; only the fact that every $u \in N^{1,p}(X)$ has a quasicontinuous representative follows (see the remark at the end of Section 4). Our proof of the claim in Remark 4.4 of [19] is considerably more complicated. We do not know if the fact claimed in Remark 4.4 in [19] remains true in the nonproper case, however we have the following result valid also in the nonproper case.

Proposition 1.2. *Let $\Omega \subset X$ be open and assume that continuous functions are dense in $N^{1,p}(X)$. Then every $u \in N^{1,p}(\Omega)$ has a representative which is quasicontinuous in Ω.*

The fact that functions in $N^{1,p}(X)$ are quasicontinuous has been quoted and used in several later papers, e.g. [2], [3], [11], [12], [13], [14], [15], [16] and [20], although it is quite possible that it has only been used in spirit rather than in any fundamental way, and that the use of a refined Sobolev space (similar to those defined on p. 96 of Heinonen–Kilpeläinen–Martio [9]) would suffice. However, it is appealing that using the Newtonian definition of Sobolev spaces one automatically obtains the refined Sobolev space, and does not have to weed out nonquasicontinuous functions later. This can also be useful, as seen for example in A. Björn [1] (which quotes the present paper). It is at least not trivial to see how the proof of Theorem 4.4 in [1] can be replaced by a proof based on standard Sobolev spaces in $\mathbf{R}^n$ (the quasicontinuity is used in the third paragraph of the proof). It thus seems that Theorem 1.1 of this note may be of use even in the context of $\mathbf{R}^n$.

As a corollary of Theorem 1.1 we obtain the following result, where C_p is the capacity associated with the space $N^{1,p}(X)$, see Definition 3.

Corollary 1.3. *Let X be proper and assume that continuous functions are dense in $N^{1,p}(X)$. Then C_p is an outer capacity, i.e. for all $E \subset X$,*

$$C_p(E) = \inf_{\substack{G \supset E \\ G \text{ open}}} C_p(G).$$

Observe that it is essential to know that C_p is an outer capacity when concluding that C_p is the usual Sobolev capacity in the case of $\mathbf{R}^n$.

The proof of Theorem 1.1 relies on the following outer regularity result for sets of zero capacity. Note that it applies to a wider class of spaces than Corollary 1.3 (which yields regularity for all sets).

Proposition 1.4. *Let X be proper and let $E \subset X$ with $C_p(E) = 0$. Then for every $\varepsilon > 0$, there is an open set $U \supset E$ with $C_p(U) < \varepsilon$.*

We point out here that Corollary 1.3 and Proposition 1.4 also hold for the relative capacity cap_p, as defined in J. Björn [4], Definition 2.10, with essentially identical proofs. This is true even in cases as in Example 5.1, when there are sets with zero cap_p-capacity, but nonzero C_p-capacity. Recall that if X is doubling and supports a p-Poincaré inequality, then C_p and cap_p have the same zero sets, see Lemma 3.3 of [4].

We end this paper in Section 5 by giving several results concerning density of Lipschitz functions.

2. Notation and preliminaries

Let $X = (X, d, \mu)$ be a metric space endowed with a metric d and a Borel measure μ satisfying $0 < \mu(B) < \infty$ for all nonempty balls B in X. We emphasize that the σ-algebra on which μ is defined is obtained by the completion of the Borel σ-algebra. We also assume that $1 \le p < \infty$.

A *curve* is a continuous mapping from an interval. We will only consider curves which are compact and rectifiable (i.e. of finite length). Such a curve can thus be parameterized by its arc length ds.

Definition 1. A nonnegative Borel function g on X is an *upper gradient* of a function $f : X \to [-\infty, \infty]$ if for all nonconstant curves $\gamma : [0, l_\gamma] \to X$,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \le \int_\gamma g\,ds \tag{2.1}$$

whenever both $f(\gamma(0))$ and $f(\gamma(l_\gamma))$ are finite, and $\int_\gamma g\,ds = \infty$ otherwise. If g is a nonnegative measurable function on X and if (2.1) holds for p-almost every curve, then g is a *p-weak upper gradient* of f.

By saying that (2.1) holds for p-almost every curve we mean that it fails only for a curve family with zero p-modulus, see Definition 2.1 in Shanmugalingam [19]. It is implicitly assumed that $\int_\gamma g\,ds$ is defined (with a value in $[0, \infty]$) for p-almost every curve.

Following Shanmugalingam [19], we define a version of Sobolev spaces on the metric space X.

Definition 2. Whenever $u \in L^p(X)$, let

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

The space $N^{1,p}(X)$ is a Banach space and a lattice, see Shanmugalingam [19]. Let us make it clear that when we write $u \in N^{1,p}(X)$ we mean that u is a representative of an equivalence class in $N^{1,p}(X)$ and thus defined everywhere.

If X is proper and $\Omega \subset X$ is open, we say that $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in N^{1,p}(\Omega')$ for all open $\Omega' \Subset \Omega$. (Recall that $E \Subset A$ if $\overline{E}$ is a compact subset of A.)

Definition 3. The *capacity* of a set $E \subset X$ is the number

$$C_p(E) = \inf \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E.

It is easily verified that the capacity is countably subadditive. The capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$ and v is a function on X, then $u \sim v$ if and only if $u = v$ outside a set of capacity zero. Moreover, Corollary 3.3 in Shanmugalingam [19] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ μ-a.e., then $u \sim v$.

The following two conditions are often assumed in the theory of Sobolev type spaces on metric measure spaces.

Definition 4. We say that μ is *doubling* if there exists a constant $C > 0$ such that for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$,

$$\mu(2B) \le C\mu(B),$$

where $\lambda B = B(x_0, \lambda r)$.

Definition 5. We say that X supports a *p-Poincaré inequality* if there exist constants $C > 0$ and $\lambda \ge 1$ such that for all balls $B \subset X$, all $u \in N^{1,p}(X)$ and all p-weak upper gradients g of u,

$$\fint_B |u - u_B| \, d\mu \le C \operatorname{diam}(B) \left(\fint_{\lambda B} g^p \, d\mu \right)^{1/p}, \tag{2.2}$$

where $u_B := \fint_B u \, d\mu := \mu(B)^{-1} \int_B u \, d\mu$.

The following theorem is proved in Shanmugalingam [19], Theorem 4.1. Note however, that the Poincaré inequality is not necessary for the conclusion of the theorem, see Examples 5.1 and 5.2. Thus the results of this paper apply to a wider class of spaces than those satisfying the doubling condition and the Poincaré inequality.

Theorem 2.1. *If μ is doubling and X supports a p-Poincaré inequality, then Lipschitz functions are dense in $N^{1,p}(X)$.*

Note that if μ is doubling, then X is proper if and only if it is complete.

Definition 6. A function $u : X \to [-\infty, \infty]$ is *quasicontinuous* if for every $\varepsilon > 0$ there is an open set $U \subset X$ such that $C_p(U) < \varepsilon$ and $u|_{X \setminus U}$ is continuous.

3. Functions with prescribed upper gradients

In this section we construct functions with prescribed upper gradients and show that under certain assumptions they are lower semicontinuous.

Definition 7. We say that a curve $\gamma : [a, b] \to X$ *connects* two points x and y if $\gamma(a) = x$ and $\gamma(b) = y$. Similarly, γ connects x and a set A if $\gamma(a) = x$ and $\gamma(b) \in A$.

Lemma 3.1. *Let $\rho : X \to [0, \infty]$ be a Borel function and $A \subset X$. Let*

$$u(x) = \inf_{\gamma} \int_{\gamma} \rho \, ds \quad \textit{and} \quad u_1(x) = \min\{1, u(x)\},$$

where the infimum is taken over all rectifiable (including constant) curves connecting x to A. If there are no such curves, we let the infimum be infinity. Then $u = u_1 = 0$ on A and ρ is an upper gradient of both u and u_1.

Proof. That $u = u_1 = 0$ on A is clear (use the constant curves). Let $x, y \in X$ be arbitrary and let γ_{xy} be a curve connecting x and y.

We may assume that $0 \le u(y) < u(x)$. If γ is a curve connecting y to A, then the curve $\gamma_{xy} + \gamma$ connects x to A and hence

$$u(x) \le \int_{\gamma_{xy}+\gamma} \rho \, ds = \int_{\gamma_{xy}} \rho \, ds + \int_{\gamma} \rho \, ds.$$

Taking infimum over all curves γ connecting y to A shows that

$$u(x) \le \int_{\gamma_{xy}} \rho \, ds + u(y),$$

i.e. ρ is an upper gradient of u. As for u_1, it is immediate that

$$|u_1(x) - u_1(y)| \le |u(x) - u(y)|,$$

from which the result follows. $\square$

The following truncation property is a strengthening of Lemma 4.3 in Shanmugalingam [19].

Lemma 3.2. *Let F be a closed subset of X such that $u : X \to [-\infty, \infty]$ is constant on F. If $\rho : X \to [0, \infty]$ is an upper gradient of u on X, then $\rho\chi_{X\setminus F}$ is also an upper gradient of u in X.*

Proof. Let $\gamma : [a, b] \to X$ be a curve connecting x and y. The cases $\gamma \subset F$ and $\gamma \subset X \setminus F$ are trivial. If γ intersects both F and $X \setminus F$, then by splitting γ into two parts if necessary, we can assume that $x \in X \setminus F$ and $y \in F$. Let $c = \inf\{t \in [a, b] : \gamma(t) \in F\}$. As F is closed, we have $\gamma([a, c)) \subset X \setminus F$ with $z = \gamma(c) \in F$, and hence

$$|u(x) - u(y)| = |u(x) - u(z)| \le \int_{\gamma|_{[a,c]}} \rho\, ds \le \int_\gamma \rho\chi_{X\setminus F}\, ds.$$

$\square$

The lemma below shows that under certain assumptions, the functions constructed in Lemma 3.1 are lower semicontinuous. This will be crucial in the proof of Proposition 1.4.

Lemma 3.3. *Let F be a closed subset of a proper space X, such that at least one of F and $X \setminus F$ is bounded. Let $\rho : X \to [0, \infty]$ be lower semicontinuous and such that $\rho \ge c > 0$ in $X \setminus F$. Let*

$$u(x) = \inf_\gamma \int_\gamma \rho\, ds,$$

where the infimum is taken over all rectifiable (including constant) curves connecting x to F. Then u is lower semicontinuous.

Proof. Clearly, $u = 0$ on F. Let $a > 0$ be arbitrary and let

$$F_a = \{x \in X : u(x) \le a\}.$$

We shall show that F_a is closed. Let $x_j \in F_a$ and $x_j \to x \in X$. If $x \in F \subset F_a$, then there is nothing to prove. We can thus assume that $x_j \notin F$ for all j.

Then there exist curves $\gamma_j : [0, \ell_{\gamma_j}] \to X$, parameterized by arc length, such that $\gamma_j(0) = x_j$, $\gamma_j(\ell_{\gamma_j}) \in F$, $\gamma_j|_{[0,\ell_{\gamma_j})} \subset X \setminus F$ and

$$\int_{\gamma_j} \rho\, ds \le a + \frac{1}{j}, \quad j = 1, 2 \dots .$$

(We use the argument in the proof of Lemma 3.2 to justify our assumption that $\gamma_j|_{[0,\ell_{\gamma_j})} \subset X \setminus F$.) Reparameterize γ_j by $\tilde\gamma_j(t) = \gamma_j(t\ell_{\gamma_j})$, so that $\tilde\gamma_j : [0,1] \to X$, $\tilde\gamma_j(0) = x_j$ and $\tilde\gamma_j(1) \in F$. Note that each $\tilde\gamma_j$ is ℓ_{γ_j}-Lipschitz and

$$\ell_{\gamma_j} \le \int_{\gamma_j} \frac{\rho}{c}\, ds \le \frac{a+1}{c}.$$

This, together with the assumptions on F, also shows that all $\tilde\gamma_j$ are equicontinuous and contained in a compact set. Thus, by the Ascoli theorem (Theorem 10.28 in Heinonen [8]), a subsequence of $\{\tilde\gamma_j\}_{j=1}^\infty$ converges uniformly to a curve $\tilde\gamma : [0,1] \to X$ with $\tilde\gamma(0) = x$ and $\tilde\gamma(1) \in F$. Since ρ is lower semicontinuous, Lemma 2.3.2 in Ohtsuka [18] (which although written only for $\mathbf{R}^n$ carries over verbatim to proper metric spaces), or page 14 of Heinonen–Koskela [10], shows that

$$\int_{\tilde\gamma} \rho\, ds \le \liminf_{j\to\infty} \int_{\gamma_j} \rho\, ds \le a,$$

i.e. $u(x) \le a$ and $x \in F_a$, which completes the proof. □

In the next example we show that the properness in Lemma 3.3 is essential.

Example 3.4. Let $F = [-1, 1] \times [-2, -1]$,

$$A = \bigcup_{j=0}^{\infty} \bigl[2^{-2j-1}, 2^{-2j}\bigr]$$

and

$$X = F \cup (A \times [-1, 0]) \cup \{(x_1, x_2) \in \mathbf{R}^2 : -1 \le x_1 \le x_2 \le 0\}.$$

We let X inherit the Euclidean metric and Lebesgue measure from $\mathbf{R}^2$. Since $(0, x) \notin X$ when $-1 < x < 0$, we see directly that X is neither complete nor proper.

Let $\rho \equiv 1$ and define u as in Lemma 3.3. It is easy to see that $u(0,0) = \sqrt{2}$ and that $u(x, 0) = 1$ when $x \in A$, and hence u is not lower semicontinuous.

4. Proofs of the main results

We first prove Proposition 1.4, and then use it to prove the other main results of this paper.

Proof of Proposition 1.4. Assume first that E is bounded. As $C_p(E) = 0$, we have $\chi_E \in N^{1,p}(X)$ and there exists an upper gradient $g \in L^p(X)$ of χ_E. The Vitali–Carathéodory theorem (Proposition 7.14 in Folland [6] applied to g^p) provides us with a lower semicontinuous function $\rho \in L^p(X)$ such that $\rho \geq g$.

Let $\varepsilon > 0$ be arbitrary. By the definition of capacity, $\mu(E) = 0$, and as $\rho + 1 \in L^p_{\mathrm{loc}}(X)$, there exists a bounded open set $V \supset E$ such that

$$\mu(V) + \int_V (\rho + 1)^p \, d\mu < \varepsilon.$$

Let

$$u(x) = \min\biggl\{1, \inf_\gamma \int_\gamma (\rho + 1) \, ds\biggr\},$$

where the infimum is taken over all rectifiable (including constant) curves connecting x to the closed set $X \setminus V$. Note that $u = 0$ on $X \setminus V$. By Lemma 3.3, u is lower semicontinuous and thus the set $U = \bigl\{x \in X : u > \frac{1}{2}\bigr\}$ is open. Note that for $x \in E$ and every curve γ connecting x to some $y \in X \setminus V$,

$$\int_\gamma (\rho + 1) \, ds \geq \int_\gamma \rho \, ds \geq \chi_E(x) - \chi_E(y) = 1,$$

i.e. $u = 1$ on E and $E \subset U$. Lemmas 3.1 and 3.2 show that the function $(\rho+1)\chi_V$ is an upper gradient of u. Since moreover $u \leq 1$ and u is lower semicontinuous and hence measurable, we have $u \in N^{1,p}(X)$ and hence

$$C_p(U) \leq 2^p \|u\|^p_{N^{1,p}(X)} \leq 2^p \biggl(\mu(V) + \int_V (\rho + 1)^p \, d\mu\biggr) < 2^p \varepsilon,$$

which proves the claim for bounded E. If E is unbounded, write E as a countable union of bounded sets E_j, $j = 1, 2 \ldots$, and using the above argument, find open sets $U_j \supset E_j$ such that $C_p(U_j) < 2^{-j}\varepsilon$. The open set $U = \bigcup_{j=1}^\infty U_j$ contains E and $C_p(U) \leq \sum_{j=1}^\infty C_p(U_j) < \varepsilon$. □

Proof of Proposition 1.2. Let $u \in N^{1,p}(\Omega)$ and $\Omega_j = \{x : \operatorname{dist}(x, X \setminus \Omega) > 1/j\}$ (if $\Omega = X$, we let $\Omega_j = X$), $j = 1, 2, \ldots$. We have thus obtained open sets such that

$$\Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2 \subset \overline{\Omega}_2 \subset \Omega_3 \subset \cdots \subset \Omega = \bigcup_{j=1}^\infty \Omega_j.$$

Choose $\eta_j \in \operatorname{Lip}(X)$ such that $\eta_j = 1$ on Ω_j and $\operatorname{supp}\eta_j \subset \Omega$. Then $u\eta_j \in N^{1,p}(X)$ and there exists a sequence of continuous functions converging to $u\eta_j$ in $N^{1,p}(X)$. The proof of Theorem 3.7 in Shanmugalingam [19] then shows that a subsequence converges to a function $\tilde{u}_j \in N^{1,p}(X)$, uniformly outside of open sets with arbitrarily small capacity and that $\tilde{u}_j = u\eta_j$ q.e. Let

$$\tilde{u} = \begin{cases} \lim_{j\to\infty} \tilde{u}_j, & \text{wherever the limit exists,} \\ 0, & \text{elsewhere.} \end{cases}$$

Then $\tilde{u}$ is quasicontinuous in Ω_j for each $j = 1, 2 \ldots$, and thus in Ω. Moreover $\tilde{u} = u$ q.e. in Ω. □

Proof of Theorem 1.1. Let $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$. As X is proper we can find open sets

$$\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset \Omega = \bigcup_{j=1}^{\infty} \Omega_j.$$

By Proposition 1.2 there is a quasicontinuous representative $\tilde{u}_j$ of u in Ω_j. By Proposition 1.4, there exist open sets with arbitrarily small capacity containing the set where $\tilde{u}_j \neq u$. Combined with the quasicontinuity of $\tilde{u}_j$ this shows that u is quasicontinuous in Ω_j for each $j = 1, 2 \ldots$, and thus in Ω. □

Proof of Corollary 1.3. Let $0 < \varepsilon < 1$ and $u \geq \chi_E$ with

$$\|u\|_{N^{1,p}(X)} \leq C_p(E)^{1/p} + \varepsilon.$$

By Theorem 1.1, u is quasicontinuous and hence there is an open set V with $C_p(V)^{1/p} < \varepsilon$ such that $u|_{X\setminus V}$ is continuous. Thus, there is an open set U such that

$$U \setminus V = \{x : u(x) > 1 - \varepsilon\} \setminus V \supset E \setminus V.$$

We can also find $v \geq \chi_V$ with $\|v\|_{N^{1,p}(X)} < \varepsilon$. Let

$$w = \frac{u}{1-\varepsilon} + v.$$

Then $w \geq 1$ on $(U \setminus V) \cup V = U \cup V$ an open set containing E. Hence

$$\begin{aligned} C_p(E)^{1/p} &\leq \inf_{\substack{G \supset E \\ G \text{ open}}} C_p(G)^{1/p} \leq C_p(U \cup V)^{1/p} \leq \|w\|_{N^{1,p}(X)} \\ &\leq \frac{1}{1-\varepsilon}\|u\|_{N^{1,p}(X)} + \|v\|_{N^{1,p}(X)} \leq \frac{1}{1-\varepsilon}(C_p(E)^{1/p} + \varepsilon) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ completes the proof. □

Remark. Here we point out where the problem with Remark 4.4 in Shanmugalingam [19] is, since the difficulty is subtle. Let E be a set with $C_p(E) = 0$ and let $u = \chi_E$. In the proof of Theorem 4.1 in [19] Lipschitz continuous approximations of u are constructed, and in this particular case these approximations are all identically 0. They are thus equal to u outside a set of capacity zero. However without Proposition 1.4, it is not possible to conclude that $u = 0$ outside of *open* sets of arbitrarily small capacity, a fact that is needed in order to obtain the quasicontinuity of u. The quasicontinuity has earlier been used to deduce that the capacity is an outer capacity, as in our proof of Corollary 1.3. Here we have had to reverse the order of argument by first obtaining Proposition 1.4.

5. Examples and density of Lipschitz functions

Theorem 4.1 in Shanmugalingam [19] shows that Lipschitz functions in $N^{1,p}(X)$ are dense whenever the measure is doubling and supports a p-Poincaré inequality. The next two examples demonstrate that the Poincaré inequality is not essential for Lipschitz functions to be dense in $N^{1,p}(X)$.

Example 5.1. If the space X contains no nonconstant rectifiable curves (e.g. if X is the von Koch snowflake curve), then $N^{1,p}(X) = L^p(X)$. It is well known that Lipschitz functions are dense in $L^p(X)$. (The basic ideas for a proof are given in Proposition 7.9 of Folland [6].) In this case the p-Poincaré inequality is obviously not supported, since $g = 0$ will be an upper gradient of any measurable function.

The next example is perhaps more illuminating than the previous example, since it contains an abundance of nonconstant rectifiable curves.

Example 5.2. Let $n \geq 2$ and write $x \in \mathbf{R}^n$ as $x = (x_1, \dots, x_n)$. Let

$$\begin{aligned} X_+ &= \{x \in \mathbf{R}^n : x_j \geq 0,\ j = 1, \dots, n\} \quad \text{and} \\ X_- &= \{x \in \mathbf{R}^n : x_j \leq 0,\ j = 1, \dots, n\}, \end{aligned}$$

equipped with the Euclidean metric and Lebesgue measure (which is doubling both on X_+ and X_-).

Let $1 \leq p \leq n$ and $X = X_+ \cup X_-$. We shall show that X does not support the p-Poincaré inequality but that Lipschitz functions are dense in $N^{1,p}(X)$. The functions

$$u_j = \min\{1, \max\{0, -j^{-1} \log |x|\}\}$$

are Lipschitz in X, $u_j(0) = 1$ and a simple calculation shows that $\|u_j\|_{N^{1,p}(X)} \to 0$, as $j \to \infty$. Hence $C_p(\{0\}) = 0$ and consequently, the family of curves in X

passing through 0 has zero p-modulus (by Lemma 3.6 in Shanmugalingam [19]). It follows that the characteristic function χ_{X_+} has zero as a p-weak upper gradient, which shows that the p-Poincaré inequality is violated.

Note, however, that both X_+ and X_- support a 1-Poincaré inequality: if $u \in N^{1,p}(X_+)$, then u and all of its upper gradients g can be extended to the whole of $\mathbf{R}^n$ by reflection as follows,

$$\bar{u}(x_1,\dots,x_n) = u(|x_1|,\dots,|x_n|) \quad \text{and} \quad \bar{g}(x_1,\dots,x_n) = g(|x_1|,\dots,|x_n|),$$

so that $\bar{g}$ is an upper gradient of $\bar{u} \in N^{1,p}(\mathbf{R}^n)$ and $\|\bar{u}\|^p_{N^{1,p}(\mathbf{R}^n)} \le 2^n \|u\|^p_{N^{1,p}(X_+)}$. The 1-Poincaré inequality for X_+ then follows from the 1-Poincaré inequality for $\mathbf{R}^n$.

To prove the density of Lipschitz functions in $N^{1,p}(X)$, let $u \in N^{1,p}(X)$ and note that both $u\chi_{X_+}$ and $u\chi_{X_-}$ belong to $N^{1,p}(X)$, due to the fact that $C_p(\{0\}) = 0$. Thus, it suffices to approximate $\tilde{u} = u\chi_{X_+}$ by Lipschitz functions in $N^{1,p}(X)$. Let $\varepsilon > 0$. Since X_+ supports a p-Poincaré inequality, there exists a Lipschitz function v on X_+ such that $\|\tilde{u} - v\|_{N^{1,p}(X_+)} < \varepsilon$. Now, note that $(v - v(0))\chi_{X_+}$ is Lipschitz on X and hence it suffices to approximate the characteristic function χ_{X_+} by Lipschitz functions in $N^{1,p}(X)$. It is easily verified that the functions

$$v_j(x) = \max\{\chi_{X_+}, u_j\}$$

will do.

In view of the preceding examples it is worth pointing out that Lipschitz functions are not always dense in $N^{1,p}(X)$.

Example 5.3. Let $X = \{z \in \mathbf{C} = \mathbf{R}^2 : |z| = 1\} \setminus \{-1\}$, then $f(z) = \arg z$ (the principal branch) belongs to $N^{1,p}(X) \setminus \overline{\operatorname{Lip}(X)}$ (for all $p \ge 1$).

Example 5.4. Let X be the slit disc $B((0,0),1) \setminus ((-1,0] \times \{0\}) \subset \mathbf{R}^2 = \mathbf{C}$. Then $f(z) = \max\{2|z| - 1, 0\} \arg z$ belongs to $N^{1,p}(X) \setminus \overline{\operatorname{Lip}(X)}$.

Open problem 5.5. Is there a metric space X equipped with a Borel measure μ satisfying $0 < \mu(B) < \infty$ for all nonempty balls B in X, such that *locally* Lipschitz functions are not dense in $N^{1,p}(X)$ for some $1 \le p < \infty$?

Proposition 5.6. *Let X be proper and assume that locally Lipschitz functions are dense in $N^{1,p}(X)$. Then $\overline{\operatorname{Lip}_c(X)} = N^{1,p}(X)$, and in particular, Lipschitz functions form a dense subclass of $N^{1,p}(X)$.*

Recall that $\operatorname{Lip}_c(A) = \{f \in \operatorname{Lip}(A) : \operatorname{supp} f \Subset A\}$.

Proof. If X is bounded, then it is compact and every locally Lipschitz function on X is Lipschitz and the claim follows.

If X is unbounded, then we may write X as an increasing union of compact sets, $X = \bigcup_{k=1}^\infty X_k$. Let $u \in N^{1,p}(X)$, $\varepsilon > 0$ and find a locally Lipschitz function v with an upper gradient $g \in L^p(X)$, such that $\|u - v\|_{N^{1,p}(X)} < \varepsilon$. Then

$$\int_{X \setminus X_k} (|v| + g)^p \, d\mu \to 0, \quad \text{as } k \to \infty.$$

Let $\eta_k(x) = \max\{1 - \operatorname{dist}(x, X_k), 0\}$. Then $v\eta_k \in \operatorname{Lip}_c(X)$ and $(|v| + g)\chi_{X \setminus X_k}$ is a p-weak upper gradient of $v - v\eta_k$. Hence

$$\|v - v\eta_k\|_{N^{1,p}(X)} \le 2^{1/p} \biggl(\int_{X \setminus X_k} (|v| + g)^p \, d\mu \biggr)^{1/p} \to 0, \quad \text{as } k \to \infty.$$

□

Definition 8. For $E \subset X$, we introduce the space of Newtonian functions with zero values outside E as follows:

$$N_0^{1,p}(E) = \{f \in N^{1,p}(X) : f = 0 \text{ in } X \setminus E\}.$$

Theorem 5.7. *Let X be proper and assume that locally Lipschitz functions are dense in $N^{1,p}(X)$. If $\Omega \subset X$ is open, then $\overline{\operatorname{Lip}_c(\Omega)} = N_0^{1,p}(\Omega)$.*

In the proof of Theorem 5.7, we shall use the following lemma, which is essentially Lemmas 4.9 and 4.10 in Shanmugalingam [20]. For the reader's convenience, we provide a shortened proof at the end of this section adapted to our situation.

Lemma 5.8. *Assume that all $u \in N^{1,p}(X)$ are quasicontinuous. Let $\Omega \subset X$ be open. Then every $u \in N_0^{1,p}(\Omega)$ can be approximated in $N^{1,p}(X)$ by bounded functions with closed bounded support in Ω.*

Proof of Theorem 5.7. Let $u \in N_0^{1,p}(\Omega)$. If $\Omega = X$, then the result follows from Proposition 5.6. Assume therefore that $\Omega \ne X$. By Lemma 5.8, we can assume that u is bounded and that $\operatorname{supp} u$ is a compact subset of Ω. Let $\varepsilon > 0$ be arbitrary, $\delta = \min\{\operatorname{dist}(\operatorname{supp} u, X \setminus \Omega), 1\}$, and choose $\eta \in \operatorname{Lip}_c(\Omega)$ so that $0 \le \eta \le 1$, $\eta = 1$ on $\operatorname{supp} u$ and that $g_\eta \equiv 2/\delta$ is an upper gradient of η. As $u \in N^{1,p}(X)$, by Proposition 5.6 there exists a Lipschitz function v on X such that $\|u - v\|_{N^{1,p}(X)}^p < \varepsilon$. Let g_v be an upper gradient of v so that $\|v\|_{L^p(X)}^p + \|g_v\|_{L^p(X)}^p \le \|v\|_{N^{1,p}(X)}^p + \varepsilon$. Then $v\eta \in \operatorname{Lip}_c(\Omega)$ and $|v|g_\eta + g_v$ is an upper gradient of $v(1-\eta)$. Thus we get

$$\begin{aligned}\|v-v\eta\|^p_{N^{1,p}(X)} &\le \int_{X\setminus \operatorname{supp} u} (|v|^p + (|v|g_\eta + g_v)^p)\,d\mu \\ &\le \frac{4^p}{\delta^p}(\|v\|^p_{N^{1,p}(X\setminus \operatorname{supp} u)} + \varepsilon) \le \frac{4^p}{\delta^p}(\|u-v\|^p_{N^{1,p}(X)} + \varepsilon) < \frac{2\cdot 4^p\varepsilon}{\delta^p}.\end{aligned}$$

□

Theorem 5.9. *Let X be proper and assume that locally Lipschitz functions are dense in $N^{1,p}(X)$. If $\Omega \subset X$ is open, $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$, and $\varepsilon > 0$, then there exists a locally Lipschitz function $v : \Omega \to \mathbf{R}$ such that $\|u-v\|_{N^{1,p}(\Omega)} < \varepsilon$.*

We cannot expect Lipschitz functions to always be dense in $N^{1,p}(\Omega)$, see Examples 5.3 and 5.4 above.

Proof. Let $\varepsilon > 0$. As X is proper, there are open sets $\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset \Omega$ such that $\Omega = \bigcup_{j=1}^\infty \Omega_j$. Choose $\eta_j \in \operatorname{Lip}_c(\Omega_{j+1})$ so that $\eta_j = 1$ on Ω_j and $0 \le \eta_j \le 1$ everywhere. Let $u_1 = u\eta_1$ and $u_j = (u - \sum_{i=1}^{j-1} u_i)\eta_j$, $j = 2,3,\ldots$. It is easily shown by induction that

$$u - \sum_{i=1}^{j} u_i = u(1-\eta_1)(1-\eta_2)\cdots(1-\eta_j),$$

and hence $u = \sum_{j=1}^\infty u_j$ in Ω.

Now, for each j, we have $u_j \in N^{1,p}_0(\Omega_{j+1}\setminus \overline{\Omega}_{j-1})$, where we let $\Omega_0 = \varnothing$. By Theorem 5.7, there exists $v_j \in \operatorname{Lip}_c(\Omega_{j+1}\setminus\overline{\Omega}_{j-1})$ such that $\|u_j - v_j\|_{N^{1,p}(\Omega)} < 2^{-j}\varepsilon$.

Let $v = \sum_{j=1}^\infty v_j$. Note that for each $x \in \Omega$, only three terms in the sum are nonzero, and hence v is locally Lipschitz in Ω. Finally,

$$\|u-v\|_{N^{1,p}(\Omega)} \le \sum_{j=1}^\infty \|u_j - v_j\|_{N^{1,p}(\Omega)} < \varepsilon,$$

which concludes the proof. □

Proof of Lemma 5.8. To start with, we make several obvious reductions. Note that $u \in N^{1,p}_0(\Omega) \subset N^{1,p}(X)$. We can write $X = \bigcup_{n=1}^\infty B(x_0,n)$ for some $x_0 \in \Omega$. Then the sequence $u\eta_n$ is in $N^{1,p}_0(\Omega)$ with $u\eta_n \to u$ in $N^{1,p}(X)$, where η_n is a 1-Lipschitz function on X with $0 \le \eta_n \le 1$, $\eta_n = 1$ on $B(x_0,n)$, and $\eta_n = 0$ on $X \setminus B(x_0, n+1)$. Hence, without loss of generality we may assume that u has bounded support in X. Next, as both $u_+ = \max\{u,0\}$ and $u_- = \min\{u,0\}$ belong to $N^{1,p}_0(\Omega)$, we can assume that $u \ge 0$. Clearly, $\min\{u,k\} \to u$ in $N^{1,p}(X)$, as

$k \to \infty$, and we can assume that u is bounded. Similarly, $u_\varepsilon := \max\{u-\varepsilon, 0\} \to u$ in $N^{1,p}(X)$ as $\varepsilon \to 0$.

By assumption, u is quasicontinuous in X, and so there exist open sets U_j, $j = 1, 2, \dots$, with $\lim_{j\to\infty} C_p(U_j) = 0$, such that $u|_{X\setminus U_j}$ is continuous. It follows that for each j, the set $\{x \in X \setminus U_j : u(x) < \varepsilon\}$ is open in $X \setminus U_j$ and hence

$$V_j = U_j \cup \{x \in X : u(x) < \varepsilon\}$$

is open in X. Thus, $X \setminus V_j$ is closed with $X \setminus V_j \subset \Omega$. Moreover, as u has bounded support in X and $u \geq \varepsilon$ in $X \setminus V_j$, this set is also bounded. Now, by the choice of U_j, there exist w_j with upper gradients g_j, such that $0 \leq w_j \leq 1$, $w_j = 1$ on U_j, $\lim_{j\to\infty} \|w_j\|_{N^{1,p}(X)} = \lim_{j\to\infty} \|g_j\|_{L^p(X)} = 0$, and $w_j \to 0$ μ-a.e in X. Let $u_{\varepsilon,j} = (1-w_j)u_\varepsilon$, and $g \in L^p(X)$ be an upper gradient of u_ε. Then $u_\varepsilon g_j + w_j g$ is a p-weak upper gradient of $u_\varepsilon - u_{\varepsilon,j} = w_j u_\varepsilon$, and hence

$$\|u_\varepsilon - u_{\varepsilon,j}\|^p_{N^{1,p}(X)} \leq 2^{p-1}\biggl(\int_X (w_j u_\varepsilon)^p \, d\mu + \int_X (u_\varepsilon g_j)^p \, d\mu + \int_X (w_j g)^p \, d\mu\biggr).$$

Since u_ε is bounded, the first two integrals on the right-hand side tend to zero as $j \to \infty$. As for the last integral, we have $w_j \to 0$ μ-a.e. in X and as $w_j g \leq g \in L^p(X)$, dominated convergence shows that the last integral tends to zero as $j \to \infty$.

A diagonal argument now shows that among the functions $u_{\varepsilon,j}$, $j = 1, 2 \dots$, $\varepsilon > 0$, we can find a subsequence $u_k := u_{\varepsilon_k, j_k}$ such that $u_k \to u$ in $N^{1,p}(X)$, as $k \to \infty$.

To complete the proof, we observe that each $u_{\varepsilon,j}$ has closed bounded support in Ω. Indeed, if $u_{\varepsilon,j}(x) \neq 0$, then $u(x) > \varepsilon$ and thus $x \in X \setminus V_j$, which is a closed bounded subset of Ω. $\square$

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Department of Mathematics, Linköpings universitet, SE-581 83 Linköping, Sweden
E-mail address: anbjo@mai.liu.se

Department of Mathematics, Linköpings universitet, SE-581 83 Linköping, Sweden
E-mail address: jabjo@mai.liu.se

Department of Mathematical Sciences, University of Cincinnati, P.O. Box 210025, Cincinnati, OH 45221-0025, U.S.A.
E-mail address: nages@math.uc.edu