MEASURABILITY OF EQUIVALENCE CLASSES AND MEC_p-PROPERTY IN METRIC SPACES

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Abstract. We prove that a locally compact metric space that supports a doubling measure and a weak p-Poincaré inequality for some $1 \le p < \infty$ is a MEC_p-space. The methods developed for this purpose include measurability considerations and lead to interesting consequences. For example, we verify that each extended real valued function having a p-integrable upper gradient is locally p-integrable.

1. Introduction and main results

From the analytical point of view, the concept of a rectifiable path connected set is crucial in the study of metric spaces. It is well known that if the gradient of a Sobolev function in $\mathbb{R}^n$ equals zero almost everywhere, then the function is constant. This is not valid in general metric spaces with the notation of upper gradient given below in Definition 1.1. Indeed, it is evident from Definition 1.1 that if there are no rectifiable paths in the metric space, then 0 is an upper gradient of any function - even if the function is not constant. However, as stated in [18], it turns out that the MEC_p-property of a metric space (see Definition 1.3), guaranteeing that almost all points of the space belong to the same rectifiable path connected component, implies that each function which has 0 as an upper gradient, or more generally as a p-weak upper gradient (see [18, Definition 2.3]), is constant. This leads us to the natural question of which metric spaces admit the MEC_p-property.

For the purpose of studying quasi-conformal maps in certain metric spaces, Heinonen and Koskela [10] considered the following notion of an upper gradient.

Definition 1.1. Given a metric space (X, d) with a Borel measure μ, let u be an extended real valued function defined on X. A non-negative Borel function ρ is said to be an *upper gradient* of u if for all compact rectifiable paths $\gamma : I \to X$ ($I \subset \mathbb{R}$ is compact) the following inequality holds:

$$|u(x) - u(y)| \le \int_\gamma \rho \, ds,$$

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where x and y denote the endpoints of the path γ. Note that the right hand side of the above inequality should be infinite whenever at least one of $|u(x)|$ and $|u(y)|$ is infinite.

Remark 1.2. Throughout this paper we will consider only outer measures and simply refer to them as measures. So a measure is defined on the power set and not just only on some σ-algebra. For more information on this simplification, see [13, p. 8].

If the function ρ defines a metric in X, that is, if the expression

$$d_\rho(x,y) = \inf_\gamma \int_\gamma \rho\, ds,$$

where the infimum is taken over all compact rectifiable paths γ connecting x to y, gives a metric on X, then Definition 1.1 can be re-interpreted to state that u is a 1-Lipschitz mapping with respect to the metric d_ρ.

In the theory of Sobolev spaces one usually restricts attention to L^p-functions. Hence it is useful to know when it is true that every non-negative Borel measurable $\rho \in L^p(X)$ defines a d_ρ-quasi metric in a set $X_\rho \subset X$ with $\mu(X \setminus X_\rho) = 0$. Metric spaces satisfying this condition are said to admit the MEC$_p$-property (see Definition 1.3). It is evident from Definitions 1.1 and 1.3 that if X satisfies the MEC$_p$-property, then whenever $\rho \in L^p(X)$ is an upper gradient of a function u on X, the set of points where u is infinite must be contained in the exceptional set $X \setminus X_\rho$. Such a set is very small from the point of view of potential theory.

Throughout, (X,d) is a metric space with a σ-finite Borel measure μ. The triple (X,d,μ) is called a metric measure space. A path $\gamma : I \to X$ is said to be compact if $I \subset \mathbb{R}$ is a compact set. Given x, $y \in X$, let Γ_{xy} be the set of all compact rectifiable paths in X connecting x to y. Note that a constant path is also a compact rectifiable path. The length of a path γ is denoted by $\ell(\gamma)$.

Definition 1.3. A Borel function $\rho : X \to [0,\infty]$ defines an equivalence relation $\sim_\rho$ as follows: For $x, y \in X$ we have $x \sim_\rho y$ if there is $\gamma \in \Gamma_{xy}$ such that $\int_\gamma \rho\, ds < \infty$. We use the notation $[x]_\rho = \{y \in X : y \sim_\rho x\}$ to denote the equivalence classes of $x \in X$. Let $1 \le p < \infty$. A metric measure space X is said to admit the *main equivalence class* property with respect to p, abbreviated as MEC$_p$-property, if for each non-negative Borel function $\rho \in L^p(X)$ there is a point $x \in X$ such that $\mu(X \setminus [x]_\rho) = 0$. We call this equivalence class $[x]_\rho$ the *main equivalence class* of ρ.

The fact that all Euclidean domains have the MEC$_p$-property for all $p \ge 1$ was first noticed by Ohtsuka [14]. Clearly, the work of Ohtsuka also shows that smooth Riemann manifolds admit the MEC$_p$-property for all $p \ge 1$. This property was abstracted to the metric space setting by Shanmugalingam in [16, 17].

In this note we address the question of how generally the MEC$_p$-property is valid in metric spaces. It appears that all metric spaces that support a doubling measure (see Definition 1.4) and satisfy an analytic property called a weak p-Poincaré inequality for some $1 \leq p < \infty$ (see Definition 1.5) have the MEC$_p$-property. This is the content of one of our main results, Theorem 1.6. As an immediate consequence of it we see that Heisenberg groups and the metric spaces constructed by Bourdon and Pajot [2] as well as Laakso [12] admit the MEC$_p$-property. As far as we know, this has been unknown until now.

Definition 1.4. We say that the measure μ on X is *doubling* if there is a positive constant C_μ such that

$$\mu(B(x,2r)) \leq C_\mu \mu(B(x,r))$$

for every $x \in X$ and $r > 0$. Here $B(x,r)$ is an open ball with center at x and with radius $r > 0$.

Definition 1.5. Let $1 \leq p < \infty$. We say that the metric measure space X supports a weak p-Poincaré inequality if there exist constants $\tau \geq 1$ and $C_p \geq 1$ such that for all $r > 0$ and $x \in X$, for all μ-measurable functions $f \in L^1(B(x,r))$ defined on X, and for all upper gradients ρ of f we have

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq C_p r \left(\frac{1}{\mu(B(x,\tau r))} \int_{B(x,\tau r)} \rho^p \, d\mu \right)^{\frac{1}{p}},$$

where $f_{B(x,r)} = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, d\mu$ is the integral average of f on the ball $B(x,r)$.

Our main result concerning MEC$_p$-spaces is as follows:

Theorem 1.6. *Let X be a locally compact metric space that supports a doubling Borel measure μ which is non-trivial and finite on balls. If X supports a weak p-Poincaré inequality for some $1 \leq p < \infty$, then X is a* MEC$_p$*-space.*

In [18, Theorem 2.17] a similar result was first claimed, but the proof given there is not complete because it failed to prove that the equivalence classes are measurable. This problem is rectified by Theorem 1.8 of this paper.

Remark 1.7. Given a complete separable metric space Z, a set $A \subset Z$ is said to be analytic if there exist a complete separable metric space Y and a continuous function $f : Y \to Z$ such that $f(Y) = A$. Hence the continuous image of an analytic set is analytic. In particular, a continuous image of a Borel set is analytic, since Borel sets are analytic ([11, Theorem 14.2]). By Lusin's theorem (see [11, Theorem 21.10]), analytic subsets of Z are ν-measurable for any σ-finite Borel measure ν on Z. Countable unions and countable intersections of analytic sets are analytic.

Theorem 1.8. *Let X be a complete separable metric space equipped with a σ-finite Borel measure μ. If $\rho\colon X \to [0,\infty]$ is a Borel function, then $[x]_\rho$ is analytic for all $x \in X$. In particular, $[x]_\rho$ is μ-measurable for all $x \in X$.*

With the choice $\rho \equiv 1$ we have an immediate consequence:

Corollary 1.9. *Let X be a complete separable metric space equipped with a σ-finite Borel measure μ. Then the rectifiable path connected components of X are μ-measurable.*

Furthermore, the proof of Theorem 1.8 yields the following corollary:

Corollary 1.10. *Let X be a complete separable metric space equipped with a σ-finite Borel measure μ, and let $\rho\colon X \to [0,\infty]$ be a Borel function. Then for each $x_0 \in X$, the function $u\colon X \to [0,\infty]$, defined for all $x \in X$ by*

$$u(x) = \inf\Big\{ \int_\gamma \rho\, ds : \gamma \in \Gamma_{x_0 x} \Big\},$$

is measurable with respect to the σ-algebra generated by analytic sets, and therefore, it is μ-measurable.

Weaker versions of Corollary 1.10 have appeared earlier in the literature. For example, in [10] it is stated that u is continuous if X is quasi-convex and ρ is bounded.

It should be noted that not all metric spaces admit the MEC_p-property. For example, the metric space, obtained by gluing two planar triangular regions at one vertex point and using the length metric obtained from the Euclidean metric of the two triangular regions, is not a MEC_p-space when $1 \le p \le 2$. It should be also noted that the converse of Theorem 1.6 is not true. Indeed, the metric space X obtained by removing a radial slit from the unit disc $\mathbb{D}$ in the plane, that is, $X = \mathbb{D} \setminus [0,1] \subset \mathbb{R}^2$, is easily seen to be a MEC_p-space whenever $p \ge 1$, but never supports a weak p-Poincaré inequality. While this is not a complete metric space, one can modify it to obtain a complete metric space that admits the MEC_p-property whenever $p \ge 1$ but does not support a weak p-Poincaré inequality for certain values of p. For example, let X be the metric space achieved by considering the length metric induced by the Euclidean distance metric on the set obtained by removing the two open disks $B((-1,0),1)$ and $B((1,0),1)$ from $\mathbb{R}^2$. Such a space has MEC_p-property whenever $p \ge 1$, but fails to have a weak p-Poincaré inequality whenever $1 \le p \le 2$.

The methods from the proofs of Proposition 3.2 and Theorem 1.6 turn out to be quite powerful for other purposes as well. Indeed, they give the following surprising result according to which, under the assumptions of Theorem 1.6, all we need to know to conclude that a function belongs to $L^p_{loc}(X)$ is that it has an upper gradient in $L^p(X)$. Recall that $u \in L^p_{loc}(X)$ if and only if for all $x \in X$ there is a neighborhood V_x of x such that $u \in L^p(V_x)$.

Theorem 1.11. *Let X be a complete metric space that supports a doubling Borel measure μ which is non-trivial and finite on balls. Assume that X supports a weak p-Poincaré inequality for some $1 \le p < \infty$. If an extended real valued function*

$u\colon X \to [-\infty, \infty]$ *has a* p*-integrable upper gradient, then* u *is measurable and locally* p*-integrable.*

This paper is organized as follows: In the next section we state and verify three auxiliary results whereas Section 3 contains the proofs of the results introduced in this section. For the convenience of the reader, a version of a quasi-convexity result, needed in the proofs of Proposition 3.2 and Theorem 1.6, is included as an appendix. Indeed, in Section 4 we show that if a locally compact metric space supports a doubling measure and a weak p-Poincaré inequality, then it is quasi-convex.

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2. Auxiliary results

Given any metric space (X, d), we use the notation $(\widehat{X}, \widehat{d})$ for the completion of X which is complete and unique up to an isometry. Note that (X, d) is a subspace of $(\widehat{X}, \widehat{d})$ and X is dense in $\widehat{X}$. For our purposes, the crucial observation is that the essential features of X are inherited by $\widehat{X}$. Indeed, supposing that there is a doubling Borel measure on X which is non-trivial and finite on balls, we may extend it to $\widehat{X}$ such that $\widehat{X} \setminus X$ has zero measure and the extended measure has the same properties as the original one. Also, if X supports a weak p-Poincaré inequality for some $1 \leq p < \infty$, then $\widehat{X}$ does too, see for example [1, Proposition 7.1].

We proceed by stating a series of lemmas which will be needed in the next section.

Remark 2.1. In the following lemma, we consider all paths, not only rectifiable ones. The reason is that in the proof of Theorem 1.8 and in Remark 3.1 we need to study a complete metric space of paths and the space of rectifiable paths is not complete under the supremum norm.

The integral of a Borel function ρ over a rectifiable path $\gamma : I \to X$ is usually defined via the path length parametrization γ_0, that is, $\int_\gamma \rho\, ds = \int_0^{\ell(\gamma)} \rho \circ \gamma_0(t)\, dt$. There is an alternative definition of path integrals that extends to non-rectifiable paths as well. Namely if $\gamma : I \to X$ is a path, let F be the set of all closed subintervals $C \subset I$. Define $\zeta\colon F \to [0, \infty]$ by setting $\zeta([a, b]) = \ell(\gamma|_{[a,b]})$. The usual Carathéodory construction now yields a Borel regular measure μ_γ defined on I. The measure has the property that $\mu_\gamma([a, b]) = \ell(\gamma|_{[a,b]})$. Define $\int_\gamma \rho\, ds = \int_I \rho \circ \gamma(t)\, d\mu_\gamma(t)$. If γ is a rectifiable path, then this definition of path integral coincides with the previous definition. Observe also that when ρ is continuous the path integral defined in this way may be calculated as the supremum of lower Riemannian sums.

Lemma 2.2. *Let $\gamma_i\colon [0,1] \to X$ be a sequence of paths such that $\gamma_i \to \gamma$ uniformly. Then for each lower semicontinuous function $\rho\colon X \to [0,\infty]$ we have*

$$\int_\gamma \rho\, ds \leq \liminf_{i\to\infty} \int_{\gamma_i} \rho\, ds.$$

Proof. If ρ is continuous, then the claim follows from the definition of the path integral and the compactness of $\gamma([0,1])$. If ρ is lower semicontinuous, there exists a sequence (g_j) of continuous functions such that $g_j \nearrow \rho$ as $j \to \infty$. By the monotone convergence theorem we have

$$\int_\gamma \rho\, ds = \liminf_{j\to\infty} \int_\gamma g_j\, ds \leq \liminf_{j\to\infty}\ \liminf_{i\to\infty} \int_{\gamma_i} g_j\, ds \leq \liminf_{i\to\infty} \int_{\gamma_i} \rho\, ds.$$

□

The following lemma is a version of the Vitali-Carathéodory theorem tailored for our purposes. The difference between the Vitali-Carathéodory theorem and the following lemma is that we do not assume that the function f is everywhere finite.

Lemma 2.3. *Let X be a locally compact metric space that supports a Borel measure μ which is non-trivial and finite on balls. Assume that $f \in L^p(X)$ is a non-negative extended real valued Borel function. Then for every $\varepsilon > 0$ there is a lower semicontinuous function $g \in L^p(X)$ such that $g(x) \geq f(x)$ for all $x \in X$ and $||g - f||_p \leq \varepsilon$.*

Proof. One may approximate f with appropriate simple or continuous functions in the similar manner as in the proof of [5, Proposition 7.14]. □

The last result of this section serves as a base of our measurability considerations in the next section. It is a modification of [11, Theorem 11.6].

Lemma 2.4. *Let X be a metric space and let $\mathcal{Y}$ be a class of functions $g\colon X \to [0,\infty]$ such that the following properties are valid:*

(a) *If $g\colon X \to [0,\infty]$ is continuous, then $g \in \mathcal{Y}$.*
(b) *If (g_i) is an increasing sequence of functions in $\mathcal{Y}$ converging up to g, then $g \in \mathcal{Y}$.*
(c) *If $r, s \in \mathbb{R}_+$ and $g, f \in \mathcal{Y}$, then $rg + sf \in \mathcal{Y}$.*
(d) *If $g \in \mathcal{Y}$ and $0 \leq g \leq 1$, then $1 - g \in \mathcal{Y}$.*

Then $\mathcal{Y}$ contains all Borel functions $g\colon X \to [0,\infty]$.

Proof. The proof is a simplified version of that of [11, Theorem 11.6]. Using (a) and (b), one first verifies that the characteristic functions of open sets belong to

$\mathcal{Y}$. Note that finite intersections of open sets are open. Let $\mathcal{S}$ be the collection of all subsets of X whose characteristic functions belong to $\mathcal{Y}$. Then by (d) the family $\mathcal{S}$ is closed under complements, and by (b) it is closed under countable disjoint unions, and as we have noted, open sets belong to $\mathcal{S}$; therefore by [11, Theorem 10.1(iii)] (also called the π–λ theorem and states that the smallest collection of subsets of X that contains all open sets and is closed under complementation and countable disjoint unions is the Borel class), we see that $\mathcal{S}$ contains all Borel subsets of X. Thus the characteristic functions of Borel sets are in $\mathcal{Y}$. Finally, an application of (b) and (c) completes the proof. □

3. Proofs of main results

In this section we prove the results of Section 1. We proceed in a slightly different order here because, when verifying Proposition 3.2 and Theorem 1.6, we need the measurability result, Theorem 1.8.

Proof of Theorem 1.8. Let $\rho : X \to [0,\infty]$ be a Borel function and fix $x_0 \in X$. The space

$$Y = \{\gamma\colon [0,1] \to X : \gamma \text{ is a path with } \gamma(0) = x_0\}$$

equipped with the metric

$$d_\infty(\gamma, \tilde{\gamma}) = \sup_{t\in[0,1]} d(\gamma(t), \tilde{\gamma}(t))$$

is a complete separable metric space. This follows from [11, Theorem 4.19] combined with the fact that every subset of a separable metric space is separable.

Consider the mapping $\varphi_g\colon Y \to [0,\infty]$ defined by

$$\varphi_g(\gamma) = \int_\gamma g\, ds.$$

If g is continuous, then Lemma 2.2 implies that φ_g is lower semicontinuous, and therefore a Borel function. Thus choosing $g \equiv 1$ we have that $Y_0 = \varphi_1^{-1}([0,\infty))$ is a Borel set. From now on we will restrict the mappings φ_g to Y_0. We proceed by checking that the assumptions (a)-(d) of Lemma 2.4 are valid for the class

$$\mathcal{Y} = \{g\colon X \to [0,\infty] : \varphi_g\colon Y_0 \to [0,\infty] \text{ is a Borel map}\}.$$

We already saw that (a) is valid. Letting $\gamma \in Y$ and $g_i \in \mathcal{Y}$ be such that $g_i \nearrow g$ pointwise, we obtain by the monotone convergence theorem

$$\varphi_g(\gamma) = \int_\gamma g\, ds = \lim_{i\to\infty} \int_\gamma g_i\, ds = \lim_{i\to\infty} \varphi_{g_i}(\gamma).$$

Hence, φ_g is a Borel function, since it is a limit of Borel functions, and (b) is satisfied. The items (c) and (d) follow from the linearity of the integral operator.

Thus the assumptions of Lemma 2.4 are satisfied, and it follows that $\mathcal{Y}$ contains all non-negative Borel functions. In particular, $\varphi_\rho\colon Y_0 \to [0,\infty]$ is a Borel function. Defining $\pi\colon Y \to X$ as $\pi(\gamma) = \gamma(1)$ for all $\gamma \in Y$, the choice for metric in Y guarantees that π is continuous. Therefore, $[x_0]_\rho = \pi(\varphi_\rho^{-1}([0,\infty)))$ is an analytic set by Remark 1.7. $\square$

Remark 3.1. In Theorem 1.8, the metric space X does not need to be complete. It is sufficient to assume that X is separable and an open subset of the completion $\widehat{X}$ of X. Indeed, fix $x_0 \in X$, and define

$$Y = \{\gamma\colon [0,1] \to X : \gamma \text{ is a path with } \gamma(0) = x_0\}$$

and

$$\widehat{Y} = \{\gamma\colon [0,1] \to \widehat{X} : \gamma \text{ is a path with } \gamma(0) = x_0\}.$$

Since $\widehat{X}$ is complete and separable the set $\widehat{Y}$ has the same properties. Moreover, the openness of X guarantees that Y is an open subset of $\widehat{Y}$. Given a Borel function $\rho\colon X \to [0,\infty]$, define a Borel map $\widehat{\rho}\colon \widehat{X} \to [0,\infty]$ by $\widehat{\rho}(x) = \rho(x)$ if $x \in X$, and $\widehat{\rho}(x) = 0$ otherwise. Letting $\varphi_{\widehat{\rho}}\colon \widehat{Y}_0 \to [0,\infty]$ be as in the proof of Theorem 1.8, we see that $\varphi_{\widehat{\rho}}^{-1}([0,\infty))$ is a Borel subset of $\widehat{Y}$ which, in turn, implies that $\varphi_{\widehat{\rho}}^{-1}([0,\infty)) \cap Y$ is a Borel set since Y is open. Finally, if $\pi\colon \widehat{Y} \to \widehat{X}$ is defined by $\pi(\gamma) = \gamma(1)$ for all $\gamma \in \widehat{Y}$, then we conclude, similarly as in the proof of Theorem 1.8, that $[x_0]_\rho = \pi(\varphi_{\widehat{\rho}}^{-1}([0,\infty)) \cap Y)$ is analytic.

Proof of Corollary 1.10. Let $\mathcal{P}$ be the set of all compact paths in X equipped with the same metric as Y in the proof of Theorem 1.8. Let $\mathcal{P}_{x_0} \subset \mathcal{P}$ be the set of all paths starting from x_0. By Lemma 2.2 the function $\Phi\colon \mathcal{P} \to [0,\infty]$, $\Phi(\gamma) = \ell(\gamma)$, is lower semicontinuous, and therefore

$$G = \cup_{x \in X} \Gamma_{x_0 x} = \Phi^{-1}([0,\infty)) \cap \mathcal{P}_{x_0}$$

is a Borel set. Letting $\rho\colon X \to [0,\infty]$ be a Borel map and defining functions φ_ρ and π as in the proof of Theorem 1.8, gives $u^{-1}([0,a)) = \pi(\varphi_\rho^{-1}([0,a)) \cap G)$ for all real numbers $a > 0$. From the above consideration we know that φ_ρ is a Borel map and π is continuous. The claim follows since $\pi(\varphi_\rho^{-1}([0,a)) \cap G)$ is analytic, as verified in the proof of Theorem 1.8. $\square$

Before giving the proof of Theorem 1.6, we will prove a corresponding result with slightly stronger assumptions. Observe that in a complete metric space X the existence of a doubling Borel measure which is non-trivial and finite on balls implies that X is separable, and closed bounded subsets of X are compact, in particular, X is locally compact. The reason for stating Proposition 3.2 as a separate result is that we need the methods from its proof when verifying Theorem 1.6.

Proposition 3.2. *Let X be a complete metric space that supports a doubling Borel measure μ which is non-trivial and finite on balls. If X supports a weak p-Poincaré inequality for some $1 \le p < \infty$, then X is a* MEC_p*-space.*

Proof. Let $\rho \in L^p(X)$ be a non-negative Borel function. In order to verify the MEC$_p$-property we have to show the existence of $x \in X$ such that $\mu(X \setminus [x]_\rho) = 0$. By Lemma 2.3, there is a lower semicontinuous function $\tilde{\rho} \in L^p(X)$ such that $\tilde{\rho} \geq \rho$ everywhere. Noticing that $[x]_{\tilde{\rho}} \subset [x]_\rho$ for every $x \in X$, it suffices to prove that there exists $x \in X$ such that $\mu(X \setminus [x]_{\tilde{\rho}}) = 0$.

For $m \in \mathbb{N}$, define

$$S_m = \{x \in X : M(\tilde{\rho}^p)(x) \leq m^p\}.$$

Here M is the non-centered maximal function operator defined for $f \in L^1_{loc}(X)$ as

$$Mf(x) = \sup\Big\{\frac{1}{\mu(B)} \int_B |f|\, d\mu : B \text{ is a ball containing } x\Big\}.$$

Since $\tilde{\rho}^p \in L^1(X)$, [8, Theorem 2.2] implies that $\mu(X \setminus \cup_m S_m) = 0$. Let m_0 be the smallest integer for which $S_{m_0} \neq \emptyset$. Fix $x_0 \in S_{m_0}$. We will verify that for every $y \in \cup_{m \geq m_0} S_m$ there is $\gamma \in \Gamma_{x_0 y}$ with the property $\int_\gamma \tilde{\rho}\, ds < \infty$. This shows that $\cup_{m \geq m_0} S_m \subset [x_0]_{\tilde{\rho}}$, and by the choice of m_0, we have $\mu(X \setminus \cup_{m \geq m_0} S_m) = 0$ which, in turn, will imply the claim.

Defining for all $k \in \mathbb{N}$ a lower semicontinuous function $\tilde{\rho}_k = \min\{\tilde{\rho}, k\}$, set

$$\begin{aligned} u_k(x) &= \inf\Big\{\ell(\gamma) + \int_\gamma \tilde{\rho}_k\, ds : \gamma \in \Gamma_{x_0 x}\Big\} \text{ and} \\ u(x) &= \inf\Big\{\ell(\gamma) + \int_\gamma \tilde{\rho}\, ds : \gamma \in \Gamma_{x_0 x}\Big\} \end{aligned}$$

with the interpretation that the infimum of an empty set is infinite. Our claim is that $u(y) < \infty$ for every $y \in \cup_{m \geq m_0} S_m$. Note that u is measurable by Corollary 1.10, but nothing else is known about it. A priori it could be infinite in a set of positive measure. Trying directly to prove that u is finite almost everywhere is therefore difficult and that is why we use the functions u_k.

Since the complete metric space X supports a doubling measure and a weak p-Poincaré inequality, it is quasi-convex with a constant C_q which only depends on constants associated with the measure and the Poincaré inequality (see Appendix: Lemma 4.1). Recall that quasi-convexity means that for every pair of points $z, y \in X$ there is $\gamma \in \Gamma_{zy}$ such that $\ell(\gamma) \leq C_q d(z, y)$.

Take $z, y \in X$ and let $\varepsilon > 0$. By quasi-convexity $u_k(z)$, $u_k(y) < \infty$, and we may assume that $u_k(z) \geq u_k(y)$. By the definition of u_k, for all $\varepsilon > 0$ there is a path $\gamma_y \in \Gamma_{x_0 y}$ such that

$$u_k(y) \geq \ell(\gamma_y) + \int_{\gamma_y} \tilde{\rho}_k\, ds - \varepsilon.$$

Thus noticing that

$$u_k(z) \leq \ell(\gamma_y \cup \gamma_{yz}) + \int_{\gamma_y \cup \gamma_{yz}} \tilde{\rho}_k\, ds$$

for all $\gamma_{yz} \in \Gamma_{yz}$, we obtain

$$|u_k(z) - u_k(y)| \leq u_k(z) - \ell(\gamma_y) - \int_{\gamma_y} \tilde{\rho}_k \, ds + \varepsilon \leq \ell(\gamma_{yz}) + \int_{\gamma_{yz}} \tilde{\rho}_k \, ds + \varepsilon.$$

By choosing $\gamma_{yz} \in \Gamma_{yz}$ so that $\ell(\gamma) \leq C_q d(z,y)$ and remembering that $\tilde{\rho}_k \leq k$, we see that u_k is a $C_q(k+1)$-Lipschitz function.

Next we will show that the restriction of u_k to S_m is a $C(m+1)$-Lipschitz function, where C does not depend on k. From the above calculation we deduce that for each $\gamma \in \Gamma_{yz}$ we have

$$|u_k(z) - u_k(y)| \leq \int_\gamma (\tilde{\rho}_k + 1) \, ds \leq \int_\gamma (\tilde{\rho} + 1) \, ds.$$

This shows that $\tilde{\rho} + 1$ is an upper gradient for u_k. Fix $z, y \in S_m$. For $i \in \mathbb{Z}$, set $B_i = B(z, 2^{-i} d(z,y))$ when $i \geq 1$, $B_0 = B(z, 2d(z,y))$, and $B_i = B(y, 2^i d(z,y))$ when $i \leq -1$. In what follows we use the notation $\tau B(x,r) = B(x, \tau r)$. In the first inequality of the following estimation we use the fact that, since u_k is continuous, all points are its Lebesgue points. Combining the weak p-Poincaré inequality with the doubling condition gives the third inequality. Finally, the fourth one comes from the Minkowski inequality whereas the fifth one follows from the definition of S_m:

$$\begin{aligned} |u_k(z) - u_k(y)| &\leq \sum_{i \in \mathbb{Z}} |(u_k)_{B_i} - (u_k)_{B_{i+1}}| \\ &\leq \sum_{i \in \mathbb{Z}} \frac{1}{\mu(B_i)} \int_{B_i} |(u_k) - (u_k)_{B_{i+1}}| \, d\mu \\ &\leq C_\mu C_p d(z,y) \sum_{i \in \mathbb{Z}} 2^{-|i|} \left(\frac{1}{\mu(\tau B_i)} \int_{\tau B_i} (\tilde{\rho} + 1)^p \, d\mu \right)^{\frac{1}{p}} \qquad (3.1) \\ &\leq C_\mu C_p d(z,y) \sum_{i \in \mathbb{Z}} 2^{-|i|} \left(1 + \left(\frac{1}{\mu(\tau B_i)} \int_{\tau B_i} \tilde{\rho}^p \, d\mu \right)^{\frac{1}{p}} \right) \\ &\leq C_\mu C_p d(z,y)(1+m) \sum_{i \in \mathbb{Z}} 2^{-|i|} \\ &\leq C(m+1) d(z,y), \end{aligned}$$

where C is a constant depending only on C_μ and C_p. Hence on S_m, u_k is a $C(m+1)$-Lipschitz function for all k. Notice that $u_k \leq u_{k+1}$ and therefore we may define

$$v(x) = \sup_k u_k(x) = \lim_{k \to \infty} u_k(x).$$

Thus v is a $C(m+1)$-Lipschitz function on S_m. Since $v(x_0) = 0$ and $x_0 \in S_m$ when $m \geq m_0$, we have that $v(x) < \infty$ for every $x \in \cup_{m \geq m_0} S_m$.

Our claim reduces to showing that $u(x) \le v(x)$ for $x \in \cup_{m \ge m_0} S_m$. For this, fix $m \ge m_0$ and $x \in S_m$. For each k there is $\gamma_k \in \Gamma_{x_0 x}$ such that

$$\ell(\gamma_k) + \int_{\gamma_k} \tilde\rho_k \, ds \le u_k(x) + \frac{1}{k} \le C(m+1)d(x, x_0) + \frac{1}{k}.$$

This implies that $\ell(\gamma_k) \le C(m+1)d(x,x_0) + 1 =: M$ for every k. Thus, by reparametrization, we may assume that γ_k is an M-Lipschitz function and $\gamma_k \colon [0,1] \to \overline{B}(x_0, M)$ for all k. Since X is complete and doubling, and therefore proper (that is, closed balls are compact), we may use the Ascoli-Arzela theorem to obtain a subsequence (γ_k) (which we denote by the same subscripts as the original one) and $\gamma \colon [0,1] \to X$ such that $\gamma_k \to \gamma$ uniformly.

For each k_0, the function $1+\tilde\rho_{k_0}$ is lower semicontinuous, and therefore Lemma 2.2 and the fact that $(\tilde\rho_k)$ is an increasing sequence of functions imply

$$\begin{aligned}
\ell(\gamma) + \int_\gamma \tilde\rho_{k_0} \, ds &= \int_\gamma (1+\tilde\rho_{k_0}) \, ds \le \liminf_{k\to\infty} \int_{\gamma_k} (1+\tilde\rho_{k_0}) \, ds \\
&\le \liminf_{k\to\infty} \int_{\gamma_k} (1+\tilde\rho_k) \, ds.
\end{aligned}$$

Using the monotone convergence theorem on the left hand side and letting k_0 tend to infinity yields

$$\ell(\gamma) + \int_\gamma \tilde\rho \, ds \le \liminf_{k\to\infty} \int_{\gamma_k} (1+\tilde\rho_k) \, ds.$$

Since $\gamma \in \Gamma_{x_0 x}$ we have

$$\begin{aligned}
u(x) &\le \ell(\gamma) + \int_\gamma \tilde\rho \, ds \le \liminf_{k\to\infty} \int_{\gamma_k} (1+\tilde\rho_k) \, ds \\
&\le \liminf_{k\to\infty} (u_k(x) + \frac{1}{k}) \le v(x)
\end{aligned}$$

completing the proof. □

Similar methods serve as the base of the verification of Theorem 1.6.

Proof of Theorem 1.6. Let $\widehat{X}$ be the completion of X. We extend the measure μ to $\widehat{X}$ according to the discussion at the beginning of Section 2. The extension will still be denoted by μ. Note that the claim does not follow directly from Proposition 3.2 since there might be paths connecting different equivalence classes of X via $\widehat{X} \setminus X$.

First we will show that there is x_0 such that $\mu([x_0]_\rho) > 0$. For this let $\rho \in L^p(X)$ be a non-negative Borel function. Extend ρ by zero to $\widehat{X} \setminus X$. Let $\tilde\rho \in L^p(\widehat{X})$ be a lower semicontinuous function given by Lemma 2.3 such that $\|\tilde\rho - \rho\|_p < 1$ and $\tilde\rho(x) \ge \rho(x)$ for all $x \in \widehat{X}$. Setting for all $m \in \mathbb{N}$

$$\widetilde{S}_m = \{x \in \widehat{X} : M(\tilde\rho^p)(x) \le m^p\},$$

we obtain, similarly as in the proof of Proposition 3.2, that $\mu(\widehat{X}\setminus(\cup_m\widetilde{S}_m))=0$. For $m\in\mathbb{N}$, define $S_m=\widetilde{S}_m\cap X$. Since μ is non-trivial and $\mu(\widehat{X}\setminus X)=0$ we may pick m_0 such that $\mu(S_{m_0})>0$.

Let $x_0\in S_{m_0}$ be a point of density for S_{m_0}. As before, define for all $x\in\widehat{X}$

$$u(x)=\inf\Big\{\ell(\gamma)+\int_\gamma\tilde{\rho}\,ds:\gamma\in\Gamma_{x_0x}(\widehat{X})\Big\},$$

where $\Gamma_{x_0x}(\widehat{X})$ is the set of all rectifiable paths in $\widehat{X}$ connecting x_0 to x. Recall that Γ_{x_0x} is the set of corresponding paths in X. From the proof of Proposition 3.2 we see that u is a $C_1(m_0+1)$-Lipschitz function on $\widetilde{S}_{m_0}$, and therefore, on S_{m_0} as well. (Observe that, using the notation of the proof of Proposition 3.2, we have $u(x)=v(x)$ for all $x\in S_{m_0}$.)

Since x_0 is a point of density of S_{m_0} we have that $\mu(S_{m_0}\cap B(x_0,r))>0$ for all $r>0$. Furthermore, X is locally compact, and therefore there exists r_0 such that $B(x_0,r_0)\subset X$. Setting $r=(3C_1(m_0+1))^{-1}r_0$, we obtain

$$u(y)=|u(y)-u(x_0)|\le C_1(m_0+1)d(y,x_0)$$

for all $y\in S_{m_0}\cap B(x_0,r)$. Thus there is $\gamma\in\Gamma_{x_0y}(\widehat{X})$ such that $\ell(\gamma)\le 2C_1(m_0+1)r\le\frac{2}{3}r_0$. This gives $\gamma\subset B(x_0,r_0)\subset X$, and so $\gamma\in\Gamma_{x_0y}$. Moreover, $\int_\gamma\tilde{\rho}\,ds<\infty$, implying that $S_{m_0}\cap B(x_0,r)\subset[x_0]_{\tilde{\rho}}\subset[x_0]_\rho$, where the equivalence classes are defined in X. Thus $\mu([x_0]_\rho)>0$.

It remains to prove that $\mu(X\setminus[x_0]_\rho)=0$. Assume to the contrary that $\mu(X\setminus[x_0]_\rho)>0$. Since X is locally compact and μ is a doubling Borel measure which is non-trivial and finite on balls, the space X is separable and an open subset of $\widehat{X}$. From Remark 3.1 we know that $[x_0]_\rho$ is μ-measurable. Moreover, for all $\varepsilon>0$, the mapping $\varepsilon\rho$ is an upper gradient of the characteristic function $\chi_{[x_0]_\rho}$ of the set $[x_0]_\rho$. Choosing $R>0$ sufficiently large so that both $\mu(B(x_0,R)\cap[x_0]_\rho)>0$ and $\mu(B(x_0,R)\setminus[x_0]_\rho)>0$ and applying the weak p-Poincaré inequality to the function-weak upper gradient pair $(\chi_{[x_0]_\rho},\varepsilon\rho)$ gives a contradiction since the left hand side of the inequality is positive whereas the right hand side tends to zero as $\varepsilon\to 0$. Therefore $X\setminus[x_0]_\rho$ must be of zero measure, and the proof is done. □

The proof of the following result employs similar techniques to that of Proposition 3.2.

Proof of Theorem 1.11. Let ρ be a p-integrable upper gradient of u. As in the proof of Proposition 3.2, apply Lemma 2.3 to produce a lower semicontinuous function $\tilde{\rho}\in L^p$ such that $\tilde{\rho}\ge\rho$ pointwise. Define

$$f(x,y)=\inf\Big\{l(\gamma)+\int_\gamma\tilde{\rho}\,ds:\gamma\in\Gamma_{xy}\Big\}.$$

As before, we see that for $x,y\in S_m=\{x\in X:M(\tilde{\rho}^p)(x)\le m^p\}$ we have

$$f(x,y)\le C(m+1)d(x,y).$$

Moreover, for $x, y \in S_m$,

$$|u(x) - u(y)| \leq \inf\Big\{\int_\gamma \rho\, ds : \gamma \in \Gamma_{xy}\Big\} \leq \inf\Big\{\ell(\gamma) + \int_\gamma \tilde{\rho}\, ds : \gamma \in \Gamma_{xy}\Big\} = f(x,y) \leq C(m+1)d(x,y). \tag{3.2}$$

Hence, $u\colon S_m \to [-\infty, \infty]$ is a Lipschitz function, and therefore a Borel function. Thus $u\colon \cup_m S_m \to [-\infty, \infty]$ is a Borel function, and since $\mu(X \setminus \cup_m S_m) = 0$, we see that u is measurable.

To prove the local p-integrability of u, we first observe that under our assumptions there exist constants $\lambda \geq 1$ and $\tilde{C}_p \geq 1$ such that for all $r > 0$ and $x \in X$, for all μ-measurable functions $f \in L^1(B(x,r))$ defined on X, and for all upper gradients ρ of f we have

$$\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f - f_{B(x,r)}|^p\, d\mu\right)^{\frac{1}{p}} \leq \tilde{C}_p r \left(\frac{1}{\mu(B(x,\lambda r))}\int_{B(x,\lambda r)} \rho^p\, d\mu\right)^{\frac{1}{p}} \tag{3.3}$$

by arguments in [6] and [7].

Next fix $x_0 \in S_{m_0}$, where m_0 is the smallest integer such that $S_{m_0} \neq \emptyset$. Notice that $|u(x_0)| < \infty$ (see the definition of the upper gradient and (3.2)). As in the proof of Proposition 3.2, for all $k \in \mathbb{N}$ consider the lower semicontinuous functions $\tilde{\rho}_k = \min\{\tilde{\rho}, k\}$, and set

$$\begin{aligned} v_k(x) &= \inf\Big\{\ell(\gamma) + \int_\gamma \tilde{\rho}_k\, ds : \gamma \in \Gamma_{x_0 x}\Big\} \text{ and} \\ v(x) &= \inf\Big\{\ell(\gamma) + \int_\gamma \tilde{\rho}\, ds : \gamma \in \Gamma_{x_0 x}\Big\}. \end{aligned}$$

By the proof of Proposition 3.2, we have $v(x) = \sup_k v_k(x) = \lim_k v_k(x)$ for all $x \in \cup_m S_m$. Let $r > 0$ and set $B_i = B_i(x_0, 2^{-i}r)$ for $i = 0, 1, \ldots$. Since v_k is continuous at x_0 and $v_k(x_0) = 0$, by an argument similar to the one that led to the chain of inequalities (3.1) we can obtain

$$\begin{aligned} \frac{1}{\mu(B(x_0,r))}\int_{B(x_0,r)} v_k\, d\mu &\leq \sum_{i=0}^{\infty} |(v_k)_{B_i} - (v_k)_{B_{i+1}}| \\ &\leq C_p C_\mu \sum_{i=0}^{\infty} 2^{-i} r \left(1 + \left(\frac{1}{\mu(\tau B_i)}\int_{\tau B_i} \tilde{\rho}^p\, d\mu\right)^{\frac{1}{p}}\right) \\ &\leq C_p C_\mu r \left(1 + (M\tilde{\rho}^p(x_0))^{\frac{1}{p}}\right) < \infty, \end{aligned} \tag{3.4}$$

since $x_0 \in S_{m_0}$. Using equations (3.3) and (3.4) and the fact $(a+b)^p \le 2^p(a^p+b^p)$ for any positive numbers a and b, we get

$$\begin{aligned}\frac{1}{\mu(B(x_0,r))}\int_{B(x_0,r)} v_k^p\,d\mu &\le \frac{2^p}{\mu(B(x_0,r))}\int_{B(x_0,r)} |v_k-(v_k)_{B(x_0,r)}|^p\,d\mu \\ &\quad +2^p(v_k)^p_{B(x_0,r)} \\ &\le 2^p\tilde{C}_p^p r^p\frac{1}{\mu(B(x_0,\lambda r))}\int_{B(x_0,\lambda r)}(\tilde\rho+1)^p\,d\mu \\ &\quad +2^p\left(C_pC_\mu r\left(1+(M\tilde\rho^p(x_0))^{\frac1p}\right)\right)^p<\infty.\end{aligned}$$

Notice that the upper bound for the mean values of v_k^p's does not depend on k.

By the monotone convergence theorem we see that

$$\begin{aligned}\left(\frac{1}{\mu(B(x_0,r))}\int_{B(x_0,r)}|u|^p\,d\mu\right)^{\frac1p} &\le \left(\frac{1}{\mu(B(x_0,r))}\int_{B(x_0,r)}|u-u(x_0)|^p\,d\mu\right)^{\frac1p} \\ &\quad +|u(x_0)| \\ &\le \left(\frac{1}{\mu(B(x_0,r))}\int_{B(x_0,r)}v^p\,d\mu\right)^{\frac1p}+|u(x_0)| \\ &= \lim_{k\to\infty}\left(\frac{1}{\mu(B(x_0,r))}\int_{B(x_0,r)}v_k^p\,d\mu\right)^{\frac1p}+|u(x_0)|.\end{aligned}$$

The above calculations imply that the first term is finite and we know that $|u(x_0)|<\infty$. Hence we get that $u\in L^p(B(x_0,r))$. Since $r>0$ was arbitrary, we obtain the claim. □

4. Appendix: quasi-convexity

It is folklore that if a metric measure space admits a weak Poincaré inequality and possesses additional miscellaneous properties, then it is quasi-convex. The proof of this fact can be found in [3], [7], and [19]. For the convenience of the reader we include one form of this folklore.

Lemma 4.1. *Let (X,d) be a locally compact metric space with a doubling Borel measure μ that is non-trivial and finite on balls and admits a weak p-Poincaré inequality for some $1\le p<\infty$. Then there exists $C>0$, depending only on the constants of the Poincaré inequality and the doubling condition, so that for each pair of points $x,y\in X$ there exists $\gamma\in\Gamma_{xy}$ with $\ell(\gamma)\le Cd(x,y)$.*

Proof. Since X admits a weak p-Poincaré inequality, it is connected. We first consider the case where X is complete. For each $\varepsilon>0$, consider the equivalence relation $\sim_\varepsilon$ given by $x\sim_\varepsilon y$ if and only if there exists a finite ε-chain connecting x to y, i.e. a finite sequence $x_0,x_1,\dots,x_n$ of points so that $x_0=x$, $x_n=y$, and $d(x_i,x_{i+1})<\varepsilon$ for each i. Then the equivalence classes are open, and since X is connected there is only one equivalence class, X itself. This means that every pair of points in X can be connected with a finite ε-chain. Hence, for a fixed point $x_0 \in X$ and for each $\varepsilon > 0$, we can define the function

$$f_\varepsilon(x) = \inf\Big\{\sum_{i=1}^{n} d(x_i, x_{i-1}) \,:\, x_0, \ldots, x_n \text{ is a finite } \varepsilon\text{-chain connecting } x_0 \text{ to } x\Big\}.$$

When $d(x,y) < \varepsilon$, we see that $|f_\varepsilon(x) - f_\varepsilon(y)| \le d(x,y)$. Hence, f_ε is a locally 1-Lipschitz function, in particular, every point is a Lebesgue point of f_ε and the function $\rho = 1$ is an upper gradient of f_ε. A similar argument as in the proof of Proposition 3.2 (see (3.1)) then gives us that for each $\varepsilon > 0$, f_ε is a globally C-Lipschitz function where C depends only on the data of X. Moreover, for each $\varepsilon > 0$, $f_\varepsilon(x_0) = 0$. Hence the function

$$f(x) = \sup_{\varepsilon > 0} f_\varepsilon(x) = \lim_{\varepsilon \downarrow 0} f_\varepsilon(x)$$

is also a C-Lipschitz function with $f(x_0) = 0$.

We now claim that if $f(x) < M$, then there exists a 1-Lipschitz path $\gamma : [0,M] \to X$ so that $\gamma(0) = x_0$ and $\gamma(M) = x$. Indeed, let x be such that $f(x) < M$. For each $i \in \mathbb{N}$, let $\varepsilon_i = \frac{1}{2^i}$. Then for each i, there exists a finite ε_i-chain $x_0^i, x_1^i, \ldots, x_{m_i}^i$ connecting x_0 to x so that

$$\sum_{j=1}^{m_i} d(x_{j-1}^i, x_j^i) \le M.$$

Let

$$T_i = \sum_{j=1}^{m_i} d(x_{j-1}^i, x_j^i).$$

View X as a subset of

$$\mathcal{B}(X) = \{f \colon X \to \mathbb{R} \,:\, f \text{ is bounded}\}$$

via the isometric embedding $\iota(y) = g_y$ where $g_y(z) = d(z,y) - d(z,x_0)$. Here $\mathcal{B}(X)$ is equipped with the sup-norm. To simplify the notation, we omit the embedding ι from now on. Since X is complete, we may view X as a closed subset of the Banach space $\mathcal{B}(X)$. For each i, define $\gamma_i : [0, T_i] \to \mathcal{B}(X)$ as the 1-Lipschitz path that connects the successive points $x_0 = x_0^i, x_1^i, \ldots, x_{m_i}^i = x$ via line segments. Extend $\gamma_i : [0,M] \to \mathcal{B}(X)$ by setting $\gamma(t) = x$ for $t \ge T_i$. Then for each i, γ_i is a 1-Lipschitz function. Note that for each i and each $0 \le t \le M$, $d(\gamma_i(t), x_0) \le M$, $\gamma_i(0) = x_0$ and $\gamma_i(M) = x$. Let

$$Y = \big(\overline{\bigcup_i \gamma_i([0,M])}\big) \cup \iota(\overline{B(x_0, M)}).$$

Since X is locally compact and closed (in $\mathcal{B}(X)$), $\iota(\overline{B(x_0,M)})$ is compact. Thus it is a straightforward task to see that Y is compact. Apply the Ascoli-Arzela theorem to the sequence (γ_i) to produce a subsequence (γ_{i_j}) which converges uniformly to a 1-Lipschitz path $\gamma : [0, M] \to Y$. Clearly, $\gamma(0) = x_0$ and $\gamma(M) = x$. Finally, $\gamma([0, M]) \subset X$, since X is closed and $\operatorname{dist}(X, \gamma_i(t)) \leq \varepsilon_i$ for all $0 \leq t \leq M$ and $i \in \mathbb{N}$.

Tying this together, we see that if $f(x) < M$, then there exists a rectifiable path in X connecting x_0 to x with length no more than M. Now $f(x_0) = 0$ and f is a C-Lipschitz function. Thus for each x there exists $\gamma \in \Gamma_{x_0 x}$ such that $\ell(\gamma) \leq Cd(x, x_0)$. As x_0 was arbitrary, we conclude that X is quasi-convex.

We now handle the situation where X is only locally compact. Let $\widehat{X}$ be the completion of X and view X as a subset of $\widehat{X}$. Since X is locally compact we see that X is an open subset of $\widehat{X}$. Extend the measure μ to $\widehat{X}$ by setting $\mu(\widehat{X} \setminus X) = 0$. Then $\widehat{X}$ equipped with the doubling measure μ admits a weak p-Poincaré inequality. In particular, $\widehat{X}$ is quasi-convex. Since X is locally compact and hence is an open subset of $\widehat{X}$, X is locally quasi-convex. Create the equivalence relation on X via $x \sim y$ if and only if there exists $\gamma \in \Gamma_{xy}$. Since X is locally quasi-convex, the equivalence classes are open. That X is connected implies that there is only one equivalence class. Hence X is rectifiably path connected.

Fix $x_0 \in X$. Define the function

$$g(x) = \inf \left\{ \ell(\gamma) \,:\, \gamma \in \Gamma_{x_0 x} \right\}.$$

Since X is locally quasi-convex, we see as in the proof of Proposition 3.2 that g is a locally Lipschitz function. Hence every point in X is a Lebesgue point of g. It follows from the definition of g that the function $\rho = 1$ is an upper gradient of g. As before in the proof of Proposition 3.2 we see that g is a C-Lipschitz function with C depending only on the constants of the Poincaré inequality and the doubling condition. In particular, for each $x \in X$ there exists $\gamma \in \Gamma_{x_0 x}$ with $\ell(\gamma) \leq Cd(x, x_0)$. As x_0 was arbitrary we conclude that X is quasi-convex. □

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