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INTERPOLATION PROPERTIES OF BESOV SPACES DEFINED ON METRIC SPACES

AMIRAN GOGATISHVILI, PEKKA KOSKELA AND NAGESWARI SHANMUGALINGAM

Abstract. Let $X = (X, d, \mu)$ be a doubling metric measure space. For $0 < \alpha < 1$, $1 \le p, q < \infty$, we define semi-norms

$$\|f\|_{B^\alpha_{p,q}(X)} = \left(\int_0^\infty \left(\int_X \fint_{B(x,t)} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x) \right)^{\frac{q}{p}} \frac{dt}{t^{\alpha q+1}} \right)^{\frac{1}{q}}.$$

When $q = \infty$ the usual change from integral to supremum is made in the definition. The Besov space $B^\alpha_{p,q}(X)$ is the set of those functions f in $L^p_{\text{loc}}(X)$ for which the semi-norm $\|f\|_{B^\alpha_{p,q}(X)}$ is finite. We will show that if a doubling metric measure space (X, d, μ) supports a $(1, p)$-Poincaré inequality, then the Besov space $B^\alpha_{p,q}(X)$ coincides with the real interpolation space $(L^p(X), KS^{1,p}(X))_{\alpha,q}$, where $KS^{1,p}(X)$ is the Sobolev space defined by Korevaar and Schoen [KS]. This results in (sharp) imbedding theorems. We further show that our definition of a Besov space is equivalent with the definition given by Bourdon and Pajot [BP], and establish a trace theorem.

1. Introduction

The Besov spaces $B^\alpha_{p,q}(\mathbb{R}^n)$ consisting of functions of smoothness order α occur naturally in many fields of analysis. Their applications often require a knowledge of interpolation properties, i.e. a description of the spaces which arise when the real method of interpolation is applied to a pair of spaces. Of the many different possible definitions for the Besov spaces, a central one is given in terms of a modulus of smoothness. We will adopt this approach in the metric setting.

Given $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, the L^p- modulus of smoothness $\omega(f, t)_p$, $t > 0$, of f is defined as

$$\omega(f, t)_p = \sup_{|h| \le t} \|\Delta_h(f, \cdot)\|_{L^p(\mathbb{R}^n)},$$

where $|h|$ is the Euclidean length of the vector h and $\Delta_h(f, x) = f(x+h) - f(x)$. As a general metric space possesses no group structure, a modification to this definition is needed. Let

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us first consider the modified modulus given by

$$w(f,t)_p = \left(\fint_{B(0,t)} \|\Delta_h(f,\cdot)\|^p_{L^p(\mathbb{R}^n)}\, dh \right)^{\frac{1}{p}},$$

Using subadditivity of $\|\Delta_h(f,\cdot)\|_{L^p(\mathbb{R}^n)}$, with respect to f, it is easy to show that $\omega(f,t)_p$ and $w(f,t)_p$ are equivalent, i.e. $C_1\omega(f,t)_p \le w(f,t)_p \le C_2\omega(f,t)_p$, with constants $C_1, C_2 > 0$ which depend only on p and n. Applying the Fubini theorem and a change of variables we obtain a form of our modified modulus of smoothness that does not rely on the group structure of $\mathbb{R}^n$:

$$\begin{aligned} w(f,t)_p &= \left(\int_{\mathbb{R}^n} \fint_{B(0,t)} |f(x+h) - f(x)|^p\, dh\, dx \right)^{1/p} \\ &= \left(\int_{\mathbb{R}^n} \fint_{B(x,t)} |f(y) - f(x)|^p\, dy\, dx \right)^{1/p}. \end{aligned}$$

Now let $X = (X, d, \mu)$ be a doubling metric measure space. Motivated by the above Euclidean case we set

$$E_p(f,t) := \left(\int_X \fint_{B(x,t)} |f(x) - f(y)|^p\, d\mu(y)\, d\mu(x) \right)^{1/p}$$

and define the Besov space $B^\alpha_{p,q}(X)$, $0 \le \alpha < \infty$, $0 < q \le \infty$, $1 \le p < \infty$, as the set of those functions in $L^p_{\text{loc}}(X)$ for which the semi-norm $\|f\|_{B^\alpha_{p,q}(X)}$ is finite. Here

$$\|f\|_{B^\alpha_{p,q}(X)} = \begin{cases} \left(\int_0^\infty (t^{-\alpha} E_p(f,t))^q \frac{dt}{t} \right)^{\frac{1}{q}}, & 1 \le q < \infty, \\ \sup_{t>0} t^{-\alpha} E_p(f,t), & q = \infty. \end{cases} \tag{1}$$

Our definition is rather concrete and gives the usual Besov space in the Euclidean setting. Moreover, it has very recently been shown by Müller and Yang [MY] that our definition here coincides with the definition based on test functions and used earlier by Han [H], Han and Yang [HY] and Yang [Y], provided that X, besides being doubling, also satisfies a reverse doubling condition. We show in Section 5 that, under a p-Poincaré inequality assumption, the Besov space $B^\alpha_{p,q}(X)$, $0 < \alpha < 1$, is realized as the real interpolation space $(L^p(X), KS^{1,p}(X))_{\alpha,q}$ between the corresponding $L^p(X)$ and Sobolev spaces. This is proved by showing that $E_p(u,t)$ is equivalent to the K-functional between $L^p(X)$ and $KS^{1,p}(X)$. Consequently, interpolation allows one to obtain embedding theorems. For brevity, we only give a simple example of such a theorem in Section 5; many further results can be obtained in a similar manner. Our Poincaré inequality assumption covers many interesting settings [HaK], [KZ], and in fact implies the reverse doubling assumption in [MY]. It is by now well understood that, under a certain Poincaré inequality assumption, various seemingly different definitions for a Sobolev space turn out to be equivalent; see Section 3. One of the consequences of the results in this paper, in comparison with [MY], is that many definitions for a Besov space that seem at first to be different also turn out to be equivalent under the assumption of Poincaré inequality. The Poincaré inequality is not inherited by subsets of X, but the trace of the Sobolev spaces (of functions on X) to a regular subset results in the Besov spaces on the subset, see Section 6.

If $\alpha \geq 1$ and $q < \infty$ in the above definition of $B^{\alpha}_{p,q}(X)$, then functions u in this space must have the property that $\liminf_{t\to 0} E_p(u,t)/t = 0$. In Section 5 we show that, consequently, under the Poincaré inequality assumption, $B^{\alpha}_{p,q}(X)$ only contains constant functions. This extends also to the case $q = \infty$ exactly when $\alpha > 1$.

Let us close this introduction by pointing out that a strongly related definition for a Besov space was considered in [BP] (also see [HaM], [P]). To be precise, they relied on the norm

$$\|f\|_{\mathcal{B}^{\alpha}_{p}(X)} = \left(\int_X \int_X \frac{|f(x)-f(y)|^p}{d(x,y)^{\alpha p}\mu(B(x,d(x,y)))} d\mu(x)d\mu(y)\right)^{\frac{1}{p}} \tag{2}$$

for $\alpha = Q/p$ on an Ahlfors Q-regular space. We will prove in Section 5 that, on a doubling metric space X, the norm (2) also generates $B^{\alpha}_{p,p}(X)$.

2. The K method of real interpolation

We recall only the essential definitions; for details we refer the reader to [BL], [N], and [BS].

A pair (X_0, X_1) of Banach spaces X_0 and X_1 is called a compatible couple if there is some Hausdorff topological vector space in which each of X_0 and X_1 is continuously embedded. For each compatible couple (X_0, X_1), the sum $X_0 + X_1$ and intersection $X_0 \cap X_1$ are Banach spaces under the norms

$$\|f\|_{X_0+X_1} = \inf\{\|f_0\|_{X_0} + \|f_1\|_{X_1} \,:\, f = f_0 + f_1\} \tag{3}$$

and

$$\|f\|_{X_0\cap X_1} = \max\{\|f\|_{X_0}, \|f\|_{X_1}\}, \tag{4}$$

respectively.

Let (X_0, X_1) be a compatible couple of Banach spaces. The K-functional is defined for each $f \in X_0 + X_1$ and $t > 0$ by

$$K(f,t,X_0,X_1) := \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} \,:\, f_0 \in X_0, f_1 \in X_1 \text{ with } f = f_0 + f_1\}.$$

Notice that K is an increasing function of t. Let (X_0, X_1) be a compatible couple and suppose $0 < \vartheta < 1$, $1 \leq q < \infty$ or $0 \leq \vartheta \leq 1$, $q = \infty$. Then the interpolation space $(X_0, X_1)_{\vartheta,q}$ consists of all f in $X_0 + X_1$ for which

$$\|f\|_{\vartheta,q} = \begin{cases} \left(\int_0^\infty (t^{-\vartheta}K(f,t,X_0,X_1))^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 0 < \vartheta < 1,\ 1 \leq q < \infty, \\ \sup_{t>0} t^{-\vartheta}K(f,t,X_0,X_1), & 0 \leq \vartheta \leq 1,\ q = \infty, \end{cases} \tag{5}$$

is finite.

It follows from the definition of the K-functional that each linear operator which is bounded on X_0 and X_1 is also bounded on $(X_0, X_1)_{\vartheta,q}$ for $0 < \vartheta < 1$ (see [BS], Chapter 5, Theorem 1.12).

The definition of $(X_0, X_1)_{\vartheta,q}$ in (5) could well be extended to the case $\vartheta > 1$. However, for the concrete K-functional considered in Section 4, the resulting space would only contain constant functions (see the discussion in Section 5).

While the theory of interpolation spaces, as given in [BS], assume X_0 and X_1 to be Banach spaces, this is in general not necessary. Indeed, it suffices for X_0 and X_1 be seminormed spaces, see [N]. The paper [N] extends the interpolation theory to a pair (X_0, X_1) of seminormed spaces that are continuously embedded in a Hausdorff topological vector space. The embedding of X_0 and X_1 in such a topological space is merely to ensure that the spaces $X_0 + X_1$ and $X_0 \cap X_1$ make sense. In this note, we will consider X_0 and X_1 to be seminormed spaces of measurable functions on a metric measure space X; in such a situation, the two spaces $X_0 + X_1$ and $X_0 \cap X_1$ always make sense. Specifically, we will consider X_0 to be the collection of all measurable functions $f : X \to [-\infty, \infty]$ such that $\int_X |f|^p \, d\mu$ is finite, and X_1 to be the collection of all measurable functions $f : X \to [-\infty, \infty]$ such that the seminorm $\|f\|_{\mathcal{KS}^{1,p}(X)}$, defined in Section 3 below, is finite. This seminorm is not in general a norm on X_1, as non-zero constant functions on X have zero seminorm, but non-zero $L^p(X)$-norm if $\mu(X)$ is finite. To take care of this fact, we consider X_0 and X_1 to be collections of functions rather than collections of equivalence classes of functions (for example, $X_0 \neq L^p(X)$). For this choice of X_0 and X_1, the two spaces $X_0 + X_1$ and $X_0 \cap X_1$ make sense, and as in [BS] we can consider the K-functional and the corresponding interpolation spaces. We leave it to the reader to verify that the proof of Proposition 4.2 in [BS] can be modified to show that if $0 < \theta < 1$ and $1 \le q_1 \le q_2 \le \infty$, then $(X_0, X_1)_{\theta, q_1}$ embeds into $(X_0, X_1)_{\theta, q_2}$ (this is because as a function of t, $K(f, t, X_0, X_1)$ is an increasing function); that the Holmstedt result (Theorem 2.1 of [BS]) holds; and hence the reiteration theorem (Theorem 2.4 of [BS]) holds in our setting. It is also elemetrary to check that if $T : X_0 + X_1 \to Y_0 + Y_1$ is linear transformation with $T : X_0 \to Y_0$ and $T : X_1 \to Y_1$ bounded, then $T : (X_0, X_1)_{\theta,q} \to (Y_0, Y_1)_{\theta,q}$ is also bounded.

3. Sobolev spaces on metric measure spaces

We say that $X = (X, d, \mu)$ is a doubling metric measure space if (X, d) is a metric space and μ a doubling measure on X, i.e. a positive Borel measure for which there exists a constant C_d such that

$$0 < \mu(B(x, 2r)) \le C_d \mu(B(x, r))$$

for all $x \in X$, $r > 0$; $B(x, r)$ denotes the set of all points $y \in X$ such that $d(x, y) < r$. If $B = B(x, r)$, we denote by τB, $\tau > 0$ its concentric dilate $B(x, \tau r)$. An iteration of the above inequality shows that there are constants C and s depending only on C_d such that

$$\mu(B(y, R)) \le C \left(\frac{R}{r}\right)^s \mu(B(x, r)), \tag{6}$$

whenever $x \in B(y, R)$ and $0 < r \le R \le 2 \operatorname{diam}(X)$.

As usual, if $A \subset X$ is μ-measurable, then $L^p(A)$ is the space of μ-measurable functions f such that the norm $\|f\|_{L^p(A)} = \left(\int_A |f|^p d\mu\right)^{\frac{1}{p}}$ for $1 \le p < \infty$, or $\|f\|_{L^\infty(A)} = \operatorname{ess\,sup}_A |f|$ is finite. $L^1_{\rm loc}(X)$ will denote functions that are p-integrable on all balls.

A Borel measurable function $g : X \to [0, \infty)$ is an upper gradient (cf. [HeK], [KM]) of a function $f : X \to \bar{\mathbb{R}}$ if

$$|f(\gamma(0)) - f(\gamma(1))| \le \int_\gamma g \, ds \tag{7}$$

for every rectifiable curve $\gamma : [0, 1] \to X$.

The Sobolev space $N^{1,p}(X)$, defined by Shanmugalingam in [S], consists of the functions $f \in L^p(X)$ having an $g \in L^p(X)$. The space $N^{1,p}(X)$ is a Banach space with the norm

$$\|f\|_{N^{1,p}(X)} = \|f\|_{L^p(X)} + \inf \|g\|_{L^p(X)},$$

where the infimum is taken over all upper gradients $g \in L^p(X)$ of f. The seminorm

$$\|f\|_{DN^{1,p}(X)} = \inf \|g\|_{L^p(X)}$$

where the infimum is over all upper gradients g of f, is of interest to us. The Dirichlet-Newton space $DN^{1,p}(X)$ consist of all measurable functions f for which $\|f\|_{DN^{1,p}(X)}$ is finite.

A pair $f \in L^q_{\text{loc}}(X)$ and measurable function $g \geq 0$ satisfies a (q,p)-Poincaré inequality if there are constants $C > 0$ and $\tau \geq 1$ such that

$$\left(\fint_B |f(y) - f_B|^q d\mu(y) \right)^{\frac{1}{q}} \leq C \operatorname{rad}(B) \left(\fint_{\tau B} g^p \right)^{\frac{1}{p}} \tag{8}$$

for all balls $B = B(x,r) \subset X$. Here $\fint$ denotes the average value of the integral and $f_B = \fint_B f \, d\mu$. If (8) holds for all measurable functions and their upper gradients with fixed constants $C > 0$ and $\tau \geq 1$, then X supports a (q,p)-Poincaré inequality. Such a space is necessarily connected.

Denote by $P^{1,p}(X)$ the set of functions $f \in L^1_{\text{loc}}(X)$ for which there exists a function $0 \leq g \in L^p(X)$ such that

$$\fint_B |f(y) - f_B| d\mu(y) \leq \operatorname{rad}(B) \left(\fint_{\tau B} g^p \right)^{\frac{1}{p}}$$

for all balls $\tau B = B(x, \tau r) \subset X$. The semi-norm on $P^{1,p}(X)$ is given by $||f||_{P^{1,p}(X)} = \inf ||g||_{L^p(X)}$, where the infimum is taken over all functions g that satisfy the above inequality.

We say that $f \in M^{1,p}(X)$ if there exists $g \in L^p(X)$ such that for a.e. x,y the inequality

$$|f(x) - f(y)| \leq d(x,y)(g(x) + g(y)) \tag{9}$$

holds. The semi-norm on $M^{1,p}(X)$ is defined by $\|f\|_{M^{1,p}(X)} = \inf \|g\|_{L^p(X)}$ where the infimum is taken over all g satisfying the above inequality. In $\mathbb{R}^n$ this definition yields the usual Sobolev space and the semi-norm is equivalent to the usual semi-norm (see [Ha]).

The Sobolev space of Korevaar and Schoen consists of those functions for which the following semi-norm of f is finite:

$$\|f\|_{KS^{1,p}(X)} := \limsup_{\epsilon \to 0} \left(\int_X \left(\fint_{B(x,\epsilon)} \frac{|f(x) - f(y)|^p}{\epsilon^p} d\mu(y) \right) d\mu(x) \right)^{1/p} = \limsup_{\epsilon \to 0} \frac{E_p(f,\epsilon)}{\epsilon}. \tag{10}$$

When X is a Riemannian manifold this definition yields the usual Sobolev space and the quantity in (10) is equivalent to the usual semi-norm (see [KS]).

We have the following inclusions (cf.[KM]): $M^{1,p}(X) \subset P^{1,p}(X) \subset KS^{1,p}(X) \subset N^{1,p}(X)$. It is an interesting question when it is possible to replace limsup by sup in the definition (10). One can give simple examples of doubling (even Ahlfors regular) metric spaces on which this is not possible. Indeed, $X = [0,1] \cup [2,3]$ equipped with the Lebesgue measure and the usual distance is such an example.

We define

$$\|f\|_{\mathcal{K}S^{1,p}(X)} := \sup_{\epsilon>0}\left(\int_X\left(\fint_{B(x,\epsilon)}\frac{|f(x)-f(y)|^p}{\epsilon^p}\,d\mu(y)\right)d\mu(x)\right)^{1/p} = \sup_{\epsilon>0}\frac{E_p(f,\epsilon)}{\epsilon}.$$

A function f is said to be in $\mathcal{K}S^{1,p}(X)$ if $\|f\|_{\mathcal{K}S^{1,p}(X)}<\infty$. In general $\mathcal{K}S^{1,p}(X)\subset KS^{1,p}(X)$.

If X supports a $(1,p)$-Poincaré inequality, $p>1$, for $N^{1,p}(X)$, then

$$M^{1,p}(X)=P^{1,p}(X)=KS^{1,p}(X)=\mathcal{K}S^{1,p}(X)=DN^{1,p}(X)$$

and

$$M^{1,p}(X)\cap L^p(X)=P^{1,p}(X)\cap L^p(X)=KS^{1,p}(X)\cap L^p(X)=\mathcal{K}S^{1,p}(X)\cap L^p(X)=N^{1,p}(X)$$

with equivalent semi-norms. This follows by combining results from [KM] and [KZ] and noting that the Poincaré inequality implies the density of Lipschitz functions in $N^{1,p}(X)$. There are no concrete geometric characterizations for a doubling space X to support a Poincaré inequality. It is would be interesting to know if the weaker conclusion that there be a constant C such that

$$\|f\|_{\mathcal{K}S^{1,p}(X)}\le C\|f\|_{KS^{1,p}(X)} \tag{11}$$

would allow for a concrete characterization.

4. Calculation of the K-functional and the Besov space

The main result of this paper is the following description of the K-functional for the couple $(L^p(X),KS^{1,p}(X))$.

Theorem 4.1. *Suppose that* (11) *holds. There exist constants $c_1,c_2>0$ such that for all $t>0$,*

$$c_1E_p(f,t)\le K(f,t;L^p(X),KS^{1,p}(X))\le c_2E_p(f,t). \tag{12}$$

Proof. First suppose $f=g+h$ with $g\in L^p(X)$ and $h\in KS^{1,p}(X)$. Since $u\to E_p(u,t)$ is subadditive,

$$E_p(f,t)\le E_p(g,t)+E_p(h,t).$$

$$\begin{aligned}E_p(g,t)^p &\le 2^{p-1}\int_X|g(x)|^p\,d\mu(x)+2^{p-1}\int_X\fint_{B(x,t)}|g(y)|^p\,d\mu(y)\,d\mu(x)\\ &\le 2^{p-1}\int_X|g(x)|^p\,d\mu(x)+2^{p-1}\int_X|g(y)|^p\int_{B(y,t)}\frac{d\mu(x)}{\mu(B(x,t))}\,d\mu(y).\end{aligned}$$

Using the doubling property of the measure and the fact that $B(y,t)\subset B(x,2t)$ whenever $x\in B(y,t)$ we conclude that

$$\int_{B(y,t)}\frac{d\mu(x)}{\mu(B(x,t))}\le C\int_{B(y,t)}\frac{d\mu(x)}{\mu(B(x,2t))}\le C\int_{B(y,t)}\frac{d\mu(x)}{\mu(B(y,t))}=C.$$

Therefore we have $E_p(g,t)^p\le C\int_X|g(x)|^p\,d\mu(x)$. The estimate $E_p(h,t)\le Ct\|h\|_{KS^{1,p}}$ is simply a consequence of the definition and assumption (11).

Consequently,

$$E_p(f,t) \le C(\|g\|_{L^p} + t\|h\|_{KS^{1,p}}).$$

Taking an infimum over such decompositions gives the left hand side estimate in (12).

For the right hand inequality in (12), fix $t > 0$, and let $\{B_i = B(x_i, t/6)\}_i$ be a cover of X such that $\{B(x_i, t/30)\}_i$ is pairwise disjoint and $\sum_i \chi_{B(x_i,t)} \le C$ with $C > 0$ independent of t (but depends on the doubling constant). There is a collection $\{\varphi_i\}$ of functions $X \to \mathbb{R}$ such that

(1) each φ_i is Ct^{-1} Lipschitz.
(2) $0 \le \varphi_i \le 1$ for all i.
(3) $\varphi_i(x) = 0$ for $x \in X \setminus 2B_i$ for all i.
(4) $\sum_i \varphi_i(x) = 1$ for all $x \in X$.

A collection $\{\varphi_i\}$ as above is called a partition of unity with respect to $\{B_i = B(x_i, t/6)\}_i$. (for the existence, see e.g. [KST] or [CW]). Now let h be defined by

$$h(x) = \sum_i f_{B_i}\varphi_i(x).$$

Let $g(x) = f(x) - h(x)$. By property 4 we have that

$$g(x) = \sum_i (f(x) - f_{B_i})\varphi_i(x).$$

By the bounded overlap property $\sum_i \chi_{B(x_i,t)} \le C$, there exists a positive integer N, which depends only on the overlap constant C, so that we may partition the collection $\{B_i = B(x_i, t/6)\}_i$ into families $\mathcal{B}_1, \dots, \mathcal{B}_N$, so that the balls in the family $\mathcal{B}_j = \{B_{i,j}\}_i$ have the property that $\{3B_{i,j}\}_i$ is a pairwise disjoint family.

$$\begin{aligned}\|g\|^p_{L^p(X)} \le& C\sum_{j=1}^N \sum_i \int_X |f(x) - f_{B_{i,j}}|^p (\varphi_{i,j}(x))^p \, d\mu(x)\\ \le& C\sum_{j=1}^N \sum_i \int_{2B_{i,j}} |f(x) - f_{B_{i,j}}|^p \, d\mu(x).\end{aligned}$$

Using the doubling property of the measure and the fact that

$$B_{i,j} \subset B(y, t/3) \quad \text{and} \quad B(y, t/3) \subset 3B_{i,j} \quad \text{for all} \quad y \in B_{i,j},$$

we have that

$$\int_{2B_{i,j}} |f(x) - f_{B_{i,j}}|^p \, d\mu(x) \le C \int_{2B_{i,j}} \fint_{B(y,t/3)} |f(x) - f(y)|^p \, d\mu(x) d\mu(y).$$

Then, we obtain

$$\begin{aligned}
\|g\|^p_{L^p(X)} \le & C\sum_{j=1}^N \sum_i \int_{2B_{i,j}} \fint_{B(y,t/3)} |f(x)-f(y)|^p\, d\mu(x)\, d\mu(y) \\
\le & C\int_X \fint_{B(y,t)} |f(x)-f(y)|^p\, d\mu(x)\, d\mu(y) \\
\le & CE_p(f,t).
\end{aligned} \tag{13}$$

We now want to estimate h in the $KS^{1,p}$ norm. Let $y \in B(x,t/3)$. Using the properties of the functions φ_i, we have that

$$|h(x)-h(y)| = \big|\sum_j (f_{B(x,t/3)} - f_{B_j})(\varphi_j(x)-\varphi_j(y))\big|.$$

Let $J = \{j : B(x,t/3) \cap B(x_j,t/3) \neq \emptyset\}$. Then $\operatorname{card} J \le C$ for some constant that depends only on the doubling constant C_d, and $B(x_i,t/3) \subset B(x,t)$.

Since $|f_{B(x,t/3)} - f_{B_j}| \le C\fint_{B(x,t)} |f(y)-f_{B(x,t)}|\, d\mu(y)$, when $j \in J$, we obtain that

$$|h(x)-h(y)| \le C\frac{d(x,y)}{t} \fint_{B(x,t)} |f(y)-f_{B(x,t)}|\, d\mu(y) \quad \text{for all} \quad y \in B(x,t/3).$$

Hence integration gives for $0<\epsilon<t/6$,

$$\begin{aligned}
\fint_{B(x,\epsilon)} |h(x)-h(y)|^p\, d\mu(y) \le & C\frac{\epsilon^p}{t^p}\left(\fint_{B(x,t)} |f(y)-f_{B(x,t)}|\, d\mu(y)\right)^p \\
\le & C\frac{\epsilon^p}{t^p}\fint_{B(x,t)} |f(y)-f_{B(x,t)}|^p\, d\mu(y).
\end{aligned}$$

Therefore, by using the Hölder inequality we have the estimate

$$
\begin{aligned}
\|h\|_{KS_{1,p}(X)} &\le Ct^{-1}\left(\int_X \fint_{B(x,t)} |f(y)-f_{B(x,t)}|^p\,d\mu(y)d\mu(x)\right)^{1/p}\\
&\le Ct^{-1}\left(\int_X \fint_{B(x,t)} |f(y)-f(x)|^p\,d\mu(y)d\mu(x)\right)^{1/p}\\
&+ Ct^{-1}\left(\int_X |f(x)-f_{B(x,t)}|^p\,d\mu(x)\right)^{1/p}\\
&\le Ct^{-1}\left(\int_X \fint_{B(x,t)} |f(y)-f(x)|^p\,d\mu(y)d\mu(x)\right)^{1/p}\\
&+ Ct^{-1}\left(\int_X \left(\fint_{B(x,t)} |f(y)-f(x)|\,d\mu(y)\right)^p d\mu(x)\right)^{1/p}\\
&\le Ct^{-1}\left(\int_X \fint_{B(x,t)} |f(y)-f(x)|^p\,d\mu(y)d\mu(x)\right)^{1/p}\\
&\le Ct^{-1}E_p(f,t).
\end{aligned}
$$

This together with (13) proves the right hand estimate in (12). □

We continue with a version of Theorem 4.1 for the non-homogeneous Sobolev space $KS^{1,p}(X)\cap L^p(X)$ that is equipped with the norm given by (4). Notice that in this case a lower bound $c\min(1,t)\|f\|_p$ is immediate from the definition and does not require (11); consequently the interpolation space from (5) would only contain the zero function when $\vartheta \ge 1$. This is not the case when the homogeneous Sobolev space $KS^{1,p}(X)$ is used, as seen for example for the one-dimensional metric space $X=[0,1]\cup[2,3]$.

Theorem 4.2. *Suppose that* (11) *holds. There exist constants* $c_1, c_2 > 0$ *such that for all* $t>0$

(14)
$$c_1(\min(1,t)\|f\|_p + E_p(f,t)) \le K(f,t;L^p(X),KS^{1,p}(X)\cap L^p(X)) \le c_2(\min(1,t)\|f\|_p + E_p(f,t)),$$

Proof. By the definition of the K-functional as an infimum,

$$K(f,t;L^p,KS^{1,p}) \le K(f,t;L^p,KS^{1,p}\cap L^p)$$

and

$$\min(1,t)\|f\|_p \le K(f,t;L^p,KS^{1,p}\cap L^p),$$

and hence we obtain the left hand side estimate in (14) from (12). The right hand side estimate in (14) follows from the proof of Theorem 4.1 by noticing that

$$K(f,t;L^p,KS^{1,p}\cap L^p)\le 2K(f,t;L^p,KS^{1,p})+t\|f\|_{L^p}$$

and, observing from the definition of a K-functional that

$$K(f,t;L^p,KS^{1,p}\cap L^p)\le \|f\|_{L^p}.$$

□

Having determined the K-functional between $L^p(X)$ and $KS^{1,p}(X)$ in terms of $E_p(f,t)$, it is now routine to identify the corresponding (α,q) interpolation spaces as Besov spaces.

Corollary 4.3. *Assume that metric space is such that* (11) *holds. Let* $1\le p<\infty$. *If* $0<\alpha<1$ *and* $1\le q\le\infty$, *then*

$$B^\alpha_{p,q}(X)=(L^p(X),KS^{1,p}(X))_{\alpha,q}$$

and

$$B^\alpha_{p,q}(X)\cap L^p(X)=(L^p(X),KS^{1,p}(X)\cap L^p(X))_{\alpha,q}$$

with equivalent norms. Furthermore,

$$B^0_{p,\infty}(X)=L^p(X)\ \text{and}\ B^1_{p,\infty}(X)\cap L^p(X)=KS^{1,p}(X)\cap L^p(X).$$

By the reiteration theorem (see [BS], Chapter 5, Theorem 2.4), Corollary 4.3 immediately yields an interpolation theorem:

Theorem 4.4. *Suppose that* (11) *holds for a doubling metric space* X. *Let* $1\le p<\infty$, $0<\alpha_0,\alpha_1,\theta<1$ *and* $1\le q,q_0,q_1\le\infty$, *or* $0<\alpha_0,\theta<1$, $\alpha_1=0$, $1\le q,q_0\le\infty$, *and* $q_1=\infty$. *Then*

$$\begin{aligned}\left(B^{\alpha_0}_{p,q_0}(X),\,B^{\alpha_1}_{p,q_1}(X)\right)_{\theta,q}&=B^\alpha_{p,q}(X),\quad \alpha=(1-\theta)\alpha_0+\theta\alpha_1,\quad \alpha_0\ne\alpha_1,\\ \left(KS^{1,p}(X),\,B^{\alpha_1}_{p,q_1}(X)\right)_{\theta,q}&=B^\alpha_{p,q}(X),\quad \alpha=(1-\theta)+\theta\alpha_1,\end{aligned}$$

with equivalent norms.

Under the assumption that X be Ahlfors regular, the interpolation result Theorem 4.4 has been established by Yang [Y]. An analogous version of the above interpolation theorem has been established by Han, Müller and Yang [HMY, Theorem 8.1 and Theorem 8.3] for Besov type spaces constructed using frames and approximations of the identity in general metric measure spaces equipped with a doubling measure that in addition has a reverse doubling property. Since under the reverse doubling condition of the measure the Besov spaces constructed in [HMY] coincide with the Besov spaces considered in this paper ([MY]), their result subsumes the above theorem if the measure on X satisfies a reverse doubling condition. Given the construction of Besov spaces considered in [HMY], their proof of the interpolation theorem requires a more intricate argument.

Remarks 4.5. If our metric measure space X supports a $(1,p)$-Poincaré inequality, then (11) holds and all the Sobolev spaces discussed in Section 2 coincide, and we can use any of them instead of $KS^{1,p}(X)$.

5. Properties of the Besov space

We begin by showing that our interpolation theorem implies a generalization of the Sobolev embedding theorem for Besov spaces.

The Lorentz spaces $L^{p,q}(X)$, consist of measurable functions f of finite norm $\|f\|_{L^{p,q}(X)} = \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L^q(0,\infty)}$, where f^* is the *non-increasing rearrangement* of a measurable function f on X, defined by

$$f^*(t) = \inf\{\lambda > 0 : \mu(\{x \in X : |f(x)| > \lambda\}) \leq t\}, \quad t \in [0,\infty).$$

Theorem 5.1. *Suppose that (X,μ) is Ahlfors s-regular in the large with $s > 1$ from (6): that is, there exists $r_0 > 0$ and $x_0 \in X$ such that whenever $r > r_0$,*

$$r^s/C \leq \mu(B(x_0,r)) \leq Cr^s.$$

Suppose further that $1 \leq q \leq \infty$, $0 < \alpha < 1$, $1 < p < s$, and that X supports a $(1,p)$-Poincaré inequality. Then there is a constant C such that

$$\inf_{c\in\mathbb{R}} \|f - c\|_{L^{\overline{p},q}(X)} \leq C\|f\|_{B^\alpha_{p,q}(X)},$$

where $\overline{p} = \frac{sp}{s-p\alpha}$.

We note here that if X is a graph, then X is not s-Ahlfors regular, but can be s-Ahlfors regular in the large.

Proof. By the Sobolev embedding theorem (Theorem 5.1) from [HaK] we have that whenever $f \in KS^{1,p}(X)$,

$$\left(\fint_{B(x_0,R)} |f - f_{B(x_0,R)}|^{p*} d\mu \right)^{\frac{1}{p*}} \leq Cr \left(\fint_{B(x_0,CR)} g^p d\mu \right)^{\frac{1}{p}},$$

where $p* = \frac{sp}{s-p}$ whenever g is an upper gradient of a function $f \in L^p(X)$. Using the Ahlfors regularity of μ we conclude that for $R > r_0$,

$$\left(\int_{B(x_0,R)} |f - f_{B(x_0,R)}|^{p*} d\mu \right)^{\frac{1}{p*}} \leq C \left(\int_{B(x_0,CR)} g^p d\mu \right)^{\frac{1}{p}}.$$

Let $B_k = B(x_0,k)$ for $k \geq 1$. Then

$$\left(\int_{B_1} |f - f_{B_k}|^{p*} d\mu \right)^{1/p*} \leq \left(\int_{B_k} |f - f_{B_k}|^{p*} d\mu \right)^{1/p*} \leq C \left(\int_X g^p d\mu \right)^{1/p}.$$

It follows that the sequence $(f - f_{B_k})$ is bounded in $L^{p*}(B_1)$. As $f \in L^{p*}(B_1)$, it follows that the sequence (f_{B_k}) of real numbers is bounded, and hence has a convergent subsequence that converges to a finite number c. Now

$$\left(\int_{B_k} |f - c|^{p*} d\mu \right)^{1/p*} \leq \left(\int_{B_k} |f - f_{B_k}|^{p*} d\mu \right)^{1/p*} + |f_{B_k} - c| \leq C \left(\int_X g^p d\mu \right)^{1/p} + |f_{B_k} - c|,$$

and letting $k \to \infty$ through the indices corresponding to our subsequence, we conlude that

$$\|f - c\|_{L^{p*}(X)} \leq C\|f\|_{P^{1,p}(X)}.$$

Because $\frac{1-\alpha}{p}+\frac{\alpha}{p*}=\frac{1}{\overline{p}}$, the usual interpolation theorem for Lebesgue spaces states that $L^{\overline{p},q}(X)=(L^p(X),L^{p*}(X))_{\alpha,q}$ with equivalence of norms (see [BL], Theorem 5.2.1). Using Corollary 4.3, whenever $f\in B^\alpha_{p,q}(X)$,

$$\begin{aligned}\inf_{c\in\mathbb{R}}\|f-c\|_{L^{\overline{p},q}(X)}&\le C\inf_c\|f-c\|_{(L^p(X),L^{p*}(X))_{\alpha,q}}\le C\inf_c\|f-c\|_{(L^p(X),P^{1,p}(X))_{\alpha,q}}\\&\approx C\inf_c\|f-c\|_{(L^p(X),KS^{1,p}(X))_{\alpha,q}}\approx C\|f\|_{B^\alpha_{p,q}(X)}.\end{aligned}$$

□

For simplicity, we refrain from giving further consequences of our interpolation theorem. Hoewever, we wish to discuss the restriction $\alpha<1$ in Theorem 5.1. We claim that, under the $(1,p)$-Poincaré inequality assumption, $B^\alpha_{p,q}(X)$ only contains constant functions when both $\alpha\ge 1$ and $q<\infty$ hold and also when both $\alpha>1$ and $q=\infty$ hold. To see this one reasons as follows.

Under these conditions, the definition of $B^\alpha_{p,q}(X)$ implies that

$$\liminf_{t\to 0}\frac{E_p(f,t)}{t}=0$$

when $f\in B^\alpha_{p,q}(X)$. Consider the approximating function h constructed in the second part of the proof of Theorem 4.1, and denote it h_t (as it indeed depends on t.) One easily sees that $h_t\to f$ in L^p_{loc} when $t\to 0$. On the other hand, from the proof of Theorem 4.5 in [KM] one observes that h_t has an upper gradient g_t so that

$$||g_t||_{L^p}\le Ct^{-1}E_p(f,t),$$

where C is independent of t. Applying the $(1,p)$-Poincaré inequality to h_t, taking the limit inferior with respect to t and inserting the fact that X must be connected, we conclude that f must be constant.

We conclude this section by showing that the Besov spaces considered by Bourdon and Pajot [BP] identify with our spaces when we choose $q=p$.

Theorem 5.2. *Let $1\le p<\infty$, $\alpha>0$. Then*

$$\|f\|_{B^\alpha_{p,p}(X)}\approx\|f\|_{\mathcal{B}^\alpha_p(X)}.$$

Proof. We first observe that for each $x\in X$ and $k\in\mathbb{Z}$, doubling of our measure μ gives us the estimate

$$\begin{aligned}&\int_{2^k}^{2^{k+1}}\fint_{B(x,t)}|f(x)-f(y)|^p d\mu(y)\,\frac{dt}{t^{1+\alpha p}}\\&\qquad\le\int_{2^k}^{2^{k+1}}\frac{1}{2^{k(1+\alpha p)}}\frac{1}{\mu(B(x,2^k))}\int_{B(x,2^{k+1})}|f(x)-f(y)|^p\,d\mu(y)\,dt\\&\qquad\le\frac{2^k}{2^{k(1+\alpha p)}}\frac{1}{\mu(B(x,2^k))}\int_{B(x,2^{k+1})}|f(x)-f(y)|^p\,d\mu(y)\\&\qquad\le\frac{2^{\alpha p}}{2^{(k+1)\alpha p}}\frac{C}{\mu(B(x,2^{k+1}))}\int_{B(x,2^{k+1})}|f(x)-f(y)|^p\,d\mu(y)\\&\qquad=C\,2^{-(k+1)\alpha p}\fint_{B(x,2^{k+1})}|f(x)-f(y)|^p\,d\mu(y).\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{2^k}^{2^{k+1}} \fint_{B(x,t)} |f(x)-f(y)|^p d\mu(y)\,\frac{dt}{t^{1+\alpha p}} \\
&\qquad \geq \frac{2^k}{2^{(k+1)(1+\alpha p)}}\,\frac{1}{\mu(B(x,2^{k+1}))}\int_{B(x,2^k)} |f(x)-f(y)|^p d\mu(y) \\
&\qquad \geq \frac{2^{-k\alpha p}}{C} \fint_{B(x,2^k)} |f(x)-f(y)|^p d\mu(y).
\end{aligned}$$

Thus,

$$\int_0^\infty \fint_{B(x,t)} |f(x)-f(y)|^p d\mu(y)\,\frac{dt}{t^{1+\alpha p}} \approx \sum_{k\in\mathbb{Z}} 2^{-k\alpha p} \fint_{B(x,2^k)} |f(x)-f(y)|^p\, d\mu(y). \tag{15}$$

By the Fubini theorem,

$$\begin{aligned}
\|f\|^p_{B^\alpha_{p,p}} &= \int_0^\infty t^{-\alpha p} E_p(f,t)^p \frac{dt}{t} \\
&= \int_0^\infty \int_X \fint_{B(x,t)} |f(x)-f(y)|^p d\mu(y) d\mu(x) \frac{dt}{t^{1+\alpha p}} \\
&= \int_X \int_0^\infty \fint_{B(x,t)} |f(x)-f(y)|^p d\mu(y) \frac{dt}{t^{1+\alpha p}}\, d\mu(x).
\end{aligned}$$

Therefore by (15),

$$\begin{aligned}
\|f\|^p_{B^\alpha_{p,p}} &\approx \int_X \sum_{k\in\mathbb{Z}} 2^{-k\alpha p} \fint_{B(x,2^k)} |f(x)-f(y)|^p d\mu(y) d\mu(x) \\
&\approx \int_X \sum_{k\in\mathbb{Z}} 2^{-k\alpha p} \frac{1}{\mu(B(x,2^k))} \sum_{i=-\infty}^{k} \int_{B(x,2^i)\setminus B(x,2^{i-1})} |f(x)-f(y)|^p d\mu(y) d\mu(x) \\
&\approx \int_X \sum_{i\in\mathbb{Z}} \int_{B(x,2^i)\setminus B(x,2^{i-1})} |f(x)-f(y)|^p d\mu(y) d\mu(x) \sum_{k=i}^{\infty} 2^{-k\alpha p} \frac{1}{\mu(B(x,2^k))} \\
&\approx \int_X \sum_{i\in\mathbb{Z}} 2^{-i\alpha p} \frac{1}{\mu(B(x,2^i))} \int_{B(x,2^i)\setminus B(x,2^{i-1})} |f(x)-f(y)|^p d\mu(y) d\mu(x) \\
&\approx \int_X \sum_{i\in\mathbb{Z}} \int_{B(x,2^i)\setminus B(x,2^{i-1})} \frac{|f(x)-f(y)|^p}{d(x,y)^{\alpha p}\mu(B(x,d(x,y)))} d\mu(y) d\mu(x) \\
&\approx \int_X \int_X \frac{|f(x)-f(y)|^p}{d(x,y)^{\alpha p}\mu(B(x,d(x,y)))} d\mu(y) d\mu(x) \\
&= \|f\|^p_{\mathcal{B}^\alpha_p}.
\end{aligned}$$

$\square$

6. Besov spaces and a trace theorem

Recall from the discussion of Section 3 and (9) that for $p > 1$, if X supports a $(1,p)$-Poincaré inequality, then a function $f \in L^p(X)$ belongs to the Newton-Sobolev class $N^{1,p}(X)$ if and only if there is a non-negative function $g \in L^p(X)$ such that for a.e. $x, y \in X$, the inequality $|f(x) - f(y)| \le d(x,y)[g(x) + g(y)]$ holds. A version of this inequality holds for Besov functions as well, as demonstrated by the following two lemmata. See [HaM] for a version in the Euclidean setting.

Lemma 6.1. *Let (X, d, μ) be a doubling metric measure space with diam$(X) = R_0$, and $1 \le p < \infty$. If $f \in B^\alpha_{p,p}(X)$, then there is a non-negative function $g \in L^p(X)$ such that for μ-a.e. $x, y \in X$,*

$$|f(x) - f(y)| \le d(x,y)^\alpha\, [g(x) + g(y)].$$

Proof. Let $x, y \in X$, $R = d(x,y)$, and for $r > 0$ and $z \in X$, let $B_r(z) := B(z,r)$. By the doubling property of μ,

$$\begin{aligned}|f(x) - f(y)| &\le |f(x) - f_{B_R(x)}| + |f(y) - f_{B_R(x)}| \\ &\le \fint_{B_R(x)} |f(x) - f(z)|\mu(z) + C \fint_{B_{2R}(y)} |f(y) - f(z)|\mu(z) \\ &\le C\, d(x,y)^\alpha \left(\frac{\fint_{B_R(x)} |f(x) - f(z)|\mu(z)}{R^\alpha} + \frac{\fint_{B_{2R}(y)} |f(y) - f(z)|\mu(z)}{(2R)^\alpha} \right).\end{aligned}$$

With g defined by

$$g(x) = \sup\left\{ \frac{\fint_{B_r(x)} |f(x) - f(z)|\mu(z)}{r^\alpha} \;:\; 0 < r < 3R_0 \right\},$$

we see that for all $x, y \in X$,

$$|f(x) - f(y)| \le C d(x,y)^\alpha\, [g(x) + g(y)].$$

It suffices now to show that $g \in L^p(X)$. To see this, note that if $x \in X$ and $r > 0$, by choosing an integer k such that $2^{k-1} < r \le 2^k$,

$$\begin{aligned}\frac{\fint_{B_r(x)} |f(x) - f(y)|\mu(y)}{r^\alpha} &\le \frac{C}{(2^k)^\alpha} \fint_{B(x,2^k)} |f(y) - f(x)|\, d\mu(y) \\ &\le C \left(2^{-kp\alpha} \fint_{B(x,2^k)} |f(y) - f(x)|^p\, d\mu(y) \right)^{1/p},\end{aligned}$$

and so

$$g(x)^p \le C \sum_{k \in \mathbb{Z}} 2^{-kp\alpha} \fint_{B(x,2^k)} |f(y) - f(x)|^p\, d\mu(y).$$

Now by (15),

$$\int_X g^p\, d\mu \le C \int_X C \sum_{k \in \mathbb{Z}} 2^{-kp\alpha} \fint_{B(x,2^k)} |f(y) - f(x)|^p\, d\mu(y) \le C\, \|f\|^p_{B^\alpha_{p,p}} < \infty.$$

□

The next lemma provides a partial converse of the above result.

Lemma 6.2. *Let $1 \le p < \infty$ and $diam(X) = R_0 < \infty$. Suppose that $f \in L^1_{loc}(X)$ and $g \in L^p(X)$ such that for μ-a.e. $x, y \in X$,*

$$|f(x) - f(y)| \le d(x,y)^\beta\, [g(x) + g(y)].$$

Then for all $0 < \alpha < \beta$, f belongs to the Besov space $B^\alpha_{p,p}(X)$.

Proof. By Theorem 5.2,

$$\|f\|^p_{B^\alpha_{p,p}} \approx \int_X \int_X \frac{|f(x) - f(y)|^p}{d(x,y)^{\alpha p}\mu(B(x, d(x,y)))}\, d\mu(y)\, d\mu(x).$$

By the assumption on f and by the doubling property of μ,

$$\begin{aligned}\int_X \frac{|f(x) - f(y)|^p}{d(x,y)^{\alpha p}\mu(B(x, d(x,y)))}\, d\mu(y) &\le C \int_X \frac{d(x,y)^{\beta p}\,[g(x)^p + g(y)^p]}{d(x,y)^{\alpha p}\mu(B(x, d(x,y)))}\, d\mu(y) \\ &\le Cg(x)^p \int_X \frac{d(x,y)^{p(\beta-\alpha)}}{\mu(B(x, d(x,y)))}\, d\mu(y) + C\int_X d(x,y)^{p(\beta-\alpha)}\, \frac{g(y)^p}{\mu(B(y, d(x,y)))}\, d\mu(y).\end{aligned}$$

Therefore, with

$$C_0(x) := \int_X \frac{d(x,y)^{p(\beta-\alpha)}}{\mu(B(x, d(x,y)))}\, d\mu(y),$$

an employment of Fubini's theorem now yields

$$\|f\|^p_{B^\alpha_{p,p}} \le C \int_X C_0(x) g(x)^p\, d\mu(x).$$

Since $g \in L^p(X)$, it suffices to prove that the function C_0 is bounded on X. To see this, let $k_0 \in \mathbb{Z}$ with $2^{k_0-1} < R_0 \le 2^{k_0}$. Then for $x \in X$, by the doubling proeprty of μ again,

$$\begin{aligned}C_0(x) = \int_X \frac{d(x,y)^{p(\beta-\alpha)}}{\mu(B(x, d(x,y)))}\, d\mu(y) &= \sum_{k\in\mathbb{Z}} \int_{B(x,2^k)\setminus B(x,2^{k-1})} \frac{d(x,y)^{p(\beta-\alpha)}}{\mu(B(x, d(x,y)))}\, d\mu(y) \\ &\le \sum_{k=-\infty}^{k_0} 2^{kp(\beta-\alpha)}\, \frac{\mu(B(x,2^k) \setminus B(x, 2^{k-1}))}{\mu(B(x, 2^{k-1}))} \\ &\le C \sum_{k=-\infty}^{k_0} 2^{kp(\beta-\alpha)},\end{aligned}$$

and the last sum is finite since $\beta - \alpha > 0$. □

The rest of this section is devoted to exploring traces of Newton-Sobolev functions in regular compact subsets of a complete metric space X equipped with a Radon doubling measure μ supporting a $(1,p)$-Poincaré inequality for some $1 < p < \infty$. Recall from [KZ] that there exists $1 \le q < p$ such that $(1,q)$-Poincaré inequality holds as well.

Let $K \subset X$ be a compact set. If $f \in N^{1,p}(X)$ and $x, y \in K$, then by the $(1,q)$-Poincaré inequality, if x and y are both Lebesgue points of f and $0 \le \lambda < 1$,

$$\begin{aligned}
|f(x) - f(y)| &\le \sum_{k\in\mathbb{Z}} |f_{B_k} - f_{B_{k-1}}| \\
&\le C \sum_{k\in\mathbb{Z}} \fint_{B_k} |f - f_{B_k}|\, d\mu \\
&\le d(x,y)\, C \sum_{k\in\mathbb{Z}} 2^{-|k|} \left(\fint_{\lambda B_k} g_f^q \, d\mu \right)^{1/q} \\
&\le d(x,y)^{1-\lambda} C \sum_{k\in\mathbb{Z}} 2^{-|k|(1-\lambda)} \left(2^{-|k|} d(x,y)\right)^{\lambda} \left(\fint_{\tau B_k} g_f^q \, d\mu \right)^{1/q} \\
&\le d(x,y)^{1-\lambda} C \left(\sum_{k\in\mathbb{Z}} 2^{-(1-\lambda)|k|} \right) [M_{\lambda,q} g_f(x) + M_{\lambda,q} g_f(y)],
\end{aligned}$$

where

$$M_{\lambda,q} g(x) = \sup_{0<r<2\mathrm{diam}(K)} r^{\lambda} \left(\fint_{B(x,r)} g^q \, d\mu \right)^{1/q}$$

is a fractional maximal function of g. If f is a continuous function in $N^{1,p}(X)$, we now have for all $x, y \in K$,

$$|f(x) - f(y)| \le d(x,y)^{1-\lambda}\, C\, [M_{\lambda,q} g_f(x) + M_{\lambda,q} g_f(y)]. \tag{16}$$

In what follows, $s > 0$ is the number given by (6) for the measure μ.

Lemma 6.3. *Suppose that K is Ahlfors γ-regular, and set $\nu = \mathcal{H}^{\gamma}|_K$. If $0 \le s - \gamma \le \lambda q < s$, then $M_{\lambda,q} : L^q(X,\mu) \to wk - L^{q\gamma/(s-\lambda q)}(K,\nu)$ is bounded; that is, $M_{\lambda,q}$ is of weak type $(q, q\gamma/(s-\lambda q))$.*

Proof. For $t > 0$ let $E_t = \{x \in K : M_{\lambda,q} g(x) > t\}$. For each $x \in E_t$ let $0 < r_x < 2\mathrm{diam}(K)$ such that

$$\frac{1}{t^q} \int_{B(x,r_x)} g^q \, d\mu > \frac{\mu(B(x,r_x))}{r_x^{\lambda q}}.$$

By (6), with $R = 5R_0$,

$$r_x^{s-\lambda q} < \frac{C}{t^q} \int_{B(x,r_x)} g^q \, d\mu.$$

The family of balls $B(x, r_x)$, as x ranges over points in E_t, covers E_t; by the 5-covering theorem we can find a pairwise disjoint countable subfamily $\{B(x_i, r_i)\}_{i\in I}$ such that $E_t \subset$

$\bigcup_{i\in I} B(x_i, 5r_i)$. Now, as $\gamma/(s-\lambda q) \geq 1$,

$$\begin{aligned}\nu(E_t) \leq \sum_{i\in I} \nu(B(x_i, 5r_i)) &\leq C \sum_{i\in I} r_i^\gamma \\ &\leq C \sum_{i\in I} t^{-q\gamma/(s-\lambda q)} \left(\int_{B(x_i,r_i)} g^q\, d\mu\right)^{\gamma/(s-\lambda q)} \\ &\leq C t^{-q\gamma/(s-\lambda q)} \left(\sum_{i\in I} \int_{B(x_i,r_i)} g^q\, d\mu\right)^{\gamma/(s-\lambda q)} \\ &\leq t^{-q\gamma/(s-\lambda q)} \left(\int_X g^q\, d\mu\right)^{\gamma/(s-\lambda q)},\end{aligned}$$

where we used the pairwise disjointness property of the balls $B(x_i, r_i)$ in the last line. □

Lemma 6.4. *Under the assumption of Lemma 6.3, $M_{\lambda,q}$ is of strong type $(s/\lambda, \infty)$; that is, $M_{\lambda,q} : L^{s/\lambda}(X, \mu) \to L^\infty(K, \nu)$ is bounded.*

Proof. Let $x \in K$ and $0 < r < 2\text{diam}(K)$. Then by Hölder's inequality and by the fact that $q < s/\lambda$,

$$\begin{aligned} r^\lambda \left(\fint_{B(x,r)} g^q\, d\mu\right)^{1/q} &\leq r^\lambda \left(\fint_{B(x,r)} g^{s/\lambda}\, d\mu\right)^{\lambda/s} \\ &= \frac{r^\lambda}{\mu(B(x,r))^{\lambda/s}} \|g\|_{L^{s/\lambda}(X)} \\ &\leq C\|g\|_{L^{s/\lambda}(X)},\end{aligned}$$

where we used (6) to obtain the last inequality. □

Now by Marcinkiewicz interpolation theorem (see for example Corollary 4.14 in page 226 of [BS]), if there exists $0 < \theta < 1$ such that

$$\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{s/\lambda} \text{ and } \frac{1}{p^*} = \frac{1-\theta}{q\gamma/(s-\lambda q)},$$

then $M_{\lambda,q} : L^p(X) \to L^{p^*}(K, \nu)$ boundedly. Solving the above for θ and then p^*, we see that if $q < p < s$ by taking q sufficiently close to p (which we can by the Hölder inequality), and $0 < \lambda < \min\{1, s(1+q-p)/q\}$ sufficiently small, we can make the choices of θ and p^* as follows:

$$\theta = \frac{(p-q)s}{s-\lambda q} \text{ and } p^* = \frac{q\gamma}{q(s-\lambda) - s(p-1)}, \tag{17}$$

if $q(\lambda + \gamma) \geq s$, we obtain the desired boundedness of $M_{\lambda,q} : L^p(X) \to L^{p^*}(K, \nu)$. Note that by our assumptions on s and γ, we must have $\gamma \leq s$.

Recall that as X supports a $(1, p)$-Poincaré inequality, Lipschitz functions, and hence continuous functions, form a dense subclass of $N^{1,p}(X)$. Finally, by Lemma 6.2 and (16), we have the following trace theorem.

Theorem 6.5. *Suppose X supports a $(1,q)$-Poincaré inequality for some $1 \le q < \infty$ and $p > q$. If $s/(\gamma+1) < p < sq/[(s-\gamma)(\gamma+1)]$, $\max\{0,(s-\gamma)/q\} < \lambda < \min\{1, s(1+q-p)/q\}$, and p^* is given by (17), then λ satisfies $0 \le s-\gamma \le \lambda q < s$, and so for any $0 < \alpha < 1-\lambda$, there is a bounded trace operator*

$$Tr : N^{1,p}(X) \to B^{\alpha}_{p^*,p^*}(K)$$

such that whenever f is a continuous function from $N^{1,p}(X)$, $Tr(f) = f|_K$.

With the choice in the above theorem, we also see that $\gamma > s-p$. Hence by the results in [KiMa], every subset of K with positive ν-measure has positive p-capacity with respect to the Newton-Sobolev space $N^{1,p}(X)$. Thus by the results of [KiL] (which state that p-capacity almost every point is a Lebesgue point for such Sobolev functions), for all $f \in N^{1,p}(X)$, $Tr(f)(x) = f(x)$ for ν-a.e. $x \in K$.

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Amiran Gogatishvili: Institute of Mathematics of the Academy of Sciences of the Czech Republic, 115 67 Praha 1, Czech Republic.
E-mail address: gogatish@math.cas.cz

Pekka Koskela: Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), FIN-40014 Jyväskylä, Finland.
E-mail address: pkoskela@maths.jyu.fi

Nageswari Shanmugalingam: Department of Mathematical Sciences, University of Cincinnati, P.O. Box 210025, Cincinnati, Ohio 45221-0025, U.S.A.
E-mail address: nages@math.uc.edu