

# BV functions and Besov spaces associated with Dirichlet spaces

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## Abstract

We develop a theory of bounded variation functions and Besov spaces in abstract Dirichlet spaces which unifies several known examples and applies to new situations, including fractals.

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# Introduction

## Foreword

While the theory developed here is far from complete, it is the hope of the authors that the readers would find this monograph to provide a reasonable material introducing the topic, with many open questions. Most of the results here are new, but those that are not have been marked with references to the original papers or textbooks where they appeared.

## Bird's eyeview

Before describing our results in details, due to the length of the paper, it may be useful to indicate some of the main ideas. Let  $(X, \mu, \mathcal{E}, \mathcal{F} = \mathbf{dom}(\mathcal{E}))$  be a symmetric Dirichlet space, that is,  $X$  is a topological measure space equipped with the Radon measure  $\mu$ ,  $\mathcal{E}$  a closed Markovian bilinear form on  $L^2(X, \mu)$ , and  $\mathcal{F}$  the collection of all functions  $u \in L^2(X)$  with  $\mathcal{E}(u, u)$  finite. The book [FOT11] contains a good introduction to the theory of Dirichlet forms. Let  $\{P_t\}_{t \in [0, \infty)}$  denote the Markovian semigroup associated with  $(X, \mu, \mathcal{E}, \mathcal{F})$ . Let  $p \geq 1$  and  $\alpha \geq 0$ . For  $f \in L^p(X, \mu)$ , we define the Besov type seminorm:

$$\|f\|_{p, \alpha} = \sup_{t > 0} t^{-\alpha} \left( \int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p},$$

and introduce the Besov space

$$\mathbf{B}^{p, \alpha}(X) = \{f \in L^p(X, \mu) : \|f\|_{p, \alpha} < +\infty\}.$$

Note that if  $0 < \beta < \alpha$ , then  $\mathbf{B}^{p, \alpha}(X) \subset \mathbf{B}^{p, \beta}(X)$ . The critical parameter

$$\alpha_p^*(X) = \inf\{\alpha > 0 : \mathbf{B}^{p, \alpha}(X) \text{ is trivial}\}$$

will be of special interest in our study. By trivial, we mean here that for  $\alpha > \alpha_p^*(X)$ , any function  $f \in \mathbf{B}^{p, \alpha}(X)$  is constant. For instance, we will see that for strongly local forms with a nice intrinsic metric in the sense of [Stu94], one typically has  $\alpha_p^*(X) = \frac{1}{2}$ , while for strongly local forms with no intrinsic distance but with sub-Gaussian heat kernel estimates one typically has  $\alpha_p^*(X) > \frac{1}{2}$  when  $1 \leq p < 2$  and  $\alpha_p^*(X) < \frac{1}{2}$ , when  $p > 2$ . When  $p = 2$ , one always has  $\alpha_2^*(X) = \frac{1}{2}$ .

While all the spaces  $\mathbf{B}^{p, \alpha}(X)$  are of interest and shall be discussed in the monograph, we shall pay a special attention to the case  $p = 1$ . In particular, we will see that in many situations it may be possible and natural to define a space of bounded variation functions on  $X$  as the space  $\mathbf{B}^{1, \alpha_1^*(X)}(X)$ . For instance, on complete Riemannian manifolds we have that  $\alpha_1^*(X) = \frac{1}{2}$  and  $\mathbf{B}^{1, \alpha_1^*(X)}(X)$  is the usual space of bounded variation functions. On the Sierpinski gaskets  $\alpha_1^*(X) = \frac{d_H}{d_W}$ , where  $d_H$  is the Hausdorff dimension of  $X$  and  $d_W$  its walk dimension. Finding the exact value of  $\alpha_1^*(X)$  is an open question for the Sierpinski carpets.

Under a weak curvature condition on the Dirichlet space (weak Bakry-Émery type estimate (0.2)), a main observation of our work is that for functions in the space  $\mathbf{B}^{1, \alpha_1^*(X)}(X)$ , one has the following estimate valid for all  $t$ 's:

$$(0.1) \quad \|P_t f - f\|_{L^1(X, \mu)} \leq C t^{\alpha_1^*(X)} \limsup_{s \rightarrow 0} s^{-\alpha_1^*(X)} \int_X P_s(|f - f(y)|)(y) d\mu(y).$$

Therefore, by using the approach of Michel Ledoux to heat flow in the study of isoperimetry [Led96], one deduces that for  $f \in \mathbf{B}^{1,\alpha_1^*(X)}(X)$ , the quantity  $\limsup_{s \rightarrow 0} s^{-\alpha_1^*(X)} \int_X P_s(|f - f(y)|)(y) d\mu(y)$  plays the role of the total variation of  $f$ . For instance, in the case where  $X$  is a complete Riemannian manifold, one has

$$\limsup_{s \rightarrow 0} s^{-1/2} \int_X P_s(|f - f(y)|)(y) d\mu(y) \simeq \int_X |\nabla f|(y) d\mu(y),$$

which coincides with the usual notion of variation. However, the quantity

$$\limsup_{s \rightarrow 0} s^{-\alpha_1^*(X)} \int_X P_s(|f - f(y)|)(y) d\mu(y)$$

is well-defined and possible to estimate in a class of Dirichlet spaces well beyond Riemannian manifolds, like for instance strongly local metric spaces satisfying a doubling condition and supporting a 2-Poincaré inequality, fractal spaces, non-local Dirichlet spaces and even some infinite dimensional examples.

With (0.1) in hand, and adapting ideas from [BCLSC95], we are then able to deduce several Sobolev and isoperimetric inequalities with sharp exponents holding in a very general framework.

We summarize some of the classes of spaces, studied in this paper, in the table below. The parameter  $d_H$  indicates the Hausdorff dimension of the space and  $d_W$  its walk dimension. The parameter  $\kappa$  is the Hölder regularity exponent of the heat semigroup in the assumed weak Bakry-Émery estimate:

$$(0.2) \quad |P_t f(x) - P_t f(y)| \leq C \frac{d(x,y)^\kappa}{t^{\kappa/d_W}} \|f\|_{L^\infty(X,\mu)}.$$

Examples		
Dirichlet space $X$	$\alpha_1^*(X)$	Characterization of $\mathbf{B}^{1,\alpha_1^*}(X)$
1) Strongly local Dirichlet metric space with doubling and 2-Poincaré	$\frac{1}{2}$	$BV(X)$
2) Local Dirichlet metric space with sub-Gaussian heat kernel estimates	$1 - \frac{\kappa}{d_W}$	$\sup_{r>0} \iint_{d(x,y)<r} \frac{ f(x)-f(y) }{r^{d_H+d_W-\kappa}} d\mu(x) d\mu(y) < +\infty$
3) a) Non-local Dirichlet metric space with heat kernel estimates and $d_W \leq 1$	1	$\iint \frac{ f(x)-f(y) }{d(x,y)^{d_H+d_W}} d\mu(x) d\mu(y) < +\infty$
b) Non-local Dirichlet metric space with heat kernel estimates and $d_W > 1$	$\frac{1}{d_W}$	$\sup_{r>0} \iint_{d(x,y)<r} \frac{ f(x)-f(y) }{r^{d_H+1}} d\mu(x) d\mu(y) < +\infty$

TABLE 1. Summary of examples.

We point out that the case (2), in the above table, of a local Dirichlet metric space with sub-Gaussian heat kernel estimates, we actually only prove that  $1 - \frac{\kappa}{d_W}$  is an upper bound of the critical exponent  $\alpha_1^*(X)$ . We make the conjecture that  $\alpha_1^*(X) = 1 - \frac{\kappa}{d_W}$  should hold in some generality, see the discussion after Remark 5.51. The conjecture is proved in some concrete examples like the Sierpinski gasket, although the question is open for the Sierpinski carpet.

### Summary of results and structure of the monograph

The paper is divided into two parts. The first part develops the theory of Besov spaces and related Sobolev embeddings in the general framework of abstract Dirichlet spaces. The second part particularizes to four big classes of Dirichlet spaces: Local regular Dirichlet spaces with absolutely continuous energy measures, doubling and 2-Poincaré (like Riemannian manifolds with non negative Ricci curvature or more generally the  $\mathbf{RCD}(0, +\infty)$  metric measure spaces), Local regular Dirichlet spaces with sub-Gaussian heat kernel estimates (like fractal spaces), Non-local regular Dirichlet spaces (like Dirichlet forms associated with jump processes), and Local quasi-regular and infinite dimensional Dirichlet spaces (like the Wiener space). We now further describe these constituent parts of the paper.



**Part 1.**

**Chapter 1.** In Chapter 1, we lay the foundations of the paper. Let  $(X, \mu, \mathcal{E}, \mathcal{F} = \mathbf{dom}(\mathcal{E}))$  be a symmetric Dirichlet space. Let  $p \geq 1$  and  $\alpha \geq 0$ . For  $f \in L^p(X, \mu)$ , we define the Besov type seminorm:

$$\|f\|_{p,\alpha} = \sup_{t>0} t^{-\alpha} \left( \int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p},$$

where  $P_t$  is the Markovian semigroup associated with  $\mathcal{E}$ . We then introduce the space

$$\mathbf{B}^{p,\alpha}(X) = \{f \in L^p(X, \mu), \|f\|_{p,\alpha} < +\infty\},$$

and prove that it is a Banach space, which is reflexive for  $p > 1$ . The value  $\alpha = \frac{1}{2}$  plays a special role throughout the paper since  $\mathbf{B}^{2,1/2}(X)$  is always equal to the domain  $\mathcal{F}$  of the Dirichlet form. A first class of examples is studied: We prove that if  $\mathcal{E}$  arises from a smooth Hörmander's type diffusion operator on a manifold, then for every  $p \geq 1$ ,  $\mathbf{B}^{p,1/2}(X)$  contains the space of smooth and compactly supported functions with the continuous embedding

$$\left( \int_X \Gamma(f, f)(x)^{p/2} d\mu(x) \right)^{1/p} \leq 2 \left( \frac{\Gamma(\frac{1+p}{2})}{\sqrt{\pi}} \right)^{1/p} \|f\|_{p,1/2},$$

where  $\Gamma$  is the *carré du champ* operator of the Dirichlet form  $(\mathcal{E}, \mathcal{F})$  (however,  $\Gamma((1+p)/2)$  is the standard Gamma function). On the other hand, in Corollary 1.13, we will prove that if the Kusuoka measure is singular with respect to  $\mu$ , like in the Sierpinski gasket, then  $\mathbf{B}^{p,1/2}(X)$  contains only constant functions when  $p > 2$ . This dichotomy already shows that the properties of the spaces  $\mathbf{B}^{p,\alpha}(X)$  are closely related to the energy measures  $\Gamma(f, g)$  being absolutely continuous or not. This dichotomy will be investigated in great details in the second part of the monograph.

An important, very general, theorem proved in Chapter 1 is the following:

**THEOREM 0.1.** *Let  $1 < p \leq 2$ . There exists a constant  $C_p > 0$  such that for every  $f \in L^p(X, \mu)$  and  $t \geq 0$*

$$\|P_t f\|_{p,1/2} \leq \frac{C_p}{t^{1/2}} \|f\|_{L^p(X, \mu)}.$$

*In particular, when  $1 < p \leq 2$ ,  $P_t : L^p(X, \mu) \rightarrow \mathbf{B}^{p,1/2}(X)$  is bounded for  $t > 0$ .*

The proof of the theorem will follow from some nice ideas originally due to Nick Dungey [**Dun08**] and then developed in [**CCH, CCFR17**]. As a corollary, we obtain:

**PROPOSITION 0.2.** *Let  $1 < p \leq 2$ . Let  $L$  be the infinitesimal generator of  $\mathcal{E}$  and  $\mathcal{L}_p$  be the domain of  $L$  in  $L^p(X, \mu)$ , (that is,  $\mathcal{L}_p$  is the domain of the generator of the strongly continuous semigroup  $P_t : L^p(X, \mu) \rightarrow L^p(X, \mu)$ ). Then*

$$\mathcal{L}_p \subset \mathbf{B}^{p,1/2}(X)$$

*and for every  $f \in \mathcal{L}_p$ ,*

$$(0.3) \quad \|f\|_{p,1/2}^2 \leq C_p \|Lf\|_{L^p(X, \mu)} \|f\|_{L^p(X, \mu)}.$$

We note that due to Corollary 1.13, one cannot hope for an extension of Theorem 0.1 to the range  $p > 2$  in the general case.

We define then the  $L^p$  Besov critical exponent of  $(X, \mu, \mathcal{E}, \mathcal{F})$  as

$$\alpha_p^*(X) = \inf\{\alpha > 0, \mathbf{B}^{p,\alpha}(X) \text{ is trivial}\}.$$

Let us recall that the Dirichlet form  $\mathcal{E}$  is said to be regular if  $\mathcal{F} \cap C_0(X)$  is dense in  $\mathcal{F}$  for the domain norm and uniformly dense in  $C_0(X)$  (see Chapter 1 in [**FOT11**]). The form  $\mathcal{E}$  is said to be irreducible if  $f \in \mathcal{F}$  with  $\mathcal{E}(f, f) = 0$  implies that  $f$  is constant. We prove the following result:

**PROPOSITION 0.3.** *Assume that  $\mathcal{E}$  is regular and irreducible. Then*

$$(1) \quad \alpha_2^*(X) = \frac{1}{2};$$

- (2)  $p \rightarrow \alpha_p^*(X)$  is non increasing;
- (3) For  $p \geq 2$ ,  $\alpha_p^*(X) \leq \frac{1}{2}$ ;
- (4) For  $1 \leq p \leq 2$ ,  $\frac{1}{2} \leq \alpha_p^*(X) \leq \frac{1}{p}$ .

In particular (and this is of course not surprising),  $\mathbf{B}^{p,\alpha}(X)$  is always trivial if  $\alpha > 1$ . Note from Corollary 1.13 that if for every non-constant  $f \in \mathcal{F}$  the energy measure  $\nu_f$  is singular with respect to  $\mu$ , then  $\alpha_p^*(X) < \frac{1}{2}$  for  $p > 2$ . We recall that for  $f \in \mathcal{F}$ , its energy measure  $\nu_f$  is defined by

$$2\mathcal{E}(fg, f) - \mathcal{E}(f^2, g) = \int_X 2g d\nu_f.$$

**Chapter 2.** Chapter 2 is devoted to the study of Sobolev embeddings associated to our Besov spaces, in the presence of ultracontractive estimates for the semigroup. It will follow ideas from [BCLSC95] (see also [SC02]) and the key lemma is the following simple, but powerful, observation that functions in the Besov spaces satisfy a pseudo-Poincaré inequality:

LEMMA 0.4. *Let  $p \geq 1$  and  $\alpha > 0$ . For every  $f \in \mathbf{B}^{p,\alpha}(X)$ , and  $t \geq 0$ ,*

$$\|P_t f - f\|_{L^p(X, \mu)} \leq t^\alpha \|f\|_{p,\alpha}.$$

The importance of such pseudo-Poincaré inequalities in Sobolev embeddings was explicitly noted in [SC02] and [Led03]. In particular, the technique used by M. Ledoux in [Led03] to obtain improved Gagliardo-Nirenberg type inequalities may likely be adapted to our general framework. However, to be concise, we restricted ourselves to the Sobolev embeddings in the presence of ultracontractive estimates.

A general result of the paper is the following weak-type Sobolev inequality:

THEOREM 0.5. *Assume that  $\{P_t\}_{t \in (0, \infty)}$  admits a heat kernel  $p_t(x, y)$  satisfying, for some  $C > 0$  and  $\beta > 0$ ,*

$$p_t(x, y) \leq Ct^{-\beta}$$

*for  $\mu \times \mu$ -a.e.  $(x, y) \in X \times X$  for each  $t \in (0, +\infty)$ . Let  $0 < \alpha < \beta$  and  $1 \leq p < \frac{\beta}{\alpha}$ . Then there exists a constant  $C_{p,\alpha} > 0$  such that for every  $f \in \mathbf{B}^{p,\alpha}(X)$ ,*

$$\sup_{s \geq 0} s \mu(\{x \in X, |f(x)| \geq s\})^{\frac{1}{q}} \leq C_{p,\alpha} \|f\|_{p,\alpha},$$

where  $q = \frac{p\beta}{\beta - p\alpha}$ .

One may consider the above to be a very weak version of the Mazya capacity type inequality. The case  $p = 1$  is particularly interesting in the previous theorem and yields an isoperimetric type inequality.

THEOREM 0.6. *Assume that  $\{P_t\}_{t \in (0, \infty)}$  admits a heat kernel  $p_t(x, y)$  satisfying, for some  $C > 0$  and  $\beta > 0$ ,*

$$p_t(x, y) \leq Ct^{-\beta}$$

*for  $\mu \times \mu$ -a.e.  $(x, y) \in X \times X$  for each  $t \in (0, +\infty)$ . Let  $0 < \alpha < \beta$ . There exists a constant  $C_{\text{iso}} > 0$ , such that for every subset  $E \subset X$  with  $1_E \in \mathbf{B}^{p,\alpha}(X)$*

$$\mu(E)^{\frac{\beta - \alpha}{\beta}} \leq C_{\text{iso}} \|1_E\|_{1,\alpha}.$$

We will see that these two theorems are interesting due to their level of generality, since we are able to characterize  $\mathbf{B}^{p,\alpha}(X)$  in many situations and will therefore be able to prove Sobolev inequalities with sharp exponents.

**Chapter 3.** Chapter 3 studies the analogue of the Sobolev embeddings of the Besov spaces for Dirichlet forms whose semigroups are not ultracontractive but satisfy supercontractive or hypercontractive type estimates.

In Section 3.1, we assume that  $\mathcal{E}$  satisfies a Poincaré inequality. Under this assumption only, by following ideas of M. Ledoux [Led94] we are then able to prove an analogue of Buser's inequality, and thus linear isoperimetry:

**THEOREM 0.7.** *Let  $\alpha \in (0, 1]$ . We define the  $\alpha$ -Cheeger's constant of  $X$  by*

$$h_\alpha = \inf \frac{\|\mathbf{1}_E\|_{1,\alpha}}{\mu(E)},$$

where the infimum runs over all measurable sets  $E$  such that  $\mu(E) \leq \frac{1}{2}$  and  $\mathbf{1}_E \in \mathbf{B}^{1,\alpha}(X)$ . Then,

$$h_\alpha \geq (1 - e^{-1})\lambda_1^\alpha,$$

where  $\lambda_1$  is the spectral gap of  $\mathcal{E}$ .

In Section 3.2, we prove that if  $\mathcal{E}$  satisfies a log-Sobolev inequality, then we can improve Buser's inequality, and prove that  $X$  supports a Gaussian isoperimetric inequality, that is,

**THEOREM 0.8.** *There exists a constant  $C > 0$  such that for every set  $E \subset X$  with  $\mathbf{1}_E \in \mathbf{B}^{1,1/2}(X)$ , one has:*

$$\mu(E)\sqrt{-\ln \mu(E)} \leq C\|\mathbf{1}_E\|_{1,1/2},$$

where the constant  $C$  explicitly depends on the log-Sobolev constant of  $\mathcal{E}$ .

The main class of examples to which the previous theorem apply are infinite-dimensional spaces like the ones studied in Chapter 7.

## Part 2.

**Chapter 4.** The Chapter 4 will be devoted to strongly local Dirichlet spaces with absolutely continuous energy measures  $\Gamma(u, u)$ , also known as carré du champ, as extensively studied in the literature (see for instance the foundational articles by K.T. Sturm [Stu94, Stu95, Stu96]). The important assumption that is essential in this chapter is that the intrinsic distance associated to  $\Gamma(u, u)$  defines the topology of  $X$ . Such Dirichlet spaces are called strictly local and are extensively studied in the literature (see e.g. [BSC00, KZ12, KSZ14, LSV09, LSV11, KST04] and references therein).

This chapter has multiple objectives. A first objective is to develop the notion of bounded variation function in this general framework. We achieve this task in Sections 4.1 and 4.2. Our basic definition of bounded variation function in a strictly local Dirichlet space is the following:

**DEFINITION 0.9.** *We set  $u \in L^1(X, \mu)$  to be in  $BV(X)$  if there is a sequence of locally Lipschitz functions  $u_k \in L^1(X, \mu)$  such that  $u_k \rightarrow u$  in  $L^1(X, \mu)$  and*

$$\liminf_{k \rightarrow \infty} \int_X |\nabla u_k| d\mu < \infty,$$

where  $|\nabla u|$  is the square-root of the Radon-Nikodym derivative  $d\Gamma(u, u)/d\mu$ . For  $u \in BV(X)$  and open sets  $U \subset X$ , we set

$$\|Du\|(U) = \inf_{u_k \in \mathcal{C}(U), u_k \rightarrow u \text{ in } L^1(U)} \liminf_{k \rightarrow \infty} \int_U |\nabla u_k| d\mu,$$

and then for sets  $A \subset X$  we define

$$\|Du\|(A) = \inf\{\|Du\|(O) : A \subset O \text{ and } O \text{ is open in } X\}.$$

We then have the following result whose proof is based on that of [Mir03].

**THEOREM 0.10.** *If  $X$  is complete and  $u \in BV(X)$ , then  $\|Du\|$  is a Radon outer measure on  $X$ .*

A standard major tool in the study of bounded variation functions is the co-area formula. Such formula is established in Lemma 4.9.

It should be emphasized that while the notion of BV function in some strictly local metric settings has already been studied in the literature (see [MMS16] and the references therein), much of the current theory on functions of bounded variation in the metric setting requires a 1-Poincaré inequality. In this chapter we will *not* require the support of 1-Poincaré inequality, only the weaker 2-Poincaré inequality, but in some of the analysis we will need an additional requirement called in this monograph the weak Bakry-Émery curvature condition: namely that whenever  $u \in \mathcal{F} \cap L^\infty(X, \mu)$ ,

$$(0.4) \quad \|\nabla P_t u\|_{L^\infty(X)}^2 \leq \frac{C}{t} \|u\|_{L^\infty(X)}^2.$$

The weak Bakry-Émery curvature condition is satisfied in the following examples:

- Complete Riemannian manifolds with non-negative Ricci curvature and more generally, the  $RCD(0, +\infty)$  spaces ([CJKS17, Jia15]).
- Carnot groups (see [BB16])
- Complete sub-Riemannian manifolds with generalized non-negative Ricci curvature (see [BB12, BK14])
- Metric graphs with finite number of edges (see [BK17])

Several statements equivalent to the weak Bakry-Émery curvature condition are given in Theorem 1.2 of the recent work [CJKS17]. There are some metric measure spaces equipped with a doubling measure supporting a 2-Poincaré inequality but without the above Bakry-Émery condition, see for example [KRS03]. For instance, it should be noted, that in the setting of complete sub-Riemannian manifolds with generalized non-negative Ricci curvature in the sense of [BG17], while the weak Bakry-Émery curvature condition is known to be satisfied (see [BB12, BK14]), the 1-Poincaré inequality has not been proven yet (though the 2-Poincaré is known, see [BBG14]).

A major result in Chapter 4, is then the following theorem that makes the connection with Part 1 of the monograph:

**THEOREM 0.11.** *Assume that the weak Bakry-Émery curvature condition (0.4) is satisfied. Then,  $\mathbf{B}^{1,1/2}(X) = BV(X)$  with comparable seminorms. Moreover, there exist constants  $c, C > 0$  such that for every  $u \in BV(X)$*

$$c \limsup_{s \rightarrow 0} s^{-1/2} \int_X P_s(|u - u(y)|)(y) d\mu(y) \leq \|Du\|(X) \leq C \liminf_{s \rightarrow 0} s^{-1/2} \int_X P_s(|u - u(y)|)(y) d\mu(y).$$

We note that by combining Theorems 0.11 and 0.13 we deduce for instance the following corollary:

**COROLLARY 0.12.** *Assume that the weak Bakry-Émery curvature condition (0.4) is satisfied. If the volume growth condition  $\mu(B(x, r)) \geq C_1 r^Q$  is satisfied, then there exists a constant  $C_2 > 0$  such that for every  $f \in BV(X)$ ,*

$$\|f\|_{L^q(X, \mu)} \leq C_2 \|Df\|(X)$$

where  $q = \frac{Q}{Q-1}$ .

A second objective in the chapter is to study all of the Besov spaces  $\mathbf{B}^{p,\alpha}(X)$ ,  $p \geq 1$ . We will first prove the following result that gives a metric characterization of our Besov spaces:

**THEOREM 0.13.** *For  $1 \leq p < \infty$  and  $0 \leq \alpha < \infty$  we have*

$$\mathbf{B}^{p,\alpha/2}(X) = \left\{ u \in L^p(X, \mu) : \sup_{t > 0} \left( \int_X \int_{B(x,t)} \frac{|u(y) - u(x)|^p}{t^{\alpha p} \mu(B(x,t))} d\mu(y) d\mu(x) \right)^{1/p} < +\infty \right\}$$

with comparable seminorms.

We will then be interested in comparing the Besov seminorm  $\|\cdot\|_{p,1/2}$  to Sobolev seminorms and our result is the following:

THEOREM 0.14.

- Let  $p > 1$ . There exists a constant  $C > 0$  such that for every  $u \in \mathbf{B}^{p,1/2}(X) \cap \mathcal{F}$ ,

$$\|\nabla u\|_{L^p(X,\mu)} \leq C\|u\|_{p,1/2}$$

- Assume furthermore that the strong Bakry-Émery estimate is satisfied:  $|\nabla P_t u| \leq C P_t |\nabla u|$ . Then, for every  $p > 1$ , there exists a constant  $C > 0$  such that for every  $u \in L^p(X, \mu) \cap \mathcal{F}$  with  $|\nabla u| \in L^p(X, \mu)$ ,

$$\|u\|_{p,1/2} \leq C\|\nabla u\|_{L^p(X,\mu)}.$$

This theorem will allow us to prove an analogue of Theorem 0.1 for the range  $p > 2$  and to study the  $L^p$  Besov critical exponent

$$\alpha_p^*(X) = \inf\{\alpha > 0 : \mathbf{B}^{p,\alpha}(X) \text{ is trivial}\}.$$

More precisely, we prove:

THEOREM 0.15. Assume that the strong Bakry-Émery estimate is satisfied. Let  $p > 1$ . There exists a constant  $C_p > 0$  such that for every  $f \in L^p(X, \mu)$  and  $t \geq 0$

$$\|P_t f\|_{p,1/2} \leq \frac{C_p}{t^{1/2}} \|f\|_{L^p(X,\mu)}.$$

As a consequence, for every  $p \geq 1$ ,  $\alpha_p^*(X) = \frac{1}{2}$  and  $\mathbf{B}^{p,1/2}(X)$  is dense in  $L^p(X, \mu)$ .

**Chapter 5.** In Chapter 5 we apply our analysis to strongly local Dirichlet spaces which do not have absolutely continuous energy measures  $\Gamma(u, u)$ . Thus these spaces are not strictly local. The main class of examples we are interested in are fractal spaces where typically  $\Gamma(u, u)$  and  $\mu$  are mutually singular. This fact was first observed by Kusuoka in [Kus89] (see also [BBST99, Hin16, Hin13a, Hin13b]). The general framework of analysis and probability on fractals can be found in [Bar98, Gri03, Kig01, Kig12, Str03b, Str06].

In a first general result, we recover results from [PP10] and obtain the following characterization of the spaces  $\mathbf{B}^{p,\alpha}(X)$ .

THEOREM 0.16 ([PP10]). Let  $(X, \mu, \mathcal{E}, \mathcal{F})$  be a symmetric Dirichlet space and let  $d$  be a metric on  $X$  compatible with the topology of  $X$ . We assume that  $(X, d)$  is Ahlfors  $d_H$ -regular and that  $\{P_t\}_{t \in (0, \infty)}$  admits a heat kernel  $p_t(x, y)$  satisfying, for some  $c_3, c_4, c_5, c_6 \in (0, \infty)$  and  $d_W \in (1, \infty)$ ,

$$c_5 t^{-d_H/d_W} \exp\left(-c_6 \left(\frac{d(x, y)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) \leq p_t(x, y) \leq c_3 t^{-d_H/d_W} \exp\left(-c_4 \left(\frac{d(x, y)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right)$$

for  $\mu \times \mu$ -a.e.  $(x, y) \in X \times X$  for each  $t \in (0, +\infty)$ . Let  $p \geq 1$  and  $\alpha \geq 0$ . We have

$$\mathbf{B}^{p, \frac{\alpha}{d_W}}(X) = \left\{ f \in L^p(X, \mu) : \sup_{r>0} \frac{1}{r^{\alpha+d_H/p}} \left( \iint_{\Delta_r} |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p} < \infty \right\}$$

with comparable norms.

In the above and the sequel, for  $r > 0$  the set  $\Delta_r$  denotes the collection of all  $(x, y) \in X \times X$  for which  $d(x, y) < r$ .

We note that by combining Theorems 0.6 and 0.16 one immediately obtains:

COROLLARY 0.17. Let  $X$  be an Ahlfors  $d_H$ -regular space that satisfies sub-Gaussian heat kernel estimates as in Theorem 0.16. Then, one has the following weak type Besov space embedding. Let  $0 < \delta < d_H$ . Let  $1 \leq p < \frac{d_H}{\delta}$ . There exists a constant  $C_{p,\delta} > 0$  such that for every  $f \in \mathbf{B}^{p,\delta/d_W}(X)$ ,

$$\sup_{s \geq 0} s \mu(\{x \in X, |f(x)| \geq s\})^{\frac{1}{q}} \leq C_{p,\delta} \sup_{r > 0} \frac{1}{r^{\delta+d_H/p}} \left( \iint_{\Delta_r} |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p}$$

where  $q = \frac{pd_H}{d_H - p\delta}$ . Furthermore, for every  $0 < \delta < d_H$ , there exists a constant  $C_{\text{iso},\delta}$  such that for every measurable  $E \subset X$ ,  $\mu(E) < +\infty$ ,

$$(0.5) \quad \mu(E)^{\frac{d_H - \delta}{d_H}} \leq C_{\text{iso},\delta} \sup_{r>0} \frac{1}{r^{\delta+d_H}} (\mu \otimes \mu) \{(x, y) \in E \times E^c : d(x, y) < r\}.$$

The number  $\delta$  in the previous corollary plays the role of the upper codimension of the boundary of  $E$ . In Section 5.3, we will construct several explicit examples of sets with fractal boundaries for which  $\delta \neq 1$  and

$$\sup_{r>0} \frac{1}{r^{\delta+d_H}} (\mu \otimes \mu) \{(x, y) \in E \times E^c : d(x, y) < r\} < +\infty$$

We then discuss in detail existence of sets for which their indicator functions are in  $\mathbf{B}^{1,\alpha}(X)$ , and the related question of the density of  $\mathbf{B}^{1,\alpha}(X)$  in  $L^1(X, \mu)$ . Related co-area type formulas are studied.

We will then introduce the analogue of the weak Bakry-Émery estimate 0.4 in this framework of Dirichlet spaces with sub-Gaussian heat kernel estimates. We will see that a convenient analogue to work with is the following:

$$(0.6) \quad |P_t f(x) - P_t f(y)| \leq C \frac{d(x, y)^\kappa}{t^{\kappa/d_W}} \|f\|_{L^\infty(X, \mu)}, \quad f \in L^\infty(X, \mu), \quad t \geq 0.$$

The parameter  $\kappa$  is the Hölder regularity exponent. We will show that the weak Bakry-Émery condition is satisfied with  $\kappa = 1 - \frac{d_H}{d_W}$  in many examples, like the infinite Sierpinski gaskets or infinite Sierpinski carpets. An important consequence of the weak Bakry-Émery estimate is the continuity of the heat semigroup in the Besov spaces with  $p > 2$ .

**THEOREM 0.18.** *Let  $X$  be an Ahlfors  $d_H$ -regular space that satisfies sub-Gaussian heat kernel estimates and  $BE(\kappa)$  with  $0 < \kappa \leq \frac{d_W}{2}$ . Then, for any  $p \geq 2$ , there exists a constant  $C > 0$  such that for every  $t > 0$  and  $f \in L^p(X, \mu)$*

$$\|P_t f\|_{p, (1-\frac{2}{p})\frac{\kappa}{d_W} + \frac{1}{p}} \leq \frac{C}{t^{(1-\frac{2}{p})\frac{\kappa}{d_W} + \frac{1}{p}}} \|f\|_{L^p(X, \mu)}.$$

*In particular, for  $t > 0$ ,  $P_t : L^p(X, \mu) \rightarrow \mathbf{B}^{p, (1-\frac{2}{p})\frac{\kappa}{d_W} + \frac{1}{p}}(X)$  is bounded.*

The theorem provides the analogue of Theorem 0.1 to the range  $p > 2$  and allows us to study the  $L^p$  Besov critical exponents. As before, for  $p \geq 1$ , if we denote

$$\alpha_p^*(X) = \inf \{ \alpha > 0 : \mathbf{B}^{p,\alpha}(X) \text{ is trivial} \},$$

we obtain then:

**THEOREM 0.19.** *Let  $X$  be an Ahlfors  $d_H$ -regular space that satisfies sub-Gaussian heat kernel estimates and  $BE(\kappa)$  with  $0 < \kappa \leq \frac{d_W}{2}$ . The following inequalities hold:*

- For  $1 \leq p \leq 2$ ,

$$\frac{1}{2} \leq \alpha_p^*(X) \leq \left(1 - \frac{2}{p}\right) \frac{\kappa}{d_W} + \frac{1}{p}.$$

- For  $p \geq 2$ ,

$$\left(1 - \frac{2}{p}\right) \frac{\kappa}{d_W} + \frac{1}{p} \leq \alpha_p^*(X) \leq \frac{1}{2}.$$

In the end we discuss generalized Riesz transforms, sets of finite perimeter, and Sobolev and isoperimetric inequalities. Our main motivation is to connect these classical analysis notions with the heat kernel estimates, which in the context of fractals are mostly studied in probability theory.

**Chapter 6.** Many arguments appearing in the study of local Dirichlet spaces turn out to be easily adaptable to obtain a non-local version of several results from Chapter 5. In this chapter we consider non-local and regular Dirichlet spaces whose associated semigroup has a heat kernel satisfying the estimates

$$(0.7) \quad c_5 t^{-\frac{d_H}{d_W}} \left(1 + c_6 \frac{d(x, y)}{t^{1/d_W}}\right)^{-d_H - d_W} \leq p_t(x, y) \leq c_3 t^{-\frac{d_H}{d_W}} \left(1 + c_4 \frac{d(x, y)}{t^{1/d_W}}\right)^{-d_H - d_W}$$

for some  $c_3, \dots, c_6 \in (0, \infty)$  and  $0 < d_W \leq d_H + 1$ .

Following the same structure as Chapter 5 we start by discussing in Section 6.1 some metric properties of the space  $\mathbf{B}^{p, \alpha}(X)$  in terms of other Besov type spaces appearing in the literature, see e.g. [Gri03]. In contrast to the local case, the characterization of the Besov space given below, c.f. Theorem 6.1, is confined to  $\alpha \in [0, 1/p)$ . Some situations when  $\alpha \geq 1/p$  are analyzed in Proposition 6.2.

**THEOREM 0.20.** *Let the non-local Dirichlet space  $(X, d, \mu, \mathcal{E}, \mathcal{F})$  have an associated heat kernel satisfying (0.7). For any  $p \in [1, \infty)$  and  $\alpha \in [0, 1/p)$  we have*

$$\|f\|_{p, \alpha} \simeq \sup_{r > 0} \frac{1}{r^{\alpha d_W + d_H/p}} \left( \iint_{\Delta_r} |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p}.$$

Sections 6.2 and 6.3 provide coarea type estimates of the Besov norm, c.f. Theorem 6.4 as well as a characterization of sets with finite perimeter in Theorem 6.8 that is analogous to their local counterparts in Chapter 5. In this case the results require  $0 < \alpha \leq \min\{d_H/d_W, 1\}$  but also include situations where  $d_W < 1$ . The question about the critical exponent  $\alpha_1^*(X)$  for the space  $\mathbf{B}^{1, \alpha_1^*(X)}(X)$  is examined in Section 6.4 under an extra assumption on the underlying space  $(X, d, \mu)$ , c.f. Assumption 6.14, where we prove in Theorem 6.15 that

**THEOREM 0.21.** *Under Assumption 6.14, we have*

$$\alpha_1^*(X) = \begin{cases} 1 & \text{if } d_W \leq 1, \\ \frac{1}{d_W} & \text{if } d_W > 1 \end{cases}$$

and the space  $\mathbf{B}^{1, \alpha_1^*(X)}(X)$  is characterized as displayed in Table 1.

This chapter finishes with a discussion in Section 6.5 concerning a special type of Sobolev and isoperimetric inequalities which we can treat in the non-local setting. Assuming that the heat semigroup is transient and the Dirichlet form  $(\mathcal{E}, \mathcal{F})$  is regular, general arguments from Dirichlet form theory, see e.g. [FOT11, Section 2.4] [CF12, Section 2.1] provide capacity conditions that yield a Sobolev type inequality of the form

$$\left( \int_X |f|^{2\kappa} d\mu \right)^{2\kappa} \leq C \sqrt{\mathcal{E}(f, f)}$$

for some  $\kappa \geq 1$  and  $C > 0$ , c.f. Theorem 6.18. Moreover, combining the results from previous sections in this chapter and with Sections 2.1-2.3 in Chapter 2, we obtain the corresponding weak Sobolev, capacity and isoperimetric inequalities.

**Chapter 7.** Finally, we also have some results in the setting of infinite dimensional Banach spaces. Due to the technical difficulty of the task (see [FH01]), our goal in the chapter is not to develop the general theory but rather to point out some research directions showing that our approach is also suitable to handle infinite-dimensional spaces.

We prove in particular that if  $E$  is a set of finite perimeter in the Wiener space, or more generally in a path space endowed with a Gibbs measure as in Kawabi [Kaw08] or Kawabi-Röckner [KR07], then one must have  $1_E \in \mathbf{B}^{1, 1/2}(X)$ . More precisely, we establish the following result:

**THEOREM 0.22.** *Let  $X$  be one of the Dirichlet spaces studied in Sections 7.1 or 7.2. Let  $E$  be a measurable set with finite measure. The following are equivalent:*

- (1)  $E$  has a finite perimeter;

(2) The limit  $\lim_{t \rightarrow 0^+} \int_X \|DP_t 1_E\|_{\mathcal{H}_x} d\mu(x)$  exists.

Moreover, if  $E$  is a set satisfying one of the above conditions, then  $1_E \in \mathbf{B}^{1,1/2}(X)$  and

$$\|1_E\|_{1,1/2} \leq 2\sqrt{2}P(E, X) = 2\sqrt{2} \lim_{t \rightarrow 0^+} \int_X \|DP_t 1_E\|_{\mathcal{H}_x} d\mu(x).$$

In these cases, the isoperimetric inequality from Theorem 0.6 does not apply, but we prove instead a Poincaré and a Gaussian isoperimetric inequality. In particular we prove:

**THEOREM 0.23.** *Let  $(X, \mathcal{E}, \mathcal{F}, \mu)$  be the path space with Gibbs measure described in Section 7.2. Assume that the interaction potential  $U$  is strictly convex in the sense that there exists a constant  $a > 0$  such that  $\nabla^2 U \geq a$ . Then  $X$  supports a Gaussian isoperimetric inequality, that is, there exists a constant  $C > 0$  such that for every set  $E \subset X$  with finite perimeter  $P(E, X)$ , one has:*

$$\mu(E)\sqrt{-\ln \mu(E)} \leq CP(E, X).$$



## Part 1

# General theory of Besov spaces related to Dirichlet spaces



## Besov spaces related to a Dirichlet space

### 1.1. Definitions

Let  $(X, \mu, \mathcal{E}, \mathcal{F} = \mathbf{dom}(\mathcal{E}))$  be a symmetric Dirichlet space, that is,  $X$  is a topological measure space equipped with the Radon measure  $\mu$ ,  $\mathcal{E}$  a closed Markovian bilinear form on  $L^2(X, \mu)$ , and  $\mathcal{F}$  the collection of all functions  $u \in L^2(X, \mu)$  with  $\mathcal{E}(u, u)$  finite. The book [FOT11] is a standard reference on the theory of Dirichlet forms. Let  $\{P_t\}_{t \in [0, \infty)}$  denote the Markovian semigroup associated with  $(X, \mu, \mathcal{E}, \mathcal{F})$ . From classical theory (see for instance Theorems 1.4.1 and 1.4.2 in [Dav89]), the semigroup  $P_t$  lets  $L^1(X, \mu) \cap L^\infty(X, \mu)$  invariant and may be extended to a positive, contraction semigroup on  $L^p(X, \mu)$ ,  $1 \leq p \leq +\infty$ , that we shall still denote by  $P_t$ . Moreover, for  $1 \leq p < +\infty$ ,  $P_t$  is strongly continuous and for  $1 < p < +\infty$ ,  $P_t$  is a bounded analytic semigroup on  $L^p(X, \mu)$  with angle  $\theta_p = \frac{\pi}{2} \left(1 - \left|\frac{2}{p} - 1\right|\right)$ . In this monograph, we always assume that  $P_t$  is conservative, i.e.  $P_t 1 = 1$ .

Our basic definition of the Besov seminorm is the following:

DEFINITION 1.1. *Let  $p \geq 1$  and  $\alpha \geq 0$ . For  $f \in L^p(X, \mu)$ , we define the Besov seminorm:*

$$\|f\|_{p,\alpha} = \sup_{t>0} t^{-\alpha} \left( \int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p}.$$

Observe that if  $P_t$  admits a heat kernel  $p_t(x, y)$ , then

$$\int_X P_t(|f - f(y)|^p)(y) d\mu(y) = \int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y),$$

but the seminorm  $\|\cdot\|_{p,\alpha}$  is well defined even if  $P_t$  does not have a heat kernel.

Our goal in this paper is to study the Besov space

$$(1.1) \quad \mathbf{B}^{p,\alpha}(X) = \{f \in L^p(X, \mu) : \|f\|_{p,\alpha} < +\infty\},$$

and specially  $\mathbf{B}^{1,\alpha}(X)$  and its connection to the notion of bounded variation function. The norm on  $\mathbf{B}^{p,\alpha}(X)$  is defined as:

$$\|f\|_{\mathbf{B}^{p,\alpha}(X)} = \|f\|_{L^p(X,\mu)} + \|f\|_{p,\alpha}.$$

EXAMPLE 1.2. *If  $X = \mathbb{R}^n$  and  $\mathcal{E}$  is the standard Dirichlet form on  $\mathbb{R}^n$ , that is, for  $f, g \in W^{1,2}(\mathbb{R}^n)$  we have*

$$\mathcal{E}(f, g) = \int_{\mathbb{R}^n} \langle \nabla f(x), \nabla g(x) \rangle dx, \quad f, g \in W^{1,2}(\mathbb{R}^n),$$

*then  $\mathbf{B}^{p,\alpha}(X)$  coincides with the Besov-Nikol'skii space  $B_{p,\infty}^{2\alpha}(\mathbb{R}^n)$  that consists of functions  $f \in L^p(\mathbb{R}^n, dx)$  such that*

$$\sup_{h \in \mathbb{R}^n, h \neq 0} \frac{\|f(\cdot + h) - f(\cdot)\|_p}{h^{2\alpha}} < +\infty.$$

*We refer for instance to [AS61] and [Tai64] (Theorems 4 and 4\*) for several equivalent descriptions of those spaces. In particular, we observe therefore that  $\mathbf{B}^{1,1/2}(\mathbb{R}^n)$  is the space of bounded variation functions  $BV(\mathbb{R}^n)$  and that  $\mathbf{B}^{2,1/2}(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n)$  is the domain of  $\mathcal{E}$ . The observation  $\mathbf{B}^{1,1/2}(\mathbb{R}^n) = BV(\mathbb{R}^n)$  will be generalized in the framework of metric spaces with a weak Bakry-Émery type non-negative curvature condition and the observation  $\mathbf{B}^{1,1/2}(\mathbb{R}^n) = \mathcal{F}$  is a general fact, see Proposition 1.10*

### 1.2. A preliminary class of examples

Before discussing general properties of the spaces  $\mathbf{B}^{p,\alpha}(X)$ , it may be instructive to quickly look at a preliminary large class of examples in a smooth setting.

**PROPOSITION 1.3.** *Assume that  $X$  is a smooth manifold. Let  $L = V_0 + \sum_{i=1}^d V_i^2$  be a Hörmander's type operator on  $X$ , where the  $V_i$ 's are smooth vector fields. Let us assume that  $L$  is essentially self-adjoint on  $C_0^\infty(X)$  in  $L^2(X, \mu)$  for some Radon measure  $\mu$  on  $X$ . Consider the Dirichlet space  $(X, \mu, \mathcal{E}, \mathcal{F})$  obtained by closing the pre-Dirichlet form*

$$\mathcal{E}(f, g) = \int_X \Gamma(f, g) d\mu(x), \quad f, g \in C_0^\infty(X),$$

where  $\Gamma(f, g)$  is the carré du champ operator defined by  $\Gamma(f, g) = \frac{1}{2}(L(fg) - fLg - gLf)$ . Assume that the associated semigroup  $P_t$  is conservative. Then, for every  $p \geq 1$ ,

$$C_0^\infty(X) \subset \mathbf{B}^{p,1/2}(X)$$

and one has for every  $f \in C_0^\infty(X)$ , and open set  $A \subset X$ ,

$$\lim_{t \rightarrow 0} t^{-1/2} \left( \int_A P_t(|f - f(x)|^p)(x) d\mu(x) \right)^{1/p} = 2 \left( \frac{\Gamma(\frac{1+p}{2})}{\sqrt{\pi}} \right)^{1/p} \left( \int_A \Gamma(f, f)(x)^{p/2} d\mu(x) \right)^{1/p}.$$

**REMARK 1.4.** *In the previous setting one has therefore for  $f \in C_0^\infty(X)$ ,*

$$\left( \int_X \Gamma(f, f)(x)^{p/2} d\mu(x) \right)^{1/p} \leq C_p \|f\|_{p,1/2}.$$

If  $P_t$  satisfies moreover the Bakry-Émery estimate  $\sqrt{\Gamma(P_t f)} \leq C P_t \sqrt{\Gamma(f)}$ , then we will see (Chapter 4, Section 4.5) that we have a converse inequality

$$\|f\|_{p,1/2} \leq c_p \left( \int_X \Gamma(f, f)(x)^{p/2} d\mu(x) \right)^{1/p},$$

for  $p = 1$ .

**REMARK 1.5.** *Proposition 1.3 indicates that at a high level of generality, one may expect the Besov spaces  $\mathbf{B}^{p,1/2}(X)$ ,  $1 \leq p < +\infty$  to be closely related to the various notions of Sobolev spaces that have been defined on metric measure spaces (see for instance [Sha00]). While in this paper we shall only be concerned with the study of all the Besov spaces  $\mathbf{B}^{p,\alpha}(X)$ , the comparison between Sobolev spaces and Besov spaces will be made in Chapter 4 in the framework of Dirichlet spaces with absolutely continuous energy measures. In the framework of Chapter 5, it will be interesting to compare our results with the recent preprint [HKM18] on Sobolev spaces and calculus of variations on fractals. Such a comparison will be the subject of future study.*

**EXAMPLE 1.6.** *If  $X = \mathbb{R}^n$  and  $\mathcal{E}$  is the standard Dirichlet form on  $\mathbb{R}^n$ , it is natural to expect that for every  $p \geq 1$ , and every  $f \in \mathbf{B}^{p,1/2}(X)$*

$$\lim_{t \rightarrow 0} t^{-1/2} \left( \int_{\mathbb{R}^n} P_t(|f - f(x)|^p)(x) dx \right)^{1/p} = 2 \left( \frac{\Gamma(\frac{1+p}{2})}{\sqrt{\pi}} \right)^{1/p} \left( \int_{\mathbb{R}^n} |\nabla f(x)|^p dx \right)^{1/p}.$$

The case  $p = 1$  is proved in [MPPP07], but we did not find it in the literature for  $p > 1$ ,  $p \neq 2$ .

**PROOF OF PROPOSITION 1.3.** We use here a probabilistic argument. For  $x \in X$ , we denote by  $(B_t^x)_{t \geq 0}$  the  $L$ -Brownian motion on  $X$  started from  $x$ , that is the diffusion with generator  $L$ . It can be constructed as the solution of a stochastic differential equation in Stratonovich form:

$$dB_t^x = V_0(B_t^x) dt + \sqrt{2} \sum_{i=1}^d V_i(B_t^x) \circ d\beta_t^i$$

where  $\beta$  is a  $d$ -dimensional Brownian motion. Let  $f \in C_0^\infty(X)$ . The process

$$M_t^f = f(B_t^x) - f(x) - \int_0^t Lf(B_s^x) ds$$

is a square integrable martingale that can be written

$$M_t^f = \sqrt{2} \sum_{i=1}^d \int_0^t (V_i f)(B_s^x) d\beta_s^i.$$

We have then

$$\begin{aligned} P_t(|f - f(x)|^p)(x) &= \mathbb{E}(|f(B_t^x) - f(x)|^p) \\ &= \mathbb{E}\left(\left|M_t^f + \int_0^t Lf(B_s^x) ds\right|^p\right). \end{aligned}$$

Observe now that  $\frac{1}{\sqrt{t}} \int_0^t Lf(B_s^x) ds$  almost surely converges to 0 when  $t \rightarrow 0$ . Therefore,  $\frac{1}{\sqrt{t}} M_t^f$  converges in all  $L^p$ 's to the Gaussian random variable  $\sqrt{2} \sum_{i=1}^d (V_i f)(x) \beta_1^i$ . Since  $f$  has a compact support, one deduces that

$$\lim_{t \rightarrow 0} t^{-1/2} \left( \int_A P_t(|f - f(x)|^p)(x) d\mu(x) \right)^{1/p} = C_p \left( \int_A \Gamma(f, f)(x)^{p/2} d\mu(x) \right)^{1/p},$$

with  $C_p = \sqrt{2} \mathbb{E}(|N|^p)^{1/p} = 2 \left( \frac{\Gamma(\frac{1+p}{2})}{\sqrt{\pi}} \right)^{1/p}$ , where  $N$  denotes a Gaussian random variable with mean 0 and variance 1.  $\square$

### 1.3. Basic properties of the Besov spaces

PROPOSITION 1.7. *For  $p \geq 1$  and  $\alpha \geq 0$ ,  $\mathbf{B}^{p,\alpha}(X)$  is a Banach space.*

PROOF. Let  $f_n$  be a Cauchy sequence in  $\mathbf{B}^{p,\alpha}(X)$ . Let  $f$  be the  $L^p$  limit of  $f_n$ . From Minkowski inequality and conservativeness of  $P_t$ , one has

$$\begin{aligned} &\left| \left( \int_X P_t(|f_n - f_n(y)|^p)(y) d\mu(y) \right)^{1/p} - \left( \int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} \right| \\ &\leq \left( \int_X P_t(|(f_n - f) - (f_n(y) - f(y))|^p)(y) d\mu(y) \right)^{1/p} \\ &\leq \left( \int_X P_t(|f_n - f|^p)(y) d\mu(y) \right)^{1/p} + \left( \int_X P_t(|f_n(y) - f(y)|^p)(y) d\mu(y) \right)^{1/p} \\ &\leq 2 \|f - f_n\|_{L^p(X,\mu)}. \end{aligned}$$

Therefore

$$\lim_{n \rightarrow +\infty} \left( \int_X P_t(|f_n - f_n(y)|^p)(y) d\mu(y) \right)^{1/p} = \left( \int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p},$$

from which we deduce that

$$\left( \int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} \leq t^\alpha \lim_{n \rightarrow +\infty} \|f_n\|_{p,\alpha}.$$

Therefore  $f \in \mathbf{B}^{p,\alpha}(X)$  and  $\|f\|_{p,\alpha} \leq \lim_{n \rightarrow +\infty} \|f_n\|_{p,\alpha}$ . Similarly,

$$\|f - f_m\|_{p,\alpha} \leq \lim_{n \rightarrow +\infty} \|f_n - f_m\|_{p,\alpha}$$

and taking the limit  $m \rightarrow +\infty$  finishes the proof.  $\square$

The following basic observation shall be useful in the sequel:

LEMMA 1.8. *Let  $p \geq 1, \alpha \geq 0$ .*

$$\mathbf{B}^{p,\alpha}(X) = \left\{ f \in L^p(X, \mu) : \limsup_{t \rightarrow 0} t^{-\alpha} \left( \int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} < +\infty \right\}.$$

*In particular if  $\beta > \alpha$ ,  $\mathbf{B}^{p,\beta}(X) \subset \mathbf{B}^{p,\alpha}(X)$ . Moreover, for  $f \in \mathbf{B}^{p,\alpha}(X)$ , one has for every  $t > 0$ ,*

$$\|f\|_{p,\alpha} \leq \frac{2}{t^\alpha} \|f\|_{L^p(X)} + \sup_{s \in (0,t]} s^{-\alpha} \left( \int_X P_s(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p}.$$

PROOF. Obviously, if  $f \in \mathbf{B}^{p,\alpha}(X)$ , then

$$\limsup_{t \rightarrow 0} t^{-\alpha} \left( \int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} \leq \|f\|_{p,\alpha}.$$

Conversely, if  $\limsup_{t \rightarrow 0} t^{-\alpha} \left( \int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} < +\infty$ , then, for some  $\varepsilon > 0$ ,

$$\sup_{t \in (0,\varepsilon]} t^{-\alpha} \left( \int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} < +\infty$$

For  $t > \varepsilon$ , since  $|f(x) - f(y)|^p \leq 2^{p-1}(|f(x)|^p + |f(y)|^p)$  and the semigroup is conservative (and hence  $P_t \mathbf{1}(x) = 1$  for all  $x \in X$ ), one has then

$$t^{-\alpha} \left( \int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} \leq 2\varepsilon^{-\alpha} \|f\|_{L^p(X)}.$$

□

Both the following lemma and Proposition 1.10 are well-known, but we do not know where in the literature they appear in this form. In the standard source [FOT11] the corresponding results rely on a topological assumption, see footnote on page of [FOT11]. The argument given below is similar to that in Lemma 2.3.2.1 of [BH91].

LEMMA 1.9. *For  $t > 0$ ,  $f \in L^2(X, \mu)$  and a bounded function  $g$ , writing  $\langle \cdot, \cdot \rangle$  for the  $L^2(X, \mu)$  inner product*

$$\frac{1}{t} \int_X g(y) P_t(|f - f(y)|^2)(y) d\mu(y) = \frac{2}{t} \langle (I - P_t)f, fg \rangle - \frac{1}{t} \langle (I - P_t)f^2, g \rangle.$$

PROOF. Consider a simple function  $f = \sum_{j=1}^n a_j \mathbf{1}_{A_j}$  where the  $A_j$  are pairwise disjoint and measurable and compute (using pairwise disjointness and the symmetry of  $P_t$  on  $L^2$ ) that

$$\begin{aligned} & \int_X g(y) P_t(|f - f(y)|^2) d\mu(y) \\ &= \sum_{j,k,l,m} \int \left( a_j a_l P_t(\mathbf{1}_{A_j} \mathbf{1}_{A_l})(y) - a_j a_m P_t(\mathbf{1}_{A_j})(y) \mathbf{1}_{A_m}(y) \right. \\ & \quad \left. - a_k a_l \mathbf{1}_{A_k}(y) P_t(\mathbf{1}_{A_l})(y) + a_k a_m \mathbf{1}_{A_k}(y) \mathbf{1}_{A_m}(y) \right) g(y) d\mu(y) \\ &= \sum_j a_j^2 \left( \langle P_t \mathbf{1}_{A_j}, g \rangle + \langle \mathbf{1}_{A_j}, g \rangle \right) - 2 \sum_{j,l} a_j a_l \langle P_t \mathbf{1}_{A_j}, g \mathbf{1}_{A_l} \rangle \\ &= \sum_{j,l} a_j a_l \left( 2 \langle (I - P_t) \mathbf{1}_{A_j}, g \mathbf{1}_{A_l} \rangle - \langle (I - P_t)(\mathbf{1}_{A_j} \mathbf{1}_{A_l}), g \rangle \right) \\ &= 2 \langle (I - P_t)f, fg \rangle - \langle (I - P_t)f^2, g \rangle. \end{aligned}$$

The result then follows by density of simple functions in  $L^2(X, \mu)$  and  $L^p$  boundedness of  $P_t$  for  $t > 0$  for  $p = 1, 2$ ; for the former see [FOT11, page 33]. □

PROPOSITION 1.10.  $2\mathcal{E}(f) = \|f\|_{2,1/2}^2$  and hence  $\mathbf{B}^{2,1/2}(X) = \mathcal{F}$ .

PROOF. Apply Lemma 1.9 with  $g \equiv 1$ . Since  $(I - P_t)$  is symmetric and  $(I - P_t)1 = 0$  this simplifies to

$$(1.2) \quad \frac{1}{2t} \int_X P_t(|f - f(x)|^2) d\mu(x) = \frac{1}{t} \langle (I - P_t)f, f \rangle.$$

However it is standard (see [BH91] Proposition 1.2.3) that the right side of 1.2 is positive and decreasing in  $t$  and has limit  $\mathcal{E}(f)$  as  $t \downarrow 0$  if  $f \in \mathcal{F}$ , from which this limit is the supremum and the result follows.  $\square$

PROPOSITION 1.11. *Assume that  $\mathcal{E}$  has the property that  $\mathcal{E}(f, f) = 0$  implies  $f$  constant. Then, any  $f \in \mathbf{B}^{p,\alpha}(X)$  with  $1 \leq p \leq 2$  and  $\alpha > 1/p$  is constant.*

PROOF. Let  $f \in \mathbf{B}^{p,\alpha}(X)$  with  $1 \leq p \leq 2$ . For  $n \geq 0$ , one considers  $f_n(x) = f(x)$  if  $|f(x)| \leq n$  and  $f_n(x) = 0$  otherwise. Since  $|f_n(x) - f_n(y)| \leq |f(x) - f(y)|$  for every  $x, y \in X$ , it is clear that  $f_n \in \mathbf{B}^{p,\alpha}(X)$ . One has then

$$P_t(|f_n - f_n(x)|^2) = P_t(|f_n - f_n(x)|^{2-p} |f_n - f_n(x)|^p) \leq 2^{2-p} \|f_n\|_\infty^{2-p} P_t(|f_n - f_n(x)|^p).$$

Therefore

$$\frac{1}{2t} \int_X P_t(|f_n - f_n(x)|^2) d\mu(x) \leq 2^{1-p} t^{\alpha p - 1} \|f_n\|_\infty^{2-p} \|f_n\|_{p,\alpha}^p.$$

This implies

$$\lim_{t \rightarrow 0} \frac{1}{2t} \int_X P_t(|f_n - f_n(x)|^2) d\mu(x) = 0.$$

Thus  $f_n \in \mathcal{F}$  and  $\mathcal{E}(f_n, f_n) = 0$ . This implies that  $f_n$  is constant for every  $n$ , thus  $f$  is constant.  $\square$

PROPOSITION 1.12. *If  $\mathcal{E}$  is regular and  $f \in \mathbf{B}^{p,1/2}(X)$  for  $p > 2$  then the energy measure  $\nu_f$  (in the sense of Beurling-Deny) is absolutely continuous with respect to  $\mu$ .*

PROOF. Let  $p > 2$ ,  $f \in \mathbf{B}^{p,1/2}(X)$  and  $g \in L^{p/(p-2)}(X, \mu) \cap L^\infty(X, \mu) \cap \mathcal{F}$ ,  $g \geq 0$ . From Hölder's inequality we have

$$\begin{aligned} \frac{1}{t} \int_X |g(y)| P_t(|f - f(y)|^2) d\mu(y) &\leq \frac{1}{t} \int_X |g(y)| (P_t(|f - f(y)|^p))^{2/p} d\mu(y) \\ &\leq \frac{1}{t} \left( \int_X P_t(|f - f(y)|^p) d\mu(y) \right)^{2/p} \|g\|_{p/(p-2)} \\ &\leq \|f\|_{p,1/2}^2 \|g\|_{p/(p-2)} \end{aligned}$$

Now use the result of Lemma 1.9 and take the limit as  $t \downarrow 0$  to obtain

$$\|f\|_{p,1/2}^2 \|g\|_{p/(p-2)} \geq \lim_{t \downarrow 0} \frac{2}{t} \langle (I - P_t)f, fg \rangle - \frac{1}{t} \langle (I - P_t)f^2, g \rangle = 2\mathcal{E}(fg, f) - \mathcal{E}(f^2, g) = \int_X 2g d\nu_f$$

where, as in the previous result, the limit is by Proposition 1.2.3 of [BH91]. The final equality is from the definition of  $\nu_f$ , Definition 4.1.2 of [BH91].

In particular, if  $E_1 \subset E_2$  are of finite  $\mu$  measure and  $\mathbf{1}_{E_1} \leq g \leq \mathbf{1}_{E_2}$  then we obtain

$$\nu_f(E_1) \leq \int_X g d\nu_f \leq \frac{1}{\sqrt{2}} \|f\|_{p,1/2}^2 (\mu(E_2))^{(p-2)/p}.$$

Regularity of  $\mathcal{E}$  then supplies sufficiently many functions  $g \in \mathcal{F}$  to conclude that  $\nu_f$  is absolutely continuous with respect to  $\mu$ , see [BH91, FOT11, CF12].  $\square$

There are spaces on which the Kusuoka measure (and hence  $\nu_f$  for any non-constant  $f \in \mathcal{F}$ ) is singular to  $\mu$ , see [Kus82, BBST99]. For such spaces  $\mathbf{B}^{p,1/2}(X)$  is therefore trivial when  $p > 2$ .

COROLLARY 1.13. *If the Kusuoka measure is singular to  $\mu$  then  $\mathbf{B}^{p,1/2}(X)$  contains only constant functions when  $p > 2$ .*

### 1.4. Pseudo-Poincaré inequalities

The following pseudo-Poincaré inequalities will play an important role in many parts of this monograph.

LEMMA 1.14. *Let  $p \geq 1$  and  $\alpha > 0$ . For every  $f \in \mathbf{B}^{p,\alpha}(X)$ , and  $t \geq 0$ ,*

$$\|P_t f - f\|_{L^p(X,\mu)} \leq t^\alpha \|f\|_{p,\alpha}.$$

PROOF. From conservativeness of the semigroup and Hölder's inequality, we have

$$\begin{aligned} \|P_t f - f\|_{L^p(X,\mu)} &= \left( \int_X |P_t f(x) - f(x)|^p d\mu(x) \right)^{1/p} \\ &= \left( \int_X |P_t(f - f(x))(x)|^p d\mu(x) \right)^{1/p} \\ &\leq \left( \int_X P_t(|f - f(x)|^p)(x) d\mu(y) \right)^{1/p}. \end{aligned}$$

□

REMARK 1.15. *Triebel [Tri78] (Section 1.13.6) introduced the interpolation spaces:*

$$(L^p(X, \mu), \mathcal{E})_{\alpha, \infty} = \left\{ u \in L^p(X, \mu) : \sup_{t>0} t^{-\alpha} \|P_t u - u\|_{L^p(X, \mu)} < +\infty \right\}.$$

*From the previous lemma, it is therefore clear that  $\mathbf{B}^{p,\alpha}(X) \subset (L^p(X, \mu), \mathcal{E})_{\alpha, \infty}$ . However, it may not be true that  $\mathbf{B}^{p,\alpha}(X) = (L^p(X, \mu), \mathcal{E})_{\alpha, \infty}$ , even when  $X = \mathbb{R}^n$ , see Remark 4.5 in [MPPP07] and [Tai64] (Theorems 4 and 4\*).*

The following lemma will be useful:

LEMMA 1.16. *Let  $L$  the generator of  $\mathcal{E}$ . Let  $p > 1$  and  $\alpha > 0$ . Then, there exists a constant  $C > 0$  such that for every  $f \in \mathbf{B}^{p,\alpha}(X)$  and  $t \geq 0$ ,*

$$\|LP_t f\|_{L^p(X,\mu)} \leq C \frac{\|f\|_{p,\alpha}}{t^{1-\alpha}}.$$

PROOF. Since  $\lim_{t \rightarrow +\infty} \|LP_t f\|_{L^p(X,\mu)} = 0$ , we have

$$\begin{aligned} \|LP_t f\|_{L^p(X,\mu)} &= \left\| \sum_{k=1}^{+\infty} (LP_{2^k t} f - LP_{2^{k-1} t} f) \right\|_{L^p(X,\mu)} \\ &\leq \sum_{k=1}^{+\infty} \|LP_{2^k t} f - LP_{2^{k-1} t} f\|_{L^p(X,\mu)} \\ &\leq \sum_{k=1}^{+\infty} \|LP_{2^{k-1} t} (P_{2^{k-1} t} f - f)\|_{L^p(X,\mu)} \\ &\leq \sum_{k=1}^{+\infty} \frac{1}{2^{k-1} t} \|P_{2^{k-1} t} f - f\|_{L^p(X,\mu)} \\ &\leq C \sum_{k=1}^{+\infty} \frac{(2^{k-1} t)^\alpha}{2^{k-1} t} \|f\|_{p,\alpha} \\ &\leq C \frac{\|f\|_{p,\alpha}}{t^{1-\alpha}}, \end{aligned}$$

where we used in the proof the fact that since  $P_t$  is an analytic semigroup, we have for any  $g \in L^p(X, \mu)$ ,  $\|LP_t g\|_{L^p(X,\mu)} \leq \frac{C}{t} \|g\|_{L^p(X,\mu)}$ . □



In a very general framework, one can resort to (Hille-Yosida) spectral theory to define the fractional powers of a closed operator  $A$  on a Banach space via the following formula

$$(-A)^s x = \frac{\sin \pi s}{\pi} \int_0^\infty \lambda^{s-1} (\lambda I - A)^{-1} (-A)x \, d\lambda,$$

for every  $x \in D(A)$ . In fact, using Bochner's subordination one can express the fractional powers of  $A$  also in terms of the heat semi-group  $P_t = e^{tA}$  via the following formula, see (5) on p. 260 in [Yos95],

$$(1.3) \quad (-A)^s x = -\frac{s}{\Gamma(1-s)} \int_0^\infty t^{-s-1} [P_t x - x] \, dt.$$

Let us denote by  $L$  the generator of  $\mathcal{E}$  and, for  $0 < s \leq 1$ , by  $\mathcal{L}_p^s$  the domain of the operator  $(-L)^s$  in  $L^p(X, \mu)$ ,  $1 \leq p < +\infty$ . One has then the following proposition:

PROPOSITION 1.17. *Let  $\alpha \in (0, 1]$ ,  $p \geq 1$  and  $0 < s < \alpha$ . Then*

$$\mathbf{B}^{p,\alpha}(X) \subset \mathcal{L}_p^s,$$

and there exists a constant  $C = C_{p,s,\alpha}$  such that for every  $\mathbf{B}^{p,\alpha}(X)$ ,

$$\|(-L)^s f\|_{L^p(X,\mu)} \leq C \|f\|_{L^p(X,\mu)}^{1-\frac{s}{\alpha}} \|f\|_{p,\alpha}^{\frac{s}{\alpha}}.$$

In particular,  $(-L)^s : \mathbf{B}^{p,\alpha}(X) \rightarrow L^p(X, \mu)$  is bounded.

PROOF. Let  $f \in \mathbf{B}^{p,\alpha}(X)$ . One shall prove that the integral  $\int_0^\infty t^{-s-1} [P_t f - f] \, dt$  is finite, and therefore that  $f \in \mathcal{L}_p^s$ . For  $\delta > 0$ , one has

$$\begin{aligned} \left\| \int_0^\infty t^{-s-1} [P_t f - f] \, dt \right\|_{L^p(X,\mu)} &\leq \int_0^\infty t^{-s-1} \|P_t f - f\|_{L^p(X,\mu)} \, dt \\ &\leq \int_0^\delta t^{-s-1} \|P_t f - f\|_{L^p(X,\mu)} \, dt + \int_\delta^\infty t^{-s-1} \|P_t f - f\|_{L^p(X,\mu)} \, dt \\ &\leq \|f\|_{p,\alpha} \int_0^\delta t^{-s-1+\alpha} \, dt + 2\|f\|_{L^p(X,\mu)} \int_\delta^\infty t^{-s-1} \, dt \\ &\leq \|f\|_{p,\alpha} \frac{\delta^{\alpha-s}}{\alpha-s} + 2\|f\|_{L^p(X,\mu)} \frac{\delta^{-s}}{s}. \end{aligned}$$

The result follows then by optimizing  $\delta$ . □

### 1.5. Reflexivity of the Besov spaces

For any  $p \geq 1$  and  $\alpha > 0$ , recall the norm on  $\mathbf{B}^{p,\alpha}(X)$ :

$$\|f\|_{\mathbf{B}^{p,\alpha}(X)} = \|f\|_{L^p(X,\mu)} + \|f\|_{p,\alpha},$$

where

$$\|f\|_{p,\alpha} = \sup_{t>0} t^{-\alpha} \left( \int_X P_t(|f - f(y)|^p)(y) \, d\mu(y) \right)^{1/p}.$$

In the following proof, for convenience we use an equivalent norm, still denoted by  $\|\cdot\|_{\mathbf{B}^{p,\alpha}(X)}$ , defined as follows

$$\|f\|_{\mathbf{B}^{p,\alpha}(X)} = \left( \|f\|_{L^p(X,\mu)}^p + \|f\|_{p,\alpha}^p \right)^{\frac{1}{p}}.$$

LEMMA 1.18 (Clarkson type inequalities). *Let  $f, g \in \mathbf{B}^{p,\alpha}(X)$ . Let  $1 < p < \infty$  and  $q$  be its conjugate. If  $2 \leq p < \infty$ , then*

$$(1.4) \quad \left\| \frac{f+g}{2} \right\|_{\mathbf{B}^{p,\alpha}(X)}^p + \left\| \frac{f-g}{2} \right\|_{\mathbf{B}^{p,\alpha}(X)}^p \leq \frac{1}{2} \|f\|_{\mathbf{B}^{p,\alpha}(X)}^p + \frac{1}{2} \|g\|_{\mathbf{B}^{p,\alpha}(X)}^p.$$

If  $1 < p \leq 2$ , then

$$(1.5) \quad \left\| \frac{f+g}{2} \right\|_{\mathbf{B}^{p,\alpha}(X)}^q + \left\| \frac{f-g}{2} \right\|_{\mathbf{B}^{p,\alpha}(X)}^q \leq \left( \frac{1}{2} \|f\|_{\mathbf{B}^{p,\alpha}(X)}^p + \frac{1}{2} \|g\|_{\mathbf{B}^{p,\alpha}(X)}^p \right)^{q-1}.$$

The proof is essentially the same as in the  $L^p$  case (see for instance [AF03, Theorem 2.38]), so we omit it for concision. As a corollary, we immediately obtain

**COROLLARY 1.19.** *For any  $p > 1$  and  $\alpha > 0$ ,  $\mathbf{B}^{p,\alpha}(X)$  is a reflexive Banach space.*

**PROOF.** Since  $\mathbf{B}^{p,\alpha}(X)$  is a Banach space, from Milman-Pettis theorem it suffices to show that  $\mathbf{B}^{p,\alpha}(X)$  is uniformly convex. That is, for every  $0 < \varepsilon \leq 2$ , there exists  $\delta > 0$  such that for any  $f, g \in \mathbf{B}^{p,\alpha}(X)$  with  $\|f\|_{\mathbf{B}^{p,\alpha}(X)} = \|g\|_{\mathbf{B}^{p,\alpha}(X)} = 1$  and  $\|f - g\|_{\mathbf{B}^{p,\alpha}(X)} \geq \varepsilon$ , then

$$\left\| \frac{f+g}{2} \right\|_{\mathbf{B}^{p,\alpha}(X)} \leq 1 - \delta.$$

This can be seen as follows.

If  $2 \leq p < \infty$ , then (1.4) implies that

$$\left\| \frac{f+g}{2} \right\|_{\mathbf{B}^{p,\alpha}(X)}^p \leq 1 - \frac{\varepsilon^p}{2^p}.$$

If  $1 < p \leq 2$ , then (1.5) implies that

$$\left\| \frac{f+g}{2} \right\|_{\mathbf{B}^{p,\alpha}(X)}^q \leq 1 - \frac{\varepsilon^q}{2^q}.$$

□

## 1.6. Interpolation inequalities

We have the following basic interpolation inequalities.

**PROPOSITION 1.20.** *Let  $\theta \in [0, 1]$ ,  $1 \leq p, q < +\infty$  and  $\alpha, \beta > 0$ . Let us assume  $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r}$  and  $\alpha = \theta\beta + (1-\theta)\gamma$ . Then,  $\mathbf{B}^{q,\beta}(X) \cap \mathbf{B}^{r,\gamma}(X) \subset \mathbf{B}^{p,\alpha}(X)$  and for any  $f \in \mathbf{B}^{q,\beta}(X) \cap \mathbf{B}^{r,\gamma}(X)$ ,*

$$\|f\|_{p,\alpha} \leq \|f\|_{q,\beta}^\theta \|f\|_{r,\gamma}^{1-\theta}.$$

**PROOF.** Let  $f \in \mathbf{B}^{q,\beta}(X) \cap \mathbf{B}^{r,\gamma}(X)$ . One has for every  $t > 0$

$$t^{-\alpha} \left( \int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} = t^{-\theta\beta - (1-\theta)\gamma} \left( \int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p}.$$

Then, from Hölder's inequality

$$\begin{aligned} \int_X P_t(|f - f(y)|^p)(y) d\mu(y) &= \int_X P_t(|f - f(y)|^{p\theta + p(1-\theta)})(y) d\mu(y) \\ &\leq \left( \int_X P_t(|f - f(y)|^q)(y) d\mu(y) \right)^{\frac{p\theta}{q}} \left( \int_X P_t(|f - f(y)|^r)(y) d\mu(y) \right)^{\frac{p(1-\theta)}{r}}. \end{aligned}$$

One deduces

$$\begin{aligned} &t^{-\alpha} \left( \int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} \\ &\leq t^{-\theta\beta} \left( \int_X P_t(|f - f(y)|^q)(y) d\mu(y) \right)^{\frac{\theta}{q}} t^{-(1-\theta)\gamma} \left( \int_X P_t(|f - f(y)|^r)(y) d\mu(y) \right)^{\frac{1-\theta}{r}}. \end{aligned}$$

Taking the supremum over  $t > 0$  finishes the proof. □

As a special case of the previous result, note that since  $\mathbf{B}^{2,1/2}(X) = \mathcal{F}$  we easily deduce:

**COROLLARY 1.21.** *Let  $1 < p \leq 2$  and  $q$  be its conjugate ( $\frac{1}{p} + \frac{1}{q} = 1$ ). Let  $0 < \alpha < 1$ . Then, for any  $f \in \mathcal{F} \cap \mathbf{B}^{p,\alpha}(X)$  and  $g \in \mathcal{F} \cap \mathbf{B}^{q,1-\alpha}(X)$ , it holds that*

$$|\mathcal{E}(f, g)| \leq \|f\|_{p,\alpha} \|g\|_{q,1-\alpha}.$$

### 1.7. Continuity of the heat semigroup in the Besov spaces

Our goal in this section is to study the continuity properties of the semigroup  $P_t$  in the Besov spaces with range  $1 < p \leq 2$  and parameter  $\alpha = \frac{1}{2}$ . As corollaries we will deduce several important properties of the Besov spaces themselves. The main result is the following:

**THEOREM 1.22.** *Let  $1 < p \leq 2$ . There exists a constant  $C_p > 0$  such that for every  $f \in L^p(X, \mu)$  and  $t \geq 0$*

$$\|P_t f\|_{p,1/2} \leq \frac{C_p}{t^{1/2}} \|f\|_{L^p(X, \mu)}.$$

*In particular  $P_t : L^p(X, \mu) \rightarrow \mathbf{B}^{p,1/2}(X)$  is bounded for  $t > 0$ .*

A key intermediate result is the following inequality that will follow from nice ideas originally due to Nick Dungey [Dun08] and then developed in [CCH, CCFR17].

**LEMMA 1.23.** *Let  $1 < p \leq 2$ . There exists a constant  $C_p > 0$  such that for every non-negative  $f \in L^p(X, \mu)$  and  $t > 0$*

$$\left( \int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} \leq C_p \|f\|_{L^p(X, \mu)}^{1/2} \|P_t f - f\|_{L^p(X, \mu)}^{1/2}.$$

**PROOF.** Let  $1 < p \leq 2$  and  $t > 0$  be fixed in the following proof. The constant  $C$  in the following will denote a positive constant depending only on  $p$  that may change from line to line. For simplicity of notation, we shall assume in this proof that  $P_t$  admits a measurable heat kernel  $p_t(x, y)$ , however the argument can readily be extended if this is not the case, by using the heat kernel measure instead. For  $\alpha, \beta \geq 0$ , set

$$\gamma_p(\alpha, \beta) := p\alpha(\alpha - \beta) - \alpha^{2-p}(\alpha^p - \beta^p)$$

and for a non-negative function  $f \in L^p(X, \mu)$

$$\begin{aligned} \Gamma_p(f)(x) &:= pf(x) \int_X p_t(x, y)(f(x) - f(y)) d\mu(y) - f^{2-p}(x) \int_X p_t(x, y)(f^p(x) - f^p(y)) d\mu(y) \\ &= \int_X p_t(x, y) \gamma_p(f(x), f(y)) d\mu(y). \end{aligned}$$

Note that from Lemma 3.5 in [CCH], one has for any  $\alpha, \beta \geq 0$

$$(p-1)(\alpha - \beta)^2 \leq \gamma_p(\alpha, \beta) + \gamma_p(\beta, \alpha) \leq p(\alpha - \beta)^2$$

and that, similarly to [Dun08], page 122, one has  $\Gamma_p(f) \geq 0$ . Then the same argument as in [CCH], Lemma 3.6, gives

$$\begin{aligned} \int_X \int_X p_t(x, y) |f(x) - f(y)|^p d\mu(x) d\mu(y) &\leq C \int_X \int_X p_t(x, y) (\gamma_p(f(x), f(y)) + \gamma_p(f(y), f(x)))^{p/2} d\mu(x) d\mu(y) \\ &\leq C \int_X \int_X p_t(x, y) \left( \gamma_p^{p/2}(f(x), f(y)) + \gamma_p^{p/2}(f(y), f(x)) \right) d\mu(x) d\mu(y) \\ &= C \int_X \int_X p_t(x, y) \gamma_p^{p/2}(f(x), f(y)) d\mu(x) d\mu(y) \\ &\leq C \int_X \left( \int_X p_t(x, y) \gamma_p(f(x), f(y)) d\mu(y) \right)^{p/2} d\mu(x) \\ &= C \int_X \Gamma_p^{p/2}(f)(x) d\mu(x). \end{aligned}$$

We then follow the proof of Theorem 1 in [CCH] (see also Theorem 1.3 in [Dun08]). Let  $u(s, x) = e^{-s\Delta_t} f(x)$  where  $\Delta_t = \mathbf{Id} - P_t$ . Note that

$$\begin{aligned}\Gamma_p(u) &= pu(u - P_t u) - u^{2-p}(u^p - P_t(u^p)) \\ &= pu\Delta_t u - u^{2-p}\Delta_t(u^p) \\ &= -pu\partial_s u - u^{2-p}\Delta_t(u^p) \\ &= -u^{2-p}(\partial_s + \Delta_t)u^p.\end{aligned}$$

Set now

$$J(s, x) = -(\partial_s + \Delta_t)u^p(s, x),$$

so that

$$\Gamma_p(u) = u^{2-p}J.$$

Note that since  $u \geq 0$  and  $\Gamma_p(u) \geq 0$ , one has  $J \geq 0$ . One has then from Hölder's inequality

$$\begin{aligned}\int_X \Gamma_p^{p/2}(u) d\mu &= \int_X u^{p(2-p)/2} J^{p/2} d\mu \\ &\leq \left( \int_X u^p d\mu \right)^{\frac{2-p}{2}} \left( \int_X J d\mu \right)^{p/2}.\end{aligned}$$

One computes then

$$\int_X J d\mu = - \int_X \partial_s(u^p) d\mu = -p \int_X u^{p-1} \partial_s u d\mu = p \int_X u^{p-1} \Delta_t u d\mu.$$

Thus, we have from Hölder's inequality

$$\int_X J d\mu \leq p \|u\|_{L^p(X, \mu)}^{p-1} \|\Delta_t u\|_{L^p(X, \mu)}^p.$$

From the definition of  $\Delta_t$  one concludes therefore

$$\left( \int_X \int_X p_t(x, y) |u(s, x) - u(s, y)|^p d\mu(x) d\mu(y) \right)^{1/p} \leq C \|u(s, \cdot)\|_{L^p(X, \mu)}^{1/2} \|P_t u(s, \cdot) - u(s, \cdot)\|_{L^p(X, \mu)}^{1/2}.$$

Letting  $s \rightarrow 0^+$  yields from Fatou lemma

$$\left( \int_X \int_X p_t(x, y) |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p} \leq C \|f\|_{L^p(X, \mu)}^{1/2} \|P_t f - f\|_{L^p(X, \mu)}^{1/2}$$

□

We are now ready for the proof of Theorem 1.22.

PROOF. Let  $f \in L^p(X, \mu)$ . We can assume  $f \geq 0$ . If not, it is enough to decompose  $f$  as  $f^+ - f^-$  with  $f^+ = \max\{f, 0\}$  and  $f^- = \max\{-f, 0\}$ . Let  $s, t > 0$ , applying Lemma 1.23 to  $P_s f$ , one obtains

$$\left( \int_X P_t(|P_s f - P_s f(y)|^p)(y) d\mu(y) \right)^{1/p} \leq C_p \|P_s f\|_{L^p(X, \mu)}^{1/2} \|P_{t+s} f - P_s f\|_{L^p(X, \mu)}^{1/2}.$$

Note that  $\|P_s f\|_{L^p(X,\mu)} \leq \|f\|_{L^p(X,\mu)}$  and that

$$\begin{aligned} \|P_{t+s}f - P_s f\|_{L^p(X,\mu)} &= \left\| \int_0^t LP_{s+u}f du \right\|_{L^p(X,\mu)} \\ &= \left\| \int_0^t P_u LP_s f du \right\|_{L^p(X,\mu)} \\ &\leq \int_0^t \|P_u LP_s f\|_{L^p(X,\mu)} du \\ &\leq t \|LP_s f\|_{L^p(X,\mu)} \\ &\leq \frac{t}{s} \|f\|_{L^p(X,\mu)}, \end{aligned}$$

where in the last step we used analyticity of the semigroup. One concludes

$$\left( \int_X P_t (|P_s f - P_s f(y)|^p)(y) d\mu(y) \right)^{1/p} \leq C_p \left( \frac{t}{s} \right)^{1/2} \|f\|_{L^p(X,\mu)}.$$

Dividing both sides by  $\sqrt{t}$  and taking the supremum over  $t > 0$  finishes the proof of Theorem 1.22.  $\square$

We now collect several corollaries of Theorem 1.22. The first one is the following:

**COROLLARY 1.24.** *For  $1 < p \leq 2$ ,  $\mathbf{B}^{p,1/2}(X)$  is dense in  $L^p(X, \mu)$ .*

**PROOF.** Let  $1 < p \leq 2$ . Since for any  $f \in L^p(X, \mu)$ ,  $P_t f \in \mathbf{B}^{p,1/2}(X)$  and  $P_t f \rightarrow f$  when  $t \rightarrow 0$  (see for instance [Dav89, Theorem 1.4.1]), one concludes that  $\mathbf{B}^{p,1/2}(X)$  is dense in  $L^p(X, \mu)$ .  $\square$

We then have the following second corollary of Theorem 1.22.

**PROPOSITION 1.25.** *Assume that  $\mathcal{E}$  is regular. Let  $2 \leq p < +\infty$ . For every  $f \in L^p(X, \mu)$ , and  $t \geq 0$ ,*

$$\|P_t f - f\|_{L^p(X,\mu)} \leq C_p t^{1/2} \liminf_{s \rightarrow 0} s^{-1/2} \left( \int_X P_s (|f - f(y)|^p)(y) d\mu(y) \right)^{1/p}$$

**PROOF.** Let  $f \in L^p(X, \mu)$  and  $g \in L^q(X, \mu) \cap \mathcal{F}$  where  $q$  is the conjugate exponent of  $p$ . One has for  $t \geq 0$ ,

$$\int_X (P_t f - f) g d\mu = \int_0^t \mathcal{E}(P_s f, g) ds.$$

The idea is now to bound  $\mathcal{E}(g, P_s f)$  by using an approximation of  $\mathcal{E}$ . For  $\tau \in (0, \infty)$  we set

$$(1.6) \quad \mathcal{E}_\tau(u, u) := \frac{1}{\tau} \langle (I - P_\tau)u, u \rangle.$$

We have then  $\mathcal{E}(P_s f, g) = \lim_{\tau \rightarrow 0} \mathcal{E}_\tau(P_s f, g)$ . Note now that  $\mathcal{E}_\tau(P_s f, g) = \mathcal{E}_\tau(f, P_s g)$  and that from Hölder inequality (applied as in the proof of Proposition 1.20)

$$\begin{aligned} 2|\mathcal{E}_\tau(f, P_s g)| &\leq \tau^{-1/2} \left( \int_X P_\tau (|P_s g - P_s g(y)|^q)(y) d\mu(y) \right)^{1/q} \tau^{-1/2} \left( \int_X P_\tau (|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} \\ &\leq \tau^{-1/2} \left( \int_X P_\tau (|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} \|P_s g\|_{q,1/2} \\ &\leq C_p \tau^{-1/2} \left( \int_X P_\tau (|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} s^{-1/2} \|g\|_{L^q(X,\mu)}. \end{aligned}$$

One has therefore

$$\left| \int_X (P_t f - f) g d\mu \right| \leq C_p t^{1/2} \|g\|_{L^q(X,\mu)} \liminf_{s \rightarrow 0} s^{-1/2} \left( \int_X P_s (|f - f(y)|^p)(y) d\mu(y) \right)^{1/p},$$

and we conclude by  $L^p - L^q$  duality.  $\square$

COROLLARY 1.26. *Assume that  $\mathcal{E}$  is regular. Let  $2 \leq p < +\infty$  and  $\alpha > 1/2$ . If  $f \in \mathbf{B}^{p,\alpha}(X)$  then  $\mathcal{E}(f, f) = 0$ .*

PROOF. Indeed, for  $f \in \mathbf{B}^{p,\alpha}(X)$  with  $\alpha > 1/2$  one has

$$\liminf_{s \rightarrow 0} s^{-1/2} \left( \int_X P_s(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} = 0,$$

so that for every  $t \geq 0$ ,  $P_t f = f$ , and thus  $\mathcal{E}(f, f) = 0$ .  $\square$

We finally point out the following other corollary of Theorem 1.22.

PROPOSITION 1.27. *Let  $1 < p \leq 2$ . Let  $L$  be the generator of  $\mathcal{E}$  and  $\mathcal{L}_p$  be the domain of  $L$  in  $L^p(X, \mu)$ . Then*

$$\mathcal{L}_p \subset \mathbf{B}^{p,1/2}(X)$$

and for every  $f \in \mathcal{L}_p$ ,

$$(1.7) \quad \|f\|_{p,1/2}^2 \leq C \|Lf\|_{L^p(X,\mu)} \|f\|_{L^p(X,\mu)}.$$

PROOF. Write for  $\lambda > 0$

$$R_\lambda f = (L - \lambda)^{-1} f = \int_0^\infty e^{-\lambda t} P_t f dt.$$

Consequently

$$\|R_\lambda f\|_{p,1/2} \leq \int_0^\infty e^{-\lambda t} \|P_t f\|_{p,1/2} dt \leq \int_0^\infty e^{-\lambda t} \frac{C}{t^{1/2}} \|f\|_{L^p(X,\mu)} dt \leq C \lambda^{-1/2} \|f\|_{L^p(X,\mu)}.$$

It follows that

$$\|f\|_{p,1/2} \leq C \lambda^{-1/2} \|(L - \lambda)f\|_p \leq C(\lambda^{-1/2} \|Lf\|_{L^p(X,\mu)} + \lambda^{1/2} \|f\|_{L^p(X,\mu)}).$$

Taking  $\lambda = \|Lf\|_{L^p(X,\mu)} \|f\|_{L^p(X,\mu)}^{-1}$ , we then get the result.  $\square$

### 1.8. Besov critical exponents

Let  $p \geq 1$ . We define the  $L^p$  Besov critical exponent of  $X$  as

$$\alpha_p^*(X) = \inf\{\alpha > 0 : \mathbf{B}^{p,\alpha}(X) \text{ is trivial}\}.$$

$\mathbf{B}^{p,\alpha}(X)$  trivial means that any  $f \in \mathbf{B}^{p,\alpha}(X)$  is constant. We have then the following result:

PROPOSITION 1.28. *Assume that  $\mathcal{E}$  is regular and irreducible, i.e.  $\mathcal{E}$  has the property that  $\mathcal{E}(f, f) = 0$  implies  $f$  constant, then*

- (1)  $\alpha_2^*(X) = \frac{1}{2}$ ;
- (2)  $p \rightarrow \alpha_p^*(X)$  is non increasing;
- (3) For  $p \geq 2$ ,  $\alpha_p^*(X) \leq \frac{1}{2}$ ;
- (4) For  $1 \leq p \leq 2$ ,  $\frac{1}{2} \leq \alpha_p^*(X) \leq \frac{1}{p}$ .

PROOF.

- (1) Since  $\mathbf{B}^{2,1/2}(X) = \mathcal{F}$ , one has  $\alpha_2^*(X) \geq \frac{1}{2}$ . Then, from Corollary 1.26, one has  $\alpha_2^*(X) \leq \frac{1}{2}$ .
- (2) For simplicity of notation, we assume in the proof that  $P_t$  has a measurable heat kernel  $p_t(x, y)$ , but the proof easily extends to the general case. It suffices to show that  $\alpha_p^*(X) \leq \alpha_q^*(X)$  if  $q \leq p$ . Indeed, first notice that for any  $a, b > 0$  such that  $a \neq b$ , there holds

$$|a^p - b^p| |a^q - b^q|^{-1} \leq \frac{p}{q} \max\{a^q, b^q\}^{\frac{p}{q}-1}.$$

Equivalently,

$$|a^p - b^p|^q \leq \left(\frac{p}{q}\right)^q \max\{a^q, b^q\}^{p-q} |a^q - b^q|^q.$$

Next by Hölder's inequality and Fubini's theorem, we have

$$\begin{aligned}
& \int_X \int_X p_t(x, y) | |f(x)|^p - |f(y)|^p |^q d\mu(x) d\mu(y) \\
& \leq \left(\frac{p}{q}\right)^q \int_X \int_X p_t(x, y) (|f(x)|^{p-q} + |f(y)|^{p-q}) | |f(x)|^q - |f(y)|^q |^q d\mu(x) d\mu(y) \\
& \leq 2 \left(\frac{p}{q}\right)^q \int_X |f(x)|^{p-q} \int_X p_t(x, y) | |f(x)|^q - |f(y)|^q |^q d\mu(y) d\mu(x) \\
& \leq 2 \left(\frac{p}{q}\right)^q \int_X |f(x)|^{p-q} \left( \int_X p_t(x, y) | |f(x)|^q - |f(y)|^q |^p d\mu(y) \right)^{q/p} d\mu(x) \\
& \leq 2 \left(\frac{p}{q}\right)^q \| |f|^q \|_p^{p-q} \left( \int_X \int_X p_t(x, y) | |f(x)|^q - |f(y)|^q |^p d\mu(x) d\mu(y) \right)^{q/p} \\
& \leq 2 \left(\frac{p}{q}\right)^q \| |f|^q \|_p^{p-q} t^{\alpha q} \| |f|^q \|_{p, \alpha}^q,
\end{aligned}$$

which implies that

$$\| |f|^p \|_{q, \alpha}^q \leq 2 \left(\frac{p}{q}\right)^q \| |f|^q \|_p^{p-q} \| |f|^q \|_{p, \alpha}^q.$$

Hence  $\alpha_p^*(X) \leq \alpha_q^*(X)$ .

(3) This is Corollary 1.26.

(4) This follows from Theorem 1.22 and Proposition 1.11.

□

REMARK 1.29. *In view of the duality given by Corollary 1.21, it is natural to conjecture that under suitable conditions one may have*

$$\alpha_p^*(X) + \alpha_q^*(X) = 1$$

*if  $p$  and  $q$  are conjugate, i.e. satisfy  $\frac{1}{p} + \frac{1}{q} = 1$ .*

REMARK 1.30. *We will see in Chapters 4 and 5 that for local Dirichlet forms the limit*

$$\alpha_\infty(X) = \lim_{p \rightarrow +\infty} \alpha_p(X)$$

*is closely related to a Hölder regularity property in space of the heat semigroup. If the conjecture in Remark 1.29 is true, then classical interpolation theory suggests that it is reasonable to expect that for every  $p \geq 1$ :*

$$\alpha_p^*(X) = \frac{1}{p} + \left(1 - \frac{2}{p}\right) \alpha_\infty(X).$$

EXAMPLE 1.31. *For strongly local Dirichlet forms with absolutely continuous energy measures, we will see in Chapter 4 that one generically has  $\alpha_p^*(X) = \frac{1}{2}$  for every  $p \geq 1$ .*





## Sobolev and isoperimetric inequalities

In this chapter, we are interested in Sobolev type embeddings (the case  $p = 1$  corresponds to isoperimetric type results) for the Besov spaces defined in the previous chapter.

Let  $(X, \mu, \mathcal{E}, \mathcal{F} = \mathbf{dom}(\mathcal{E}))$  be a symmetric Dirichlet space. Let  $\{P_t\}_{t \in [0, \infty)}$  denote the Markovian semigroup associated with  $(X, \mu, \mathcal{E}, \mathcal{F})$ . Throughout the chapter, we shall assume that  $P_t$  admits a measurable heat kernel  $p_t(x, y)$ .

Let  $p \geq 1$  and  $\alpha \geq 0$ . As before, we define the Besov type seminorm:

$$\|f\|_{p,\alpha} = \sup_{t>0} t^{-\alpha} \left( \int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p}$$

and define

$$\mathbf{B}^{p,\alpha}(X) = \{f \in L^p(X, \mu) : \|f\|_{p,\alpha} < +\infty\}.$$

Throughout the chapter, we assume that  $\{P_t\}_{t \in (0, \infty)}$  is conservative and admits a heat kernel  $p_t(x, y)$  satisfying, for some  $C > 0$  and  $\beta > 0$ ,

$$(2.1) \quad p_t(x, y) \leq Ct^{-\beta}$$

for  $\mu \times \mu$ -a.e.  $(x, y) \in X \times X$ , and for each  $t \in (0, +\infty)$ .

Our goal in this chapter is to prove global Sobolev embeddings with sharp exponents for the space  $\mathbf{B}^{p,\alpha}(X)$  and one of the main results will be the following weak-type Sobolev inequality and the corresponding isoperimetric inequality:

**THEOREM 2.1.** *Let  $0 < \alpha < \beta$ . Let  $1 \leq p < \frac{\beta}{\alpha}$ . There exists a constant  $C_{p,\alpha} > 0$  such that for every  $f \in \mathbf{B}^{p,\alpha}(X)$ ,*

$$\sup_{s \geq 0} s \mu(\{x \in X : |f(x)| \geq s\})^{\frac{1}{q}} \leq C_{p,\alpha} \|f\|_{p,\alpha},$$

where  $q = \frac{p\beta}{\beta - p\alpha}$ . Therefore, there exists a constant  $C_{\text{iso}} > 0$ , such that for every subset set  $E \subset X$  with  $\mathbf{1}_E \in \mathbf{B}^{1,\alpha}(X)$

$$\mu(E)^{\frac{\beta - \alpha}{\beta}} \leq C_{\text{iso}} \|\mathbf{1}_E\|_{1,\alpha}.$$

### 2.1. Weak type Sobolev inequality

We follow and adapt to our setting a general approach to Sobolev inequalities developed in [BCLSC95] (see also [SC02]). The pseudo-Poincaré inequality proved in Lemma 1.14 plays a fundamental role here.

**LEMMA 2.2.** *Let  $1 \leq p, q < +\infty$  and  $\alpha > 0$ . There exists a constant  $C_{p,q,\alpha} > 0$  such that for every  $f \in \mathbf{B}^{p,\alpha}(X) \cap L^q(X, \mu)$  and  $s \geq 0$ ,*

$$\sup_{s \geq 0} s^{1+q\frac{\alpha}{\beta}} \mu(\{x \in X : |f(x)| > s\})^{\frac{1}{p}} \leq C_{p,q,\alpha} \|f\|_{p,\alpha} \|f\|_{L^q(X,\mu)}^{q\frac{\alpha}{\beta}}.$$

**PROOF.** We adapt an argument given in the proof of Theorem 9.1 in [BCLSC95]. Let  $f \in \mathbf{B}^{p,\alpha}(X)$  and denote

$$F(s) = \mu(\{x \in X : |f(x)| > s\})$$

We have then

$$F(s) \leq \mu(\{x \in X : |f(x) - P_t f(x)| > s/2\}) + \mu(\{x \in X : |P_t f(x)| > s/2\}).$$

Now, from the heat kernel upper bound

$$p_t(x, y) \leq \frac{C}{t^\beta}, \quad t > 0,$$

one deduces, that for  $g \in L^1(X, \mu)$ ,

$$|P_t g(x)| \leq \frac{C}{t^\beta} \|g\|_{L^1(X, \mu)}.$$

Since  $P_t$  is a contraction in  $L^\infty(X, \mu)$ , by the Riesz-Thorin interpolation one gets that

$$|P_t f(x)| \leq \frac{C^{1/q}}{t^{\beta/q}} \|f\|_{L^q(X, \mu)}.$$

Therefore, for  $s = 2 \frac{C^{1/q}}{t^{\beta/q}} \|f\|_{L^q(X, \mu)}$ , one has

$$\mu(\{x \in X : |P_t f(x)| > s/2\}) = 0.$$

On the other hand, from Theorem 1.14,

$$\mu(\{x \in X : |f(x) - P_t f(x)| > s/2\}) \leq 2^p s^{-p} t^{p\alpha} \|f\|_{p, \alpha}^p.$$

We conclude that

$$F(s)^{1/p} \leq \tilde{C} s^{-1-q \frac{\alpha}{\beta}} \|f\|_{\alpha, p} \|f\|_{L^q(X, \mu)}^{\frac{\alpha q}{\beta}}.$$

□

As a corollary, we are now ready to prove the weak Sobolev inequality.

**THEOREM 2.3.** *Let  $0 < \alpha < \beta$ . Let  $1 \leq p < \frac{\beta}{\alpha}$ . There exists a constant  $C_{p, \alpha} > 0$  such that for every  $f \in \mathbf{B}^{p, \alpha}(X)$ ,*

$$\sup_{s \geq 0} s \mu(\{x \in X : |f(x)| \geq s\})^{\frac{1}{q}} \leq C_{p, \alpha} \|f\|_{p, \alpha},$$

where  $q = \frac{p\beta}{\beta - p\alpha}$ .

**PROOF.** Let  $f \in \mathbf{B}^{p, \alpha}(X)$  be a non-negative function. For  $k \in \mathbb{Z}$ , we denote

$$f_k = (f - 2^k)_+ \wedge 2^k.$$

Observe that  $f_k \in L^p(X, \mu)$  and  $\|f_k\|_{L^p(X, \mu)} \leq \|f\|_{L^p(X, \mu)}$ . Moreover, for every  $x, y \in X$ ,  $|f_k(x) - f_k(y)| \leq |f(x) - f(y)|$  and so  $\|f_k\|_{p, \alpha} \leq \|f\|_{p, \alpha}$ . We also note that  $f_k \in L^1(X, \mu)$ , with

$$\|f_k\|_{L^1(X, \mu)} = \int_X |f_k| d\mu \leq 2^k \mu(\{x \in X : f(x) \geq 2^k\}).$$

We now use Lemma 2.2 to deduce:

$$\begin{aligned} \sup_{s \geq 0} s^{1 + \frac{\alpha}{\beta}} \mu(\{x \in X : f_k(x) > s\})^{\frac{1}{p}} &\leq C_{p, \alpha} \|f_k\|_{p, \alpha} \|f_k\|_{L^1(X, \mu)}^{\frac{\alpha}{\beta}} \\ &\leq C_{p, \alpha} \|f_k\|_{p, \alpha} (2^k \mu(\{x \in X : f(x) \geq 2^k\}))^{\frac{\alpha}{\beta}}. \end{aligned}$$

In particular, by choosing  $s = 2^k$  we obtain

$$2^{k(1 + \frac{\alpha}{\beta})} \mu(\{x \in X : f(x) \geq 2^{k+1}\})^{\frac{1}{p}} \leq C_{p, \alpha} \|f_k\|_{p, \alpha} (2^k \mu(\{x \in X : f(x) \geq 2^k\}))^{\frac{\alpha}{\beta}}.$$

Let

$$M(f) = \sup_{k \in \mathbb{Z}} 2^k \mu(\{x \in X : f(x) \geq 2^k\})^{1/q}$$

where  $q = \frac{p\beta}{\beta - p\alpha}$ . Using the fact that  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\beta}$  and the previous inequality we obtain:

$$2^k \mu(\{x \in X : f(x) \geq 2^{k+1}\})^{\frac{1}{p}} \leq 2^{-\frac{kq\alpha}{\beta}} C_{p, \alpha} \|f\|_{p, \alpha} M(f)^{\frac{q\alpha}{\beta}}.$$

and

$$2^k \mu(\{x \in X : f(x) \geq 2^{k+1}\})^{\frac{1}{q}} \leq C_{p, \alpha}^{\frac{p}{q}} \|f\|_{p, \alpha}^{p/q} M(f)^{\frac{p\alpha}{\beta}}.$$

Therefore

$$M(f)^{1-\frac{p\alpha}{\beta}} \leq 2C_{p,\alpha}^{\frac{p}{q}} \|f\|_{p,\alpha}^{p/q}.$$

One concludes

$$M(f) \leq 2^{q/p} C_{p,\alpha} \|f\|_{p,\alpha}.$$

This easily yields:

$$\sup_{s \geq 0} s \mu(\{x \in X : f(x) \geq s\})^{\frac{1}{q}} \leq 2^{1+q/p} C_{p,\alpha} \|f\|_{p,\alpha}.$$

Let now  $f \in \mathbf{B}^{p,\alpha}(X)$ , which is not necessarily non-negative. From the previous inequality applied to  $|f|$ , we deduce

$$\sup_{s \geq 0} s \mu(\{x \in X : |f(x)| \geq s\})^{\frac{1}{q}} \leq 2^{1+q/p} C_{p,\alpha} \|f\|_{p,\alpha} \leq 2^{1+q/p} C_{p,\alpha} \|f\|_{p,\alpha}.$$

□

## 2.2. Capacitary estimates

It is well-known that Sobolev inequalities are related to capacitary estimates. Let  $p \geq 1$  and  $0 < \alpha < \beta$ . For a measurable set  $A \subset X$ , we define its  $(\alpha, p)$  capacity:

$$\mathbf{Cap}_p^\alpha(A) = \inf\{\|f\|_{\alpha,p}^p : f \in \mathbf{B}^{\alpha,p}(X), \mathbf{1}_A \leq f \leq 1\}.$$

We have the following theorem:

**THEOREM 2.4.** *Let  $0 < \alpha < \beta$ . Let  $1 \leq p < \frac{\beta}{\alpha}$ . There exists a constant  $C_{p,\alpha} > 0$  such that for every measurable set  $A \subset X$ ,*

$$\mu(A)^{1-\frac{p\alpha}{\beta}} \leq C_{p,\alpha} \mathbf{Cap}_p^\alpha(A).$$

**PROOF.** This is an immediate corollary of Proposition 2.3. □

## 2.3. Isoperimetric inequality

Let  $E \subset X$  be a measurable set with finite measure. We will say that  $E$  has a finite  $\alpha$ -perimeter if  $\mathbf{1}_E \in \mathbf{B}^{1,\alpha}(X)$ . In that case, we will denote

$$P_\alpha(E) = \|\mathbf{1}_E\|_{1,\alpha}.$$

**PROPOSITION 2.5.** *Let  $0 < \alpha < \beta$ . There exists a constant  $C_{\text{iso}} > 0$ , such that for every subset  $E \subset X$  with finite  $\alpha$ -perimeter*

$$\mu(E)^{\frac{\beta-\alpha}{\beta}} \leq C_{\text{iso}} P_\alpha(E).$$

**PROOF.** Observe that we have

$$\|P_t \mathbf{1}_E - \mathbf{1}_E\|_{L^1(X,\mu)} = 2 \left( \mu(E) - \int (P_{t/2} \mathbf{1}_E)^2 d\mu \right).$$

Indeed,

$$\begin{aligned} \|P_t \mathbf{1}_E - \mathbf{1}_E\|_{L^1(X,\mu)} &= \int_E (1 - P_t \mathbf{1}_E) d\mu + \int_{E^c} P_t(\mathbf{1}_E) d\mu \\ &= \int_E (1 - P_t \mathbf{1}_E) d\mu + \int_E (P_t \mathbf{1}_{E^c}) d\mu \\ &= 2 \left( \mu(E) - \int_E P_t(\mathbf{1}_E) d\mu \right) \\ &= 2 \left( \mu(E) - \|P_{\frac{t}{2}}(\mathbf{1}_E)\|_{L^2(X,\mu)}^2 \right), \end{aligned}$$

where the last inequality is due to the fact that

$$\int_E P_t \mathbf{1}_E d\mu = \int (P_{t/2} \mathbf{1}_E)^2 d\mu.$$

We now note that

$$\begin{aligned} \int (P_{t/2} \mathbf{1}_E)^2 d\mu &\leq \left( \int_E \left( \int p_{t/2}(x, y)^2 d\mu(y) \right)^{\frac{1}{2}} d\mu(x) \right)^2 \\ &= \left( \int_E p_t(x, x)^{\frac{1}{2}} d\mu(x) \right)^2 \leq \frac{A}{t^\beta} \mu(E)^2. \end{aligned}$$

for some constant  $A > 0$ . Combining these equations yields

$$\mu(E) \leq Bt^\alpha P_\alpha(E) + \frac{C}{t^\beta} \mu(E)^2, \quad t > 0,$$

for some positive constants  $B, C$ . Optimizing in  $t$  concludes the proof.  $\square$

In the limiting case  $\alpha = \beta$ , the previous proof yields the following:

**COROLLARY 2.6.** *There exists a constant  $C_{\text{iso}} > 0$ , such that for every subset  $E \subset X$  with finite  $\beta$ -perimeter and  $\mu(E) > 0$ ,*

$$P_\beta(E) \geq C_{\text{iso}}.$$

#### 2.4. Strong Sobolev inequality

**DEFINITION 2.7.** *We say that the Dirichlet space satisfies the property  $(P_{p,\alpha})$  if there exists a constant  $C > 0$  such that for every  $f \in \mathbf{B}^{p,\alpha}(X)$ ,*

$$\|f\|_{p,\alpha} \leq C \liminf_{t \rightarrow 0} t^{-\alpha} \left( \int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p}$$

**REMARK 2.8.** *In the Chapter 4 framework, property  $(P_{p,\alpha})$  is satisfied when  $p = 1, \alpha = 1/2$ , see Theorem 4.11.*

Our main theorem is the following:

**THEOREM 2.9.** *Assume that the Dirichlet space satisfies the property  $(P_{p,\alpha})$  and that  $\beta$  satisfies (2.1). Let  $0 < \alpha < \beta$ . Let  $1 \leq p < \frac{\beta}{\alpha}$ . There exists a constant  $C_{p,\alpha,\beta} > 0$  such that for every  $f \in \mathbf{B}^{p,\alpha}(X)$ ,*

$$\|f\|_{L^q(X,\mu)} \leq C_{p,\alpha,\beta} \|f\|_{p,\alpha},$$

where  $q = \frac{p\beta}{\beta - p\alpha}$ .

Note that in the standard Euclidean setting of  $\mathbb{R}^n$  the Sobolev embedding theorem holds as above with  $\beta = n$ . To show that the weak type inequality implies the desired Sobolev inequality, we will need another cutoff argument and the following lemma is needed.

**LEMMA 2.10.** *For  $f \in \mathbf{B}^{p,\alpha}(X)$ ,  $f \geq 0$ , denote  $f_k = (f - 2^k)_+ \wedge 2^k$ ,  $k \in \mathbb{Z}$ . There exists a constant  $C > 0$  such that for every  $f \in \mathbf{B}^{p,\alpha}(X)$ ,*

$$\left( \sum_{k \in \mathbb{Z}} \|f_k\|_{p,\alpha}^p \right)^{1/p} \leq C \|f\|_{p,\alpha}.$$

**PROOF.** By a similar type of argument, as in the the proof of Lemma 7.1 in [BCLSC95], one has for some constant  $C_p > 0$ ,

$$\sum_{k \in \mathbb{Z}} \int_X \int_X |f_k(x) - f_k(y)|^p p_t(x, y) d\mu \leq C_p \int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu.$$

As a consequence of property  $(P_{p,\alpha})$ ,

$$\left( \sum_{k \in \mathbb{Z}} \|f_k\|_{p,\alpha}^p \right)^{1/p} \leq C'_p \|f\|_{p,\alpha}.$$

and the proof is complete.  $\square$

We can now conclude the proof of Theorem 2.9.

**PROOF OF THEOREM 2.9.** Let  $f \in \mathbf{B}^{p,\alpha}(X)$ . We can assume  $f \geq 0$ . As before, denote  $f_k = (f - 2^k)_+ \wedge 2^k$ ,  $k \in \mathbb{Z}$ . From Lemma 2.3 applied to  $f_k$ , we see that

$$\sup_{s \geq 0} s \mu(\{x \in X : |f_k(x)| \geq s\})^{\frac{1}{q}} \leq C_{p,\alpha} \|f_k\|_{p,\alpha}$$

In particular for  $s = 2^k$ , we get

$$2^k \mu(\{x \in X : f(x) \geq 2^{k+1}\})^{\frac{1}{q}} \leq C_{p,\alpha} \|f_k\|_{p,\alpha}.$$

Therefore,

$$\sum_{k \in \mathbb{Z}} 2^{kq} \mu(\{x \in X : f(x) \geq 2^{k+1}\}) \leq C_{p,\alpha}^q \sum_{k \in \mathbb{Z}} \|f_k\|_{p,\alpha}^q.$$

Since  $q \geq p$ , one has  $\sum_{k \in \mathbb{Z}} \|f_k\|_{p,\alpha}^q \leq (\sum_{k \in \mathbb{Z}} \|f_k\|_{p,\alpha}^p)^{q/p}$ . Thus, from the previous lemma

$$\sum_{k \in \mathbb{Z}} 2^{kq} \mu(\{x \in X : f(x) \geq 2^{k+1}\}) \leq C_{p,\alpha}^q \|f\|_{p,\alpha}^q.$$

Finally, we observe that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} 2^{kq} \mu(\{x \in X : f(x) \geq 2^{k+1}\}) &\geq \frac{q}{2^{q+1} - 2^q} \sum_{k \in \mathbb{Z}} \int_{2^{k+1}}^{2^{k+2}} s^{q-1} \mu(\{x \in X : f(x) \geq s\}) ds \\ &\geq \frac{1}{2^{q+1} - 2^q} \|f\|_{L^q(X,\mu)}^q. \end{aligned}$$

The proof is thus complete.  $\square$



## Cheeger constant and Gaussian isoperimetry

While the previous chapter was devoted to Sobolev inequalities on Dirichlet spaces for which the semigroup satisfies ultracontractive estimates, the present chapter is devoted to situations where the Dirichlet form satisfies a Poincaré inequality or a log-Sobolev inequality. Let  $(X, \mu, \mathcal{E}, \mathcal{F} = \mathbf{dom}(\mathcal{E}))$  be a symmetric Dirichlet space. Let  $\{P_t\}_{t \in [0, \infty)}$  denote the Markovian semigroup associated with  $(X, \mu, \mathcal{E}, \mathcal{F})$ . As usual, we assume that  $P_t$  is conservative.

### 3.1. Buser's type inequality for the Cheeger constant of a Dirichlet space

In the context of a smooth compact Riemannian manifold with Riemannian measure  $\mu$ , Cheeger [Che70] introduced the following isoperimetric constant

$$h = \inf \frac{\mu(\partial A)}{\mu(A)},$$

where the infimum runs over all open subsets  $A$  with smooth boundary  $\partial A$  such that  $\mu(A) \leq \frac{1}{2}$ . Cheeger's constant can be used to bound from below the first non zero eigenvalue of the manifold. Indeed, it is proved in [Che70] that

$$\lambda_1 \geq \frac{h^2}{4}.$$

Buser [Bus82] then proved that if the Riemannian Ricci curvature of the manifold is non-negative, then we actually have

$$\lambda_1 \leq Ch^2$$

where  $C$  is a universal constant depending only on the dimension. Buser's inequality was reproved by Ledoux [Led94] using heat semigroup techniques. Under proper assumptions, by using the tools we introduced, Ledoux' technique can essentially reproduced in our general framework of Dirichlet spaces.

We assume in this section that  $\mathcal{E}$  satisfies a spectral gap inequality and normalize the measure  $\mu$  in such a way that  $\mu(X) = 1$ . For  $\alpha \in (0, 1]$ , we define the  $\alpha$ -Cheeger's constant of  $X$  by

$$h_\alpha = \inf \frac{\|\mathbf{1}_E\|_{1, \alpha}}{\mu(E)}$$

where the infimum runs over all measurable sets  $E$  such that  $\mu(E) \leq \frac{1}{2}$  and  $\mathbf{1}_E \in \mathbf{B}^{1, \alpha}(X)$ . We denote by  $\lambda_1$  the spectral gap of  $\mathcal{E}$ .

**THEOREM 3.1.** *We have  $h_\alpha \geq (1 - e^{-1})\lambda_1^\alpha$ .*

PROOF. Let  $A$  be a set with  $P_\alpha(A) := \|\mathbf{1}_A\|_{1,\alpha} < +\infty$ . As before, by symmetry and stochastic completeness of the semigroup, we have

$$\begin{aligned} \|\mathbf{1}_A - P_t \mathbf{1}_A\|_{L^1(X,\mu)} &= \int_A (1 - P_t \mathbf{1}_A) d\mu + \int_{A^c} P_t(\mathbf{1}_A) d\mu \\ &= \int_A (1 - P_t \mathbf{1}_A) d\mu + \int_A (P_t \mathbf{1}_{A^c}) d\mu \\ &= 2 \left( \mu(A) - \int_A P_t(\mathbf{1}_A) d\mu \right) \\ &= 2 \left( \mu(A) - \|P_{\frac{t}{2}}(\mathbf{1}_A)\|_{L^2(X,\mu)}^2 \right). \end{aligned}$$

By definition, we have

$$\|P_t \mathbf{1}_A - \mathbf{1}_A\|_{L^1(X,\mu)} \leq t^\alpha P_\alpha(A).$$

We deduce that

$$\mu(A) \leq \frac{1}{2} t^\alpha P_\alpha(A) + \|P_{\frac{t}{2}}(\mathbf{1}_A)\|_{L^2(X,\mu)}^2.$$

Now, by the spectral theorem,

$$\|P_{\frac{t}{2}}(\mathbf{1}_A)\|_{L^2(X,\mu)}^2 = \mu(A)^2 + \|P_{\frac{t}{2}}(\mathbf{1}_A - \mu(A))\|_{L^2(X,\mu)}^2 \leq \mu(A)^2 + e^{-\lambda_1 t} \|\mathbf{1}_A - \mu(A)\|_{L^2(X,\mu)}^2$$

This yields

$$\mu(A) \leq \frac{1}{2} t^\alpha P_\alpha(A) + \mu(A)^2 + e^{-\lambda_1 t} \|\mathbf{1}_A - \mu(A)\|_{L^2(X,\mu)}^2.$$

Equivalently, one obtains

$$\frac{1}{2} t^\alpha P_\alpha(A) \geq \mu(A)(1 - \mu(A))(1 - e^{-\lambda_1 t}).$$

Therefore,

$$h_\alpha \geq \sup_{t>0} \left( \frac{1 - e^{-\lambda_1 t}}{t^\alpha} \right).$$

□

As already noted in [BK17], let us observe that it is known that the Cheeger lower bound on  $\lambda_1$  may be obtained under further assumptions on the Dirichlet space  $(X, d, \mathcal{E})$ . Indeed, assume that  $\mathcal{E}$  is strictly local with a carré du champ  $\Gamma$ , that Lipschitz functions are in the domain of  $\mathcal{E}$  and that  $\sqrt{\Gamma(f)}$  is an upper gradient in the sense that for any Lipschitz function  $f$ ,

$$\sqrt{\Gamma(f)}(x) = \limsup_{d(x,y) \rightarrow 0} \frac{|f(x) - f(y)|}{d(x,y)}.$$

In that case, if  $A$  is a closed set of  $X$ , one defines its Minkowski exterior boundary measure by

$$\mu_+(A) = \liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\mu(A_\varepsilon) - \mu(A)),$$

where  $A_\varepsilon = \{x \in X, d(x, A) < \varepsilon\}$ . We can then define a Cheeger's constant of  $X$  by

$$h_+ = \inf \frac{\mu_+(E)}{\mu(E)}$$

where the infimum runs over all closed sets  $E$  such that  $\mu(E) \leq \frac{1}{2}$ . Then, according to Theorem 8.5.2 in [BGL14], one has

$$\lambda_1 \geq \frac{h_+^2}{4}.$$



### 3.2. Log-Sobolev and Gaussian isoperimetric inequalities

We assume in this section that  $\mathcal{E}$  satisfies a log-Sobolev inequality and normalize the measure  $\mu$  in such a way that  $\mu(X) = 1$ .

We define the Gaussian isoperimetric constant of  $X$  by

$$k = \inf \frac{\|\mathbf{1}_E\|_{1,1/2}}{\mu(E)\sqrt{-\ln \mu(E)}}$$

where the infimum runs over all sets  $E$  such that  $\mu(E) \leq \frac{1}{2}$  and  $\mathbf{1}_E \in \mathbf{B}^{1,1/2}(X)$ . We denote by  $\rho_0$  the log-Sobolev constant of  $X$ , that is the best constant in the inequality

$$(3.1) \quad \int f^2 \ln f^2 d\mu - \int f^2 d\mu \ln \int f^2 d\mu \leq \frac{1}{\rho_0} \mathcal{E}(f, f).$$

Following an argument of M. Ledoux [Led94], one obtains:

**THEOREM 3.2.**

$$\rho_0 \leq C_{\text{Ledoux}} k^2$$

where  $C_{\text{Ledoux}}$  is a numerical constant.

**PROOF.** Let  $A$  be a measurable set such that  $P(A) := \|\mathbf{1}_A\|_{1,1/2} < +\infty$ . By the same computations as before we have

$$\mu(A) \leq \frac{1}{2} \sqrt{t} P(A) + \|P_{\frac{t}{2}}(\mathbf{1}_A)\|_{L^2(X, \mu)}^2.$$

Now we can use the hypercontractivity constant to bound  $\|P_{\frac{t}{2}}(\mathbf{1}_A)\|_2^2$ . Indeed, from Gross' theorem it is well known that the logarithmic Sobolev inequality

$$\int f^2 \ln f^2 d\mu - \int f^2 d\mu \ln \int f^2 d\mu \leq \frac{1}{\rho_0} \mathcal{E}(f, f),$$

is equivalent to hypercontractivity property

$$\|P_t f\|_{L^q(X, \mu)} \leq \|f\|_{L^p(X, \mu)}$$

for all  $f$  in  $L^p(X, \mu)$  whenever  $1 < p < q < \infty$  and  $e^{\rho_0 t} \geq \sqrt{\frac{q-1}{p-1}}$ .

Therefore, with  $p(t) = 1 + e^{-2\rho_0 t} < 2$ , we get,

$$\begin{aligned} \sqrt{t} P(A) &\geq 2 \left( \mu(A) - \mu(A)^{\frac{2}{p(t)}} \right) \\ &\geq 2\mu(A) \left( 1 - \mu(A)^{\frac{1-e^{-2\rho_0 t}}{1+e^{-2\rho_0 t}}} \right). \end{aligned}$$

By using then the computation page 956 in [Led94], one deduces that if  $A$  is a set which has a finite  $P(A)$  and such that  $0 \leq \mu(A) \leq \frac{1}{2}$ , then

$$P(A) \geq \tilde{C} \sqrt{\rho_0} \mu(A) \left( \ln \frac{1}{\mu(A)} \right)^{\frac{1}{2}},$$

where  $\tilde{C}$  is a numerical constant. □



## Part 2

# Specific settings



## Strictly local Dirichlet spaces with doubling measure and 2-Poincaré inequality

In this chapter we define the space  $BV(X)$  of functions of bounded variation when  $\mathcal{E}$  is strictly local (see the exact assumptions in Section 4.1). We shall then prove that under a weak Bakry-Émery curvature dimension condition, one actually has  $BV(X) = \mathbf{B}^{1,1/2}(X)$  with comparable seminorms. One of the key tools used there is the co-area formula for BV functions; we establish this formula in Lemma 4.9 of the present chapter. We will also study in that framework the spaces  $\mathbf{B}^{p,\alpha}(X)$  introduced in Chapter 1 and relate them to Besov spaces previously considered in the literature. Finally, applications to Sobolev and isoperimetric inequalities are given.

### 4.1. Local Dirichlet spaces and standing assumptions

We will assume in this section that  $(X, \mu)$  is a topological space equipped with a Radon measure  $\mu$ , and that there is a strongly local Markovian Dirichlet form  $\mathcal{E}$  on  $X$ . As  $\mathcal{E}$  is strongly local, it is represented by a signed Radon measure  $\Gamma(u, v)$  for each  $u, v \in \mathcal{F}$  such that whenever  $\varphi$  is a bounded function in  $\mathcal{F}$  and  $u, v \in \mathcal{F}$ , we have

$$\mathcal{E}(\varphi u, v) + \mathcal{E}(\varphi v, u) - \mathcal{E}(\varphi, uv) = 2 \int_X \varphi d\Gamma(u, v).$$

We then also have  $\mathcal{E}(u, v) = \int_X d\Gamma(u, v)$ , see for example [Stu94]. With respect to  $\mathcal{E}$  we can define the following *intrinsic metric*  $d_{\mathcal{E}}$ :

$$d_{\mathcal{E}}(x, y) = \sup\{u(x) - u(y) : u \in \mathcal{F} \cap C(X) \text{ and } \Gamma(u, u) \leq \mu\}.$$

Here by  $\Gamma(u, u) \leq \mu$  we mean that for each Borel set  $A \subset X$  we have  $\Gamma(u, u)(A) = \int_A d\Gamma(u, u) \leq \mu(A)$ . In general there is no reason why  $d_{\mathcal{E}}$  is a metric on  $X$  (for it could be infinite for a given pair of points  $x, y$  or zero for some distinct pair of points), but *suppose that*  $d_{\mathcal{E}}$  is a metric on  $X$  such that the topology generated by  $d_{\mathcal{E}}$  coincides with the topology on  $X$  and that balls with respect to  $d_{\mathcal{E}}$  have compact closures in this topology. Then by [Stu94, Lemma 1, Lemma 1'], functions  $\varphi$  of the form  $x \mapsto d_{\mathcal{E}}(x, y)$  for each  $y \in X$ , and maps  $\varphi$  of the form  $x \mapsto (r - d_{\mathcal{E}}(x, y))_+$  for each  $r > 0$  and  $y \in X$  are in  $\mathcal{F}_{loc}(X)$  with  $d\Gamma(\varphi, \varphi)/d\mu \leq 1$ . So we will also assume in this section that  $d_{\mathcal{E}}$  gives a metric on  $X$  that is commensurate with the topology on  $X$ , and that with respect to this metric the measure  $\mu$  is *doubling*, that is, there is a constant  $C > 0$  such that whenever  $x \in X$  and  $r > 0$ , we have  $\mu(B(x, 2r)) \leq C \mu(B(x, r))$ .

**LEMMA 4.1.** *Let  $f : X \rightarrow \mathbb{R}$  be locally Lipschitz continuous with respect to the metric  $d_{\mathcal{E}}$ . Then  $f \in \mathcal{F}_{loc}(X)$  with  $\Gamma(f, f) \ll \mu$ . If  $f$  is locally  $L$ -Lipschitz, then  $\Gamma(f, f) \leq L \mu$ .*

**PROOF.** Let  $Q$  be a countable dense subset of  $X$ . Let  $U \subset X$  be a bounded open set and let  $\{q_i\}_{i \in I \subset \mathbb{N}}$  be an enumeration of  $Q \cap U$ . Note that  $Q \cap U$  is dense in  $U$ . For each  $i \in I$  let  $\psi_i(x) = d_{\mathcal{E}}(x, q_i)$ . Then as explained above,  $\psi_i \in \mathcal{F}(U)$  with  $\Gamma(\psi_i, \psi_i) \leq \mu$ . For  $j \in I$  set

$$f_j(x) := \inf\{f(q_i) + L \psi_i(x) : i \in I \text{ with } i \leq j\},$$

where  $L \geq 0$  is the Lipschitz constant of  $f$  in  $U$ . The above functions are inspired by the proof of the McShane extension theorem (see for example [Hei01]). By the lattice properties of Dirichlet form, it is

seen that each  $f_j \in \mathcal{F}(U)$  with  $d\Gamma(f_j, f_j) \leq L$ . Furthermore,  $f_j$  are  $L$ -Lipschitz in  $U$  with  $f_j(q_i) = f(q_i)$  for  $i \in I$  with  $i \leq j$ .

We can see that  $f_j \rightarrow f$  monotonically and hence (as  $f$  and  $f_j$  are bounded in  $U$  because  $U$  is bounded)  $f_j \rightarrow f$  in  $L^2(U)$ , with  $d\Gamma(f, f)/d\mu \leq L$  on  $U$ .  $\square$

Thus locally Lipschitz functions  $f$  with respect to  $d_{\mathcal{E}}$  satisfy

- (1)  $f \in \mathcal{F}_{loc}(X)$ ,
- (2)  $\Gamma(f, f)$  is absolutely continuous with respect to the underlying measure  $\mu$ ; we denote its Radon-Nikodym derivative by  $|\nabla f|^2$ , that is,  $|\nabla f|$  is the square-root of its Radon-Nikodym derivative.

We say that  $(X, \mu, \mathcal{E}, \mathcal{F})$  supports a 1-Poincaré inequality if there are constants  $C, \lambda > 0$  such that whenever  $B$  is a ball in  $X$  (with respect to the metric  $d_{\mathcal{E}}$ ) and  $u \in \mathcal{F}$ , we have

$$\frac{1}{\mu(B)} \int_B |u - u_B| d\mu \leq C \operatorname{rad}(B) \frac{1}{\mu(\lambda B)} \int_{\lambda B} |\nabla u| d\mu.$$

Of course, the 1-Poincaré inequality does not make sense if  $\mathcal{E}$  does not satisfy the condition of strong locality. We say that  $(X, \mu, \mathcal{E}, \mathcal{F})$  supports 2-Poincaré inequality if there are constants  $C, \lambda > 0$  such that whenever  $B$  is a ball in  $X$  (with respect to the metric  $d_{\mathcal{E}}$ ) and  $u \in \mathcal{F}$ , we have

$$\frac{1}{\mu(B)} \int_B |u - u_B| d\mu \leq C \operatorname{rad}(B) \left( \frac{1}{\mu(\lambda B)} \int_{\lambda B} d\Gamma(u, u) \right)^{1/2}.$$

The requirement that  $\mathcal{E}$  supports a 1-Poincaré inequality is significantly a stronger requirement than that of 2-Poincaré inequality. Much of the current theory on functions of bounded variation in the metric setting require a 1-Poincaré inequality. In this chapter we will *not* require the support of 1-Poincaré inequality, only the weaker 2-Poincaré inequality, but in some of the analysis we would need an additional requirement called the weak Bakry-Émery curvature condition, see the discussion below.

Should the 2-Poincaré inequality be satisfied, a standard argument, due to Semmes, tells us that locally Lipschitz continuous functions forms a dense subclass of  $\mathcal{F}$ , where  $\mathcal{F}$  is equipped with the norm

$$\|u\|_{\mathcal{F}} := \|u\|_{L^2(X)} + \sqrt{\mathcal{E}(u, u)},$$

see for example [HKST15, Chapter 8].

**Standing assumptions for this chapter:** For this chapter we will assume that  $\mathcal{E}$  is a strongly local Markovian Dirichlet form that induces a metric  $d = d_{\mathcal{E}}$  with respect to which  $\mu$  is doubling and supports a 2-Poincaré inequality. We will also assume that the class of locally Lipschitz continuous functions on  $X$  forms a dense subclass of  $L^1(X)$  and that  $X$  is complete with respect to  $d_{\mathcal{E}}$ . As a consequence of the doubling property of  $\mu$  it follows that closed and bounded subsets of  $X$  are compact.

The work of Saloff-Coste [SC92] tells us that the above standing assumptions are equivalent to the property that there is an associated heat kernel function  $p_t(x, y)$  on  $[0, \infty) \times X \times X$  for which there are constants  $c_1, c_2, C > 0$  such that whenever  $t > 0$  and  $x, y \in X$ ,

$$(4.1) \quad \frac{1}{C} \frac{e^{-c_1 d(x, y)^2/t}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} \leq p_t(x, y) \leq C \frac{e^{-c_2 d(x, y)^2/t}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}}.$$

The above inequality is called the *Gaussian bounds* for the heat kernel. Examples of spaces that support such a Dirichlet form include complete Riemannian manifolds with non-negative Ricci curvature, Carnot groups and other complete sub-Riemannian manifolds, and doubling metric measure spaces that support a 2-Poincaré inequality with respect to the upper gradient structure of Heinonen and Koskela (see [HKST15]).

In some parts of this chapter we need a condition that guarantees that the space is not negatively curved with respect to the Dirichlet form. To do so, we will assume that the metric space  $(X, d_{\mathcal{E}}, \mu)$  satisfies a weak Bakry-Émery curvature condition, namely that whenever  $u \in \mathcal{F} \cap L^\infty(X)$ ,

$$(4.2) \quad \|\nabla P_t u\|_{L^\infty(X)}^2 \leq \frac{C}{t} \|u\|_{L^\infty(X)}^2.$$

Here  $P_t u$  denotes the heat extension of  $u$  via the heat semigroup associated with the Dirichlet form  $\mathcal{E}$ , see [FOT11].

EXAMPLE 4.2. The weak Bakry-Émery curvature condition is satisfied in the following examples:

- Complete Riemannian manifolds with non-negative Ricci curvature and more generally, the  $RCD(0, +\infty)$  spaces ([Jia15]).
- Carnot groups (see [BB16]).
- Complete sub-Riemannian manifolds with generalized non-negative Ricci curvature (see [BB12, BK14]).
- Metric graphs with finite number of edges (see [BK17]).

Several statements equivalent to the weak Bakry-Émery curvature condition are given in Theorem 1.2 in [CJKS17]. There are some metric measure spaces equipped with a doubling measure supporting a 2-Poincaré inequality but without the above Bakry-Émery condition, see for example [KRS03]. For instance, it should be noted, that in the setting of complete sub-Riemannian manifolds with generalized non-negative Ricci curvature in the sense of [BG17], while the weak Bakry-Émery curvature condition is known to be satisfied (see [BB12, BK14]), the 1-Poincaré inequality has not been proven yet (though the 2-Poincaré is known, see [BBG14]).

## 4.2. Construction of BV class using the Dirichlet form

In this section we use the Dirichlet form and the associated family  $\Gamma(\cdot, \cdot)$  of measures to construct a BV class of functions on  $X$ . To do so, we only need  $\mu$  to be a doubling measure on  $X$  for  $d_{\mathcal{E}}$  and that the class of locally Lipschitz functions to be dense in  $L^1(X)$ . So in this section we do *not* need the Poincaré inequality nor do we need the weak Bakry-Émery curvature condition.

We now set the *core* of the Dirichlet form,  $\mathcal{C}(X)$ , to be the class of all  $f \in \mathcal{F}_{loc}(X)$  such that  $\Gamma(f, f) \ll \mu$ . In this case, we can also set  $u \in L^1(X)$  to be in  $BV(X)$  if there is a sequence of local Lipschitz functions  $u_k \in L^1(X)$  such that  $u_k \rightarrow u$  in  $L^1(X)$  and

$$\liminf_{k \rightarrow \infty} \int_X |\nabla u_k| d\mu < \infty.$$

For  $u \in BV(X)$  and open sets  $U \subset X$ , we set

$$\|Du\|(U) = \inf_{u_k \in \mathcal{C}(U), u_k \rightarrow u \text{ in } L^1(U)} \liminf_{k \rightarrow \infty} \int_U |\nabla u_k| d\mu,$$

and then for sets  $A \subset X$  we define

$$\|Du\|(A) = \inf\{\|Du\|(O) : A \subset O \text{ and } O \text{ is open in } X\}.$$

LEMMA 4.3. *If  $u, v \in BV(X)$  and  $\eta$  is a Lipschitz continuous function on  $X$  with  $0 \leq \eta \leq 1$  on  $X$ , then  $\eta u + (1 - \eta)v \in BV(X)$  with*

$$\|D(\eta u + (1 - \eta)v)\|(X) \leq \|Du\|(X) + \|Dv\|(X) + \int_X |u - v| |\nabla \eta| d\mu.$$

PROOF. From Lemma 4.1 we already know that given such  $\eta$ ,  $\eta \in \mathcal{F}_{loc}$  with  $|\nabla \eta| \in L^\infty(X)$ .

We can choose sequences  $u_k, v_k \in \mathcal{F}(X)$  such that  $u_k \rightarrow u$  and  $v_k \rightarrow v$  in  $L^1(X)$  and  $\int_X |\nabla u_k| d\mu \rightarrow \|Du\|(X)$  and  $\int_X |\nabla v_k| d\mu \rightarrow \|Dv\|(X)$  as  $k \rightarrow \infty$ . Now an application of the properties of Dirichlet

form (see [Stu94, page 190] as well as the references therein) to the functions  $\eta u_k + (1 - \eta)v_k$  tells us that

$$\begin{aligned} \|D(\eta u + (1 - \eta)v)\|(X) &\leq \liminf_{k \rightarrow \infty} \int_X |\nabla[\eta u_k + (1 - \eta)v_k]| d\mu \\ &\leq \liminf_{k \rightarrow \infty} \left( \int_X \eta |\nabla u_k| d\mu + \int_X (1 - \eta) |\nabla v_k| d\mu + \int_X |u_k - v_k| |\nabla \eta| d\mu \right). \end{aligned}$$

Now using the facts that  $0 \leq \eta \leq 1$  and that  $u_k - v_k \rightarrow u - v$  in  $L^1(X)$  we obtain the required inequality.  $\square$

We first list some elementary properties of  $\|Du\|$ .

LEMMA 4.4. *Let  $U$  and  $V$  be two open subsets of  $X$ . Then*

- (1)  $\|Du\|(\emptyset) = 0$ ;
- (2)  $\|Du\|(U) \leq \|Du\|(V)$  if  $U \subset V$ ,
- (3)  $\|Du\|(\bigcup_i U_i) = \sum_i \|Du\|(U_i)$  if  $\{U_i\}_i$  is a pairwise disjoint subfamily of open subsets of  $X$ .

PROOF. We will only prove the third property here, as the other two are quite direct consequences of the definition of  $\|Du\|$ . Since any function  $f \in \mathcal{F}(\bigcup_i U_i)$  has restrictions  $u_i = f|_{U_i} \in \mathcal{F}(U_i)$  with  $\int_{\bigcup_i U_i} |\nabla f| d\mu = \sum_i \int_{U_i} |\nabla u_i| d\mu$ , it follows that

$$\|Du\|(\bigcup_i U_i) \geq \sum_i \|Df\|(U_i).$$

In the above we also used the fact that as  $f$  gets closer to  $u$  in the  $L^1(\bigcup_i U_i)$  sense,  $u_i$  gets closer to  $u$  in the  $L^1(U_i)$  sense.

To prove the reverse inequality, for  $\varepsilon > 0$  we can choose locally Lipschitz continuous  $u_i \in \mathcal{F}(U_i)$  for each  $i$  such that

$$\int_{U_i} |u - u_i| d\mu < 2^{-i-2}\varepsilon$$

and

$$\int_{U_i} |\nabla u_i| d\mu < \|Du\|(U_i) + 2^{-i-2}\varepsilon.$$

Now the function  $f_\varepsilon = \sum_i u_i \mathbf{1}_{U_i}$  is in  $\mathcal{F}(\bigcup_i U_i)$  because the  $U_i$  are pairwise disjoint open sets, and  $\mathcal{E}$  is local. Therefore

$$\int_{\bigcup_i U_i} |u - f_\varepsilon| d\mu \leq \sum_i \int_{U_i} |u - u_i| d\mu \leq \frac{\varepsilon}{2}$$

and

$$\int_{\bigcup_i U_i} |\nabla f_\varepsilon| d\mu = \sum_i \int_{U_i} |\nabla u_i| d\mu \leq \frac{\varepsilon}{2} + \sum_i \|Df\|(U_i).$$

From the first of the above two inequalities it follows that  $\lim_{\varepsilon \rightarrow 0^+} f_\varepsilon = u$  in  $L^1(\bigcup_i U_i)$ , and therefore

$$\|Du\|(\bigcup_i U_i) \leq \liminf_{\varepsilon \rightarrow 0^+} \left( \frac{\varepsilon}{2} + \sum_i \|Du\|(U_i) \right) = \sum_i \|Du\|(U_i).$$

$\square$

We use the above definition of  $\|Du\|$  on open sets to consider the following Caratheodory construction.

DEFINITION 4.5. *For  $A \subset X$ , we set*

$$\|Du\|^*(A) := \inf\{\|Du\|(O) : O \text{ is open subset of } X, A \subset O\}.$$

By the second property listed in the above lemma, we note that if  $A$  is an open subset of  $X$ , then  $\|Du\|^*(A) = \|Du\|(A)$ . With this observation, we re-name  $\|Du\|^*(A)$  as  $\|Du\|(A)$  even when  $A$  is not open.



**4.2.1. Radon measure property of BV energy.** The following lemma is due to De Giorgi and Letta [DGL77, Theorem 5.1], see also [AFP00, Theorem 1.53].

LEMMA 4.6. *If  $\nu$  is a non-negative function on the class of all open subsets of  $X$  such that*

- (1)  $\nu(\emptyset) = 0$ ,
- (2) *if  $U_1 \subset U_2$  and they are both open sets, then  $\nu(U_1) \leq \nu(U_2)$ ,*
- (3)  $\nu(U_1 \cup U_2) \leq \nu(U_1) + \nu(U_2)$  *for open sets  $U_1, U_2$  in  $X$ ,*
- (4) *if  $U_1 \cap U_2$  is empty and  $U_1, U_2$  are open, then  $\nu(U_1 \cup U_2) = \nu(U_1) + \nu(U_2)$ ,*
- (5) *for open sets  $U$*

$$\nu(U) = \sup\{\nu(V) : V \text{ is bounded and open in } X \text{ with } \bar{V} \subset U\}.$$

*Then the Caratheodory extension of  $\nu$  to all subsets of  $X$  gives a Borel regular outer measure on  $X$ .*

The proof of the following theorem is based on that of [Mir03].

THEOREM 4.7. *If  $X$  is complete and  $f \in BV(X)$ , then  $\|Df\|$  is a Radon outer measure on  $X$ .*

PROOF. For simplicity of notation we will assume that  $X$  is itself bounded. Thanks to the lemma of De Giorgi and Letta (Lemma 4.6), it suffices to verify that  $\|Du\|$  satisfies the five conditions set forth in Lemma 4.6. By Lemma 4.4, we know that  $\|Du\|$  satisfies Conditions 1, 2, and Condition 4. Thus it suffices for us to verify Condition 3 and Condition 5. We will first show the validity of Condition 5, and use it (or rather, it's proof) to show that Condition 3 holds. We will do so for bounded open subsets of  $X$ ; a simple modification (by truncating  $U_\delta$  by balls) would complete the proof for unbounded sets; we leave this part of the extension as an exercise.

**Proof of Condition 5:** From the monotonicity condition 2, it suffices to prove that

$$\|Df\|(U) \leq \sup\{\|Df\|(V) : V \text{ is open in } X, \bar{V} \text{ is a compact subset of } U\}.$$

For  $\delta > 0$  we set

$$U_\delta = \{x \in U : \text{dist}(x, X \setminus U) > \delta\}.$$

For  $0 < \delta_1 < \delta_2 < \text{diam}(U)/2$ , let  $V = U_{\delta_1}$  and  $W = U \setminus \bar{U}_{\delta_2}$ . Then  $V$  and  $W$  are open subsets of  $U$ , and the closure of  $V$  is a compact subset of  $U$ . Note also that  $U = V \cup W$  and that  $\partial V \cap \partial W$  is empty. Thus we can find a Lipschitz function  $\eta$  on  $U$  that can be used as a "needle+thread" to stitch Sobolev functions on  $V$  to Sobolev functions on  $W$  as follows to obtain a Sobolev function on  $U$ :  $0 \leq \eta \leq 1$  on  $U$ ,  $\eta = 1$  on  $V \setminus W = \bar{U}_{\delta_2}$ ,  $\eta = 0$  on  $W \setminus V = U \setminus U_{\delta_1}$ , and

$$\text{Lip } \eta \leq \frac{2}{\delta_2 - \delta_1} \mathbf{1}_{V \cap W}.$$

Now, for  $v \in \mathcal{F}(V)$  and  $w \in \mathcal{F}(W)$  we set  $u = \eta v + (1 - \eta)w$ . As we have the Leibniz rule (see [Stu94]), we can see that  $u \in \mathcal{F}(U)$  and

$$(4.3) \quad \int_U |\nabla u| d\mu \leq \int_V |\nabla v| d\mu + \int_W |\nabla w| d\mu + \frac{2}{\delta_2 - \delta_1} \int_{V \cap W} |v - w| d\mu.$$

Furthermore, whenever  $h \in L^1(U)$ , we can write  $h = \eta h + (1 - \eta)h$  to see that

$$(4.4) \quad \int_U |u - h| d\mu \leq \int_V |v - h| d\mu + \int_W |w - h| d\mu.$$

Now, we take  $v_k$  from  $\mathcal{F}(V)$  such that  $v_k \rightarrow f$  in  $L^1(V)$  and  $\lim_{k \rightarrow \infty} \int_V |\nabla v_k| d\mu = \|Df\|(V)$ , and take  $w_k \in \mathcal{F}(W)$  analogously. We then follow through by stitching together  $v_k$  and  $w_k$  into the function  $u_k$  as prescribed above. By (4.4) with  $h = f$ , we have that

$$\int_U |f - u_k| d\mu \leq \int_V |v_k - f| d\mu + \int_W |w_k - f| d\mu \rightarrow 0 \text{ as } k \rightarrow \infty.$$

It follows from (4.3) and the fact  $\int_{V \cap W} |v_k - w_k| d\mu \rightarrow 0$  as  $k \rightarrow \infty$  that

$$\|Df\|(U) \leq \liminf_{k \rightarrow \infty} \int_U |\nabla u_k| d\mu \leq \|Df\|(V) + \|Df\|(W).$$

Remembering again that the closure of  $V$  is a compact subset of  $U$ , we see that

$$\|Df\|(U) \leq \sup\{\|Df\|(V) : V \text{ is open in } X, \bar{V} \text{ is compact subset of } U\} + \|Df\|(U \setminus \overline{U_{\delta_2}}).$$

So now it suffices to prove that

$$(4.5) \quad \lim_{\delta \rightarrow 0^+} \|Df\|(U \setminus \overline{U_{\delta}}) = 0.$$

To prove this, we note first that the above limit exists as  $\|Df\|(U \setminus \overline{U_{\delta}})$  decreases as  $\delta$  decreases. We fix a strictly monotone decreasing sequence of real numbers  $\delta_k$  with  $\lim_{k \rightarrow \infty} \delta_k = 0$ , and for  $k \geq 2$  we set  $V_k := U_{\delta_{2k+3}} \setminus \overline{U_{\delta_{2k}}}$ . Observe that the family  $\{V_{2k}\}_k$  is a pairwise disjoint family of open subsets of  $X$  and that the family  $\{V_{2k+1}\}_k$  is also a pairwise disjoint family of open subsets of  $X$ .

By Lemma 4.4, we know that

$$\infty > \|Df\|(U) \geq \|Df\|\left(\bigcup_{k \geq 1} V_{2k}\right) = \sum_{k=1}^{\infty} \|Df\|(V_{2k}),$$

and

$$\infty > \|Df\|(U) \geq \|Df\|\left(\bigcup_{k \geq 1} V_{2k+1}\right) = \sum_{k=1}^{\infty} \|Df\|(V_{2k+1}).$$

It follows that for  $\varepsilon > 0$  there is some positive integer  $k_{\varepsilon} \geq 2$  such that

$$\sum_{k=k_{\varepsilon}}^{\infty} \|Df\|(V_{2k}) + \sum_{k=k_{\varepsilon}}^{\infty} \|Df\|(V_{2k+1}) < \varepsilon.$$

Now we stitch together approximations on  $V_{2k}$  to approximations on  $V_{2k+1}$ , and from there to  $V_{2k+2}$  and so on. For each  $k$  we choose a ‘‘stitching function’’  $\eta_k$  as a Lipschitz function on  $\bigcup_{j=k_{\varepsilon}}^{k+1} V_j$  such that  $0 \leq \eta_k \leq 1$ , with  $\eta_k = 1$  on  $V_k \setminus V_{k-1}$ ,  $\eta_k = 0$  on  $\bigcup_{j=k_{\varepsilon}}^{k-1} V_j \setminus V_k$ , and  $|\nabla \eta_k| \leq C_k \mathbf{1}_{V_k \cap V_{k-1}}$ .

Next, for each  $k$  we can find  $v_{k,j} \in \mathcal{F}(V_k)$  such that

$$\int_{V_k} |v_{k,j} - f| d\mu \leq \frac{2^{-k-j}}{3(1+C_k)}$$

and

$$\int_{V_k} |\nabla v_{k,j}| d\mu \leq \|Df\|(V_k) + 2^{-j-k}.$$

We now inductively stitch the functions together. To do so, we first fix  $i \in \mathbb{N}$ .

Starting with  $k = k_{\varepsilon}$ , we stitch  $u_{k,i}$  to  $u_{k+1,i}$  using  $\eta_{k+1} = \eta_{k_{\varepsilon}+1}$  to obtain  $w_{i,k} \in \mathcal{F}(V_{k_{\varepsilon}} \cup V_{k_{\varepsilon}+1})$  so that we have

$$\int_{V_{k_{\varepsilon}} \cup V_{k_{\varepsilon}+1}} |w_{i,k} - f| d\mu \leq \frac{2^{-i-k_{\varepsilon}}}{1+C_{k_{\varepsilon}+1}}$$

and

$$\int_{V_{k_{\varepsilon}} \cup V_{k_{\varepsilon}+1}} |\nabla w_{i,k}| d\mu \leq \sum_{j=k_{\varepsilon}}^{k_{\varepsilon}+1} \|Df\|(V_j) + 2^{1-i-k_{\varepsilon}}.$$

Suppose now that for some  $k \in \mathbb{N}$  with  $k \geq k_{\varepsilon} + 1$  we have constructed  $w_{i,k} \in \mathcal{F}(\bigcup_{j=k_{\varepsilon}}^k V_j)$  such that

$$\int_{\bigcup_{j=k_{\varepsilon}}^k V_j} |w_{i,k} - f| d\mu \leq \sum_{k=k_{\varepsilon}}^k \frac{2^{-i-j}}{1+C_j}$$

and

$$\int_{\bigcup_{j=k_{\varepsilon}}^k V_j} |\nabla w_{i,k}| d\mu \leq \sum_{j=k_{\varepsilon}}^k [\|Df\|(V_j) + 2^{1-i-j}].$$

Then we stitch  $w_{k+1,i}$  to  $w_{i,k}$  using  $\eta_{k+1}$  to obtain  $w_{i,k+1}$  satisfying inequalities analogous to the above two. Note that  $w_{i,k+1} = w_{i,k-1}$  on  $V_{k-1}$  for  $k \geq k_\varepsilon + 2$ . Thus, in the limit, we obtain a function  $w_i = \lim_k w_{i,k} \in \mathcal{F}(\bigcup_{k=k_\varepsilon}^\infty V_k)$  satisfying

$$\int_{\bigcup_{j=k_\varepsilon}^\infty V_j} |w_i - f| d\mu \leq \sum_{k=k_\varepsilon}^i \frac{2^{-i-j}}{1+C_j} < 2^{1-i},$$

$$\int_{\bigcup_{j=k_\varepsilon}^\infty V_j} |\nabla w_i| d\mu \leq \sum_{j=k_\varepsilon}^\infty \|Df\|(V_j) + 2^{2-i} < \varepsilon + 2^{2-i}.$$

From the first of the above two inequalities, we see that  $w_i \rightarrow f$  in  $L^1(\bigcup_{j=k_\varepsilon}^\infty V_j)$  as  $i \rightarrow \infty$ , and so from the second of the above two inequalities we obtain

$$\|Df\|(\bigcup_{j=k_\varepsilon}^\infty V_j) = \|Df\|(U \setminus \overline{U_{\delta_{k_\varepsilon}}}) \leq \liminf_{i \rightarrow \infty} \int_{\bigcup_{j=k_\varepsilon}^\infty V_j} |\nabla w_i| d\mu \leq \varepsilon.$$

This last inequality above tells us that the claim we set out to prove, namely

$$\lim_{\delta \rightarrow 0^+} \|Df\|(U \setminus \overline{U_\delta}) = 0.$$

this completes the proof of Condition 5.

**Proof of Condition 3:** By Condition 5 (which we have now proved above, so no circular argument here!), for each  $\varepsilon > 0$  we can find relatively compact open subsets  $U'_1 \Subset U_1$  and  $U'_2 \Subset U_2$  such that  $\|Df\|(U_1 \cup U_2) \leq \|Df\|(U'_1 \cup U'_2) + \varepsilon$ . We then choose a Lipschitz "stitching function"  $\eta$  on  $X$  such that  $0 \leq \eta \leq 1$  on  $X$ ,  $\eta = 1$  on  $U'_1$ ,  $\eta = 0$  on  $X \setminus U_1$ , and

$$|\nabla \eta| \leq \frac{1}{C_{U_1, U'_1}} \mathbf{1}_{U_1 \setminus U'_1}.$$

For  $u_1 \in \mathcal{F}(U_1)$  and  $u_2 \in \mathcal{F}(U_2)$ , we obtain the stitched function  $w = \eta u_1 + (1 - \eta)u_2$  and note that  $w \in \mathcal{F}(U'_1 \cup U'_2)$ . Observe that we cannot in general have  $w \in \mathcal{F}(U_1 \cup U_2)$  as  $w$  is not defined in  $U_1 \setminus (U'_1 \cup U_2)$  because  $1 - \eta$  is non-vanishing there and  $u_2$  is not defined there, for example. Then we have

$$\int_{U'_1 \cup U'_2} |\nabla w| d\mu \leq \int_{U_1} |\nabla u_1| d\mu + \int_{U_2} |\nabla u_2| d\mu + \frac{1}{C_{U_1, U'_1}} \int_{U_1 \cap U_2} |u_1 - u_2| d\mu$$

and

$$\int_{U'_1 \cup U'_2} |w - f| d\mu \leq \int_{U_1} |u_1 - f| d\mu + \int_{U_2} |u_2 - f| d\mu.$$

As before, choosing  $u_{1k}$  to be the optimal approximating sequence for  $f$  on  $U_1$  and  $u_{2,k}$  correspondingly for  $f$  on  $U_2$ , we see from the first of the above two inequalities that the stitched sequence  $w_k$  approximates  $f$  on  $U'_1 \cup U'_2$ . Therefore we obtain

$$\|Df\|(U_1 \cup U_2) \leq \varepsilon + \|Df\|(U'_1 \cup U'_2) \leq \varepsilon + \liminf_{k \rightarrow \infty} \int_{U'_1 \cup U'_2} |\nabla w_k| d\mu \leq \|Df\|(U_1) + \|Df\|(U_2) + \varepsilon.$$

Letting  $\varepsilon \rightarrow 0$  now gives the desired Condition 3.  $\square$

#### 4.2.2. Co-area formula.

**DEFINITION 4.8.** We say that a measurable set  $E \subset X$  is of finite perimeter if  $\mathbf{1}_E \in BV_{loc}(X)$  with  $\|D\mathbf{1}_E\|(X) < \infty$ . We denote by  $P(E, A) := \|D\mathbf{1}_E\|(A)$  the perimeter measure of  $E$ .

**LEMMA 4.9.** The co-area formula holds true, that is, for Borel sets  $A \subset X$  and  $u \in L^1_{loc}(X)$ ,

$$\|Du\|(A) = \int_{\mathbb{R}} P(\{u > s\}, A) ds.$$

PROOF. We first prove the formula for open sets  $A$ .

Suppose first that  $u \in BV_{loc}(X)$  with  $\|Du\|(A) < \infty$ . For  $s \in \mathbb{R}$  we set  $E_s := \{x \in X : u(x) > s\}$ . Consider the function  $m : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$m(t) = \|Du\|(A \cap E_t).$$

Then  $m$  is a monotone decreasing function, and hence is differentiable almost everywhere. Let  $t \in \mathbb{R}$  such that  $m'(t)$  exists. Then

$$m'(t) = \lim_{h \rightarrow 0^+} \frac{\|Du\|(A \cap E_t \setminus E_{t+h})}{h}.$$

Note that the functions

$$u_{t,h} := \frac{\max\{t, \min\{t+h, u\}\} - t}{h}$$

converge in  $L^1(X)$  to  $\mathbf{1}_{E_t}$  as  $h \rightarrow 0^+$ . It follows that as  $A$  is open,

$$P(E_t, A) \leq \liminf_{h \rightarrow 0^+} \|Du_{t,h}\|(A) = \liminf_{h \rightarrow 0^+} \frac{\|Du\|(A \cap E_t \setminus E_{t+h})}{h} = m'(t).$$

Note also that by this lower semicontinuity of BV energy,  $t \mapsto P(E_t, A)$  is a lower semicontinuous function, and hence is measurable; and as it is non-negative, we can talk about its integral, whether that integral is finite or not. Therefore, by the fundamental theorem of calculus for monotone functions,

$$\int_{\mathbb{R}} P(E_t, A) dt \leq \int_{\mathbb{R}} m'(t) dt \leq \lim_{s, \tau \rightarrow \infty} m(s) - m(-\tau) = \|Du\|(A).$$

The above in particular tells us that if  $u \in BV_{loc}(X)$  then almost all of its superlevel sets  $E_t$  have finite perimeter. If  $u$  is not a BV function on  $A$ , then  $\|Du\|(A) = \infty$ , and hence we also have

$$(4.6) \quad \int_{\mathbb{R}} P(E_t, A) dt \leq \|Du\|(A).$$

In particular, it also follows that  $\int_{\mathbb{R}} P(E_t, A) dt < \infty$  if  $u \in BV(X)$ .

We still continue to assume that  $A$  is open, and prove the reverse of the above inequality. If  $\int_{\mathbb{R}} P(E_t, A) dt = \infty$ , then trivially

$$\|Du\|(A) \leq \int_{\mathbb{R}} P(E_t, A) dt.$$

So we may assume without loss of generality that  $\int_{\mathbb{R}} P(E_t, A) dt$  is finite. Note also by the Markovian property of Dirichlet forms, filtered down to the level of the measure  $|\nabla f|$ , we have that

$$\|Du\|(A) = \lim_{s, \tau \rightarrow \infty} \|Du_{s, \tau}\|(A),$$

where  $u_{s, \tau} = \max\{-\tau, \min\{u, s\}\}$ . So without loss of generality we may assume that  $0 \leq u \leq 1$  for some finite  $a, b \in \mathbb{R}$ . For positive integers  $X$  we can divide  $[0, 1]$  into  $X$  equal sub-intervals  $[t_i, t_{i+1}]$ ,  $i = 0, \dots, k$  with  $t_{i+1} - t_i = 1/k$ . Then we can find  $\rho_{k,i} \in (t_i, t_{i+1})$  such that

$$\frac{1}{k} P(E_{\rho_{k,i}}, A) \leq \int_{t_i}^{t_{i+1}} P(E_s, A) ds.$$

We set

$$u_k = \sum_{j=1}^k \frac{1}{k} \mathbf{1}_{E_{\rho_{k,i}}}.$$

Then as  $|u_k - u| \leq 1/k$  on  $X$ , we have that  $u_k \rightarrow u$  in  $L^1(A)$  as  $k \rightarrow \infty$ , and so

$$(4.7) \quad \|Du\|(A) \leq \liminf_{k \rightarrow \infty} \|Du_k\|(A) = \liminf_{k \rightarrow \infty} \sum_{j=1}^k \frac{1}{k} P(E_{\rho_{k,i}}, A) \leq \int_0^1 P(E_s, A) ds.$$

Note now that by the proofs of inequalities (4.6) and (4.7), if  $A$  is an open set then  $u \in BV(A)$  if and only if  $\int_{\mathbb{R}} P(E_t, A) dt$  is finite.

Finally, we remove the requirement that  $A$  be open. By the above comment, it suffices to prove this for the case that  $u \in BV(X)$ . In this case, the maps  $A \mapsto \|Du\|(A)$  and  $A \mapsto \int_{\mathbb{R}} P(E_t, A) dt$  are both Radon measures on  $X$  that agree on open subsets of  $X$  (that is, they are equal for open  $A$ ). Hence it follows that they agree on Borel subsets of  $X$ . This completes the proof of the coarea formula.  $\square$

### 4.3. $\mathbf{B}^{1,1/2}(X)$ is equal to $BV(X)$

In this section we will assume that  $(X, d_{\mathcal{E}}, \mu)$  is doubling, supports a 2-Poincaré inequality, and satisfies a weak Bakry-Émery curvature condition (4.2). Under this condition, we will compare the  $BV$  class, constructed in Section 4.2, with the Besov class  $\mathbf{B}^{1,1/2}(X)$  from (1.1). We will also assume that for each  $u \in \mathcal{F}$  the measure  $\Gamma(u, u)$  is absolutely continuous with respect to the underlying measure  $\mu$  on  $X$ ; as in the previous section we will denote by  $|\nabla u|^2$  the Radon-Nikodym derivative of  $\Gamma(u, u)$  with respect to  $\mu$ .

In what follows,  $\mathcal{F}_1(X)$  denotes the collection of all  $u \in L^1(X) \cap \mathcal{F}_{loc}(X)$  for which  $|\nabla u| \in L^1(X)$ . This means that for each ball  $B$  in  $X$  there is a compactly supported Lipschitz function  $\varphi$  with  $\varphi = 1$  on  $B$  such that  $u\varphi \in \mathcal{F}$ ; in this case we can set  $|\nabla u| = |\nabla(u\varphi)|$  in  $B$ , thanks to the strong locality property of  $\mathcal{E}$ .

LEMMA 4.10. *Suppose that (4.2) holds. For  $u \in \mathcal{F} \cap \mathcal{F}_1(X)$ , we have that*

$$\|P_t u - u\|_{L^1(X)} \leq C \sqrt{t} \int_X |\nabla u| d\mu.$$

Hence, if  $u \in BV(X)$ , then

$$\|P_t u - u\|_{L^1(X)} \leq C \sqrt{t} \|Du\|(X).$$

PROOF. To see the first part of the claim, we note that for each  $x \in X$  and  $s > 0$ ,  $\frac{\partial}{\partial s} P_s u(x)$  exists, and so by the fundamental theorem of calculus, for  $0 < \tau < t$  and  $x \in X$ ,

$$P_t u(x) - P_\tau u(x) = \int_\tau^t \frac{\partial}{\partial s} P_s u(x) ds.$$

Thus for each compactly supported function  $\varphi \in \mathcal{F} \cap L^\infty(X)$ , by the facts that  $P_t u$  satisfies the heat equation and that  $P_s$  commutes with the infinitesimal generator (Laplacian),

$$\begin{aligned} \left| \int_X \varphi(x) [P_t u(x) - P_\tau u(x)] d\mu(x) \right| &= \left| - \int_X \int_\tau^t \varphi(x) \frac{\partial}{\partial s} P_s u(x) ds d\mu(x) \right| \\ &= \left| \int_\tau^t \int_X d\Gamma(\varphi, P_s u)(x) ds \right| \\ &= \left| \int_\tau^t \int_X d\Gamma(P_s \varphi, u)(x) ds \right| \\ &\leq \int_\tau^t \int_X |\nabla P_s \varphi| |\nabla u| d\mu ds \\ &\leq \| |\nabla P_s \varphi| \|_{L^\infty(X)} \int_\tau^t \int_X |\nabla u| ds d\mu. \end{aligned}$$

An application of (4.2) gives

$$\begin{aligned} \left| \int_X \varphi(x) [P_t u(x) - P_\tau u(x)] d\mu(x) \right| &\leq \frac{C}{\sqrt{t}} \|\varphi\|_{L^\infty(X)} \int_X \int_\tau^t |\nabla u| ds d\mu \\ &= C \frac{t - \tau}{\sqrt{t}} \|\varphi\|_{L^\infty(X)} \int_X |\nabla u| d\mu. \end{aligned}$$

As the above holds for all  $\varphi \in \mathcal{F} \cap L^\infty(X)$ , and as compactly supported Lipschitz functions are dense in  $L^\infty(X)$  (note that  $X$  is separable as closed and bounded subsets of  $X$  are compact because  $X$  is complete and supports a doubling measure), we obtain

$$\|P_t u - P_\tau u\|_{L^1(X)} \leq C \frac{t-\tau}{\sqrt{t}} \int_X |\nabla u| d\mu.$$

Now by the fact that  $P_\tau u \rightarrow u$  as  $\tau \rightarrow 0^+$  in  $L^2(X)$ , and by the fact that  $\{P_t\}_{t>0}$  has an extension as a contraction semigroup to  $L^1(X)$  (see [FOT11, Section 1.5 of page 37, Section 4.7 of page 201]), we have that

$$\|P_t u - u\|_{L^1(X)} \leq C \sqrt{t} \int_X |\nabla u| d\mu$$

as desired.

Finally, if  $u \in BV(X)$ , then we can find a sequence  $u_k \in \mathcal{F} \cap W^{1,1}(X)$  such that  $u_k \rightarrow u$  in  $L^1(X)$  and  $\lim_k \int_X |\nabla u_k| d\mu = \|Du\|(X)$ . As  $u \in L^1(X)$  and for  $t > 0$  we have that the heat kernel  $p_t(x, y)$  is bounded in  $X \times X$ , it follows that  $P_t u \in L^1(X)$  is well-defined with  $\|P_t u\|_{L^1(X)} \leq C_t \|u\|_{L^1(X)}$ , see [FOT11, Section 1.5]. Now,

$$\begin{aligned} \|P_t u - u\|_{L^1(X)} &\leq \|P_t(u - u_k)\|_{L^1(X)} + \|P_t u_k - u_k\|_{L^1(X)} + \|u_k - u\|_{L^1(X)} \\ &\leq C_t \|u - u_k\|_{L^1(X)} + C \sqrt{t} \int_X |\nabla u_k| d\mu + \|u - u_k\|_{L^1(X)} \\ &\rightarrow C \sqrt{t} \lim_{k \rightarrow \infty} \int_X |\nabla u_k| d\mu = C \sqrt{t} \|Du\|(X) \text{ as } k \rightarrow \infty. \end{aligned}$$

□

Recall the definition of  $\mathbf{B}^{1,1/2}(X)$  from (1.1):  $f$  is in this class if

$$\|f\|_{\mathbf{B}^{1,1/2}(X)} := \|f\|_{L^1(X)} + \sup_{t>0} \frac{1}{\sqrt{t}} \int_X P_t(|f - f(x)|)(x) d\mu(x) < \infty.$$

Equivalently,

$$\|f\|_{\mathbf{B}^{1,1/2}(X)} = \|f\|_{L^1(X)} + \sup_{t>0} \frac{1}{\sqrt{t}} \int_X \int_X p_t(x, y) |f(y) - f(x)| d\mu(y) d\mu(x) < \infty.$$

We now wish to prove the following theorem.

**THEOREM 4.11.** *Under the hypotheses of this section,  $\mathbf{B}^{1,1/2}(X) = BV(X)$  with comparable seminorms. Moreover, there exist constants  $c, C > 0$  such that for every  $u \in BV(X)$*

$$c \limsup_{s \rightarrow 0} s^{-1/2} \int_X P_s(|u - u(y)|)(y) d\mu(y) \leq \|Du\|(X) \leq C \liminf_{s \rightarrow 0} s^{-1/2} \int_X P_s(|u - u(y)|)(y) d\mu(y).$$

Note from the results of [MMS16, Theorem 4.1] that if the measure  $\mu$  is doubling and supports a 1-Poincaré inequality, then a measurable set  $E \subset X$  is in the BV class if

$$\liminf_{t \rightarrow 0^+} \frac{1}{\sqrt{t}} \int_{E^{\sqrt{t}} \setminus E} P_t \mathbf{1}_E d\mu < \infty.$$

Here  $E^\varepsilon = \bigcup_{x \in E} B(x, \varepsilon)$ . Note that

$$\int_{E^{\sqrt{t}} \setminus E} P_t \mathbf{1}_E d\mu \leq \int_{X \setminus E} P_t \mathbf{1}_E d\mu = \frac{1}{2} \|P_t \mathbf{1}_E - \mathbf{1}_E\|_{L^1(X)}.$$

Thus if  $\mu$  is doubling and supports a 1-Poincaré inequality, and in addition

$$\sup_{t>0} \frac{1}{\sqrt{t}} \|P_t \mathbf{1}_E - \mathbf{1}_E\|_{L^1(X)} < \infty,$$

then  $E$  is of finite perimeter. Note that we do not assume the validity of 1-Poincaré inequality, but the weaker version of (4.2).

PROOF. First we assume that  $u \in BV(X)$ . Then we know that for almost every  $t \in \mathbb{R}$  the set  $E_t$  is of finite perimeter, where

$$E_t = \{x \in X : u(x) > t\},$$

and by the co-area formula for BV functions (see for example Lemma 4.9 or [Mir03]),

$$\|Du\|(X) = \int_{\mathbb{R}} \|D\mathbf{1}_{E_t}\|(X) dt.$$

For such  $t$ , by Lemma 4.10 we know that

$$\sup_{s>0} \frac{1}{\sqrt{s}} \int_X |P_s \mathbf{1}_{E_t}(x) - \mathbf{1}_{E_t}(x)| d\mu(x) \leq C \|D\mathbf{1}_{E_t}\|(X).$$

Now, setting  $A = \{(x, y) \in X \times X : u(x) < u(y)\}$ , we have for  $s > 0$ ,

$$\begin{aligned} & \int_X \int_X p_s(x, y) |u(x) - u(y)| d\mu(x) d\mu(y) \\ &= 2 \iint_A p_s(x, y) |u(x) - u(y)| d\mu(x) d\mu(y) \\ &= 2 \iint_A \int_{u(x)}^{u(y)} p_s(x, y) dt d\mu(x) d\mu(y) \\ &= 2 \int_X \int_X \int_{\mathbb{R}} \mathbf{1}_{[u(x), u(y))}(t) \mathbf{1}_A(x, y) p_s(x, y) dt d\mu(x) d\mu(y) \\ &= 2 \int_{\mathbb{R}} \int_X \int_X \mathbf{1}_{E_t}(y) [1 - \mathbf{1}_{E_t}(x)] p_s(x, y) d\mu(x) d\mu(y) dt \\ &= 2 \int_{\mathbb{R}} \int_X P_s \mathbf{1}_{E_t}(x) [1 - \mathbf{1}_{E_t}(x)] d\mu(x) dt \\ &= 2 \int_{\mathbb{R}} \int_{X \setminus E_t} P_s \mathbf{1}_{E_t}(x) d\mu(x) dt. \end{aligned}$$

Observe that

$$\int_X |P_s \mathbf{1}_{E_t}(x) - \mathbf{1}_{E_t}(x)| d\mu(x) \geq \int_{X \setminus E_t} |P_s \mathbf{1}_{E_t}(x) - \mathbf{1}_{E_t}(x)| d\mu(x) = \int_{X \setminus E_t} P_s \mathbf{1}_{E_t}(x) d\mu(x).$$

Therefore we obtain

$$\int_X \int_X p_s(x, y) |u(x) - u(y)| d\mu(x) d\mu(y) \leq 2 \int_{\mathbb{R}} \|P_s \mathbf{1}_{E_t} - \mathbf{1}_{E_t}\|_{L^1(X)} dt.$$

An application of Lemma 4.10 now gives

$$\int_X \int_X p_s(x, y) |u(x) - u(y)| d\mu(x) d\mu(y) \leq C\sqrt{s} \int_{\mathbb{R}} \|D\mathbf{1}_{E_t}\|(X) dt,$$

whence with the help of the co-area formula we obtain

$$\|u\|_{1,1/2} \leq C \|Du\|(X),$$

that is,  $u \in \mathbf{B}^{1,1/2}(X)$ . Thus  $BV(X) \subset \mathbf{B}^{1,1/2}(X)$  boundedly.

Now we show that  $\mathbf{B}^{1,1/2}(X) \subset BV(X)$ ; this inclusion holds even when  $\mathcal{E}$  does not support a Bakry-Émery curvature condition; only a 2-Poincaré inequality and the doubling condition on  $\mu$  are needed. Now suppose that  $u \in \mathbf{B}^{1,1/2}(X)$ . Then there is some  $C \geq 0$  such that for each  $t > 0$ ,

$$\int_X \int_X p_t(x, y) |u(y) - u(x)| d\mu(y) d\mu(x) \leq C\sqrt{t}.$$

By (4.1), we have a Gaussian lower bound for the heat kernel:

$$p_t(x, y) \geq \frac{e^{-cd(x,y)^2/t}}{C \sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}}.$$

Let  $C_0 = \|u\|_{\mathbf{B}^{1,1/2}(X)} - \|f\|_{L^1(X)}$ . Therefore, setting  $\Delta_\varepsilon = \{(x, y) \in X : d(x, y) < \varepsilon\}$  for some  $\varepsilon > 0$ , we get

$$\begin{aligned} C_0 \sqrt{t} &\geq \int_X \int_X \frac{e^{-cd(x,y)^2/t}}{C \sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} |u(y) - u(x)| d\mu(y) d\mu(x) \\ &\geq C^{-1} \iint_{\Delta_\varepsilon} \frac{e^{-cd(x,y)^2/t}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} |u(y) - u(x)| d\mu(x) d\mu(y) \\ &\geq \frac{e^{-c\varepsilon^2/t}}{C} \iint_{\Delta_\varepsilon} \frac{|u(y) - u(x)|}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} d\mu(x) d\mu(y). \end{aligned}$$

With the choice of  $\varepsilon = \sqrt{t}$ , we now get

$$C_0 \varepsilon \geq \frac{1}{C} \iint_{\Delta_\varepsilon} \frac{|u(y) - u(x)|}{\sqrt{\mu(B(x, \varepsilon))\mu(B(y, \varepsilon))}} d\mu(x) d\mu(y).$$

It follows that

$$(4.8) \quad \liminf_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \iint_{\Delta_\varepsilon} \frac{|u(y) - u(x)|}{\sqrt{\mu(B(x, \varepsilon))\mu(B(y, \varepsilon))}} d\mu(x) d\mu(y) \leq C C_0 < \infty.$$

Now an argument as in the second half of the proof of [MMS16, Theorem 3.1] tells us that  $u \in BV(X)$ . We point out here that although [MMS16, Theorem 3.1] assumes that  $X$  supports a 1-Poincaré inequality, this part of the proof of [MMS16, Theorem 3.1] does not need this assumption; the argument using discrete convolution there is valid also in our setting. We also then obtain that

$$\|Du\|(X) \leq \liminf_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \iint_{\Delta_\varepsilon} \frac{|u(y) - u(x)|}{\sqrt{\mu(B(x, \varepsilon))\mu(B(y, \varepsilon))}} d\mu(x) d\mu(y) \leq \|u\|_{\mathbf{B}^{1,1/2}(X)} - \|f\|_{L^1(X)}.$$

□

REMARK 4.12. As a byproduct of this proof, we also obtain that there exists a constant  $C > 0$  such that for every  $u \in BV(X)$ ,

$$\sup_{\varepsilon > 0} \frac{1}{\varepsilon} \iint_{\Delta_\varepsilon} \frac{|u(y) - u(x)|}{\sqrt{\mu(B(x, \varepsilon))\mu(B(y, \varepsilon))}} d\mu(x) d\mu(y) \leq C \liminf_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \iint_{\Delta_\varepsilon} \frac{|u(y) - u(x)|}{\sqrt{\mu(B(x, \varepsilon))\mu(B(y, \varepsilon))}} d\mu(x) d\mu(y)$$

because both sides are comparable to  $\|Du(X)\|$ . This property of the metric space  $(X, d)$  can be viewed as an interesting consequence of the weak Bakry-Émery estimate.

We say that  $X$  supports a 1-Poincaré inequality if there are constants  $C, \lambda > 0$  such that whenever  $B$  is a ball in  $X$  and  $u \in W^{1,1}(X)$ , we have

$$\int_B |u - u_B| d\mu \leq C \text{rad}(B) \int_{\lambda B} |\nabla u| d\mu.$$

Given  $A \subset X$  we set

$$\mathcal{H}(A) := \liminf_{\varepsilon \rightarrow 0^+} \left\{ \sum_i \frac{\mu(B_i)}{\text{rad}(B_i)} : A \subset \bigcup_i B_i, \text{ and } \forall i, \text{rad}(B_i) < \varepsilon \right\}.$$



In the event that  $X$  supports a 1-Poincaré inequality, from the results of Ambrosio, Miranda and Pallara, we know that  $P(E, X) \simeq \mathcal{H}(\partial_m E)$  whenever  $E$  is of finite perimeter. Here  $\partial_m E$  consists of points  $x \in \partial E$  for which both

$$\limsup_{r \rightarrow 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0$$

and

$$\limsup_{r \rightarrow 0^+} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0.$$

In this current paper we do not assume the strong condition that  $X$  supports a 1-Poincaré inequality, but we assume that  $X$  supports a 2-Poincaré inequality; in this case, we have the double Gaussian bounds (4.1) for the heat kernel  $p_t(x, y)$ . Note that even without the assumption of 2-Poincaré inequality, we know that if  $\mathcal{H}(\partial E) < \infty$ , then  $E$  is of finite perimeter.

For  $r_0 > 0$  and  $0 < \alpha < 1$  we set  $\partial_\alpha^{r_0} E$  to be the collection of all points  $x \in \partial E$  such that for all  $0 < r \leq r_0$ ,

$$\frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > \alpha, \quad \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > \alpha.$$

Observe that  $\partial_m E = \bigcup_{0 < \alpha < 1} \bigcup_{0 < r_0 < 1} \partial_\alpha^{r_0}(E)$ . This union can be made a countable union by taking  $\alpha$  and  $r_0$  to be rational numbers.

LEMMA 4.13. *Suppose that  $\mu$  is doubling and supports a 2-Poincaré inequality, and that  $E \subset X$  with  $\|\mathbf{1}_E\|_{\mathbf{B}^{1,1/2}(X)}$  finite. Then for all  $r_0 > 0$  and  $0 < \alpha < 1$ ,*

$$\mathcal{H}(\partial_\alpha^{r_0} E) \leq \frac{C}{\alpha} P(E, X).$$

Consequently,  $\mathcal{H}|_{\partial_m E}$  is a  $\sigma$ -finite measure.

PROOF. Note that the proof of Theorem 4.11 also tells us that even without the Bakry-Émery condition (4.2), we know that if  $\mathbf{1}_E \in \mathbf{B}^{1,1/2}(X)$  then  $\mathbf{1}_E \in BV(X)$ . By the definition of  $\mathbf{B}^{1,1/2}(X)$ , we know that

$$\sup_{t > 0} \frac{1}{\sqrt{t}} \int_X \int_X p_t(x, y) |\mathbf{1}_E(x) - \mathbf{1}_E(y)| d\mu(x) d\mu(y) \leq C P(E, X).$$

Fix  $t < (r_0/3)^2$ . We cover  $\partial_\alpha^{r_0} E$  by countably many balls  $B_i$  of radius  $\sqrt{t}$  such that the balls  $5B_i$  have a bounded overlap (the bound depending solely on the doubling constant of  $\mu$ , see [Hei01] for example). Then by the doubling property of  $\mu$  and by the Gaussian lower bound for  $p_t(x, y)$  (which follows from the 2-Poincaré inequality together with the doubling property of  $\mu$ ),

$$\begin{aligned} C \sqrt{t} P(E, X) &\geq \sum_i \int_{B_i \cap E} \int_{B_i \setminus E} p_t(x, y) d\mu(x) d\mu(y) \\ &\geq C^{-1} \sum_i \int_{B_i \cap E} \int_{B_i \setminus E} \frac{e^{-C}}{\mu(B_i)} d\mu(x) d\mu(y) \\ &\geq C^{-1} \sum_i \frac{\mu(B_i \cap E) \mu(B_i \setminus E) \mu(B_i)}{\mu(B_i)^2}. \end{aligned}$$

In the above computations,  $C$  stands for various generic constants that depend only on the doubling and Poincaré constants of the space, and the value of  $C$  could change at each occurrence. Note that at least one of  $\mu(B_i \cap E)/\mu(B_i)$  and  $\mu(B_i \setminus E)/\mu(B_i)$  is larger than  $1/2$ . Now by the definition of  $\partial_\alpha^{r_0} E$  we obtain

$$C P(E, X) \geq \alpha \sum_i \frac{\mu(B_i)}{\sqrt{t}}.$$

Since  $\sqrt{t}$  is the radius of each  $B_i$ , we get

$$C P(E, X) \geq \alpha \limsup_{t \rightarrow 0^+} \sum_i \frac{\mu(B_i)}{\sqrt{t}} \geq \alpha \mathcal{H}(\partial_\alpha^{r_0} E).$$

If  $0 < r_1 < r_0$ , then

$$\partial_\alpha^{r_0} E \subset \partial_\alpha^{r_1} E \subset \partial_m E.$$

Setting

$$\partial_\alpha E := \bigcup_{0 < r_0 < 1} \partial_\alpha^{r_0} E = \bigcup_{(0,1) \cap \mathbb{Q}} \partial_\alpha^{r_0} E,$$

we now see by the continuity of measure that if the sets  $\partial_\alpha^{r_0} E$  are Borel sets, then

$$\mathcal{H}(\partial_\alpha E) \leq \frac{C}{\alpha} P(E, X).$$

To see that  $\partial_\alpha^{r_0} E$  is a Borel set we argue as follows. Recall that we assume  $\mu$  to be Borel regular. Therefore, given a  $\mu$ -measurable set  $E$  and  $r > 0$ , the function

$$x \mapsto \mu(B(x, r) \cap E)$$

is lower semicontinuous, the map

$$\varphi_{E,r}(x) := \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))}$$

is a Borel function. Hence the function  $\Phi_{E,r_0} := \inf_{r \in \mathbb{Q} \cap (0, r_0]} \varphi_{E,r}$  is also a Borel function, and hence

$$\partial_\alpha^{r_0} E = \{x \in X : \Phi_{E,r_0}(x) > \alpha\} \cap \{x \in X : \Phi_{X \setminus E, r_0}(x) > \alpha\}$$

is a Borel set. □

Should  $X$  support a 1-Poincaré inequality, then by the results of [AMP04], there is a number  $\gamma \in (0, 1/2]$  such that  $\mathcal{H}(\partial_m E \setminus \partial_\gamma E) = 0$ ; this number  $\gamma$  depends solely on the doubling and the 1-Poincaré constants. Observe that the results of [AMP04] also tells us that if  $E$  is of finite perimeter, then  $\mathcal{H}(\partial_m E) \simeq P(E, X)$ .

#### 4.4. Metric characterization of $\mathbf{B}^{p,\alpha}(X)$

Now we turn our attention to the study of Besov spaces in the metric setting. In this section, we follow the notions given in [GKS10]. For  $0 \leq \alpha < \infty$ ,  $1 \leq p < \infty$  and  $p < q \leq \infty$ , we set  $B_{p,q}^\alpha(X)$  to be the collection of all functions  $u \in L^p(X, \mu)$  for which

$$(4.9) \quad \|u\|_{B_{p,q}^\alpha(X)} := \left( \int_0^\infty \left( \int_X \int_{B(x,t)} \frac{|u(y) - u(x)|^p}{t^{\alpha p} \mu(B(x,t))} d\mu(y) d\mu(x) \right)^{q/p} \frac{dt}{t} \right)^{1/q}$$

is finite when  $q < \infty$ , and

$$(4.10) \quad \|u\|_{B_{p,\infty}^\alpha(X)} := \sup_{t > 0} \left( \int_X \int_{B(x,t)} \frac{|u(y) - u(x)|^p}{t^{\alpha p} \mu(B(x,t))} d\mu(y) d\mu(x) \right)^{1/p}$$

is finite when  $q = \infty$ . In the setting considered in the next chapter, with  $\mu$  an Ahlfors  $d_H$ -regular measure (that is,  $\mu(B(x, r)) \simeq r^{d_H}$  for all  $x \in X$  and  $0 < r < 2\text{diam}(X)$ )  $\mathfrak{B}_p^\alpha(X) = B_{p,\infty}^\alpha(X)$ .

**PROPOSITION 4.14.** *Suppose that  $\mu$  is doubling and supports a 2-Poincaré inequality. Then for  $1 \leq p < \infty$  and  $0 \leq \alpha < \infty$  we have*

$$\mathbf{B}^{p,\alpha/2}(X) = B_{p,\infty}^\alpha(X),$$

with equivalent seminorms.

PROOF. Since  $\mu$  is doubling and supports a 2-Poincaré inequality, we have the Gaussian double bound for  $p_t(x, y)$ . Hence if  $u \in \mathbf{B}^{p,\alpha}(X)$ , we then must have

$$\begin{aligned} \|u\|_{\mathbf{B}^{p,\alpha/2}(X)}^p &\geq C^{-1} \sup_{t>0} \int_X \int_X \frac{|u(y) - u(x)|^p}{t^{\alpha p/2}} \frac{e^{-C d(x,y)^2/t}}{\mu(B(x, \sqrt{t}))} d\mu(y) d\mu(x) \\ &\geq C^{-1} \sup_{\sqrt{t}>0} \int_X \int_{B(x, \sqrt{t})} \frac{|u(y) - u(x)|^p}{t^{\alpha p/2}} \frac{e^{-C d(x,y)^2/t}}{\mu(B(x, \sqrt{t}))} d\mu(y) d\mu(x) \\ &\geq C^{-1} \sup_{\sqrt{t}>0} \int_X \int_{B(x, \sqrt{t})} \frac{|u(y) - u(x)|^p}{\sqrt{t}^{\alpha p} \mu(B(x, \sqrt{t}))} d\mu(y) d\mu(x) \\ &= C^{-1} \|u\|_{B_{p,\infty}^\alpha(X)}^p, \end{aligned}$$

and from this it follows that  $\mathbf{B}^{p,\alpha/2}(X)$  embeds boundedly into  $B_{p,\infty}^\alpha(X)$ .

Now we focus on proving the converse embedding. Since  $X$  supports 2-Poincaré inequality, it is connected. Therefore by the doubling property of  $\mu$  (see [Hei01]), there are constants  $0 < Q \leq s < \infty$  and  $C \geq 1$  such that whenever  $0 < r \leq R$ ,  $x \in X$ , and  $y \in B(x, R)$ ,

$$(4.11) \quad \frac{1}{C} \left(\frac{r}{R}\right)^s \leq \frac{\mu(B(y, r))}{\mu(B(x, R))} \leq C \left(\frac{r}{R}\right)^Q.$$

Therefore,

$$\begin{aligned} &\frac{1}{t^{\alpha p/2}} \int_X \int_X |u(y) - u(x)|^p p_t(x, y) d\mu(y) d\mu(x) \\ &\leq \frac{C}{t^{\alpha p/2}} \int_X \sum_{i=-\infty}^{\infty} \int_{B(x, 2^i \sqrt{t}) \setminus B(x, 2^{i-1} \sqrt{t})} \frac{|u(y) - u(x)|^p e^{-C4^i}}{\sqrt{\mu(B(\sqrt{t}))} \mu(B(y, \sqrt{t}))} d\mu(y) d\mu(x) \\ &\leq \frac{C}{t^{\alpha p/2}} \int_X \sum_{i=-\infty}^{\infty} \int_{B(x, 2^i \sqrt{t})} \frac{|u(y) - u(x)|^p e^{-C4^i}}{\mu(B(x, 2^i \sqrt{t}))} \sqrt{\frac{\mu(B(x, 2^i \sqrt{t}))}{\mu(B(x, \sqrt{t}))}} \sqrt{\frac{\mu(B(y, 2^i \sqrt{t}))}{\mu(B(y, \sqrt{t}))}} d\mu(y) d\mu(x) \\ &\leq \frac{C}{t^{\alpha p/2}} \sum_{i=-\infty}^{\infty} e^{-C4^i} \max\{2^{is}, 2^{iQ}\} (2^i \sqrt{t})^{\alpha p} \int_X \int_{B(x, 2^i \sqrt{t})} \frac{|u(y) - u(x)|^p}{(2^i \sqrt{t})^{\alpha p} \mu(B(x, 2^i \sqrt{t}))} d\mu(y) d\mu(x) \\ &\leq C \|u\|_{B_{p,\infty}^\alpha(X)}^p \sum_{i=-\infty}^{\infty} e^{-C4^i} 2^{i\alpha p} \max\{2^{is}, 2^{iQ}\}. \end{aligned}$$

Since

$$\sum_{i=-\infty}^{\infty} e^{-C4^i} 2^{i\alpha p} \max\{2^{is}, 2^{iQ}\} \leq \sum_{i \in \mathbb{N}} e^{-C4^i} 2^{i[\alpha p + s]} + \sum_{i=0}^{\infty} 2^{-i[\alpha p + Q]} < \infty,$$

the desired bound follows.  $\square$

REMARK 4.15. While the previous theorem gave us a way to control, from above, the  $\mathcal{H}$ -measure of  $\partial_m E$  for a set  $E$  of finite perimeter, Proposition 4.14 give us a way to control, from below, the co-dimension 1 Minkowski measure of  $\partial E$ . Given set  $A \subset X$ , the co-dimension 1-Minkowski measure of  $A$  is given by

$$\mathcal{M}_{-1}(A) := \liminf_{\varepsilon \rightarrow 0^+} \frac{\mu(A_\varepsilon)}{\varepsilon},$$

where  $A_\varepsilon = \bigcup_{x \in A} B(x, \varepsilon)$ . As  $\mu$  is doubling, we see with the aid of Proposition 4.14 that if  $E \subset X$  is a measurable set such that  $\mathbf{1}_E \in \mathbf{B}^{1,1/2}(X)$ , then

$$\liminf_{t \rightarrow 0^+} \int_X \int_{B(x,t)} \frac{|\mathbf{1}_E(x) - \mathbf{1}_E(y)|}{t^{1/2} \mu(B(x,t))} d\mu(y) d\mu(x) \leq \mathcal{M}_{-1}(\partial E),$$

with the limit infimum on the left-hand side a finite number.

REMARK 4.16. *Another application of Proposition 4.14 is the following. It is in general not true that if  $\|Du\|(X) = 0$  then  $u$  is constant almost everywhere in  $X$ , even if  $X$  is connected. Should  $X$  support a 1-Poincaré inequality, the fact that  $\|Du\|(X) = 0$  implies  $u$  is constant follows immediately. We can use the above proposition to show that even if we do not have 1-Poincaré inequality, if  $X$  supports the Bakry-Émery curvature condition (4.2), then we still have the constancy of  $u$  from  $\|Du\|(X) = 0$ , for then*

$$\sup_{t>0} \int_X \int_{B(x,t)} \frac{|\mathbf{1}_E(x) - \mathbf{1}_E(y)|}{t^{1/2} \mu(B(x,t))} d\mu(y) d\mu(x) \simeq \|Du\|(X).$$

To conclude this section, we point out the following result that shows that  $1/2$  is a critical index for the spaces  $\mathbf{B}^{1,\alpha}(X)$ .

PROPOSITION 4.17. *Let  $f \in \mathbf{B}^{1,\alpha}(X)$  with  $\alpha > 1/2$ . Then,  $f$  is constant.*

PROOF. Let  $f \in \mathbf{B}^{1,\alpha}(X)$  with  $\alpha > 1/2$ . Since  $\mathbf{B}^{1,\alpha}(X) \subset \mathbf{B}^{1,1/2}(X) = BV(X)$ , we deduce that  $f$  is a BV function. Now since  $f \in \mathbf{B}^{1,\alpha}(X)$ , one has for every  $t > 0$ ,

$$\int_X \int_X p_t(x,y) |f(x) - f(y)| d\mu(x) d\mu(y) \leq t^\alpha \|f\|_{1,\alpha}.$$

By using the heat kernel lower bound as before, it implies

$$\liminf_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \iint_{\Delta_\varepsilon} \frac{|f(y) - f(x)|}{\sqrt{\mu(B(x,\varepsilon))\mu(B(y,\varepsilon))}} d\mu(x) d\mu(y) = 0.$$

Therefore  $\|Df(X)\| = 0$ , so from the previous remark  $f$  is constant.  $\square$

#### 4.5. Comparison between Sobolev and Besov seminorms

In this section, we compare the Besov and Sobolev seminorms for  $p > 1$ . The case  $p = 1$  was studied in detail in Section 4.3.

Throughout the section we will assume that  $(X, d_\mathcal{E}, \mu)$  is doubling, supports a 2-Poincaré inequality. As before, we will also assume that for each  $u \in \mathcal{F}$  the measure  $\Gamma(u, u)$  is absolutely continuous with respect to the underlying measure  $\mu$  on  $X$ ; as in the previous section we will denote by  $|\nabla u|^2$  the Radon-Nikodym derivative of  $\Gamma(u, u)$  with respect to  $\mu$ . We also assume that  $\mathcal{E}$  is regular and strictly local.

##### 4.5.1. Lower bound for the Besov seminorm.

THEOREM 4.18. *Let  $p > 1$ . There exists a constant  $C > 0$  such that for every  $u \in \mathbf{B}^{p,1/2}(X) \cap \mathcal{F}$ ,*

$$\| |\nabla u| \|_{L^p(X, \mu)} \leq C \|u\|_{p,1/2}$$

PROOF. Let  $u \in \mathbf{B}^{p,1/2}(X) \cap \mathcal{F}$ . Then as in the derivation of (4.8) we obtain that for each  $\varepsilon > 0$ ,

$$\frac{1}{\varepsilon^p} \iint_{\Delta_\varepsilon} \frac{|u(x) - u(y)|^p}{\mu(B(x,\varepsilon))} d\mu(y) d\mu(x) \leq \|u\|_{p,1/2}^p < \infty.$$

Fix  $\varepsilon > 0$  and cover  $X$  by family of balls  $B_i^\varepsilon = B(x_i^\varepsilon, \varepsilon)$  such that  $\frac{1}{2}B_i^\varepsilon$  are pairwise disjoint, and let  $\varphi_i^\varepsilon$  be a  $C/\varepsilon$ -Lipschitz partition of unity subordinate to this cover: that is,  $0 \leq \varphi_i^\varepsilon \leq 1$  on  $X$ ,  $\sum_i \varphi_i^\varepsilon = 1$  on  $X$ , and  $\varphi_i^\varepsilon = 0$  in  $X \setminus B_i^\varepsilon$ . We then set

$$u_\varepsilon := \sum_i u_{B_i^\varepsilon} \varphi_i^\varepsilon.$$

Then as each  $\varphi_i^\varepsilon$  is Lipschitz, we know that  $u_\varepsilon$  is locally Lipschitz and hence is in  $\mathcal{F}_{loc}(X)$ . For  $x, y \in B_j^\varepsilon$  we see that

$$\begin{aligned} |u_\varepsilon(x) - u_\varepsilon(y)| &\leq \sum_{i: 2B_i^\varepsilon \cap 2B_j^\varepsilon \neq \emptyset} |u_{B_i^\varepsilon} - u_{B_j^\varepsilon}| |\varphi_i^\varepsilon(x) - \varphi_i^\varepsilon(y)| \\ &\leq \frac{C d(x, y)}{\varepsilon} \sum_{i: 2B_i^\varepsilon \cap 2B_j^\varepsilon \neq \emptyset} \left( \int_{B_i^\varepsilon} \int_{B(x, 2\varepsilon)} |u(y) - u(x)|^p d\mu(y) d\mu(x) \right)^{1/p}. \end{aligned}$$

Therefore, by Lemma 4.1, we see that

$$\begin{aligned} |\nabla u_\varepsilon| &\leq \frac{C}{\varepsilon} \sum_{i: 2B_i^\varepsilon \cap 2B_j^\varepsilon \neq \emptyset} \left( \int_{B_i^\varepsilon} \int_{B(x, 2\varepsilon)} |u(y) - u(x)|^p d\mu(y) d\mu(x) \right)^{1/p} \\ &\leq C \left( \int_{2B_j^\varepsilon} \int_{B(x, 2\varepsilon)} \frac{|u(y) - u(x)|^p}{\varepsilon^p} d\mu(y) d\mu(x) \right)^{1/p}, \end{aligned}$$

and so by the bounded overlap property of the collection  $2B_j^\varepsilon$  (which is a consequence of the doubling property of  $\mu$ ),

$$\begin{aligned} \int_X |\nabla u_\varepsilon|^p d\mu &\leq \sum_j \int_{B_j^\varepsilon} |\nabla u_\varepsilon|^p d\mu \\ &\leq C \sum_j \int_{2B_j^\varepsilon} \int_{B(x, 2\varepsilon)} \frac{|u(y) - u(x)|^p}{\varepsilon^p} d\mu(y) d\mu(x) \\ &\leq C \int_X \int_{B(x, 2\varepsilon)} \frac{|u(y) - u(x)|^p}{\varepsilon^p} d\mu(y) d\mu(x) \\ &\leq C \frac{1}{\varepsilon^p} \int_{\Delta_{2\varepsilon}} \frac{|u(x) - u(y)|^p}{\mu(B(x, \varepsilon))} d\mu(y) d\mu(x) \leq 2M. \end{aligned}$$

In a similar manner, we can also show that

$$\int_X |u_\varepsilon(x) - u(x)|^p d\mu(x) \leq C\varepsilon^p \int_{\Delta_{2\varepsilon}} \frac{|u(x) - u(y)|^p}{\varepsilon^p \mu(B(x, \varepsilon))} d\mu(y) d\mu(x) \leq CM \varepsilon^p,$$

that is,  $u_\varepsilon \rightarrow u$  in  $L^p(X)$  as  $\varepsilon \rightarrow 0^+$ . So we get that if  $u \in \mathcal{F}$ , then

$$C \|u\|_{p, 1/2} \geq \|\nabla u\|_{L^p(X)}.$$

□

**4.5.2. Upper bound for the Besov seminorm.** We now turn to the proof of the upper bound for the Besov seminorm in terms of the Sobolev seminorm. We will use an additional assumption: The strong Bakry-Émery curvature condition. As before,  $P_t u$  will denote the heat extension of  $u$  via the heat semigroup associated with the Dirichlet form  $\mathcal{E}$ . We recall that from classical theory (see for instance Theorems 1.4.1 and 1.4.2 in [Dav89]), the semigroup  $P_t$  lets  $L^1(X, \mu) \cap L^\infty(X, \mu)$  invariant and may be extended to a positive, contraction semigroup on  $L^p(X, \mu)$ ,  $1 \leq p \leq +\infty$ , that we still denote by  $P_t$ . Moreover, from the same reference, for  $1 \leq p < +\infty$ ,  $P_t$  is strongly continuous (i.e.  $P_t u \rightarrow u$  in  $L^p(X, \mu)$  when  $t \rightarrow 0$ ). We will denote by  $L$  the generator of  $P_t$ .

**Assumption:** *In this subsection, in addition to the assumptions stated at the beginning of Section 4.5, we will furthermore assume the strong Bakry-Émery curvature condition: There exists a constant  $C > 0$  such that for every  $u \in \mathcal{F}$  and  $t \geq 0$  we have  $\mu$  a.e.*

$$(4.12) \quad |\nabla P_t u| \leq C P_t |\nabla u|.$$

We note that the strong Bakry-Émery curvature condition implies the weak one, see the proof of Theorem 3.3 in [BK17]. Examples where the strong Bakry-Émery estimate is satisfied include: Riemannian manifolds with non negative Ricci curvature and more generally  $\mathbf{RCD}(0, +\infty)$  spaces (see [EKS15]), some metric graphs (see [BK17]), the Heisenberg group and more generally H-type groups (see [BBBC08, Eld10])

A first important corollary of the strong Bakry-Émery estimate is the following Hamilton's type gradient estimate for the heat kernel. Such type of estimate is well-known on Riemannian manifolds with non-negative Ricci curvature (see for instance [Kot07]), but is new in our general framework.

**THEOREM 4.19.** *There exists a constant  $C > 0$  such that for every  $t > 0$ ,  $x, y \in X$ ,*

$$|\nabla_x \ln p_t(x, y)|^2 \leq \frac{C}{t} \left( 1 + \frac{d(x, y)^2}{t} \right)$$

**PROOF.** The proof proceeds in two steps.

**Step 1:** We first collect a gradient bound for the heat kernel. We observe that (4.12) implies a weaker  $L^2$  version as follows

$$|\nabla P_t u|^2 \leq C P_t(|\nabla u|^2),$$

and hence the pointwise heat kernel gradient bound (see [ACDH04, Lemma 3.3])

$$|\nabla_x p_t(x, y)| \leq \frac{C}{\sqrt{t}} \frac{e^{-cd(x, y)^2/t}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}}.$$

In particular, we note that  $|\nabla_x p_t(x, \cdot)| \in L^p(X, \mu)$  for every  $p \geq 1$ .

**Step 2:** In a second step, we prove a reverse log-Sobolev inequality for the heat kernel. Let  $\tau, \varepsilon > 0$  and  $x \in X$  be fixed in this Step 2.

We denote  $u = p_\tau(x, \cdot) + \varepsilon$ . One has, from the chain rule for strictly local forms [FOT11, Lemma 3.2.5],

$$\begin{aligned} P_t(u \ln u) - P_t u \ln P_t u &= \int_0^t \partial_s (P_s(P_{t-s} u \ln P_{t-s} u)) ds \\ &= \int_0^t LP_s(P_{t-s} u \ln P_{t-s} u) - P_s(LP_{t-s} u \ln P_{t-s} u) - P_s(LP_{t-s} u) ds \\ &= \int_0^t P_s(L(P_{t-s} u \ln P_{t-s} u)) - P_s(LP_{t-s} u \ln P_{t-s} u) - P_s(LP_{t-s} u) ds \\ &= \int_0^t P_s [L(P_{t-s} u \ln P_{t-s} u) - LP_{t-s} u \ln P_{t-s} u - LP_{t-s} u] ds \\ &= \int_0^t P_s((P_{t-s} u) |\nabla \ln P_{t-s} u|^2) ds, \end{aligned}$$

where the above computations may be justified by using the Gaussian heat kernel estimates for the heat kernel and the Gaussian upper bound for the gradient of the heat kernel obtained in Step 1. In particular, we point out that the commutation  $LP_s(P_{t-s} u \ln P_{t-s} u) = P_s(LP_{t-s} u \ln P_{t-s} u)$  is justified by noting that  $P_{t-s} u \ln P_{t-s} u - \varepsilon \ln \varepsilon$  is in the domain of  $L$  in  $L^2(X, \mu)$ . Here,  $L$  is the infinitesimal generator (the Laplacian operator) associated with  $\mathcal{E}$ .

Thus, from the strong Bakry-Émery estimate and the Cauchy-Schwarz inequality in the form of  $P_s \left( \frac{f^2}{g} \right) \geq \frac{(P_s f)^2}{P_s g}$ , we obtain

$$\begin{aligned} P_t(u \ln u) - P_t u \ln P_t u &= \int_0^t P_s \left( \frac{|\nabla P_{t-s} u|^2}{P_{t-s} u} \right) ds \\ &\geq \int_0^t \frac{(P_s |\nabla P_{t-s} u|)^2}{P_s (P_{t-s} u)} ds \\ &\geq \frac{1}{C} \frac{1}{P_t u} \int_0^t |\nabla P_t u|^2 ds \\ &\geq \frac{t}{C} \frac{1}{P_t u} |\nabla P_t u|^2 \end{aligned}$$

Coming back to the definition of  $u$ , noting that  $P_t p_\tau(x, \cdot) = P_{t+\tau}(x, \cdot)$  and applying the previous inequality with  $t = \tau$  one deduces

$$|\nabla_y \ln(p_{2t}(x, y) + \varepsilon)|^2 \leq \frac{C}{t} P_t \left[ \ln \left( \frac{M_t(x) + \varepsilon}{p_{2t}(x, \cdot) + \varepsilon} \right) \right] (y)$$

where  $M_t(x) = \sup_{y \in X} p_t(x, y)$ . Therefore, by letting  $\varepsilon \rightarrow 0$  and using the Gaussian heat kernels, one concludes

$$|\nabla_y \ln p_{2t}(x, y)|^2 \leq \frac{C}{t} \left( 1 + \frac{d(x, y)^2}{t} \right)$$

and thus

$$|\nabla_y \ln p_t(x, y)|^2 \leq \frac{C}{t} \left( 1 + \frac{d(x, y)^2}{t} \right).$$

Finally, we note that the inequality

$$|\nabla_x \ln p_t(x, y)|^2 \leq \frac{C}{t} \left( 1 + \frac{d(x, y)^2}{t} \right).$$

is obtained similarly by exchanging the roles of  $x$  and  $y$  in our proof.  $\square$

**COROLLARY 4.20.** *Let  $p > 1$ . There exists a constant  $C > 0$  such that for every  $u \in L^p(X, \mu)$ ,*

$$|\nabla P_t u| \leq \frac{C}{\sqrt{t}} P_t(|u|^p)^{1/p}.$$

**PROOF.** Let  $p > 1$ ,  $q$  the conjugate exponent and  $u \in L^p(X, \mu)$ . One has from Hölder's inequality

$$\begin{aligned} |\nabla P_t u|(x) &\leq \int_X |\nabla_x p_t(x, y)| |u(y)| d\mu(y) \\ &\leq \left( \int_X \frac{|\nabla_x p_t(x, y)|^q}{p_t(x, y)^{q/p}} d\mu(y) \right)^{1/q} P_t(|u|^p)^{1/p} \\ &\leq \left( \int_X |\nabla_x \ln p_t(x, y)|^q p_t(x, y) d\mu(y) \right)^{1/q} P_t(|u|^p)^{1/p} \end{aligned}$$

The proof follows then from Theorem 4.19 and the Gaussian upper bound for the heat kernel.  $\square$

Note that, by integrating over  $X$  the previous proposition immediately yields:

**LEMMA 4.21.** *Let  $p > 1$ . There exists a constant  $C > 0$  such that for every  $u \in L^p(X, \mu)$*

$$\| |\nabla P_t u| \|_{L^p(X, \mu)}^2 \leq \frac{C}{t} \|u\|_{L^p(X, \mu)}^2.$$

We deduce then:

LEMMA 4.22. *Let  $p > 1$ . There exists a constant  $C > 0$  such that for every  $u \in L^p(X, \mu) \cap \mathcal{F}$  with  $|\nabla u| \in L^p(X, \mu)$*

$$\|P_t u - u\|_{L^p(X, \mu)} \leq C\sqrt{t}\|\nabla u\|_{L^p(X, \mu)}$$

PROOF. With the previous lemma in hand, the proof is similar to the one in Lemma 4.10, with  $\varphi$  in  $\mathcal{F} \cap L^q(X, \mu)$  where  $p^{-1} + q^{-1} = 1$ . As compactly supported Lipschitz functions form a dense subclass of  $L^p(X, \mu)$  and also belong to  $\mathcal{F}$  from the discussion in Chapter 4, we recover the  $L^p$ -norm of  $P_t u - u$  by taking the supremum over all such  $\varphi$  with  $\int_X |\varphi|^q d\mu < \infty$ .  $\square$

LEMMA 4.23. *Let  $p > 1$ , then*

$$\left( \int_X \int_X |P_t u(x) - u(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p} \leq C\sqrt{t}\|\nabla u\|_{L^p(X)}.$$

PROOF. Let  $u \in L^p(X)$  and  $t > 0$  be fixed in the above argument. By an application of Fubini's theorem we have

$$\left( \int_X \int_X |P_t u(x) - u(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p} = \left( \int_X P_t(|P_t u(x) - u|^p)(x) d\mu(x) \right)^{1/p}.$$

The main idea now is to adapt the proof of [BBBC08, Theorem 6.2].

As above, let  $q$  be the conjugate of  $p$ . Let  $x \in X$  be fixed. Let  $g$  be a function in  $L^\infty(X, \mu)$  such that  $P_t(|g|^q)(x) \leq 1$ .

We first note that from the chain rule:

$$\begin{aligned} & \partial_s [P_s((P_{t-s}u)(P_{t-s}g))(x)] \\ &= LP_s((P_{t-s}u)(P_{t-s}g))(x) - P_s((LP_{t-s}u)(P_{t-s}g))(x) - P_s((P_{t-s}u)(LP_{t-s}g))(x) \\ &= P_s(L(P_{t-s}u)(P_{t-s}g))(x) - P_s((LP_{t-s}u)(P_{t-s}g))(x) - P_s((P_{t-s}u)(LP_{t-s}g))(x) \\ &= 2P_s(\Gamma(P_{t-s}u, P_{t-s}g)) \end{aligned}$$

Therefore, we have

$$\begin{aligned} P_t((u - P_t u(x))g)(x) &= P_t(ug)(x) - P_t u(x)P_t g(x) \\ &= \int_0^t \partial_s [P_s((P_{t-s}u)(P_{t-s}g))(x)] ds \\ &= 2 \int_0^t P_s(\Gamma(P_{t-s}u, P_{t-s}g))(x) ds \\ &\leq 2 \int_0^t P_s(|\nabla P_{t-s}u||\nabla P_{t-s}g|)(x) ds \\ &\leq 2 \int_0^t P_s(|\nabla P_{t-s}u|^p)^{1/p}(x) P_s(|\nabla P_{t-s}g|^q)^{1/q}(x) ds \end{aligned}$$

Now from the strong BE estimate and Hölder's inequality we have

$$P_s(|\nabla P_{t-s}u|^p)^{1/p}(x) \leq CP_s(P_{t-s}(|\nabla u|^p))^{1/p}(x) = CP_t(|\nabla u|^p)^{1/p}(x)$$

On the other hand

$$|\nabla P_{t-s}g|^q \leq \frac{C}{(t-s)^{q/2}} P_{t-s}(|g|^q)$$

Thus,

$$P_s(|\nabla P_{t-s}g|^q)^{1/q}(x) \leq \frac{C}{(t-s)^{1/2}} P_t(|g|^q)^{1/q}(x) \leq \frac{C}{(t-s)^{1/2}}.$$

One concludes

$$P_t((u - P_t u(x))g)(x) \leq C\sqrt{t}P_t(|\nabla u|^p)^{1/p}(x)$$



Thus by  $L^p - L^q$  duality, one concludes

$$P_t(|u - P_t u(x)|^p)(x)^{1/p} \leq C\sqrt{t}P_t(|\nabla u|^p)^{1/p}(x)$$

and finishes the proof by integration over  $X$ .  $\square$

We are now finally in position to prove the main result of the section.

**THEOREM 4.24.** *Let  $p > 1$ . There exists a constant  $C > 0$  such that for every  $u \in L^p(X, \mu) \cap \mathcal{F}$  with  $|\nabla u| \in L^p(X, \mu)$ ,*

$$\|u\|_{p,1/2} \leq C\|\nabla u\|_{L^p(X)}$$

**PROOF.** One has

$$\begin{aligned} & \left( \int_X \int_X |u(x) - u(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p} \\ & \leq \left( \int_X \int_X |u(x) - P_t u(x)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p} + \left( \int_X \int_X |P_t u(x) - u(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p} \\ & \leq \|P_t u - u\|_{L^p(X)} + \left( \int_X \int_X |P_t u(x) - u(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p} \end{aligned}$$

One has from the previous lemma

$$\|P_t u - u\|_{L^p(X)} \leq C\sqrt{t}\|\nabla u\|_{L^p(X)}$$

and the term  $\left( \int_X \int_X |P_t u(x) - u(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p}$  can be controlled by the above lemma.  $\square$

In view of Theorems 4.18 and 4.24, as well as [ACDH04, Theorem 1.4], we conclude this section by the following result on the Riesz transform.

**COROLLARY 4.25.** *Let  $p > 1$ . Then for any  $f \in L^p(X) \cap \mathcal{F}$ ,*

$$\|f\|_{p,1/2} \simeq \|\sqrt{-L}f\|_{L^p(X)}.$$

Consequently,

$$\mathbf{B}^{p,1/2}(X) = \mathcal{L}_p^{1/2},$$

where  $\mathcal{L}_p^{1/2}$  defined in Section 1.4 is the domain of the operator  $\sqrt{-L}$  in  $L^p(X, \mu)$ .

## 4.6. Critical exponents

In this section, our assumptions are the same as Subsection 4.5.2., that is, the standing assumptions of this chapter as well as the strong Bakry-Émery curvature condition (4.12). We first notice that as a consequence of Lemma 4.21 and Theorem 4.24,

**THEOREM 4.26.** *Let  $p > 1$ . There exists a constant  $C > 0$  such that for every  $f \in L^p(X, \mu)$  and  $t > 0$*

$$\|P_t f\|_{p,1/2} \leq \frac{C}{t^{1/2}} \|f\|_{L^p(X)}.$$

**REMARK 4.27.** *The bound*

$$\|P_t f\|_{p,1/2} \leq \frac{C}{t^{1/2}} \|f\|_{L^p(X)}, \quad p > 1,$$

*likely holds if one only assumes the weak Bakry-Émery estimate. This may be seen by adapting to this setting the argument yielding Theorem 5.47 in Chapter 5. For concision, we will not expand this remark.*

Similarly as in Section 1.7, we have several corollaries of the above result.

COROLLARY 4.28. *Let  $p > 1$ . Let  $L$  be the infinitesimal generator of  $\mathcal{E}$  and  $\mathcal{L}_p$  be the generator of  $L$  in  $L^p(X, \mu)$ . Then*

$$\mathcal{L}_p \subset \mathbf{B}^{p,1/2}(X)$$

and for every  $f \in \mathcal{L}_p$ ,

$$(4.13) \quad \|f\|_{p,1/2}^2 \leq C_p \|Lf\|_{L^p(X,\mu)} \|f\|_{L^p(X,\mu)}.$$

Recall that  $L$  plays the role of Laplacian in the theory of Dirichlet forms, see [FOT11].

COROLLARY 4.29. *For  $p > 1$ ,  $\mathbf{B}^{p,1/2}(X)$  is dense in  $L^p(X, \mu)$ .*

COROLLARY 4.30. *Let  $p > 1$ . For every  $f \in L^p(X, \mu)$ , and  $t > 0$ ,*

$$\|P_t f - f\|_{L^p(X,\mu)} \leq C_p t^{1/2} \liminf_{s \rightarrow 0} s^{-1/2} \left( \int_X P_s(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p}$$

COROLLARY 4.31. *Let  $p > 1$ . If  $f \in \mathbf{B}^{p,\alpha}(X)$  with  $\alpha > 1/2$  then  $\mathcal{E}(f, f) = 0$ .*

In particular, with the notation of Section 1.8, one concludes:

PROPOSITION 4.32. *For every  $p \geq 1$ ,  $\alpha_p^*(X) = 1/2$ .*

#### 4.7. Sobolev and isoperimetric inequalities

Combining the conclusions of this chapter with the results in Chapter 2, we immediately obtain the following results that encompass many of the known results found in the literature.

COROLLARY 4.33. *Suppose that  $\mu$  is doubling and supports a 2-Poincaré inequality. Assume moreover that the volume growth condition  $\mu(B(x, r)) \geq C_1 r^Q$ ,  $r \geq 0$ , is satisfied for some  $Q > 0$ . Then, one has the following weak type Besov space embedding. Let  $0 < \delta < Q$ . Let  $1 \leq p < \frac{Q}{\delta}$ . There exists a constant  $C_{p,\delta} > 0$  such that for every  $f \in \mathbf{B}^{p,\delta/2}(X)$ ,*

$$\sup_{s \geq 0} s \mu(\{x \in X, |f(x)| \geq s\})^{\frac{1}{q}} \leq C_{p,\delta} \sup_{r > 0} \frac{1}{r^{\delta+Q/p}} \left( \iint_{\{(x,y) \in X \times X | d(x,y) < r\}} |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p}$$

where  $q = \frac{pQ}{Q-p\delta}$ . Furthermore, for every  $0 < \delta < Q$ , there exists a constant  $C_{\text{iso},\delta}$  such that for every measurable  $E \subset X$ ,  $\mu(E) < +\infty$ ,

$$\mu(E)^{\frac{d_H - \delta}{d_H}} \leq C_{\text{iso},\delta} \sup_{r > 0} \frac{1}{r^{\delta+Q}} (\mu \otimes \mu) \{(x, y) \in E \times E^c : d(x, y) \leq r\}$$

PROOF. From the heat kernel upper bound, the volume growth condition  $\mu(B(x, r)) \geq C_1 r^Q$ ,  $r \geq 0$ , implies the ultracontractive estimate

$$p_t(x, y) \leq \frac{C}{t^{Q/2}}.$$

We are therefore in the framework of Chapter 2. From Theorem 2.3, one deduces therefore the following. Let  $0 < \delta < Q$ . Let  $1 \leq p < \frac{Q}{\delta}$ . There exists a constant  $C_{p,\delta} > 0$  such that for every  $f \in \mathbf{B}^{p,\delta/2}(X)$ ,

$$\sup_{s \geq 0} s \mu(\{x \in X : |f(x)| \geq s\})^{\frac{1}{q}} \leq C_{p,\delta} \|f\|_{p,\delta/2}$$

where  $q = \frac{pQ}{Q-p\delta}$ . We conclude then with Theorem 4.14.  $\square$

In the case where  $\delta = 1/2$  and the weak Bakry-Émery estimate is satisfied, then one gets a strong Sobolev inequality.

THEOREM 4.34. *Suppose that  $\mu$  is doubling and supports a 2-Poincaré inequality and that the weak Bakry-Émery estimate is satisfied. If the volume growth condition  $\mu(B(x, r)) \geq C_1 r^Q$ ,  $r \geq 0$ , is satisfied for some  $Q > 0$ , then there exists a constant  $C_2 > 0$  such that for every  $f \in BV(X)$ ,*

$$\|f\|_{L^q(X, \mu)} \leq C_2 \|Df\|(X)$$

where  $q = \frac{Q}{Q-1}$ . In particular, if  $E$  is a set with finite perimeter in  $X$ , then

$$\mu(E)^{\frac{Q-1}{Q}} \leq C_2 P(E, X).$$

PROOF. We apply Theorem 2.9. Since  $\|f\|_{1,1/2} \leq C \|Df\|(X)$ , it is enough to prove that  $(P_{1,1/2})$  is satisfied. This follows from

$$\sup_{s>0} s^{-1/2} \int_X P_s(|f - f(y)|)(y) d\mu(y) \leq C \liminf_{s \rightarrow 0} s^{-1/2} \int_X P_s(|f - f(y)|)(y) d\mu(y),$$

which is a consequence from Theorem 4.11. □



## Strongly local Dirichlet metric spaces with sub-Gaussian heat kernel estimates

We now turn to the study strongly local Dirichlet spaces which *do not have Gaussian heat kernel bounds* (4.1). The main class of examples we are interested in are fractal spaces. We refer to [Bar98, Gri03, Kig01, Kig12] for further details on the following framework/assumptions. We note that some of the most basic facts of Chapter 4, such as Lemma 4.1, do not hold true in this chapter because energy measures  $\Gamma(\cdot, \cdot)$  are not  $\mu$ -absolutely continuous. Therefore we can not use locally Lipschitz functions for our analysis, and need to develop a different set of tools.

Let  $(X, d, \mu)$  be a metric measure space. We assume that  $B(x, r) := \{y \in X \mid d(x, y) < r\}$  has compact closure for any  $x \in X$  and any  $r \in (0, \infty)$ , and that  $\mu$  is Ahlfors  $H$ -regular, i.e. there exist  $c_1, c_2, d_H \in (0, \infty)$  such that  $c_1 r^{d_H} \leq \mu(B(x, r)) \leq c_2 r^{d_H}$  for any  $r \in (0, +\infty)$ . Furthermore, we assume that on  $(X, \mu)$  there is a measurable heat kernel  $p_t(x, y)$  satisfying, for some  $c_3, c_4, c_5, c_6 \in (0, \infty)$  and  $d_W \in [2, +\infty)$ ,

$$(5.1) \quad c_5 t^{-d_H/d_W} \exp\left(-c_6 \left(\frac{d(x, y)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) \leq p_t(x, y) \leq c_3 t^{-d_H/d_W} \exp\left(-c_4 \left(\frac{d(x, y)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right)$$

for  $\mu \times \mu$ -a.e.  $(x, y) \in X \times X$  and each  $t \in (0, +\infty)$ . We refer to [PP10], assumptions (A1) to (A5), page 201, for the definition of heat kernels on metric measure spaces. The corresponding Dirichlet space will be denoted by  $(X, \mu, \mathcal{E}, \mathcal{F})$  and the corresponding semigroup by  $\{P_t, t \geq 0\}$ . We assume that  $(X, \mu, \mathcal{E}, \mathcal{F})$  is a strongly local regular symmetric Dirichlet space. The parameter  $d_H$  is the Hausdorff dimension and the parameter  $d_W$  the so-called walk dimension. It is possible to prove that if the metric space  $(X, d)$  satisfies a chain condition, then  $2 \leq d_W \leq d_H + 1$ . When  $d_W = 2$ , one speaks of Gaussian estimates and when  $d_W > 2$ , one speaks then of sub-Gaussian estimates.

In this framework, it is known that  $(X, \mu, \mathcal{E}, \mathcal{F})$  is conservative. Let  $p \geq 1$  and  $\alpha \geq 0$ . As before, we define the Besov seminorm

$$\|f\|_{p, \alpha} = \sup_{t > 0} t^{-\alpha} \left( \int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p}$$

and define

$$\mathbf{B}^{p, \alpha}(X) = \{f \in L^p(X, \mu) : \|f\|_{p, \alpha} < +\infty\}.$$

### 5.1. Metric characterization of Besov spaces

Our goal in this section is to compare the space  $\mathbf{B}^{p, \alpha}(X)$  to Besov type spaces previously considered in a similar framework (see [Gri03]). In the following, for  $r > 0$  we set

$$\Delta_r := \{(x, y) \in X \times X : d(x, y) < r\}.$$

For  $\alpha \in [0, \infty)$  and  $p \in [1, \infty)$ , we introduce the following Besov seminorm: for  $f \in L^p(X, \mu)$ ,

$$(5.2) \quad N_p^\alpha(f, r) := \frac{1}{r^{\alpha + d_H/p}} \left( \iint_{\Delta_r} |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p}$$

for  $r \in (0, \infty)$ , and

$$(5.3) \quad N_p^\alpha(f) := \sup_{r \in (0,1]} N_p^\alpha(f, r).$$

We then define the Besov space  $\mathfrak{B}_p^\alpha(X)$  by

$$(5.4) \quad \mathfrak{B}_p^\alpha(X) := \{f \in L^p(X, \mu) : N_p^\alpha(f) < \infty\}.$$

With the notation of the previous section, note that  $\|f\|_{B_{p,\infty}^\alpha(X)} \simeq \sup_{r>0} N_p^\alpha(f, r)$ . It is clear that  $\mathfrak{B}_p^{\alpha_2}(X) \subset \mathfrak{B}_p^{\alpha_1}(X)$  for  $\alpha_1, \alpha_2 \in [0, \infty)$  with  $\alpha_1 \leq \alpha_2$ . Note also that

$$(5.5) \quad \mathfrak{B}_p^\alpha(X) = \left\{ f \in L^p(X, \mu) : \limsup_{r \downarrow 0} N_p^\alpha(f, r) < \infty \right\}.$$

Indeed, if  $f \in \mathfrak{B}_p^\alpha(X)$  then  $f \in L^p(X, \mu)$  and  $\limsup_{r \downarrow 0} N_p^\alpha(f, r) \leq N_p^\alpha(f) < \infty$ . Conversely, for any  $f \in L^p(X)$  with  $\limsup_{r \downarrow 0} N_p^\alpha(f, r) < \infty$ , we have  $\sup_{r \in (0, \varepsilon]} N_p^\alpha(f, r) < \infty$  for some  $\varepsilon \in (0, \infty)$ , and for any  $r \in (\varepsilon, \infty)$  we see from  $|f(x) - f(y)|^p \leq 2^p(|f(x)|^p + |f(y)|^p)$  and  $\mu(B(x, r)) \leq c_2 r^{d_H}$  that

$$\begin{aligned} N_p^\alpha(f, r)^p &= \frac{1}{r^{p\alpha+d_H}} \iint_{\{(x,y) \in X \times X : d(x,y) < r\}} |f(x) - f(y)|^p d\mu(x) d\mu(y) \\ &\leq \frac{1}{r^{p\alpha+d_H}} \iint_{\{(x,y) \in X \times X : d(x,y) < r\}} 2^p(|f(x)|^p + |f(y)|^p) d\mu(x) d\mu(y) \\ &= \frac{2^{p+1}}{r^{p\alpha+d_H}} \int_X |f(y)|^p \mu(B(y, r)) d\mu(y) \\ &\leq \frac{2^{p+1}}{r^{p\alpha+d_H}} \int_X |f(y)|^p \cdot c_2 r^{d_H} d\mu(y) = \frac{2^{p+1} c_2}{r^{p\alpha}} \|f\|_{L^p(X, \mu)}^p \leq \frac{2^{p+1} c_2}{\varepsilon^{p\alpha}} \|f\|_{L^p(X, \mu)}^p, \end{aligned}$$

so that

$$\sup_{r \in (0, \infty)} N_p^\alpha(f, r) \leq \max\{2(2c_2)^{1/p} \varepsilon^{-\alpha} \|f\|_{L^p(\mu)}, \sup_{r \in (0, \varepsilon]} N_p^\alpha(f, r)\} < \infty.$$

Hence  $f \in \mathfrak{B}_p^\alpha(X)$ , proving (5.5). The above argument also shows that with the notation of the previous section  $B_{p,\infty}^\alpha(X) = \mathfrak{B}_p^\alpha(X)$ . The purpose of this section is to prove the following theorem.

**THEOREM 5.1.** [**PP10**, Theorem 3.2] *Let  $p \geq 1$  and  $\alpha \geq 0$ . We have  $\mathfrak{B}_p^\alpha(X) = \mathbf{B}^{p, \frac{\alpha}{d_W}}(X)$  and there exist constants  $c_{p,\alpha}, C_{p,\alpha} > 0$  such that for every  $f \in \mathfrak{B}_p^\alpha(X)$  and  $r > 0$ ,*

$$c_{p,\alpha} \sup_{s \in (0, r]} N_p^\alpha(f, s) \leq \|f\|_{p, \alpha/d_W} \leq C_{p,\alpha} \left( \sup_{s \in (0, r]} N_p^\alpha(f, s) + \frac{1}{r^\alpha} \|f\|_{L^p(X, \mu)} \right).$$

*In particular,  $\|f\|_{p, \alpha/d_W} \simeq \sup_{s \in (0, +\infty)} N_p^\alpha(f, s)$ .*

**REMARK 5.2.** *The above theorem is essentially a rephrasing of [PP10, Theorem 3.2]. However, the notion of Besov spaces given in [PP10] considers dyadic jumps in the parameter  $t$ ; hence the proof given there is slightly more complicated than ours. We also include the relatively short proof because it shall repeatedly be used in the next sections.*

PROOF. We first prove the lower bound. For  $s, t > 0$  and  $\alpha > 0$ ,

$$\begin{aligned}
& \int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \\
& \geq \int_X \int_{B(y, s)} |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \\
& \geq c_5 t^{-d_H/d_W} \int_X \int_{B(y, s)} |f(x) - f(y)|^p \exp\left(-c_6 \left(\frac{d(x, y)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) d\mu(x) d\mu(y) \\
& \geq c_5 t^{-d_H/d_W} \exp\left(-c_6 \left(\frac{s^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) s^{\alpha p + d_H} N_p^\alpha(f, s)^p.
\end{aligned}$$

Therefore we have

$$t^{-\frac{\alpha p}{d_W}} \int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \geq c_5 t^{-\frac{\alpha p}{d_W} - \frac{d_H}{d_W}} \exp\left(-c_6 \left(\frac{s^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) s^{\alpha p + d_H} N_p^\alpha(f, s)^p.$$

We now choose  $s = t^{1/d_W}$ . This yields

$$t^{-\frac{\alpha p}{d_W}} \int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \geq c_5 \exp(-c_6) N_p^\alpha(f, t^{1/d_W})^p.$$

From the definition of the  $\|\cdot\|_{p, \alpha/d_W}$  seminorm, one concludes

$$c_5 \exp(-c_6) N_p^\alpha(f, t^{1/d_W})^p \leq \|f\|_{p, \alpha/d_W}^p.$$

Since, it is true for every  $t > 0$ , the conclusion follows.

We now turn to the upper bound. Fixing  $r > 0$ , we set

$$(5.6) \quad A(t) := \int_X \int_{X \setminus B(y, r)} p_t(x, y) |f(x) - f(y)|^p d\mu(x) d\mu(y),$$

$$(5.7) \quad B(t) := \int_X \int_{B(y, r)} p_t(x, y) |f(x) - f(y)|^p d\mu(x) d\mu(y),$$

so that  $\int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) = A(t) + B(t)$ . By (5.1) and the inequality  $|f(x) - f(y)|^p \leq 2^{p-1}(|f(x)|^p + |f(y)|^p)$ ,

$$\begin{aligned}
A(t) &\leq \frac{c_3}{t^{d_H/d_W}} \int_X \int_{X \setminus B(y,r)} \exp\left(-c_4 \left(\frac{d(x,y)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) \cdot 2^p |f(y)|^p d\mu(x) d\mu(y) \\
&= \frac{2^p c_3}{t^{d_H/d_W}} \sum_{k=1}^{\infty} \int_X \int_{B(y,2^k r) \setminus B(y,2^{k-1} r)} \exp\left(-c_4 \left(\frac{d(x,y)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) |f(y)|^p d\mu(x) d\mu(y) \\
&\leq \frac{2^p c_3}{t^{d_H/d_W}} \sum_{k=1}^{\infty} \int_X \mu(B(y,2^k r)) \exp\left(-c_4 \left(\frac{2^{(k-1)d_W} r^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) |f(y)|^p d\mu(y) \\
&\leq \frac{2^p c_3}{t^{d_H/d_W}} \sum_{k=1}^{\infty} c_2 r^{d_H} 2^{kd_H} \|f\|_{L^p}^p \exp\left(-c_4 \left(\frac{r^{d_W}}{t}\right)^{\frac{1}{d_W-1}} \left(2^{\frac{d_W}{d_W-1}}\right)^{k-1}\right) \\
&= \|f\|_{L^p}^p 2^p c_2 c_3 \sum_{k=1}^{\infty} 2^{d_H} \left(\frac{r^{d_W}}{t} 2^{d_W(k-1)}\right)^{d_H/d_W} \exp\left(-2^{-\frac{d_W}{d_W-1}} c_4 \left(\frac{r^{d_W}}{t} 2^{d_W k}\right)^{\frac{1}{d_W-1}}\right) \\
&\leq \|f\|_{L^p}^p 2^{p+d_H} c_2 c_3 \sum_{k=1}^{\infty} \int_{(r^{d_W}/t)(2^{d_W})^{k-1}}^{(r^{d_W}/t)(2^{d_W})^k} s^{d_H/d_W} \exp\left(-2^{-\frac{d_W}{d_W-1}} c_4 s^{\frac{1}{d_W-1}}\right) \frac{1}{(d_W \log 2)^s} ds \\
&= \frac{2^{p+d_H} c_2 c_3}{d_W \log 2} \|f\|_{L^p}^p \int_{r^{d_W}/t}^{\infty} s^{d_H/d_W-1} \exp\left(-2^{-\frac{d_W}{d_W-1}} c_4 s^{\frac{1}{d_W-1}}\right) ds \\
(5.8) \quad &\leq c_8 \exp\left(-c_9 \left(\frac{r^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) \|f\|_{L^p(X,\mu)}^p,
\end{aligned}$$

where  $c_9 := 2^{-1-\frac{d_W}{d_W-1}} c_4$  and  $c_8 := 2^{p+d_H} c_2 c_3 (d_W \log 2)^{-1} \int_0^{\infty} s^{d_H/d_W-1} \exp(-c_9 s^{\frac{1}{d_W-1}}) ds$ .

On the other hand, for  $B(t)$ , by (5.1) we have

$$\begin{aligned}
B(t) &\leq \frac{c_3}{t^{d_H/d_W}} \int_X \int_{B(y,r)} \exp\left(-c_4 \left(\frac{d(x,y)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) |f(x) - f(y)|^p d\mu(x) d\mu(y) \\
&\leq \frac{c_3}{t^{d_H/d_W}} \sum_{k=1}^{\infty} \int_X \int_{B(y,2^{1-k} r) \setminus B(y,2^{-k} r)} \exp\left(-c_4 \left(\frac{d(x,y)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) |f(x) - f(y)|^p d\mu(x) d\mu(y) \\
&\leq \frac{c_3}{t^{d_H/d_W}} \sum_{k=1}^{\infty} \int_X \int_{B(y,2^{1-k} r)} \exp\left(-c_4 \left(\frac{2^{-kd_W} r^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) |f(x) - f(y)|^p d\mu(x) d\mu(y) \\
&\leq c_3 \sum_{k=1}^{\infty} \frac{(2^{1-k} r)^{p\alpha+d_H}}{t^{d_H/d_W}} \exp\left(-2c_9 \left(\frac{r^{d_W}}{t} 2^{d_W(1-k)}\right)^{\frac{1}{d_W-1}}\right) \frac{1}{(2^{1-k} r)^{p\alpha+d_H}} \int_X \int_{B(y,2^{1-k} r)} |f(x) - f(y)|^p d\mu(x) d\mu(y) \\
&\leq c_3 2^{p\alpha+d_H} t^{\frac{p\alpha}{d_W}} \sup_{s \in (0,r]} N_p^\alpha(f,s)^p \sum_{k=1}^{\infty} \left(\frac{r^{d_W}}{t} 2^{-d_W k}\right)^{\frac{d_H+p\alpha}{d_W}} \exp\left(-2c_9 \left(\frac{r^{d_W}}{t} 2^{-d_W(k-1)}\right)^{\frac{1}{d_W-1}}\right) \\
(5.9) \quad &\leq c_7 t^{\frac{p\alpha}{d_W}} \sup_{s \in (0,r]} N_p^\alpha(f,s)^p.
\end{aligned}$$

As a conclusion, one has

$$\int_X \int_X |f(x) - f(y)|^p p_t(x,y) d\mu(x) d\mu(y) \leq c_8 \exp\left(-c_9 \left(\frac{r^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) \|f\|_{L^p}^p + c_7 t^{\frac{p\alpha}{d_W}} \sup_{s \in (0,r]} N_p^\alpha(f,s)^p.$$

This yields

$$\sup_{t>0} t^{-\frac{p\alpha}{d_W}} \int_X \int_X |f(x) - f(y)|^p p_t(x,y) d\mu(x) d\mu(y) \leq c_7 \sup_{s \in (0,r]} N_p^\alpha(f,s)^p + \frac{c_{10}}{r^{p\alpha}} \|f\|_{L^p(X,\mu)}^p.$$



The proof is thus complete.  $\square$

## 5.2. Co-area type estimates

In the sequel, for any  $E \subset X$  we will denote by  $E_r$  the  $r$ -neighborhood of  $E$  and define the distance from  $x \in X$  to  $E$  by  $d(x, E) := \inf_{y \in E} d(x, y)$ . We will also refer to the standing assumptions for this section, outlined at the beginning of the chapter.

**THEOREM 5.3.** *Let  $X$  be Ahlfors  $d_H$ -regular with a heat kernel that satisfies the upper sub-Gaussian estimate (5.1). For  $u \in L^1(X, \mu)$  and  $s \in \mathbb{R}$ , let  $E_s(u) := \{x \in X, u(x) > s\}$ . Assume that there is  $0 < \alpha \leq \frac{d_H}{d_W}$  and  $R > 0$  such that*

$$(5.10) \quad h(u, s) = \sup_{r \in (0, R]} \frac{1}{r^{\alpha d_W}} \mu\{x \in E_s(u) : d(x, X \setminus E_s(u)) < r\}$$

is in  $L^1(\mathbb{R}, ds)$ . Then,  $u \in \mathbf{B}^{1, \alpha}(X)$  and there exist constants  $C_1, C_2 > 0$  independent of  $u$  such that

$$(5.11) \quad \|u\|_{1, \alpha} \leq C_1 R^{-\alpha} \|u\|_{L^1(X, \mu)} + C_2 \int_{\mathbb{R}} h(u, s) ds.$$

**PROOF.** As in the proof of Theorem 4.11 we have

$$\begin{aligned} & \int_{X \times X} p_t(x, y) |u(x) - u(y)| d\mu(x) d\mu(y) \\ &= 2 \int_{\mathbb{R}} \int_X \int_X \mathbf{1}_{E_s(u)^c}(x) \mathbf{1}_{E_s(u)}(y) p_t(x, y) d\mu(x) d\mu(y) ds \\ &= 2 \int_{\mathbb{R}} \int_{E_s(u)} \int_{X \setminus E_s(u)} p_t(x, y) d\mu(y) d\mu(x) ds \end{aligned}$$

Fix  $s$  and  $t$  and decompose the integral over  $E_s(u)$  as follows: For  $j \geq 1$ , let

$$F_j := \{x \in E_s(u) : 2^{j-1} t^{1/d_W} \leq d(x, X \setminus E_s(u)) < 2^j t^{1/d_W}\}$$

and  $F_0 := \{x \in E_s(u) : d(x, X \setminus E_s(u)) < t^{1/d_W}\}$ . Evidently  $E_s(u) = \cup_0^\infty F_j$ . In the following computation we use the bound  $\int_{X \setminus E_s(u)} p_t(x, y) d\mu(y) \leq 1$  for  $x \in F_0$ . For the remaining range of  $j$  and  $x \in F_j$  we instead note that  $d(x, y) \geq 2^{j-1} t^{1/d_W}$  for any  $y \in X \setminus E_s(u)$  and integrate the sub-Gaussian estimate 5.1 over this range of radii.

$$\begin{aligned} & \int_{X \times X} p_t(x, y) |u(x) - u(y)| d\mu(x) d\mu(y) \\ & \leq 2 \int_{\mathbb{R}} \sum_{j=0}^{\infty} \int_{F_j} \int_{X \setminus E_s(u)} p_t(x, y) d\mu(y) d\mu(x) ds \\ & \leq 2 \int_{\mathbb{R}} \left( \mu(F_0) + c_3 t^{-d_H/d_W} \sum_{j=1}^{\infty} \int_{F_j} \int_{X \setminus E_s(u)} \exp\left(-c_4 \left(\frac{d(x, y)^{d_W}}{t}\right)^{1/(d_W-1)}\right) d\mu(y) d\mu(x) \right) ds \\ & \leq 2 \int_{\mathbb{R}} \left( \mu(F_0) + c_3 t^{-d_H/d_W} \sum_{j=1}^{\infty} \mu(F_j) \int_{2^{j-1} t^{1/d_W}}^{\infty} \exp\left(-c_4 \left(\frac{r^{d_W}}{t}\right)^{1/(d_W-1)}\right) \mu(B(x, r)) \frac{dr}{r} \right) ds \\ (5.12) \quad & \leq 2 \int_{\mathbb{R}} \left( \mu(F_0) + c_3 \sum_{j=1}^{\infty} \mu(F_j) \int_{2^{(j-1)d_W/(d_W-1)}}^{\infty} e^{-c_4 u} u^{(d_W-1)d_H/d_W} \frac{du}{u} \right) ds \end{aligned}$$

Our hypotheses ensure that  $\mu(F_j) \leq (2^j t^{1/d_W})^{\alpha d_W} h(u, s)$  provided  $2^j t^{1/d_W} < R$ ; for notational convenience we let the largest  $j$  satisfy this be  $j = J$  and thus obtain

$$\begin{aligned} & \sum_{j=1}^J \mu(F_j) \int_{2^{(j-1)d_W/(d_W-1)}}^{\infty} e^{-c_4 u} u^{(d_W-1)d_H/d_W} \frac{du}{u} \\ & \leq t^\alpha h(u, s) \sum_{j=1}^J 2^{j\alpha d_W} \int_{2^{(j-1)d_W/(d_W-1)}}^{\infty} e^{-c_4 u} u^{(d_W-1)d_H/d_W} \frac{du}{u} \\ & \leq t^\alpha h(u, s) \int_1^{\infty} \left( \sum_{j=1}^{\infty} 2^{j\alpha d_W} \mathbb{1}_{\{u \geq 2^{(j-1)d_W/(d_W-1)}\}} \right) e^{-c_4 u} u^{(d_W-1)d_H/d_W} \frac{du}{u} \\ & \leq \tilde{C}_1 t^\alpha h(u, s) \int_1^{\infty} e^{-c_4 u} u^{(d_W-1)(d_H+\alpha d_W)/d_W} \frac{du}{u} \\ & \leq \tilde{C}_2 t^\alpha h(u, s), \end{aligned}$$

where we note that  $\tilde{C}_2 = \tilde{C}_2(c_4, \delta, d_W, d_H)$  is independent of  $R$ . For  $j \geq J+1$  we instead use that  $2^{(j-1)d_W/(d_W-1)} \geq \tilde{C}_3 R^{d_W/(d_W-1)} t^{-1/(d_W-1)}$ . Moreover, there is  $\tilde{C}_4 = \tilde{C}_4(c_4, d_W, d_H)$  such that  $R^{d_W/(d_W-1)} t^{-1/(d_W-1)} \geq \tilde{C}_4$  implies the bound

$$\int_{2^{(j-1)d_W/(d_W-1)}}^{\infty} e^{-c_4 u} u^{(d_W-1)d_H/d_W} \frac{du}{u} \leq \tilde{C}_5 R^{d_H} t^{-d_H/d_W} \exp\left(-c_4 \tilde{C}_4 R^{d_W/(d_W-1)} t^{-1/(d_W-1)}\right).$$

Using  $\sum_{j \geq J} \mu(F_j) \leq \mu(E_s(u))$  we conclude

$$\begin{aligned} & \sum_{j=J}^{\infty} \mu(F_j) \int_{2^{(j-1)d_W/(d_W-1)}}^{\infty} e^{-c_4 u} u^{(d_W-1)d_H/d_W} \frac{du}{u} \\ & \leq \tilde{C}_5 R^{d_H} t^{-d_H/d_W} \exp\left(-c_4 \tilde{C}_4 R^{d_W/(d_W-1)} t^{-1/(d_W-1)}\right) \mu(E_s(u)). \end{aligned}$$

Combining these with the estimate  $\mu(F_0) \leq t^\alpha h(u, s)$  we obtain from (5.12) that if  $R^{\frac{d_W}{d_W-1}} t^{-\frac{1}{d_W-1}} \geq \tilde{C}_4$ ,

$$\begin{aligned} & \int_{X \times X} p_t(x, y) |u(x) - u(y)| d\mu(x) d\mu(y) \\ & \leq \int_{\mathbb{R}} \tilde{C}_2 t^\alpha h(u, s) + \tilde{C}_5 R^{d_H} t^{-d_H/d_W} \exp\left(-c_4 C R^{d_W/(d_W-1)} t^{-1/(d_W-1)}\right) \mu(E_s(u)) ds \\ & \leq t^\alpha \int_{\mathbb{R}} \tilde{C}_2 h(u, s) ds + \tilde{C}_5 R^{d_H} t^{-d_H/d_W} \exp\left(-c_4 C R^{d_W/(d_W-1)} t^{-1/(d_W-1)}\right) \|u\|_{L^1(X, \mu)} \end{aligned}$$

Finally, recall from the proof of Theorem 5.1 that

$$\|u\|_{1, \alpha} \leq 2T^{-\alpha} \|u\|_{L^1(X, \mu)} + \sup_{t \in (0, T]} t^{-\alpha} \int_{X \times X} p_t(x, y) |u(x) - u(y)| d\mu(x) d\mu(y).$$

Our estimate for the integral is valid for  $t \leq T = \tilde{C}_4^{-(d_W-1)} R^{d_W}$ , hence there is  $\tilde{C}_6$  so that

$$\begin{aligned} \|u\|_{1, \alpha} & \leq \tilde{C}_6 R^{-\alpha d_W} \|u\|_1 + \tilde{C}_2 \int_{\mathbb{R}} h(u, s) ds \\ & \quad + \sup_{t \leq T} \tilde{C}_5 R^{d_H} t^{-(\alpha d_W + d_H)/d_W} \exp\left(-c_4 \tilde{C}_4 R^{d_W/(d_W-1)} t^{-1/(d_W-1)}\right) \|u\|_{L^1(X, \mu)} \\ & \leq C_1 R^{-\alpha d_W} \|u\|_{L^1(X, \mu)} + C_2 \int_{\mathbb{R}} h(u, s) ds \quad \square \end{aligned}$$

The latter result has several useful applications. Recall that a space is uniformly locally connected if there is  $C_{LC} > 0$  such that any  $x$  and  $y$  cannot be disconnected in  $B(x, C_{LC} d(x, y))$ . Note, for example, that a geodesic space is uniformly locally connected.

COROLLARY 5.4. *Let  $X$  be a uniformly locally connected space with a heat kernel that satisfies the upper sub-Gaussian estimate (5.1) and let  $u \in L^1(X, \mu)$ . Let  $E_s(u)$  be as in Theorem 5.3 and  $\partial E_s(u)$  be its topological boundary. Suppose there is  $0 < \alpha \leq \frac{d_H}{d_W}$  and  $R > 0$  such that*

$$H(u, s) = \sup_{r \in (0, R]} \frac{1}{r^{\alpha d_W}} \mu \{x \in E_s(u) : d(x, \partial E_s(u)) < r\}$$

*is in  $L^1(\mathbb{R}, ds)$ . Then,  $u \in \mathbf{B}^{1, \alpha}(X)$  and there exist constants  $C_1, C_2 > 0$  independent of  $u$  such that*

$$(5.13) \quad \|u\|_{1, \alpha} \leq C_1 R^{-\alpha d_W} \|u\|_{L^1(X, \mu)} + C_2 \int_{\mathbb{R}} H(u, s) ds.$$

PROOF. It suffices to show that  $H(u, s)$  and  $h(u, s)$  are comparable. This follows from the inclusions

$$\begin{aligned} \{x \in E_s(u) : d(x, \partial E_s(u)) < r\} &\subset \{x \in E_s(u) : d(x, X \setminus E_s(u)) < 2r\} \\ &\subset \{x \in E_s(u) : d(x, \partial E_s(u)) < 2C_{LC}r\}, \end{aligned}$$

where  $C_{LC}$  is the local connectivity constant. The first inclusion is trivial. Let us now assume that the second inclusion fails, so that there are  $x \in E_s(u)$  and  $z \in X \setminus E_s(u)$  with  $d(x, z) < 2r$  and yet  $d(x, y) > 2C_{LC}r$  for all  $y \in \partial E_s(u)$ . Clearly neither  $x$  nor  $z$  is in  $\partial E_s(u)$ , so they are in the interior of  $E_s(u)$  and the interior of  $X \setminus E_s(u)$  respectively. These sets are disjoint and open, and they cover  $B(x, 2C_{LC}d(x, z))$  because  $d(x, y) > 2C_{LC}r$  for all  $y \in \partial E_s(u)$  implies  $B(x, 2C_{LC}d(x, z)) \cap \partial E_s(u) = \emptyset$ . Thus,  $x$  and  $z$  are disconnected in  $B(x, 2C_{LC}d(x, z))$  in contradiction to uniform local connectedness.  $\square$

From the observation that  $\partial E_s(\mathbf{1}_E) = \partial E$  if  $s \in [0, 1]$  and is empty otherwise we obtain a bound that related to the inner Minkowski content of  $E$ , see Section 5.4.1.

COROLLARY 5.5. *Let  $X$  be a uniformly locally connected space with a heat kernel that satisfies the upper sub-Gaussian estimate (5.1). If  $E \subset X$  has finite measure and for some  $0 < \alpha \leq \frac{d_H}{d_W}$*

$$\sup_{r \in (0, R]} \frac{1}{r^{\alpha d_W}} \mu \{x \in E : d(x, \partial E) < r\} < \infty,$$

*then  $\mathbf{1}_E \in \mathbf{B}^{1, \alpha}(X)$  and*

$$\|\mathbf{1}_E\|_{1, \alpha} \leq C_1 R^{-\alpha d_W} \mu(E) + C_2 \sup_{r \in (0, R]} \frac{1}{r^{\alpha d_W}} \mu \{x \in E : d(x, \partial E) < r\}.$$

REMARK 5.6. *One can drop the assumption on  $X$  of being uniformly locally connected by considering an extended metric notion of the  $r$ -boundary of a set:*

$$(5.14) \quad \tilde{\partial}_r E = (E \cap E_r^c) \cup (E^c \cap E_r)$$

*This is done in Theorems 5.3 and 6.4 when  $h_R(u, s)$  is defined in (5.10) and (6.4). This is especially natural in the context of nonlocal Dirichlet forms in Chapter 6, where spaces considered can be totally disconnected.*

The next corollary considers a situation that often applies to fractals.

COROLLARY 5.7. *Let  $X$  be a uniformly locally connected space with a heat kernel that satisfies the upper sub-Gaussian estimate (5.1) and let  $u \in L^1(X, \mu)$ . Let  $E_s(u)$  be as in Theorem 5.3 and  $\partial E_s(u)$  be its topological boundary. Assume that  $\partial E_s(u)$  is finite and  $s \mapsto |\partial E_s(u)|$  is in  $L^1(\mathbb{R})$ , where  $|\partial E_s(u)|$  denotes the cardinality of  $\partial E_s(u)$ . Then,  $u \in \mathbf{B}^{1, \frac{d_H}{d_W}}(X)$  and there exists a constant  $C > 0$  independent of  $u$  such that*

$$(5.15) \quad \|u\|_{1, d_H/d_W} \leq C \int_{\mathbb{R}} |\partial E_s(u)| ds.$$

*In particular, if  $E \subset X$  is a set of finite measure whose boundary is finite, then  $\mathbf{1}_E \in \mathbf{B}^{1, \frac{d_H}{d_W}}(X)$  and*

$$\|\mathbf{1}_E\|_{1, d_H/d_W} \leq C |\partial E|.$$

PROOF. Writing  $\partial E_s(u) = \{z_1, \dots, z_{|\partial E_s(u)|}\}$ , the Ahlfors regularity condition implies that, for any  $r > 0$ ,

$$\mu((\partial E_s(u))_r) \leq \sum_{i=1}^{|\partial E_s(u)|} \mu(B(z_i, r)) \leq |\partial E_s(u)| c_2 r^{d_H}.$$

□

REMARK 5.8. *For the lower bound in this corollary one needs extra assumptions. One nice assumption is to assume that  $u$  is a finite linear combination of indicator functions of open sets with finite boundary. This is natural for the Sierpinski gasket and similar spaces. For the Sierpinski gasket type fractals, the geometry of the fractal will force  $u$  to be totally discontinuous piecewise-constant function, and so the lower bound follows from Theorem 5.18, Corollary 5.13 or Proposition 5.23 under the mild extra assumption. This will be the subject of further work.*

REMARK 5.9. *If Corollary 5.7 is applied to the Sierpinski gasket or a similar fractal, then the function  $u$  has to be discontinuous. This is because only countably many triangles have finite boundary, and any set which is not a finite union of triangles will have infinite boundary. However, on the Vicsek set or a similar fractal, there are infinitely continuous functions for which level sets are finite. This is because the Vicsek set is a topological tree. For instance, it is easy to see that the level sets of  $n$ -harmonic on the Vicsek functions are finite. For some related general theory see [Kig95].*

PROPOSITION 5.10. *Let  $X$  be Ahlfors  $d_H$ -regular with a heat kernel that satisfies the sub-Gaussian lower estimate (5.1). For  $u \in L^1(X, \mu)$  let us assume that the level sets  $E_s(u)$  are uniformly porous at uniformly small scale, i.e. there exists  $c_u > 0$  and  $R > 0$  small such that for any  $0 < r < R$ ,  $s \in \mathbb{R}$  and  $y \in E_s(u)$  with  $d(y, X \setminus E_s(u)) < r$ , there exists  $z_y \in \partial E_s(u)$  such that*

$$(5.16) \quad B(z_y, c_u r) \subset B(y, r) \cap (X \setminus E_s(u)).$$

Furthermore, define

$$(5.17) \quad \tilde{h}(u, s) := \liminf_{r \rightarrow 0^+} \frac{1}{r^{\alpha d_W}} \mu(\{y \in E_s(u) : d(y, X \setminus E_s(u)) < r\}).$$

If  $h \in L^1(\mathbb{R})$ , then there exists  $C_u > 0$  depending on the porosity constant  $c_u$  such that

$$(5.18) \quad \|\mathbf{1}_E\|_{1, \alpha} \geq C_u \int_{\mathbb{R}} \tilde{h}(u, s) ds.$$

PROOF. Applying in the proof of Theorem 5.1 the porosity condition (5.16), for any  $t > 0$  and  $0 < r < R$  we have

$$\begin{aligned} & t^{-\alpha} \int_{\mathbb{R}} \int_X \int_X p_t(x, y) |\mathbf{1}_{E_s(u)}(x) - \mathbf{1}_{E_s(u)}(y)| \mu(dx) \mu(dy) ds \\ &= 2t^{-\alpha} \int_{\mathbb{R}} \int_X \int_X p_t(x, y) \mathbf{1}_{E_s(u)^c}(x) \mathbf{1}_{E_s(u)}(y) \mu(dx) \mu(dy) ds \\ &\geq 2c_5 t^{-\alpha - \frac{d_H}{d_W}} \exp\left(-c_6 \left(\frac{r^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) \int_{\mathbb{R}} \int_{\{y \in E_s(u) : d(y, X \setminus E_s(u)) < r\}} \mu(B(y, r) \cap (X \setminus E_s(u))) \mu(dy) ds \\ &\geq 2c_5 t^{-\alpha - \frac{d_H}{d_W}} \exp\left(-c_6 \left(\frac{r^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) \int_{\mathbb{R}} \int_{\{y \in E_s(u) : d(y, X \setminus E_s(u)) < r\}} \mu(B(z_y, c_u r)) \mu(dy) ds \\ &\geq 2c_5 t^{-\alpha - \frac{d_H}{d_W}} \exp\left(-c_6 \left(\frac{r^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) c_2 c_u^{d_H} r^{d_H + \alpha d_W} \int_{\mathbb{R}} \frac{1}{r^{\alpha d_W}} \mu(\{y \in E_s(u) : d(y, X \setminus E_s(u)) < r\}) ds. \end{aligned}$$

Thus, (with  $t = r^{d_W}$ ) for any  $0 < r < \text{diam}_d E$  it holds that

$$\begin{aligned} \|\mathbf{1}_E\|_{1,\alpha} &= \sup_{t>0} t^{-\alpha} \int_X \int_X p_t(x,y) |\mathbf{1}_E(x) - \mathbf{1}_E(y)| \mu(dx) \mu(dy) \\ &\geq 2c_5 e^{-c_6} c_2 c_u^{d_H} \int_{\mathbb{R}} \frac{1}{r^{\alpha d_W}} \mu(\{y \in E_s(u) : d(y, X \setminus E_s(u)) < r\}) ds \\ &\geq C_u \liminf_{r \rightarrow 0^+} \int_{\mathbb{R}} \frac{1}{r^{\alpha d_W}} \mu(\{y \in E_s(u) : d(y, X \setminus E_s(u)) < r\}) ds \\ &\geq C_u \int_{\mathbb{R}} \liminf_{r \rightarrow 0^+} \frac{1}{r^{\alpha d_W}} \mu(\{y \in E_s(u) : d(y, X \setminus E_s(u)) < r\}) ds, \end{aligned}$$

where last inequality is from Fatou's lemma.  $\square$

The same argument as Corollary 5.5 gives Proposition 5.10 when  $X$  is uniformly locally constant with

$$(5.19) \quad \tilde{h}(u, s) = \liminf_{r \rightarrow 0^+} \frac{1}{r^{\alpha d_W}} \mu(\{y \in E_s(u) : d(y, \partial E_s(u)) < r\})$$

that we can call the  $(d_H - \alpha d_W)$ -dimensional inner lower Minkowski content of  $\partial E_s(u)$ .

### 5.3. Sets $E$ with $\mathbf{1}_E \in \mathbf{B}^{1,\alpha}(X)$ and fractional content of boundaries

We continue working under the standing assumptions of this section which are outlined at the beginning of the chapter. As before and in the sequel, for  $E \subset X$ , we will denote its  $r$ -neighborhood by  $E_r$ . In addition, for any  $r > 0$  and  $E \subset X$  we will denote  $(\partial E)_r^- := (\partial E)_r \cap E$  and  $(\partial E)_r^+ := (\partial E)_r \cap E^c$ , where  $\partial E$  is the topological boundary of  $E$ .

**5.3.1. Several characterizations of the sets  $E$  with  $\mathbf{1}_E \in \mathbf{B}^{1,\alpha}(X)$ .** With the arguments used in the proofs of the previous section in hand we can now provide different characterizations of sets of finite  $\alpha$ -perimeter.

**THEOREM 5.11.** *Under the assumptions of Theorem 5.1, for a bounded measurable set  $E \subset X$  and  $\alpha > 0$  we consider the following properties:*

- (1)  $\mathbf{1}_E \in \mathbf{B}^{1,\alpha}(X)$ ;
- (2)  $\sup_{r>0} \frac{1}{r^{\alpha d_W + d_H}} (\mu \otimes \mu)(\{(x, y) \in E \times E^c : d(x, y) < r\}) < +\infty$ ;
- (3)  $\limsup_{r \rightarrow 0^+} \frac{1}{r^{\alpha d_W + d_H}} (\mu \otimes \mu)(\{(x, y) \in E \times E^c : d(x, y) < r\}) < +\infty$ ;
- (4)  $\limsup_{r \rightarrow 0^+} \frac{1}{r^{\alpha d_W}} \mu(\{x \in E : d(x, E^c) < r\}) < +\infty$ ;
- (5)  $\limsup_{r \rightarrow 0^+} \int_{\{x \in E : d(x, E^c) < r\}} \int_{B(x,r) \cap E^c} \frac{1}{d(x,y)^{d_H + \alpha d_W}} d\mu(y) d\mu(x) < +\infty$ ;
- (6)  $\int_E \int_{E^c} \frac{1}{d(x,y)^{d_H + \alpha d_W}} d\mu(y) d\mu(x) < +\infty$ .

Then, the following relations hold:

- (i) (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3); moreover there exist constants  $c, C > 0$  independent of  $E$  such that

$$\begin{aligned} c \sup_{r>0} \frac{1}{r^{\alpha d_W + d_H}} (\mu \otimes \mu)(\{(x, y) \in E \times E^c : d(x, y) < r\}) \\ \leq \|\mathbf{1}_E\|_{1,\alpha} \leq C \sup_{r>0} \frac{1}{r^{\alpha d_W + d_H}} (\mu \otimes \mu)(\{(x, y) \in E \times E^c : d(x, y) < r\}); \end{aligned}$$

- (ii) (4)  $\Rightarrow$  (5)  $\Rightarrow$  (1);  
 (iii) (6)  $\Rightarrow$  (1);

(iv) If in addition  $E$  is porous, i.e. there exists  $c_E > 0$  such that for any  $0 < r < \text{diam}_d E$  and  $y \in E$  with  $d(y, X \setminus E) < r$ , there exists  $z_y \in \partial E$  such that

$$(5.20) \quad B(z_y, c_E r) \subset B(y, r) \cap (X \setminus E),$$

then (1)  $\Rightarrow$  (4);

The relations stated in this theorem follow from the proof of Theorem 5.1, Theorem 5.3 and Proposition 5.10 applied to  $\mathbf{1}_E$  along with some modifications of those arguments which are presented in the next paragraph. Assuming uniformly locally connectedness on the space, which includes the case of  $X$  being geodesic, we obtain as in Corollary 5.5 a characterization of sets with finite  $\alpha$ -perimeter that involves their inner Minkowski content.

**COROLLARY 5.12.** *Let  $X$  be uniformly locally connected and satisfy the assumptions of Theorem 5.1. Then, Theorem 5.11 holds with the inner neighborhood  $(\partial E)_r^-$  in place of  $\{x \in E : d(x, E^c) < r\}$ . In particular, if  $E$  is closed, the quantity*

$$(5.21) \quad \limsup_{r \rightarrow 0^+} \frac{1}{r^{\alpha d_W}} \mu(\{x \in E : d(x, E^c) < r\})$$

becomes the inner Minkowski content of  $E$ .

In the case when a set  $E \subset X$  is porous and has a finite boundary, as for instance triangular cells in the infinite Sierpinski gasket, see Figure 1, we obtain a simpler characterization.

**COROLLARY 5.13.** *If  $(X, d)$  is uniformly locally connected, for any bounded measurable porous set  $E \subset X$  whose topological boundary  $\partial E$  is finite there exist  $c, c_E > 0$  such that*

$$(5.22) \quad c_E |\partial E| \leq \|\mathbf{1}_E\|_{1, d_H/d_W} \leq c |\partial E|,$$

where  $c_E$  is the porosity constant of  $E$ .

**PROOF.** The upper bound is Corollary 5.7. Assuming  $\partial E = \{z_1, \dots, z_{|\partial E|}\}$  and using the fact that there exists  $0 < r_E < \text{diam}_d E$  such that for any  $0 < r < r_E$ ,  $B(z_i, r)$  are pairwise disjoint with

$$(\partial E)_r \supseteq \bigcup_{i=1}^{|\partial E|} B(z_i, r),$$

the lower bound follows analogously to Proposition 5.14. □

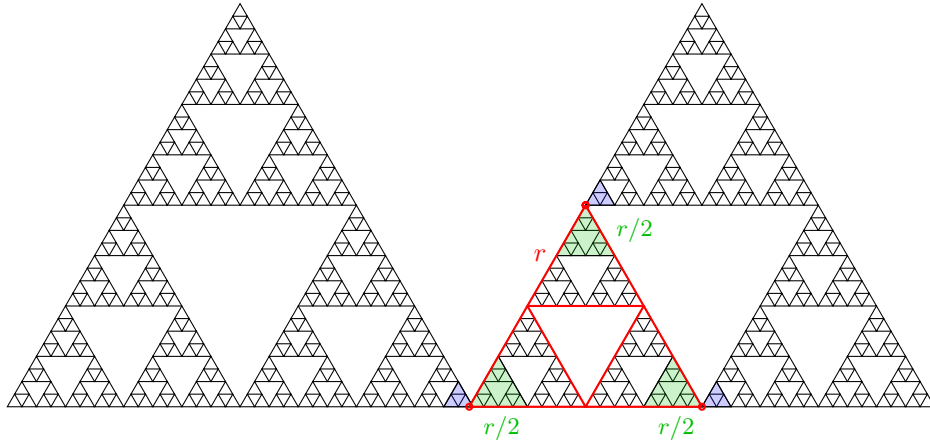


FIGURE 1. Marked red cells correspond to  $(\partial E)_r^-$ , the green colored ones to  $(\partial E)_{r/2}^-$ . In this case, any ball  $B(y, r/2)$  with  $y \in (\partial E)_{r/2}^-$  contains at least a triangular cell (colored blue) of side length  $r/4$ .

**Detailed statements and proofs for Theorem 5.11.** For the sake of clarity, in the statements in this paragraph the assumptions on the underlying space  $X$  are explicitly written.

PROPOSITION 5.14. *Let  $X$  be Ahlfors  $d_H$ -regular with a heat kernel that satisfies the sub-Gaussian lower estimate (5.1) and let  $0 < \alpha \leq \frac{d_H}{d_W}$ . For any porous set  $E \subset X$  there exists  $C_E > 0$  such that*

$$(5.23) \quad \|\mathbf{1}_E\|_{1,\alpha} \geq C_E \limsup_{r \rightarrow 0^+} \frac{1}{r^{\alpha d_W}} \mu(\{x \in E : d(x, E^c) < r\}).$$

PROOF. Applying Proposition 5.10 with  $u = \mathbf{1}_E$ , for any  $t > 0$  and  $0 < r < \text{diam}_d E$  we have

$$\begin{aligned} \|\mathbf{1}_E\|_{1,\alpha} &= \sup_{t>0} t^{-\alpha} \int_X \int_X p_t(x, y) |\mathbf{1}_E(x) - \mathbf{1}_E(y)| \mu(dx) \mu(dy) \\ &\geq 2c_5 e^{-c_6} c_2 c_E^{d_H} \frac{1}{r^{\alpha d_W}} \mu(\{y \in E : d(y, E^c) < r\}). \end{aligned}$$

Taking  $\limsup_{r \rightarrow 0^+}$  yields (5.23).  $\square$

PROPOSITION 5.15. *Let  $X$  be Ahlfors  $d_H$ -regular with a heat kernel that satisfies the sub-Gaussian upper estimate (5.1) and let  $E \subset X$  be a bounded measurable for which*

$$(5.24) \quad \limsup_{r \rightarrow 0^+} \int_{\{y \in E : d(y, E^c) < r\}} \int_{B(y, r) \cap E^c} \frac{1}{d(x, y)^{d_H + \alpha d_W}} \mu(dx) \mu(dy) < \infty$$

for some  $\alpha > 0$ . Then,  $\mathbf{1}_E \in \mathbf{B}^{1,\alpha}(X)$ .

PROOF. In order to bound  $\|\mathbf{1}_E\|_{1,\alpha}$  we follow the proof of Theorem 5.1 with some small modifications. Using the same notation, for any fixed  $t, r > 0$  we have, on the one hand

$$A(t, r) \leq c_8 \exp\left(-c_9 \left(\frac{r^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) \|\mathbf{1}_E\|_{L^1(X, \mu)} = c_8 \exp\left(-c_9 \left(\frac{r^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) \mu(E),$$

with the corresponding constants in the mentioned proof. On the other hand, following the estimate for  $B(t)$  we get

$$\begin{aligned} B(t, r) &= 2 \int_X \int_{B(y, r)} \mathbf{1}_{E^c}(x) \mathbf{1}_E(y) p_t(x, y) \mu(dx) \mu(dy) \\ &\leq \frac{2c_3}{t^{d_H/d_W}} \sum_{k=1}^{\infty} \int_X \int_{B(y, 2^{1-k}r)} \exp\left(-c_4 \left(\frac{(2^{-k}r)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) \mathbf{1}_{E^c}(x) \mathbf{1}_E(y) \mu(dx) \mu(dy) \\ &\leq 2c_3 \sum_{k=1}^{\infty} \frac{(2^{1-k}r)^{\alpha d_W + d_H}}{t^{d_H/d_W}} \exp\left(-c_4 2^{-\frac{d_W}{d_W-1}} \left(\frac{(2^{1-k}r)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) \\ &\quad \times \frac{1}{(2^{1-k}r)^{\alpha d_W + d_H}} \int_X \int_{B(y, 2^{1-k}r)} \mathbf{1}_{E^c}(x) \mathbf{1}_E(y) \mu(dx) \mu(dy) \\ &\leq 2c_3 t^\alpha \int_E \int_{B(y, r) \cap E^c} \frac{1}{d(x, y)^{\alpha d_W + d_H}} \mu(dx) \mu(dy) \sum_{k=1}^{\infty} \left(\frac{2^{1-k}r}{t^{1/d_W}}\right)^{\alpha d_W + d_H} \exp\left(-2c_9 \left(\frac{(2^{1-k}r)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) \\ &\leq c_{10} t^\alpha \int_E \int_{B(y, r) \cap E^c} \frac{1}{d(x, y)^{\alpha d_W + d_H}} \mu(dx) \mu(dy) \\ &= c_{10} t^\alpha \int_{\{y \in E : d(y, E^c) < r\}} \int_{B(y, r) \cap E^c} \frac{1}{d(x, y)^{\alpha d_W + d_H}} \mu(dx) \mu(dy), \end{aligned}$$

where  $c_{10} := c_3 2^{\alpha d_W + d_H + 1} (2d_W \log 2)^{-1} \int_0^\infty s^{\alpha + \frac{d_H}{d_W}} \exp(-2c_9 s^{\frac{1}{d_W - 1}}) ds$ . For any  $t > 0$  it thus holds that

$$\begin{aligned} & t^{-\alpha} \int_X \int_X p_t(x, y) |\mathbf{1}_E(x) - \mathbf{1}_E(y)| \mu(dx) \mu(dy) \\ & \leq t^{-\alpha} c_8 \exp\left(-c_9 \left(\frac{r^{d_W}}{t}\right)^{\frac{1}{d_W - 1}}\right) \mu(E) + c_{10} \int_{\{y \in E: d(y, E^c) < r\}} \int_{B(y, r) \cap E^c} \frac{1}{d(x, y)^{\alpha d_W + d_H}} \mu(dx) \mu(dy). \end{aligned}$$

By assumption, we can now choose  $r_\varepsilon > 0$  small enough so that

$$\sup_{t > 0} t^{-\alpha} \int_X \int_X p_t(x, y) |\mathbf{1}_E(x) - \mathbf{1}_E(y)| \mu(dx) \mu(dy) \leq \frac{c_{11}}{r_\varepsilon^{\alpha d_W}} \mu(E) + c_{10} c_\varepsilon < \infty$$

for some  $c_\varepsilon > 0$ . □

**PROPOSITION 5.16.** *Let  $X$  be Ahlfors  $d_H$ -regular with a heat kernel that satisfies the sub-Gaussian upper estimate (5.1) and let  $E \subset X$  be a bounded measurable for which*

$$(5.25) \quad \limsup_{r \rightarrow 0^+} \frac{1}{r^{\alpha d_W}} \mu(\{y \in E: d(y, E^c) < r\}) < \infty$$

for some  $0 \leq \alpha \leq \frac{d_H}{d_W}$ . Then, (5.24) holds for that set  $E$ .

**PROOF.** Let  $E \subset X$  be a bounded measurable set. For any  $r > 0$ , write

$$\begin{aligned} & \int_{\{y \in E: d(y, E^c) < r\}} \int_{B(y, r) \cap E^c} \frac{1}{d(x, y)^{d_H + \alpha d_W}} \mu(dx) \mu(dy) \\ & = \int_0^\infty (\mu \otimes \mu)(\{(x, y) \in E \times E^c: d(x, y) < r, d(x, y) < s^{-(d_H + \alpha d_W)^{-1}}\}) ds. \end{aligned}$$

Since  $r < s^{-(d_H + \alpha d_W)^{-1}}$  if and only if  $s < r^{-(d_H + \alpha d_W)}$ , the latter integral equals

$$\begin{aligned} & \int_0^{r^{-(d_H + \alpha d_W)}} (\mu \otimes \mu)(\{(x, y) \in E \times E^c: d(x, y) < r\}) ds \\ & + \int_{r^{-(d_H + \alpha d_W)}}^\infty (\mu \otimes \mu)(\{(x, y) \in E \times E^c: d(x, y) < s^{-(d_H + \alpha d_W)^{-1}}\}) ds \\ & = \int_0^{r^{-(d_H + \alpha d_W)}} \int_{\{y \in E: d(y, E^c) < r\}} \mu(B(x, r) \cap E^c) \mu(dx) ds \\ & + \int_{r^{-(d_H + \alpha d_W)}}^\infty \int_{\{y \in E: d(y, E^c) < s^{-(d_H + \alpha d_W)^{-1}}\}} \mu(B(y, s^{-(d_H + \alpha d_W)^{-1}}) \cap E^c) \mu(dy) ds. \end{aligned}$$

The Ahlfors regularity now yields

$$\begin{aligned} & \int_{\{y \in E: d(y, E^c) < r\}} \int_{B(y, r) \cap E^c} \frac{1}{d(x, y)^{d_H + \alpha d_W}} \mu(dx) \mu(dy) \\ & \leq \frac{c_2}{r^{\alpha d_W}} \mu(\{y \in E: d(y, E^c) < r\}) \\ & + c_2 \int_{r^{-(d_H + \alpha d_W)}}^\infty \mu(\{y \in E: d(y, E^c) < s^{-(d_H + \alpha d_W)^{-1}}\}) s^{-\frac{d_H}{d_H + \alpha d_W}} ds. \end{aligned}$$

Letting  $r \rightarrow 0^+$ ,  $r^{-(d_H + \alpha d_W)} \rightarrow \infty$  and (5.24) follows from (5.25). □



**5.3.2. Further results.** For the following result the examples include the finite and infinite Sierpinski gasket (Figure 4 and [BP88]) and, more generally, finite and infinite nested fractals, [Lin90, Bar98, Bar03, Bar13]. Other examples can be constructed in the class of p.c.f. self-similar sets, [Kig01, Kig12], fractafolds [ST12, Str03a] and, more generally, finitely ramified cell structures [Tep08]. To obtain similar results for fractals that are not finitely ramified, one can use the methods of [BB99, BBKT10, KY92, Kaj10].

**THEOREM 5.17.** *Let  $E \subseteq X$  be a measurable set and let  $p_t(x, y)$  satisfy the upper heat kernel estimate in (5.1). If there exists  $0 < \delta \leq d_H$  such that for all  $\varepsilon > 0$*

$$(5.26) \quad (\mu \times \mu)(\{(x, y) \in E^c \times E : d(x, y) < \varepsilon\}) \leq c_7 (\varepsilon^{d_H + \delta} + \varepsilon^{2d_H}),$$

then for all  $t > 0$  it holds that

$$(5.27) \quad \left(t^{-\delta/d_W} + t^{-d_H/d_W}\right) \|P_t \mathbf{1}_E - \mathbf{1}_E\|_{L^1(X, \mu)} \leq c_{11} < \infty,$$

where  $c_{11}$  is a constant given by (5.33) that depends on  $\delta, d_H$  and  $d_W$ .

**THEOREM 5.18.** *Let  $E \subseteq X$  be a measurable set and let  $p_t(x, y)$  satisfy the lower heat kernel estimate in (5.1). If there exist  $\delta > 0$  and  $\varepsilon > 0$  such that*

$$(5.28) \quad (\mu \times \mu)(\{(x, y) \in E^c \times E : d(x, y) < \varepsilon\}) \geq c_8 \varepsilon^{d_H + \delta},$$

then, for  $t = \varepsilon^{d_W}$ , it holds that

$$(5.29) \quad t^{-\delta/d_W} \|P_t \mathbf{1}_E - \mathbf{1}_E\|_{L^1(X, \mu)} \geq 2c_5 c_8 e^{-c_6} > 0.$$

**COROLLARY 5.19.** *If there exists  $\delta > 0$  and  $\varepsilon_0 > 0$  such that (5.28) holds for any  $0 < \varepsilon < \varepsilon_0$ , then (5.29) holds for any  $0 < t < \varepsilon^{d_W}$ .*

Localized version of Theorem 5.17.

**THEOREM 5.20.** *Let  $p_t(x, y)$  satisfy the upper heat kernel estimate (6.1) and let  $E \subset B \subset X$  be bounded measurable set. If there exists  $0 < \delta \leq d_H$  such that for all  $\varepsilon > 0$*

$$(5.30) \quad (\mu \times \mu)(\{(x, y) \in (B \cap E^c) \times E : d(x, y) < \varepsilon\}) \leq c_7 \varepsilon^{d_H + \delta},$$

then for all  $t > 0$  it holds that

$$(5.31) \quad t^{-\delta/d_W} \|P_t \mathbf{1}_E - \mathbf{1}_E\|_{L^1(B, \mu)} \leq c_{11} < \infty,$$

where  $c_{11}$  is a constant given by (5.35) that depends on  $\delta, d_H$  and  $d_W$ .

**REMARK 5.21.**

- (1) In Theorem 5.17,  $d_H$  controls the behavior of the left hand side in (5.26) when  $\varepsilon$  is large. For small  $t$  the leading term is  $t^{-\delta/d_W}$ .
- (2) The constant  $\delta$  plays the role of the (upper) co-dimension of the boundary and the constant  $c_7$  plays the role of the (possibly fractal) upper Minkowski content corresponding to that dimension (see Proposition 5.22, [Fal03, Mat95]). There is an extensive literature on the subject for fractal subsets  $E$  of a Euclidean space, such as [ADMG17, WZ13, Win15, PW14, RZ12, FK12, LPW11, and references therein]. We note that, although [BH97] does not explicitly refer to the Minkowski content, it appears in the proof of the main theorem as  $\mu^+(A)$ .
- (3) In Theorem 5.18 the constant  $\delta$  also plays the role of a (lower) co-dimension of the boundary and the constant  $c_8$  plays the role of a weaker version of the (possibly fractal) lower Minkowski content corresponding to that dimension. A useful sufficient condition for (5.28) is given in Proposition 5.23.
- (4) The values of the constant  $\delta$  can be different in Theorem 5.17 and in Theorem 5.18, so one can speak about an upper-co-dimension  $\bar{\delta}$  and about a lower-co-dimension  $\underline{\delta}$ . In general we have  $\bar{\delta} \geq \underline{\delta}$ , but in most examples we have  $\bar{\delta} = \underline{\delta}$ .

PROOF OF THEOREM 5.17. Using the fact that the semigroup  $P_t$  is conservative, we can rewrite

$$\|P_t \mathbf{1}_E - \mathbf{1}_E\|_{L^1(X, \mu)} = \|\mathbf{1}_{E^c} P_t \mathbf{1}_E\|_{L^1(X, \mu)} + \|\mathbf{1}_E P_t \mathbf{1}_{E^c}\|_{L^1(X, \mu)} = 2\|\mathbf{1}_{E^c} P_t \mathbf{1}_E\|_{L^1(X, \mu)}.$$

For any fixed  $t > 0$ , since  $p_t(x, y) \geq 0$ , we have that

$$\|\mathbf{1}_{E^c} P_t \mathbf{1}_E\|_{L^1(X, \mu)} = \int_{E^c} \int_E p_t(x, y) \mu(dy) \mu(dx) = \int_0^\infty \mu \otimes \mu(\{(x, y) \in E^c \times E : p_t(x, y) > s\}) ds.$$

Moreover, in view of (5.1),  $p_t(x, y) > s$  implies

$$d(x, y) \leq t^{\frac{1}{d_W}} \left( -c_4^{-1} \log(c_3^{-1} s t^{\frac{d_H}{d_W}}) \right)^{\frac{d_W-1}{d_W}} =: F(t, s)$$

and  $F(t, s) > 0$  if and only if  $s < c_5 t^{-\frac{d_H}{d_W}}$ . Therefore, we obtain

$$(5.32) \quad \|\mathbf{1}_{E^c} P_t \mathbf{1}_E\|_{L^1(X, \mu)} \leq c_7 \int_0^{c_3 t^{-d_H/d_W}} (F(t, s))^{d_H+\delta} ds + c_7 \int_0^{c_3 t^{-d_H/d_W}} (F(t, s))^{2d_H} ds.$$

The first integral can be estimated by

$$\begin{aligned} c_7 \int_0^{c_3 t^{-d_H/d_W}} t^{\frac{d_H+\delta}{d_W}} \left( -c_4^{-1} \log(c_5^{-1} s t^{\frac{d_H}{d_W}}) \right)^{\frac{(d_W-1)(d_H+\delta)}{d_W}} ds \\ \leq c_7 c_3 t^{\frac{\delta}{d_W}} \int_0^1 \left( -c_4^{-1} \log u \right)^{\frac{(d_W-1)(d_H+\delta)}{d_W}} du \\ = c_7 c_3 c_4^{-(d_W-1)(d_H+\delta)/d_W} \Gamma\left(\frac{(d_W-1)(d_H+\delta)}{d_W}\right) t^{\frac{\delta}{d_W}}, \end{aligned}$$

where we have used the substitution  $u = c_3^{-1} s t^{\frac{d_H}{d_W}}$ . A similar estimate is obtained for the second integral in (5.32) by substituting  $\delta$  by  $d_H$ . Thus, (5.27) holds with

$$(5.33) \quad c_{11} = 2c_3 c_7 \left\{ \Gamma\left(\frac{(d_W-1)(d_H+\delta)}{d_W}\right) c_4^{-\frac{(d_W-1)(d_H+\delta)}{d_W}} + \Gamma\left(\frac{(d_W-1)(d_H+d_H)}{d_W}\right) c_4^{-\frac{2(d_W-1)d_H}{d_W}} \right\}.$$

□

PROOF OF THEOREM 5.18. For each fixed  $t > 0$ , define

$$(5.34) \quad A_{E, t^{1/d_W}} := \{(x, y) \in E^c \times E \mid d(x, y) \leq t^{1/d_W}\}.$$

Due to the lower estimate (5.1),  $p_t(x, y) \geq c_5 \exp(-c_6) t^{-d_H/d_W}$  for any  $(x, y) \in A_{E, t^{1/d_W}}$ . Assumption (5.28) now yields

$$\begin{aligned} \|\mathbf{1}_{E^c} P_t \mathbf{1}_E\|_{L^1(X, \mu)} &= \int_E \int_{E^c} p_t(x, y) \mu(dx) \mu(dy) \geq \int_{A_{E, t^{1/d_W}}} p_t(x, y) \mu(dx) \mu(dy) \\ &\geq c_5 e^{-c_6} t^{-\frac{d_H}{d_W}} \mu \times \mu(A_{E, t^{1/d_W}}) \geq c_5 e^{-c_6} t^{-\frac{d_H}{d_W}} c_8 t^{\frac{(d_H+\delta)}{d_W}} = c_5 c_8 e^{-c_6} t^{\frac{\delta}{d_W}} \end{aligned}$$

as we wanted to prove. □

PROOF OF THEOREM 5.20. As in Theorem 5.17, it suffices to show  $t^{-\delta/d_W} \|\mathbf{1}_{E^c} P_t \mathbf{1}_E\|_{L^1(B, \mu)} \leq c_{11}$ . Moreover, we have that  $p_t(x, y) > s$  implies  $d(x, y) < F(t, s)$  as in (6.10) and  $F(s, t) > 0$  if and only if

$s < c_3 t^{-d_H/d_W}$ . Thus,

$$\begin{aligned} \|\mathbf{1}_{E^c} P_t \mathbf{1}_E\|_{L^1(B,\mu)} &= \int_{B \cap E^c} \int_E p_t(x,y) \mu(dy) \mu(dx) \\ &= \int_0^\infty \mu \times \mu(\{(x,y) \in (B \cap E^c) \times E \mid p_t(x,y) > s\}) ds \\ &\leq \int_0^\infty \mu \times \mu(\{(x,y) \in (B \cap E^c) \times E \mid d(x,y) < F(t,s)\}) ds \\ &= \int_0^\infty \mu \times \mu(\{(x,y) \in E^c \times E \mid d(x,y) \leq \int_0^{c_3 t^{-d_H/d_W}} \min\{F(t,s), d(E,B)\}\} ds, \end{aligned}$$

where  $d(E,B) := \sup_{x \in E} \inf_{y \in B} d(x,y)$ . Applying (5.30) and estimating the integral as in the proof of Theorem 5.17 yields

$$\|\mathbf{1}_{E^c} P_t \mathbf{1}_E\|_{L^1(B,\mu)} \leq c_7 c_3 t^{\frac{\delta}{d_W}} c_4^{-\frac{(d_W-1)(d_H+\delta)}{d_W}} \Gamma\left(\frac{(d_W-1)(d_H+\delta)}{d_W}\right) t^{\frac{\delta}{d_W}}$$

and (5.31) holds with

$$(5.35) \quad c_{11} = c_7 c_3 c_4^{-\frac{(d_W-1)(d_H+\delta)}{d_W}} \Gamma\left(\frac{(d_W-1)(d_H+\delta)}{d_W}\right) t^{\frac{\delta}{d_W}}.$$

□

We give next a sufficient condition for (5.26) when  $X$  is an Ahlfors  $d_H$ -regular geodesic space.

**PROPOSITION 5.22.** *Let  $X$  be Ahlfors  $d_H$ -regular and geodesic. If  $E \subseteq X$  has a compact boundary  $\partial E$  and for any  $\varepsilon > 0$  the measure of the  $\varepsilon$ -neighborhood  $(\partial E)_\varepsilon$  of  $\partial E$  satisfies*

$$(5.36) \quad \mu((\partial E)_\varepsilon) = O(\varepsilon^\delta)_{\varepsilon \rightarrow 0^+}$$

then (5.26) holds.

**PROOF.** Since the space  $X$  is geodesic, for any  $x \in E$  and  $y \in E^c$  there exists  $z = z(x,y) \in \partial E$  such that  $d(x,y) = d(x,z) + d(z,y)$ . Therefore  $x, y \in (\partial E)_\varepsilon$  if  $x \in E$ ,  $y \in E^c$  and  $d(x,y) < \varepsilon$ , so that

$$\begin{aligned} (\mu \otimes \mu)(\{(x,y) \in E^c \times E : d(x,y) < \varepsilon\}) &= \int_{E^c} \int_E \mathbf{1}_{B(x,\varepsilon)}(y) \mu(dx) \mu(dy) \\ &\leq \int_{(\partial E)_\varepsilon} \int_E \mathbf{1}_{B(x,\varepsilon)}(y) \mu(dx) \mu(dy) \leq \mu((\partial E)_\varepsilon) c_2 \varepsilon^{d_H}. \end{aligned}$$

Due to (5.36) and the fact that

$$\mu((\partial E)_\varepsilon) = O(\varepsilon^{d_H})_{\varepsilon \rightarrow \infty}$$

because  $\partial E$  is compact, the proof is complete. □

Without assuming the underlying space  $X$  to be geodesic we can give a sufficient condition ensuring (5.28) which by Theorem 5.11 provides a lower bound of the  $\alpha$ -perimeter of the set  $E \subseteq X$ .

**PROPOSITION 5.23.** *Let  $X$  be an Ahlfors  $d_H$ -regular space. Suppose that  $E \subseteq X$  has a compact boundary  $\partial E$  and for all  $n \geq 1$  there exist  $r_n > 0$  and  $x_1, \dots, x_n \in \partial E$  such that  $B_{n,k} := B(x_k, \frac{r_n}{2}) \subseteq X$ ,  $k = 1, \dots, n$ , are disjoint and satisfy*

$$(5.37) \quad \mu(B_{n,k} \cap E) \geq c_9 r_n^{d_H} \quad \text{and} \quad \mu(B_{n,k} \cap E^c) \geq c_{10} r_n^{d_H}$$

for all  $k = 1, \dots, n$ . Furthermore, assume that for some  $c_{12} > 0$  and  $d_H > \dim(\partial E) \geq 0$  we have

$$(5.38) \quad n \geq c_{12} \cdot r_n^{-\dim(\partial E)}.$$

Then, (5.28) holds with  $c_8 = c_9 c_{10} c_{12}$ ,  $\varepsilon = r_n$  and  $\delta = d_H - \dim(\partial E)$ .

PROOF. Notice that

$$\bigcup_{X=1}^n (B_{n,k} \cap E^c) \times (B_{n,k} \cap E) \subseteq \{(x, y) \in E^c \times E : d(x, y) < \varepsilon\}.$$

In view of (5.37) we thus have that

$$\mu \otimes \mu(\{(x, y) \in E^c \times E : d(x, y) < r_n\}) \geq \sum_{k=1}^n \mu(B_{n,k} \cap E) \mu(B_{k,n} \cap E^c) \geq n c_9 c_{10} r_n^{2d_H}.$$

□

REMARK 5.24. Note that (5.37) means that both  $E$  and  $E \times E$  are Ahlfors  $d_H$ -regular. We do not specifically use what notion of dimension  $\dim_H(\partial E)$  of  $\partial E$  is the most appropriate in this proposition, but usually it is the Hausdorff dimension, which in most of examples coincides with the box counting, Minkowski, and fractal self-similarity dimensions (see e.g. [Fal03, Section 3.1]).

If  $\{B_{n,k}\}_{X=1}^n$  is a disjoint cover of  $\partial E$  by sets of diameter  $r_n$ , it follows that  $n \sim r_n^{-\dim_B \partial E}$  see e.g. [Fal03, Section 3.1], where  $\dim_B$  stands for box dimension. Due to the fact that  $\dim_H \partial E \leq \dim_B \partial E$ , see [Fal03, p.46] and  $r_n$  is small, we can assume  $r_n < 1$ , hence  $n \geq c_{12} r_n^{-\dim_H \partial E}$  and

$$(\mu \otimes \mu)(\{(x, y) \in E^c \times E : d(x, y) < r_n\}) \geq c_{11} r_n^{2d_H - \dim_H \partial E} = c_{12} r_n^{d_H + \delta}.$$

#### 5.4. Density of $\mathbf{B}^{1,\alpha}(X)$ in $L^1(X, \mu)$

A noteworthy application of the results yielding Theorem 5.11 is that it actually provides sufficient conditions to ensure that  $\mathbf{B}^{1,\alpha}(X)$  is dense in  $L^1(X, \mu)$ .

THEOREM 5.25. Let  $X$  be Ahlfors  $d_H$ -regular geodesic space with sub-Gaussian upper heat kernel estimate and  $\alpha > 0$ . Assume that there is a family of bounded open sets  $E \subset X$  which generates the topology of  $X$  and such that

$$(5.39) \quad \limsup_{r \rightarrow 0^+} \frac{1}{r^{\alpha d_W}} \mu(\{x \in E : d(x, E^c) < r\}) < \infty$$

for some  $0 < \alpha \leq \frac{d_H}{d_W}$ . Then,  $\mathbf{B}^{1,\alpha}(X)$  is dense in  $L^1(X, \mu)$ .

PROOF. By Theorem 5.11 and Corollary 5.13 respectively,  $\mathbf{1}_E \in \mathbf{B}^{1,\alpha}(X)$  holds for bounded open sets and thus for simple functions. Since these are dense in  $L^1(X, \mu)$  the assertion follows. □

EXAMPLE 5.26. (i) If  $X$  is the infinite Sierpinski gasket as in [BP88] (see also Section 5.6.1), a suitable family of open sets consists of (open) triangular cells of any side length. For each such set  $E \subset X$ , the boundary  $\partial E$  consists of three points and  $(\partial E)_r^-$  is the union of three triangular cells of side length  $r$ . Hence,  $\mu(\partial E)_r^- = cr^{d_H}$  and Theorem 5.25 holds with  $\alpha = \frac{d_H}{d_W}$ .

(ii) A similar result as for the Sierpinski gasket can be expected for any nested fractal [Lin90, Bar98].

REMARK 5.27. When  $X$  is the Sierpinski carpet or a similar fractal the situation is more delicate. We mention some points here; a complete answer remains open.

(i) We believe that in this case Theorem 5.25 will not be applicable because there may not be such family of open sets. Intuitively and in view of Example 5.30, one expects open sets  $E \subset X$  to asymptotically fulfill  $\mu(\partial E)_r^- \sim r^{d_H - \dim_H \partial E}$ . Therefore, only sets of finite (or countable) boundary would produce the critical exponent but these may not generate the topology of  $X$ . In fact, in the Sierpinski carpet or the Menger sponge, there are no non-empty open sets with non-empty finite boundary [BB99]. This is true for any generalized Sierpinski carpet in the sense of Barlow and Bass.

(ii) It may still be true that  $\mathbf{B}^{1, \frac{d_H}{d_W}}(X)$  is dense in  $L^1(X, \mu)$ , only dense functions will not be generated by indicator functions of a base of open sets.

**5.4.1. Examples.** Let us now discuss several examples of different nature where the previous results apply. In particular, we can identify sets of finite  $\alpha$ -perimeter for the corresponding ranges of the parameter  $\alpha$ .

EXAMPLE 5.28 (Smooth domains in the  $d$ -dimensional Euclidean space). Consider  $X = \mathbb{R}^d$ ,  $d_H = d$ ,  $d_W = 2$ ,  $\mu = \lambda^d$  the  $d$ -dimensional Lebesgue measure. A smooth domain  $E$  with  $\partial E$  closed and  $(d-1)$ -rectifiable satisfies

$$(5.40) \quad \limsup_{r \rightarrow 0^+} \frac{1}{r} \lambda^d((\partial E)_r) = \frac{2\lambda^{d-1}(\partial E)\Gamma(\frac{1}{2})^{d-1}}{(d-1)\Gamma(\frac{d-1}{2})}$$

because the  $(d-1)$ Minkowski content of  $\partial E$  coincides with  $\lambda^{d-1}(\partial E)$ , see e.g. [Fed69, Theorem 3.2.39]. Since  $\mathbb{R}^d$  is geodesic, Theorem 5.11 (4) is satisfied with  $\alpha_{d_W} = 1$  and hence  $\mathbf{1}_E \in \mathbf{B}^{1,1/2}(\mathbb{R}^d)$ .

EXAMPLE 5.29 (Rectifiable domains in the  $d$ -dimensional Euclidean space). More generally, if  $\partial E \subset \mathbb{R}^d$  is closed and  $m$ -rectifiable, again by [Fed69, Theorem 3.2.39] we have

$$(5.41) \quad \limsup_{r \rightarrow 0^+} \frac{1}{r^{d-m}} \lambda^d((\partial E)_r) = \frac{2\lambda^m(\partial E)\Gamma(\frac{1}{2})^m}{m\Gamma(\frac{m}{2})}.$$

Thus, Theorem 5.11 (4) is satisfied with  $\alpha_{d_W} = d - m$ , which is the codimension of  $\partial E$  and  $\mathbf{1}_E \in \mathbf{B}^{1, \frac{d-m}{2}}(\mathbb{R}^d)$ .

For instance, if  $E$  is the so-called Koch snowflake domain in  $\mathbb{R}^2$  whose boundary is displayed in Figure 2, then  $d_H = 2$ ,  $d_W = 2$  and  $m = \frac{\log 4}{\log 3}$ . More precisely, from [LP06, Theorem 1] we have the tube formula

$$(5.42) \quad \lambda^2((\partial E)_r^-) = G_1(r)r^{2-\frac{\log 4}{\log 3}} + G_2(r)r^2,$$

where  $G_1, G_2$  are periodic functions. Thus, in this case Theorem 5.11 yields  $\mathbf{1}_E \in \mathbf{B}^{1,\alpha}(X)$  with  $\alpha = 1 - \frac{\log 4}{2\log 3}$ .

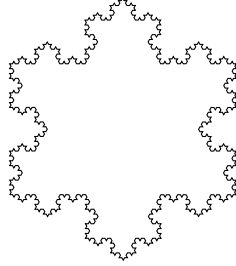


FIGURE 2. von Koch snowflake domain.

EXAMPLE 5.30 (Subsets of the Sierpinski carpet). Consider  $X$  to be the Sierpinski carpet built in the standard way on the unit square  $[0, 1]^2$ , so that  $d_H = \frac{\log 8}{\log 3}$  and  $d_W =$  unknown but unique  $> 2$ .

(1) Let  $E = X \cap [0, \frac{1}{3}]^2$  be a sub-square cell in the standard self-similar construction of  $X$ . Setting  $r = 3^{-k}$ , an inner neighborhood of the boundary consists of  $2 \cdot 3^k$  copies of a carpet  $X_r$  with side length  $r$ . Thus,

$$\mu((\partial E)_r^-) = 2 \cdot 3^k \mu(X_r) = 2 \cdot 3^k r^{\frac{\log 8}{\log 3}} = 2r^{\frac{\log 8}{\log 3} - 1}.$$

Theorem 5.11 (4) is satisfied with  $\alpha_{d_W} = \frac{\log 8}{\log 3} - 1$  and hence  $\mathbf{1}_E \in \mathbf{B}^{1, \frac{\delta}{d_W}}(X)$  with  $\delta = \frac{\log 8}{\log 3} - 1$ .

(2) If  $E = X \cap ([0, \frac{1}{2}] \times [0, 1])$  is the left half of the carpet  $X$ , a similar computation as the previous case yields  $\mathbf{1}_E \in \mathbf{B}^{1, \frac{\delta}{d_W}}(X)$  with  $\delta = \frac{\log 8}{\log 3} - \frac{\log 2}{\log 3} = \frac{2\log 2}{\log 3}$ .

Notice that in each case, the dimension of the boundary of  $E$  was 1 or  $\frac{\log 2}{\log 3}$  respectively, supporting the conjecture from Remark 5.27.

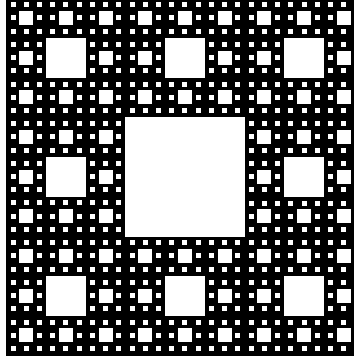


FIGURE 3. Sierpinski carpet

EXAMPLE 5.31 (Subsets of the Menger sponge). Let  $X$  be the standard Menger sponge on the unit cube  $[0, 1]^3$ . Here,  $d_H = \frac{\log 20}{\log 3}$  and  $d_W = \text{unknown but unique} > 2$ .

- (1) For  $E = X \cap [0, \frac{1}{3}]^3$  is a sub-cube cell in the standard self-similar construction of  $X$ , then  $\mathbf{1}_E \in \mathbf{B}^{1, \frac{\delta}{d_W}}(X)$  with  $\delta = \frac{\log 20}{\log 3} - 2$ .
- (2) For the left half of the cubical sponge,  $E = X \cap ([0, \frac{1}{2}] \times [0, 1]^2)$ , we have  $\mathbf{1}_E \in \mathbf{B}^{1, \frac{\delta}{d_W}}(X)$  with  $\delta = \frac{\log 20}{\log 3} - \frac{\log 4}{\log 3} = \frac{\log 5}{\log 3}$ .

### 5.5. Weak Bakry-Émery curvature condition $BE(\kappa)$

DEFINITION 5.32. We will say that  $(X, \mu, \mathcal{E}, \mathcal{F})$  satisfies the weak Bakry-Émery non-negative curvature condition  $BE(\kappa)$  if there exist a constant  $C > 0$  and a parameter  $0 < \kappa < d_W$  such that for every  $t > 0$ ,  $g \in L^\infty(X, \mu)$  and  $x, y \in X$ ,

$$(5.43) \quad |P_t g(x) - P_t g(y)| \leq C \frac{d(x, y)^\kappa}{t^{\kappa/d_W}} \|g\|_{L^\infty(X, \mu)}.$$

Several examples of spaces satisfying the condition  $BE(\kappa)$  will be given later. We first observe that the weak Bakry-Émery is related to the Hölder regularity of the heat kernel.

LEMMA 5.33. Assume that  $(X, \mu, \mathcal{E}, \mathcal{F})$  satisfies the weak Bakry-Émery condition  $BE(\kappa)$ . Then, there exist a constant  $C > 0$  such that for every  $t > 0$ ,  $x, y, z \in X$ ,

$$|p_t(x, z) - p_t(y, z)| \leq C \frac{d(x, y)^\kappa}{t^{(\kappa+d_H)/d_W}}$$

PROOF. Indeed, it is easily seen that the weak Bakry-Émery estimate is equivalent to:

$$\int_X |p_t(x, z) - p_t(y, z)| d\mu(z) \leq C \frac{d(x, y)^\kappa}{t^{\kappa/d_W}}.$$

One has then

$$\begin{aligned} |p_t(x, z) - p_t(y, z)| &= \left| \int_X (p_{t/2}(x, u) - p_{t/2}(y, u)) p_{t/2}(u, z) d\mu(u) \right| \\ &\leq \frac{C}{t^{d_H/d_W}} \int_X |p_t(x, z) - p_t(y, z)| d\mu(z) \end{aligned}$$

□

Under the weak Bakry-Émery curvature condition, the main result is the following. It can be thought as an analogue of Lemma 4.10 in a situation where we do not necessarily have a carré du champ operator.

**THEOREM 5.34.** *Assume that the weak Bakry-Émery curvature condition (5.43) is satisfied. Then, for every  $f \in \mathbf{B}^{1,1-\frac{\kappa}{d_W}}(X)$ , and  $t \geq 0$ ,*

$$\|P_t f - f\|_{L^1(X,\mu)} \leq C t^{1-\frac{\kappa}{d_W}} \limsup_{r \rightarrow 0} \frac{1}{r^{d_H+d_W-\kappa}} \iint_{\Delta_r} |f(x) - f(y)| d\mu(x) d\mu(y)$$

**PROOF.** In the proof  $C$  will denote a constant (depending only on  $\kappa, d_W, d_H$ ) that may change from line to line. Let  $g \in \mathcal{F}$ ,  $\|g\|_{L^\infty} \leq 1$  and  $f \in \mathbf{B}^{1,1-\frac{\kappa}{d_W}}(X)$ . We have

$$\left| \int_X g(f - P_t f) d\mu \right| = \left| \int_0^t \int_X g \frac{\partial P_s f}{\partial s} d\mu ds \right| = \left| \int_0^t \mathcal{E}(g, P_s f) ds \right|.$$

The idea is now to bound  $\mathcal{E}(g, P_s f)$  by using an approximation of  $\mathcal{E}$ . For  $\tau \in (0, \infty)$  and  $u \in L^2(X, \mu)$ , we set

$$(5.44) \quad \mathcal{E}_\tau(u, u) := \frac{1}{2\tau} \int_X \int_X p_\tau(x, y) |u(x) - u(y)|^2 d\mu(x) d\mu(y).$$

Note that by the symmetry and the conservativeness of  $P_t$ ,

$$\begin{aligned} \mathcal{E}_\tau(u, u) &= \frac{1}{2\tau} \int_X \int_X p_\tau(x, y) (u(x)^2 - 2u(x)u(y) + u(y)^2) d\mu(x) d\mu(y) \\ &= \frac{1}{2\tau} (\langle u, u \rangle - 2\langle u, P_\tau u \rangle + \langle u, u \rangle) \\ &= \frac{1}{\tau} \langle u - P_\tau u, u \rangle \xrightarrow{t \downarrow 0} \begin{cases} \mathcal{E}(u, u) & \text{if } u \in \mathcal{F}, \\ \infty & \text{if } u \in L^2(X, \mu) \setminus \mathcal{F}, \end{cases} \end{aligned}$$

and in particular that  $(0, \infty) \ni \tau \mapsto \mathcal{E}_\tau(u, u)$  is non-increasing. We now have

$$\begin{aligned} |\mathcal{E}_\tau(g, P_s f)| &= |\mathcal{E}_\tau(P_s g, f)| \\ &\leq \frac{1}{2\tau} \int_X \int_X p_\tau(x, y) |P_s g(x) - P_s g(y)| |f(x) - f(y)| d\mu(x) d\mu(y) \\ &\leq \frac{C}{2s^{\kappa/d_W} \tau} \int_X \int_X p_\tau(x, y) d(x, y)^\kappa |f(x) - f(y)| d\mu(x) d\mu(y). \end{aligned}$$

We can now prove, using the same arguments as in the proof of theorem 5.1 that since  $f \in \mathbf{B}^{1,1-\frac{\kappa}{d_W}}(X)$ , one has

$$(5.45) \quad \begin{aligned} &\limsup_{\tau \rightarrow 0} \frac{1}{\tau} \int_X \int_X p_\tau(x, y) d(x, y)^\kappa |f(x) - f(y)| d\mu(x) d\mu(y) \\ &\leq C \limsup_{r \rightarrow 0} \frac{1}{r^{d_H+d_W-\kappa}} \iint_{\Delta_r} |f(x) - f(y)| d\mu(x) d\mu(y). \end{aligned}$$

As a conclusion we deduce

$$|\mathcal{E}(g, P_s f)| = \lim_{\tau \rightarrow 0} |\mathcal{E}_\tau(g, P_s f)| \leq \frac{C}{s^{\kappa/d_W}} \limsup_{r \rightarrow 0} \frac{1}{r^{d_H+d_W-\kappa}} \iint_{\Delta_r} |f(x) - f(y)| d\mu(x) d\mu(y).$$

Therefore, one has

$$\left| \int_X g(f - P_t f) d\mu \right| \leq C t^{1-\frac{\kappa}{d_W}} \limsup_{r \rightarrow 0} \frac{1}{r^{d_H+d_W-\kappa}} \iint_{\Delta_r} |f(x) - f(y)| d\mu(x) d\mu(y),$$

and we conclude by  $L^1 - L^\infty$  duality.  $\square$

We now briefly comment on the significance of the parameter  $\kappa$  with two corollaries. In particular, the following lemma shows that  $\kappa$  is a critical parameter in the  $L^1$  theory of  $X$ .

COROLLARY 5.35. *Assume that  $BE(\kappa)$  is satisfied. If  $f \in \mathbf{B}^{1,\alpha}(X)$  with  $\alpha > 1 - \frac{\kappa}{d_W}$ , then  $f = 0$ .*

PROOF. Let  $f \in \mathbf{B}^{1,\alpha}(X)$  with  $\alpha > 1 - \frac{\kappa}{d_W}$ . One has then

$$\limsup_{r \rightarrow 0} \frac{1}{r^{d_H + d_W - \kappa}} \iint_{\Delta_r} |f(x) - f(y)| d\mu(x) d\mu(y) = 0.$$

Therefore, from the previous theorem, one has for every  $t > 0$ ,  $\|P_t f - f\|_{L^1(X, \mu)} = 0$ , which immediately implies  $f \in \mathcal{F}$  and  $P_t f = f$ ,  $t \geq 0$ . The sub-Gaussian heat kernel upper bound implies then  $f = 0$ .  $\square$

COROLLARY 5.36. *One has  $\kappa \leq \frac{d_W}{2}$  and thus  $1 - \frac{\kappa}{d_W} \geq \frac{1}{2}$ .*

PROOF. For  $f \geq 0$ , we have by Cauchy-Schwartz inequality

$$\begin{aligned} & \iint p_t(x, y) | |f(x)|^2 - |f(y)|^2 | d\mu(x) d\mu(y) \\ & \leq \iint p_t(x, y) (|f(x)| + |f(y)|) | |f(x)| - |f(y)| | d\mu(x) d\mu(y) \\ & \leq 2 \int |f(x)| P_t(|f - f(x)|)(x) d\mu(x) \\ & \leq 2 \|f\|_2 \left( \int P_t(|f - f(x)|^2)(x) d\mu(x) \right)^{1/2} \\ & \leq 2 \|f\|_2 t^\alpha \|f\|_{2,\alpha} \end{aligned}$$

so that  $\| |f|^2 \|_{1,\alpha} \leq 2 \|f\|_2 \|f\|_{2,\alpha}$ . Hence, if  $\alpha = \frac{1}{2}$ , one deduces that if  $f \in \mathbf{B}^{2,1/2}(X) = \mathcal{F}$ , then  $f^2 \in \mathbf{B}^{1,1/2}(X)$ . One concludes that  $\mathbf{B}^{1,1/2}(X)$  is always non-trivial, so that one must have  $1 - \frac{\kappa}{d_W} \geq \frac{1}{2}$ .  $\square$

We will see that the weak Bakry-Émery estimate can be used to improve Sobolev and isoperimetric inequalities. We show here that it also yields an improvement in Theorem 5.3.

THEOREM 5.37. *Let  $X$  be Ahlfors  $d_H$ -regular with sub-Gaussian heat kernel estimates and the  $BE(\kappa)$  estimate. Let  $u \in L^1(X, \mu)$ . For  $s \in \mathbb{R}$ , denote  $E_s(u) = \{x \in X, u(x) > s\}$ . Let  $\alpha = 1 - \frac{\kappa}{d_W}$  and assume that  $s \mapsto \limsup_{r \rightarrow 0^+} \frac{1}{r^{\alpha d_W}} \mu(\{x \in E_s(u) : d(x, X \setminus E_s(u)) < r\})$  is in  $L^1(\mathbb{R})$ . Then,  $u \in \mathbf{B}^{1,\alpha}(X)$  and there exists a constant  $C > 0$  independent from  $u$  such that:*

$$\|u\|_{1,\alpha} \leq C \int_{\mathbb{R}} \limsup_{r \rightarrow 0^+} \frac{1}{r^{\alpha d_W}} \mu(\{x \in E_s(u) : d(x, X \setminus E_s(u)) < r\}) ds.$$

*In particular, if for a set  $E \subset X$  with finite measure,  $\limsup_{r \rightarrow 0^+} \frac{1}{r^{\alpha d_W}} \mu(\{x \in E : d(x, E^c) < r\}) < +\infty$ , then  $\mathbf{1}_E \in \mathbf{B}^{1,\alpha}(X)$  and:*

$$\|\mathbf{1}_E\|_{1,\alpha} \leq C \limsup_{r \rightarrow 0^+} \frac{1}{r^{\alpha d_W}} \mu(\{x \in E : d(x, E^c) < r\}).$$

PROOF. This follows from the proof of Theorem 5.3 and the statement of Theorem 5.34.  $\square$

COROLLARY 5.38. *If in addition  $X$  is uniformly locally connected, then Theorem 5.37 holds with  $\mu((\partial E_s(u))_r^-)$  instead of  $\mu(\{x \in E_s(u) : d(x, X \setminus E_s(u)) < r\})$ .*

## 5.6. Examples of spaces satisfying the weak Bakry-Émery condition

**5.6.1. Unbounded Sierpinski gasket and unbounded Sierpinski carpet.** We show here first that the unbounded Sierpinski gasket  $(X, \mathcal{E}, \mathcal{F}, \mu)$  considered in [BP88] satisfies the weak Bakry-Émery estimate.

THEOREM 5.39. *The unbounded Sierpinski gasket satisfies  $BE(\kappa)$  with  $\kappa = d_W - d_H$ .*



PROOF. Let  $f \in L^\infty(X, \mu)$ . From Theorem 5.22 in [BP88], one has for  $t > 0$  and  $\lambda > 0$ ,

$$|P_t f(x) - P_t f(y)| \leq \frac{C}{\lambda^{d_H/d_W}} d(x, y)^{d_W - d_H} \|(L - \lambda)P_t f\|_{L^\infty(X, \mu)},$$

where  $L$  is the generator of  $\mathcal{E}$ . We now observe that

$$|(L - \lambda)P_t f(x)| \leq \int_X |\partial_t p(t, x, y) - \lambda p(t, x, y)| d\mu(y) \|f\|_{L^\infty(X, \mu)}.$$

From the sub-Gaussian upper bound for the heat kernel and its time derivative (see [BP88]), one deduces

$$\int_X |\partial_t p(t, x, y) - \lambda p(t, x, y)| d\mu(y) \leq C \left( \frac{1}{t} + \lambda \right).$$

Therefore, by choosing  $\lambda = 1/t$  one concludes:

$$|P_t f(x) - P_t f(y)| \leq C t^{d_H/d_W - 1} d(x, y)^{d_W - d_H} \|f\|_{L^\infty(X, \mu)}.$$

□

In that case, one has therefore  $1 - \frac{\kappa}{d_W} = \frac{d_H}{d_W}$ , so from Theorem 5.34 the space of interest is  $\mathbf{B}^{1, \frac{d_H}{d_W}}(X)$ . We note that from Theorem 5.25,  $\mathbf{B}^{1, \frac{d_H}{d_W}}(X)$  is dense in  $L^1(X, \mu)$ .

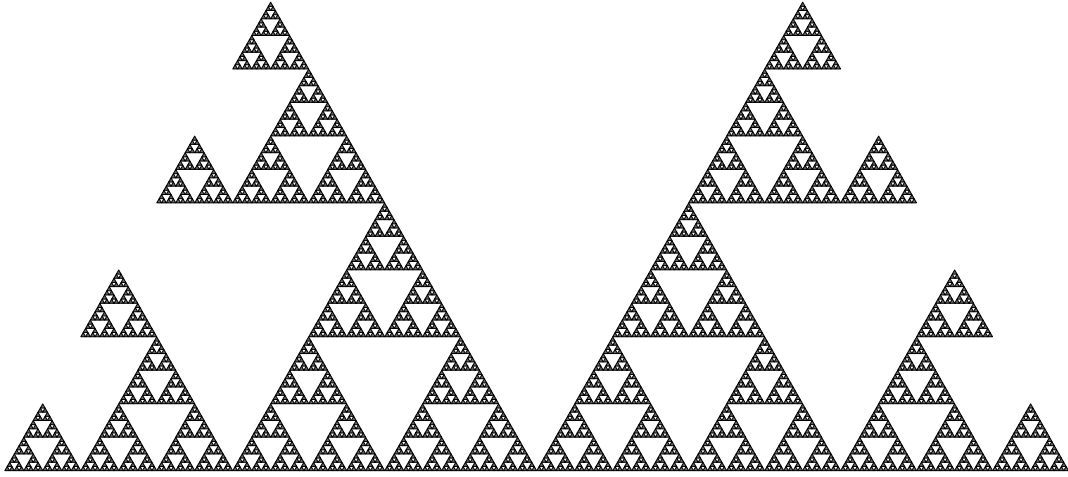


FIGURE 4. A part of an infinite Sierpinski gasket.

We now show that the unbounded Sierpinski carpet  $(X, \mathcal{E}, \mathcal{F}, \mu)$  considered in [BB92] satisfies the weak Bakry-Émery estimate.

**THEOREM 5.40.** *The unbounded Sierpinski carpet satisfies  $BE(\kappa)$  with  $\kappa = d_W - d_H$ .*

PROOF. The proof is the same as the case of Sierpinski gasket, thanks to [BB92, Theorem 4.9 (b)] and [Dav97, Theorem 4 or Corollary 5]. □

**5.6.2. Fractional metric spaces with fractional diffusion.** In [Bar98, Section 3] a class of processes called *fractional diffusions* were introduced in the context of fractional metric spaces, which are complete Ahlfors regular metric spaces with the midpoint property, c.f. [Bar98, Definition 3.2, Definition 3.5]. In this setting, applying [Bar98, Theorem 3.40] and [Dav97, Theorem 4 or Corollary 5] as in the previous example yields the corresponding weak Bakry-Émery estimate.

**THEOREM 5.41.** *A fractional metric space of dimension  $d_H \geq 1$  and associated fractional diffusion with parameters  $2 \leq d_W \leq d_H + 1$  satisfies  $BE(\kappa)$  with  $\kappa = d_W - d_H$ .*

REMARK 5.42. Notice that nested fractals ([Lin90, Bar98]), and in particular the Vicsek set, fall into this category. In the view of the discussion of Sections 5.8, we can view the case of the Vicsek set as critical since  $\kappa = 1$ . Note that the Vicsek set is a dendrite, see [Kig95].

**5.6.3. Stability of  $BE(\kappa)$  by tensorization.** The condition  $BE(\kappa)$  is stable by tensorization. This yields a lot of examples.

Consider  $(X^n, d_{X^n}, \mu^{\otimes n})$ , where the metric  $d_{X^n}$  is defined as follows: for any  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in X^n$  and  $\mathbf{x}' = (x'_1, x'_2, \dots, x'_n) \in X^n$ ,

$$d_{X^n}(\mathbf{x}, \mathbf{x}')^{\frac{d_W}{d_W-1}} = \sum_{i=1}^n d_X(x_i, x'_i)^{\frac{d_W}{d_W-1}}.$$

Let  $p_t^X(x, y)$  be the heat kernel on  $X$ . Then the heat kernel on  $X^n$  is given by

$$p_t^{X^n}(\mathbf{x}, \mathbf{y}) = p_t^X(x_1, y_1) \cdots p_t^X(x_n, y_n).$$

PROPOSITION 5.43. *if  $X$  satisfies  $BE(\kappa)$ , then  $X^n$  also satisfies  $BE(\kappa)$ .*

PROOF. Let  $f \in L^\infty(X^n, \mu^{\otimes n})$ . Then for any  $\mathbf{x}, \mathbf{x}' \in X^n$ , we have

$$\begin{aligned} \left| P_t^{X^n} f(\mathbf{x}) - P_t^{X^n} f(\mathbf{x}') \right| &\leq \left| P_t^{X^n} f(x_1, x_2, \dots, x_n) - P_t^{X^n} f(x'_1, x_2, \dots, x_n) \right| \\ &\quad + \left| P_t^{X^n} f(x'_1, x_2, \dots, x_n) - P_t^{X^n} f(x'_1, x'_2, x_3, \dots, x_n) \right| \\ &\quad + \cdots + \left| P_t^{X^n} f(x'_1, \dots, x'_{n-1}, x_n) - P_t^{X^n} f(x'_1, x'_2, \dots, x'_n) \right|. \end{aligned}$$

Notice that for any  $1 \leq i \leq n$ , since  $X$  satisfies  $BE(\kappa)$ , then

$$\begin{aligned} &\left| P_t^{X^n} f(x'_1, \dots, x'_{i-1}, x_i, \dots, x_n) - P_t^{X^n} f(x'_1, \dots, x'_i, x_{i+1}, \dots, x_n) \right| \\ &\leq \left| P_t^X (P_t^{X^{n-1}} f(x'_1, \dots, x'_{i-1}, \cdot, x_{i+1}, \dots, x_n))(x_i) - P_t^X (P_t^{X^{n-1}} f(x'_1, \dots, x'_i, \cdot, x_{i+1}, \dots, x_n))(x'_i) \right| \\ &\leq C \frac{d_X(x_i, x'_i)^\kappa}{t^{\kappa/d_W}} \left\| P_t^{X^{n-1}} f(x'_1, \dots, x'_{i-1}, \cdot, x_{i+1}, \dots, x_n) \right\|_{L^\infty(X)} \\ &\leq C \frac{d_X(x_i, x'_i)^\kappa}{t^{\kappa/d_W}} \|f\|_{L^\infty(X^n)}. \end{aligned}$$

Hence

$$\left| P_t^{X^n} f(\mathbf{x}) - P_t^{X^n} f(\mathbf{x}') \right| \leq C \frac{1}{t^{\kappa/d_W}} \left( \sum_{i=1}^n d_X(x_i, x'_i)^\kappa \right) \|f\|_{L^\infty(X^n)} \leq C \frac{d_{X^n}(\mathbf{x}, \mathbf{x}')^\kappa}{t^{\kappa/d_W}} \|f\|_{L^\infty(X^n)}.$$

(The constant  $C$  may or may not depend on  $n$ .) □

### 5.7. Continuity of the heat semigroup in the Besov spaces: the case $p \geq 2$

In this section, we study the continuity of the heat semigroup in the Besov spaces in the range  $p \geq 2$ . This complements the results of Chapter 1, under the additional weak Bakry-Émery estimate assumption. For the sake of clarity, in the statements in this paragraph the assumptions on the underlying space  $X$  are explicitly written. Throughout this section, let  $X$  be an Ahlfors  $d_H$ -regular space that satisfies sub-Gaussian heat kernel estimates and  $BE(\kappa)$  with  $0 < \kappa \leq \frac{d_W}{2}$ .

THEOREM 5.44. *For any  $p \geq 2$ , there exists a constant  $C > 0$  such that for every  $t > 0$  and  $f \in L^2(X, \mu) \cap L^\infty(X, \mu)$*

$$\|P_t f\|_{p, \kappa/d_W}^p \leq \frac{C}{t^{\frac{\kappa p}{d_W}}} \|f\|_{L^2(X, \mu)}^2 \|f\|_{L^\infty(X, \mu)}^{p-2}.$$

*In particular, for  $t > 0$ ,  $P_t f \in \mathbf{B}^{p, \frac{\kappa}{d_W}}(X)$ .*

PROOF. First, note that  $P_t f \in L^p(X, \mu) \cap \mathcal{F}$  for any  $p \geq 2$ . In order to prove that  $\|P_t f\|_{p, \kappa/d_W}$  is finite, we proceed as in the proof of Theorem 5.1 and consider, for any  $s, r > 0$ ,

$$\begin{aligned} A(s, r) &:= \int_X \int_{X \setminus B(y, r)} p_s(x, y) |P_t f(x) - P_t f(y)|^p \mu(dx) \mu(dy), \\ B(s, r) &:= \int_X \int_{X(y, r)} p_s(x, y) |P_t f(x) - P_t f(y)|^p \mu(dx) \mu(dy). \end{aligned}$$

In what follows,  $C$  will denote a positive constant that may change from line to line. On the one hand, from the proof of Theorem 5.1 we know that the upper bound on the heat kernel implies

$$A(s, r) \leq c_8 \exp\left(-c_9 \left(\frac{r^{d_W}}{s}\right)^{\frac{1}{d_W-1}}\right) \|P_t f\|_{L^p(X, \mu)}^p.$$

On the other hand, the aforementioned proof also yields

$$\begin{aligned} B(s, r) &\leq c_3 \sum_{n=1}^{\infty} \frac{(2^{1-n} r)^{p\kappa+d_H}}{s^{d_H/d_W}} \exp\left(-c_9 \left(\frac{(2^{1-n} r)^{d_W}}{s}\right)^{\frac{1}{d_W-1}}\right) \\ &\quad \times \frac{1}{(2^{1-n} r)^{p\kappa+d_H}} \int_X \int_{B(y, 2^{1-n} r)} |P_t f(x) - P_t f(y)|^p \mu(dy) \mu(dx). \end{aligned}$$

Writing  $|P_t f(x) - P_t f(y)|^p = |P_t f(x) - P_t f(y)|^{p-2} |P_t f(x) - P_t f(y)|^2$  and applying  $BE(\kappa)$  to the first factor we obtain

$$\begin{aligned} B(s, r) &\leq \frac{C \|f\|_{\infty}^{p-2}}{t^{\kappa(p-2)/d_W}} \sum_{n=1}^{\infty} \frac{(2^{1-n} r)^{p\kappa+d_H}}{s^{d_H/d_W}} \exp\left(-c_9 \left(\frac{(2^{1-n} r)^{d_W}}{s}\right)^{\frac{1}{d_W-1}}\right) \\ &\quad \times \frac{1}{(2^{1-n} r)^{2\kappa+d_H}} \int_{\{(x, y) \in X \times X : d(x, y) < 2^{1-n} r\}} |P_t f(x) - P_t f(y)|^2 \mu(dy) \mu(dx) \\ &\leq \frac{C \|f\|_{\infty}^{p-2} s^{p\kappa/d_W}}{t^{\kappa(p-2)/d_W}} \sup_{\tilde{s} \in (0, r]} (N_2^{\kappa}(P_t f, \tilde{s}))^2 \sum_{n=1}^{\infty} \left(\frac{(2^{1-n} r)^{d_W}}{s}\right)^{\frac{p\kappa+d_H}{d_W}} \exp\left(-c_9 \left(\frac{(2^{1-n} r)^{d_W}}{s}\right)^{\frac{1}{d_W-1}}\right) \\ &\leq \frac{C \|f\|_{\infty}^{p-2}}{t^{\kappa(p-2)/d_W}} s^{p\kappa/d_W} \|P_t f\|_{2, \kappa/d_W}^2, \end{aligned}$$

where the last inequality follows from the proof of the lower bound in Theorem 5.1. We have thus proved that for any  $r > 0$

$$\begin{aligned} s^{-p\kappa/d_W} \int_X \int_X p_s(x, y) |P_t f(x) - P_t f(y)|^p \mu(dx) \mu(dy) \\ \leq c_8 s^{-p\kappa/d_W} \exp\left(-c_9 \left(\frac{r^{d_W}}{s}\right)^{\frac{1}{d_W-1}}\right) \|P_t f\|_{L^p(X, \mu)}^p + \frac{C \|f\|_{\infty}^{p-2}}{t^{\kappa(p-2)/d_W}} \|P_t f\|_{2, \kappa/d_W}^2. \end{aligned}$$

This yields

$$\begin{aligned} \sup_{s>0} s^{-p\kappa/d_W} \int_X \int_X p_s(x, y) |P_t f(x) - P_t f(y)|^p \mu(dx) \mu(dy) \\ \leq \frac{C}{r^{p\kappa}} \|P_t f\|_{L^p(X, \mu)}^p + \frac{C \|f\|_{\infty}^{p-2}}{t^{\kappa(p-2)/d_W}} \|P_t f\|_{2, \kappa/d_W}^2 \end{aligned}$$

and letting  $r \rightarrow \infty$  we obtain

$$\|P_t f\|_{p, \kappa/d_W}^p \leq \frac{C \|f\|_{\infty}^{p-2}}{t^{\kappa(p-2)/d_W}} \|P_t f\|_{2, \kappa/d_W}^2.$$

We now find an upper bound for  $\|P_t f\|_{2,\kappa/d_W}$ . From Lemma 1.8, one has for  $t > 0$

$$\begin{aligned} \|P_t f\|_{2,\kappa/d_W} &\leq \frac{2}{t^{\kappa/d_W}} \|P_t f\|_{L^2(X,\mu)} + \sup_{s \in (0,t]} s^{-\kappa/d_W} \left( \int_X P_s(|P_t f - P_t f(y)|^2)(y) d\mu(y) \right)^{1/2} \\ &\leq \frac{2}{t^{\kappa/d_W}} \|f\|_{L^2(X,\mu)} + t^{\frac{1}{2} - \frac{\kappa}{d_W}} \sup_{s \in (0,t]} s^{-1/2} \left( \int_X P_s(|P_t f - P_t f(y)|^2)(y) d\mu(y) \right)^{1/2}. \end{aligned}$$

We now note that from spectral theory,

$$\sup_{s \in (0,t]} s^{-1/2} \left( \int_X P_s(|P_t f - P_t f(y)|^2)(y) d\mu(y) \right)^{1/2} \leq C \sqrt{\mathcal{E}(P_t f, P_t f)} \leq \frac{C}{t^{1/2}} \|f\|_{L^2(X,\mu)}$$

and the proof is complete.  $\square$

We first point out two corollaries.

**COROLLARY 5.45.** *For any  $p \geq 2$ , there exists a constant  $C > 0$  such that for every  $t > 0$  and  $f \in L^2(X, \mu)$*

$$\|P_t f\|_{p,\kappa/d_W}^p \leq \frac{C}{t^{\frac{\kappa p}{d_W} + \frac{(p-2)d_H}{2d_W}}} \|f\|_{L^2(X,\mu)}^p.$$

*In particular, for  $t > 0$ ,  $P_t : L^2(X, \mu) \rightarrow \mathbf{B}^{p, \frac{\kappa}{d_W}}(X)$  is bounded for every  $p \geq 2$ .*

**PROOF.** From the heat kernel upper bound, we have that  $P_t : L^2(X, \mu) \rightarrow L^\infty(X, \mu)$  is continuous with

$$\|P_t f\|_{L^\infty(X,\mu)} \leq \frac{C}{t^{\frac{d_H}{2d_W}}} \|f\|_{L^2(X,\mu)}.$$

The result then easily follows from the theorem 5.44 applied to  $P_t f$  instead of  $f$ .  $\square$

Since  $P_t$  is also bounded from  $L^2(X, \mu) \rightarrow \mathbf{B}^{1, \frac{1}{2}}(X) = \mathcal{F}$ , by using the interpolation inequalities in proposition 1.20, one obtains:

**COROLLARY 5.46.** *For any  $p \geq 2, t > 0$ ,  $P_t : L^2(X, \mu) \rightarrow \mathbf{B}^{p, (1-\frac{2}{p})\frac{\kappa}{d_W} + \frac{1}{p}}(X)$  is bounded. More precisely,*

$$\|P_t f\|_{p, (1-\frac{2}{p})\frac{\kappa}{d_W} + \frac{1}{p}} \leq \frac{C}{t^{(1-\frac{2}{p})(\frac{\kappa}{d_W} + \frac{d_H}{2d_W}) + \frac{1}{p}}} \|f\|_{L^2(X,\mu)}.$$

**PROOF.** Let  $p \geq 2$  and  $q \geq p$ . By applying proposition 1.20 with

$$\theta = \frac{\frac{1}{2} - \frac{1}{p}}{\frac{1}{2} - \frac{1}{q}}$$

and

$$\alpha = \theta \frac{\kappa}{d_W} + \frac{1}{2} (1 - \theta),$$

we get

$$\|P_t f\|_{p,\alpha} \leq \|P_t f\|_{q,\kappa/d_W}^\theta \|P_t f\|_{2,1/2}^{1-\theta}.$$

Now, from corollary 5.45,

$$\|P_t f\|_{q,\kappa/d_W} \leq \frac{C}{t^{\frac{\kappa}{d_W} + \frac{(q-2)d_H}{2qd_W}}} \|f\|_{L^2(X,\mu)},$$

and from spectral theory

$$\|P_t f\|_{2,1/2} \leq \frac{C}{\sqrt{t}} \|f\|_{L^2(X,\mu)}.$$

One deduces

$$\|P_t f\|_{p,\alpha} \leq \frac{C}{t^{\theta(\frac{\kappa}{d_W} + \frac{(q-2)d_H}{2qd_W}) + \frac{1-\theta}{2}}} \|f\|_{L^2(X,\mu)}.$$

By letting  $q \rightarrow +\infty$ , one has  $\theta \rightarrow 1 - \frac{2}{p}$  and  $\alpha \rightarrow \left(1 - \frac{2}{p}\right) \frac{\kappa}{d_W} + \frac{1}{p}$ . It follows therefore

$$\|P_t f\|_{p, (1-\frac{2}{p})\frac{\kappa}{d_W} + \frac{1}{p}} \leq \frac{C}{t^{(1-\frac{2}{p})(\frac{\kappa}{d_W} + \frac{d_H}{2d_W}) + \frac{1}{p}}} \|f\|_{L^2(X, \mu)}.$$

□

Finally, for  $p \geq 2$  we now prove the boundedness of  $P_t : L^p(X, \mu) \rightarrow \mathbf{B}^{p, (1-\frac{2}{p})\frac{\kappa}{d_W} + \frac{1}{p}}(X)$ .

**THEOREM 5.47.** *For any  $p \geq 2$ , there exists a constant  $C > 0$  such that for every  $t > 0$  and  $f \in L^p(X, \mu)$*

$$\|P_t f\|_{p, (1-\frac{2}{p})\frac{\kappa}{d_W} + \frac{1}{p}} \leq \frac{C}{t^{(1-\frac{2}{p})\frac{\kappa}{d_W} + \frac{1}{p}}} \|f\|_{L^p(X, \mu)}.$$

*In particular, for  $t > 0$ ,  $P_t : L^p(X, \mu) \rightarrow \mathbf{B}^{p, (1-\frac{2}{p})\frac{\kappa}{d_W} + \frac{1}{p}}(X)$  is bounded.*

**PROOF.** As usual, in the sequel,  $C$  is a constant that may change from line to line. From theorem 5.44, for any  $q \geq 2$ , there exists a constant  $C > 0$  such that for every  $t > 0$  and  $f \in L^2(X, \mu) \cap L^\infty(X, \mu)$

$$\|P_t f\|_{q, \kappa/d_W}^q \leq \frac{C}{t^{\frac{\kappa q}{d_W}}} \|f\|_{L^2(X, \mu)}^2 \|f\|_{L^\infty(X, \mu)}^{q-2}.$$

From the heat kernel upper bound, we have that  $P_t : L^q(X, \mu) \rightarrow L^\infty(X, \mu)$  is continuous with

$$\|P_t f\|_{L^\infty(X, \mu)} \leq \frac{C}{t^{\frac{d_H}{q d_W}}} \|f\|_{L^q(X, \mu)}.$$

Therefore, one deduces that exists a constant  $C > 0$  such that for every  $t > 0$  and  $f \in L^2(X, \mu) \cap L^q(X, \mu)$

$$\|P_t f\|_{q, \kappa/d_W}^q \leq \frac{C}{t^{\frac{\kappa q}{d_W} + \frac{(q-2)d_H}{q d_W}}} \|f\|_{L^2(X, \mu)}^2 \|f\|_{L^q(X, \mu)}^{q-2}.$$

If  $A \subset X$  is a set of finite measure, one deduces that

$$\|P_t(f \mathbf{1}_A)\|_{q, \kappa/d_W}^q \leq \frac{C \mu(A)^{1-\frac{1}{2q}}}{t^{\frac{\kappa q}{d_W} + \frac{(q-2)d_H}{q d_W}}} \|f\|_{L^q(A, \mu)}^q.$$

Thus, for every  $f \in L^q(A, \mu)$ ,

$$\|P_t(f \mathbf{1}_A)\|_{q, \kappa/d_W} \leq \frac{C \mu(A)^{\frac{1}{q} - \frac{1}{2q^2}}}{t^{\frac{\kappa}{d_W} + \frac{(q-2)d_H}{q^2 d_W}}} \|f\|_{L^q(A, \mu)}.$$

On the other hand, for every  $f \in L^2(A, \mu)$

$$\|P_t(f \mathbf{1}_A)\|_{2, 1/2} \leq \frac{C}{\sqrt{t}} \|f\|_{L^2(A, \mu)}.$$

Let now  $2 \leq p \leq q$  and

$$\theta = \frac{\frac{1}{2} - \frac{1}{p}}{\frac{1}{2} - \frac{1}{q}}$$

and

$$\alpha = \theta \frac{\kappa}{d_W} + \frac{1}{2} (1 - \theta).$$

From the Riesz-Thorin interpolation theorem one deduces that for every  $f \in L^p(A, \mu)$

$$\|P_t(f \mathbf{1}_A)\|_{p, \alpha} \leq \frac{C \mu(A)^{\frac{\theta}{q} - \frac{\theta}{2q^2}}}{t^{\theta \left( \frac{\kappa}{d_W} + \frac{(q-2)d_H}{q^2 d_W} \right) + \frac{1-\theta}{2}}} \|f\|_{L^p(A, \mu)}.$$

As before, by letting  $q \rightarrow +\infty$ , one has  $\theta \rightarrow 1 - \frac{2}{p}$  and  $\alpha \rightarrow \left(1 - \frac{2}{p}\right) \frac{\kappa}{d_W} + \frac{1}{p}$ . One deduces that there exists a constant  $C > 0$  (independent from  $A$ ) such that for every  $t > 0$  and  $f \in L^p(X, \mu)$

$$\|P_t(f\mathbf{1}_A)\|_{p, (1-\frac{2}{p})\frac{\kappa}{d_W} + \frac{1}{p}} \leq \frac{C}{t^{(1-\frac{2}{p})\frac{\kappa}{d_W} + \frac{1}{p}}} \|f\|_{L^p(A, \mu)} \leq \frac{C}{t^{(1-\frac{2}{p})\frac{\kappa}{d_W} + \frac{1}{p}}} \|f\|_{L^p(X, \mu)}.$$

The conclusion easily follows since  $A$  is arbitrary.  $\square$

We finish the section with two interesting consequence of the Besov continuity of the semigroup.

PROPOSITION 5.48. *Let  $L$  be the generator of  $\mathcal{E}$ ,  $p \geq 2$ . Denote*

$$\alpha_p^* = \left(1 - \frac{2}{p}\right) \frac{\kappa}{d_W} + \frac{1}{p}.$$

and  $\mathcal{L}_p$  be the domain of  $L$  in  $L^p(X, \mu)$ . Then

$$\mathcal{L}_p \subset \mathbf{B}^{p, \alpha_p^*}(X)$$

and for every  $f \in \mathcal{L}_p$ ,

$$(5.46) \quad \|f\|_{p, \alpha_p^*} \leq C \|Lf\|_{L^p(X, \mu)}^{\alpha_p^*} \|f\|_{L^p(X, \mu)}^{1-\alpha_p^*}.$$

PROOF. For  $\lambda > 0$ , write

$$R_\lambda f = (L - \lambda)^{-1} f = \int_0^\infty e^{-\lambda t} P_t f dt.$$

We have then

$$\|R_\lambda f\|_{p, \alpha_p^*} \leq \int_0^\infty e^{-\lambda t} \|P_t f\|_{p, \alpha_p^*} dt \leq \int_0^\infty e^{-\lambda t} \frac{C}{t^{\alpha_p^*}} \|f\|_p dt \leq C \lambda^{\alpha_p^*-1} \|f\|_p.$$

It follows that

$$\|f\|_{p, \alpha_p^*} \leq C \lambda^{\alpha_p^*-1} \|(L - \lambda)f\|_p \leq C(\lambda^{\alpha_p^*-1} \|Lf\|_p + \lambda^{\alpha_p^*} \|f\|_p).$$

Taking  $\lambda = \|Lf\|_p \|f\|_p^{-1}$ , we then get the conclusion.  $\square$

For  $p \geq 1$ , we denote:

$$\alpha_p^* = \left(1 - \frac{2}{p}\right) \frac{\kappa}{d_W} + \frac{1}{p}.$$

We have then the following theorem that complements Theorem 1.25 of Chapter 1.

THEOREM 5.49. *Let  $1 \leq p \leq 2$ . For every  $f \in \mathbf{B}^{p, \alpha_p^*}(X)$ , and  $t \geq 0$ ,*

$$\|P_t f - f\|_{L^p(X, \mu)} \leq C t^{\alpha_p^*} \limsup_{r \rightarrow 0} \frac{1}{r^{d_H + d_W \alpha_p^*}} \left( \iint_{\Delta_r} |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p}$$

PROOF. With Theorem 5.47 in hands, the proof is similar to that of Theorem 1.25.  $\square$

### 5.8. Critical exponents and generalized Riesz transforms

Let  $X$  be an Ahlfors  $d_H$ -regular space that satisfies sub-Gaussian heat kernel estimates and  $BE(\kappa)$  with  $0 < \kappa \leq \frac{d_W}{2}$ . For  $p \geq 1$ , we denote in this section:

$$\alpha_p^*(X) = \inf \{ \alpha > 0 : \mathbf{B}^{p, \alpha}(X) \text{ is trivial} \}.$$

THEOREM 5.50. *The following inequalities hold:*

- For  $1 \leq p \leq 2$ ,

$$\frac{1}{2} \leq \alpha_p^*(X) \leq \left(1 - \frac{2}{p}\right) \frac{\kappa}{d_W} + \frac{1}{p}.$$

- For  $p \geq 2$ ,

$$\left(1 - \frac{2}{p}\right) \frac{\kappa}{d_W} + \frac{1}{p} \leq \alpha_p^*(X) \leq \frac{1}{2}.$$

PROOF. Let  $p \geq 2$ . Let  $\beta_p = \left(1 - \frac{2}{p}\right) \frac{\kappa}{d_W} + \frac{1}{p}$ . Thanks, to Theorem 5.47,  $\mathbf{B}^{p, \beta_p}(X)$  is dense in  $L^p(X, \mu)$ , because for  $f \in L^p(X, \mu)$ ,  $P_t f \in \mathbf{B}^{p, \beta_p}(X)$  and  $P_t f \rightarrow f$  when  $t \rightarrow 0$ . Next, we prove that for  $\alpha > \beta_p$ , and  $1 \leq p \leq 2$ ,  $\mathbf{B}^{p, \alpha}(X)$  is trivial, which is a consequence of Theorem 5.49 since for  $f \in \mathbf{B}^{p, \alpha}(X)$  one has then

$$\limsup_{r \rightarrow 0} \frac{1}{r^{d_H + d_W \beta_p}} \left( \iint_{\Delta_r} |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p} = 0$$

and thus  $P_t f = f$ ,  $t \geq 0$ .  $\square$

REMARK 5.51. Note that for the unbounded Sierpinski gasket, from previous results (see Example 5.26), one has

$$\alpha_1^*(X) = \frac{d_H}{d_W}.$$

It is natural to conjecture that if  $X$  is an Ahlfors  $d_H$ -regular space that satisfies sub-Gaussian heat kernel estimates and  $BE(\kappa)$  with  $0 < \kappa \leq \frac{d_W}{2}$ , under possible additional conditions, we may actually always have

$$\alpha_p^*(X) = \left(1 - \frac{2}{p}\right) \frac{\kappa}{d_W} + \frac{1}{p}, \quad p \geq 1.$$

This is will be the object of a further study. We however briefly comment on this problem and its connection to a natural notion of Riesz transform.

For  $p > 1$ ,  $\alpha \in (0, 1]$  let us say that  $X$  satisfies  $(R_{p, \alpha})$  if there exists a constant  $C = C_{p, \alpha}$  such that

$$\|f\|_{p, \alpha} \leq C \|(-L)^\alpha f\|_{L^p(X, \mu)},$$

where  $f$  is in the proper domain. For instance, in the strictly local framework of Chapter 4, under the strong Bakry-Émery curvature condition,  $\|f\|_{p, 1/2}^p \sim \int_X \Gamma(f)^{p/2} d\mu$  and  $(R_{p, 1/2})$  is therefore equivalent to boundedness of the Riesz transform in  $L^p(X, \mu)$ .

One has then the following:

LEMMA 5.52. Assume that for every  $1 < p \leq 2$ ,  $(R_{p, \beta_p})$  is satisfied with  $\beta_p = \left(1 - \frac{2}{p}\right) \frac{\kappa}{d_W} + \frac{1}{p}$ . Then for every  $p \geq 1$ ,

$$\alpha_p^*(X) = \left(1 - \frac{2}{p}\right) \frac{\kappa}{d_W} + \frac{1}{p}.$$

PROOF. By analyticity of the semigroup, one has then for every  $f \in L^p(X, \mu)$ ,  $1 < p \leq 2$ ,

$$\|P_t f\|_{p, \beta_p} \leq C \|(-L)^{\beta_p} P_t f\|_{L^p(X, \mu)} \leq \frac{C}{t^{\beta_p}} \|f\|_{L^p(X, \mu)}.$$

Therefore, for  $t > 0$ ,  $P_t : L^p(X, \mu) \rightarrow \mathbf{B}^{p, \beta_p}(X)$  is bounded. In particular,  $\mathbf{B}^{p, \beta_p}(X)$  is dense in  $L^p(X, \mu)$  and we can argue as before. Details are omitted.  $\square$

## 5.9. Sets of finite perimeter

DEFINITION 5.53. Let  $X$  be an Ahlfors  $d_H$ -regular space that satisfies sub-Gaussian heat kernel estimates and  $BE(\kappa)$  with  $0 < \kappa \leq \frac{d_W}{2}$ . A set  $E \subset X$  is said to be of finite perimeter if  $\mathbf{1}_E \in \mathbf{B}^{1 - \frac{\kappa}{d_W}}(X)$ .

An important property of sets of finite perimeter is the following:

THEOREM 5.54. *There exist constants  $c, C > 0$  such that for every set  $E \subset X$  of finite perimeter,*

$$\begin{aligned} & c \sup_{r>0} \frac{1}{r^{d_W - \kappa + d_H}} (\mu \otimes \mu) \{(x, y) \in E \times E^c : d(x, y) < r\} \\ & \leq \|\mathbf{1}_E\|_{1, \alpha} \leq C \limsup_{r \rightarrow 0^+} \frac{1}{r^{d_W - \kappa + d_H}} (\mu \otimes \mu) \{(x, y) \in E \times E^c : d(x, y) < r\}; \end{aligned}$$

PROOF. This is a consequence of Theorem 5.1 and Theorem 5.34 applied to  $\mathbf{1}_E$ .  $\square$

REMARK 5.55. *In the previous theorem, note that*

$$\limsup_{r \rightarrow 0} \frac{1}{r^{d_W - \kappa + d_H}} (\mu \otimes \mu) \{(x, y) \in E \times E^c : d(x, y) \leq r\} \leq C \limsup_{r \rightarrow 0} \frac{1}{r^{d_W - \kappa}} \mu((\partial E)_r),$$

where, as everywhere in the section,  $(\partial E)_r$  denotes the  $r$ -neighborhood of  $\partial E$ .

### 5.10. Sobolev and isoperimetric inequalities

Combining the results of Part I, Chapter 2, with the results of this chapter, one immediately obtains the following corollaries. The proofs are similar to the ones in Section 4.7, so will be omitted for concision.

COROLLARY 5.56. *Let  $(X, \mu, \mathcal{E}, \mathcal{F})$  be a symmetric Dirichlet space with sub-Gaussian heat kernel estimates. Then, one has the following weak type Besov space embedding. Let  $0 < \delta < d_H$ . Let  $1 \leq p < \frac{d_H}{\delta}$ . There exists a constant  $C_{p, \delta} > 0$  such that for every  $f \in \mathbf{B}^{p, \delta/d_W}(X)$ ,*

$$\sup_{s \geq 0} s \mu(\{x \in X : |f(x)| \geq s\})^{\frac{1}{q}} \leq C_{p, \delta} \sup_{r > 0} \frac{1}{r^{\delta + d_H/p}} \left( \iint_{\Delta_r} |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p}$$

where  $q = \frac{pd_H}{d_H - p\delta}$ . Furthermore, for every  $0 < \delta < d_H$ , there exists a constant  $C_{\text{iso}, \delta}$  such that for every measurable  $E \subset X$ ,  $\mu(E) < +\infty$ ,

$$(5.47) \quad \mu(E)^{\frac{d_H - \delta}{d_H}} \leq C_{\text{iso}, \delta} \sup_{r > 0} \frac{1}{r^{\delta + d_H}} (\mu \otimes \mu) \{(x, y) \in E \times E^c : d(x, y) \leq r\}$$

In the previous corollary, when  $p = 1$  (the most interesting for isoperimetry), one may replace the sup by the limsup provided that the weak Bakry-Émery curvature condition is satisfied.

COROLLARY 5.57. *Let  $(X, \mu, \mathcal{E}, \mathcal{F})$  be a symmetric Dirichlet space with sub-Gaussian heat kernel estimates that satisfies the weak Bakry-Émery curvature condition (5.43) with a parameter  $\kappa$  such that  $d_W - d_H < \kappa < d_W$ . Let  $\delta = d_W - \kappa$ . There exists a constant  $C_{1, \delta} > 0$  such that for every  $f \in \mathbf{B}^{1, \delta/d_W}(X)$ ,*

$$\sup_{s \geq 0} s \mu(\{x \in X, |f(x)| \geq s\})^{\frac{1}{q}} \leq C_{p, \delta} \limsup_{r > 0} \frac{1}{r^{\delta + d_H}} \iint_{\Delta_r} |f(x) - f(y)| d\mu(x) d\mu(y)$$

where  $q = \frac{pd_H}{d_H - \delta}$ . Furthermore, there exists a constant  $C_{\text{iso}, \delta}$  such that for every measurable  $E \subset X$ ,  $\mu(E) < +\infty$ ,

$$(5.48) \quad \mu(E)^{\frac{d_H - \delta}{d_H}} \leq C_{\text{iso}, \delta} \limsup_{r \rightarrow 0} \frac{1}{r^{\delta + d_H}} (\mu \otimes \mu) \{(x, y) \in E \times E^c : d(x, y) \leq r\}.$$

REMARK 5.58. *In the limiting case  $\kappa = d_W - d_H$  (like the unbounded Sierpinski gasket), one has the following result. There exists a constant  $C_{\text{iso}} > 0$  such that for every measurable  $E \subset X$ ,  $0 < \mu(E) < +\infty$ ,*

$$(5.49) \quad \limsup_{r \rightarrow 0} \frac{1}{r^{2d_H}} (\mu \otimes \mu) \{(x, y) \in E \times E^c : d(x, y) \leq r\} \geq C_{\text{iso}},$$



REMARK 5.59. *In the previous theorem, if  $E \subseteq X$  has a compact topological boundary  $\partial E$*

$$\limsup_{r \rightarrow 0} \frac{1}{r^{\delta+d_H}} (\mu \otimes \mu) \{(x, y) \in E \times E^c : d(x, y) \leq r\} \leq C \limsup_{r \rightarrow 0} \frac{1}{r^\delta} \mu((\partial E)_r),$$

where  $(\partial E)_r$  denotes the  $r$ -neighborhood of  $\partial E$ .



## Non-local Dirichlet spaces with heat kernel estimates

Let  $(X, d, \mu)$  be a metric measure space. We assume that  $B(x, r) := \{y \in X \mid d(x, y) < r\}$  has compact closure for any  $x \in X$  and any  $r \in (0, \infty)$ , and that  $\mu$  is Ahlfors  $d_H$ -regular, i.e. there exist  $c_1, c_2, d_H \in (0, \infty)$  such that  $c_1 r^{d_H} \leq \mu(B(x, r)) \leq c_2 r^{d_H}$  for any  $r \in (0, +\infty)$ . Furthermore, we assume that on  $(X, \mu)$  there is a heat kernel  $p_t(x, y)$ . If the metric measure space  $(X, d, \mu)$  satisfies a certain chain condition and all its metric balls are relatively compact, it is known from [GK08, Theorem 4.1] that the sub-Gaussian heat kernel estimates (5.1) will force the associated Dirichlet form to be local. If the Dirichlet form is non-local, the mentioned result tells us that the only possible heat kernel estimates are

$$(6.1) \quad c_5 t^{-\frac{d_H}{d_W}} \left(1 + c_6 \frac{d(x, y)}{t^{1/d_W}}\right)^{-d_H - d_W} \leq p_t(x, y) \leq c_3 t^{-\frac{d_H}{d_W}} \left(1 + c_4 \frac{d(x, y)}{t^{1/d_W}}\right)^{-d_H - d_W},$$

with  $0 < d_W \leq d_H + 1$ . We refer to [GK08, GHL03, GHL14, GHH17, GK17, GL15, BGPV14] for further details and results in this direction. The present chapter is devoted to the study of the spaces  $\mathbf{B}^{p, \alpha}(X)$  in this framework of non-local Dirichlet spaces with heat kernel estimates (6.1). To compare both situations more easily, we will follow the same structure as the previous chapter about the local case.

### 6.1. Metric characterization of Besov spaces

We start by relating our Besov space  $\mathbf{B}^{p, \alpha}(X)$  with other Besov type spaces that have been considered in the literature in the non-local context; see e.g. [Gri03, Str03b].

For  $\alpha \in [0, \infty)$  and  $p \in [1, \infty)$ , recall the Besov seminorm

$$\|f\|_{p, \alpha} = \sup_{t > 0} t^{-\alpha} \left( \int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p}$$

and the Besov space

$$\mathbf{B}^{p, \alpha}(X) = \{f \in L^p(X, \mu) : \|f\|_{p, \alpha} < +\infty\}.$$

Recall also the metric Besov type seminorm defined in (5.2). That is, for  $f \in L^p(X, \mu)$ ,

$$N_p^\alpha(f, r) := \frac{1}{r^{\alpha + d_H/p}} \left( \iint_{\Delta_r} |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p}.$$

Here, as in the previous chapters, for  $r > 0$  the set  $\Delta_r$  denotes the collection of all  $(x, y) \in X \times X$  with  $d(x, y) < r$ . For any  $q \in [1, \infty)$  such that  $p \leq q$ , define

$$N_{p, q}^\alpha(f) := \left( \int_0^\infty (N_p^\alpha(f, r))^q \frac{dr}{r} \right)^{1/q},$$

and for  $q = \infty$ , define

$$N_{p, \infty}^\alpha(f) := \sup_{r > 0} N_p^\alpha(f, r).$$

The Besov space is then defined by

$$\mathfrak{B}_{p, q}^\alpha(X) = \{u \in L^p(X) : N_{p, q}^\alpha(f) < \infty\}.$$

We determine first which of these spaces coincides with  $\mathbf{B}^{p, \alpha}(X)$  when  $p \in [1, \infty)$  and  $\alpha \in [0, 1/p)$ .

**THEOREM 6.1.** *Let  $p \geq 1$  and  $0 \leq \alpha < 1/p$ . We have  $\mathfrak{B}_{p,\infty}^{\alpha d_W}(X) = \mathbf{B}^{p,\alpha}(X)$  and there exist two constants  $c, C > 0$  (depending on  $p, \alpha, d_H, d_W$ ) such that*

$$c N_{p,\infty}^{\alpha d_W}(f) \leq \|f\|_{p,\alpha} \leq C N_{p,\infty}^{\alpha d_W}(f).^1$$

**PROOF.** We follow the proof for Theorem 5.1 by first proving the lower bound. By the upper bound in (6.1), one has for  $r, t > 0$

$$\begin{aligned} & \int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \\ & \geq \frac{c}{t^{d_H/d_W}} \int_X \int_{B(y,r)} \frac{|f(x) - f(y)|^p}{\left(1 + \frac{d(x,y)}{t^{1/d_W}}\right)^{d_H+d_W}} d\mu(x) d\mu(y) \\ & \geq \frac{c}{t^{d_H/d_W}} \frac{1}{\left(1 + \frac{r}{t^{1/d_W}}\right)^{d_H+d_W}} \int_X \int_{B(y,r)} |f(x) - f(y)|^p d\mu(x) d\mu(y) \\ & \geq \frac{c}{t^{d_H/d_W}} \frac{r^{\alpha d_W p + d_H}}{\left(1 + \frac{r}{t^{1/d_W}}\right)^{d_H+d_W}} N_p^{\alpha d_W}(f, r)^p \end{aligned}$$

Choosing  $r = t^{1/d_W}$ , it easily follows that

$$\|f\|_{p,\alpha} \geq c \sup_{t>0} N_p^{\alpha d_W}(f, t^{1/d_W}) = c N_{p,\infty}^{\alpha d_W}(f).$$

We now turn to the upper bound. We set  $A(t)$  and  $B(t)$  as in (5.6), respectively (5.7), so that  $\int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) = A(t) + B(t)$ . By (6.1) and the inequality  $|f(x) - f(y)|^p \leq 2^{p-1}(|f(x)|^p + |f(y)|^p)$ ,

$$\begin{aligned} A(t) & \leq \frac{c_3}{t^{d_H/d_W}} \int_X \int_{X \setminus B(y,r)} \left(1 + c_4 \frac{d(x,y)}{t^{1/d_W}}\right)^{-d_H-d_W} \cdot 2^p |f(y)|^p d\mu(x) d\mu(y) \\ & = \frac{2^p c_3}{t^{d_H/d_W}} \sum_{k=1}^{\infty} \int_X \int_{B(y, 2^k r) \setminus B(y, 2^{k-1} r)} \left(1 + c_4 \frac{d(x,y)}{t^{1/d_W}}\right)^{-d_H-d_W} |f(y)|^p d\mu(x) d\mu(y) \\ & \leq \frac{2^p c_3}{t^{d_H/d_W}} \sum_{k=1}^{\infty} \int_X \mu(B(y, 2^k r)) \left(1 + c_4 \frac{2^{(k-1)} r}{t^{1/d_W}}\right)^{-d_H-d_W} |f(y)|^p d\mu(y) \\ & \leq \frac{2^p c_3}{t^{d_H/d_W}} \sum_{k=1}^{\infty} c_2 r^{d_H} 2^{k d_H} \|f\|_{L^p}^p \left(1 + c_4 \frac{2^{(k-1)} r}{t^{1/d_W}}\right)^{-d_H-d_W} \\ & = C \|f\|_{L^p}^p \sum_{k=1}^{\infty} \left(\frac{2^{k-1} r}{t^{1/d_W}}\right)^{d_H} \left(1 + \frac{c_4}{2} \frac{2^k r}{t^{1/d_W}}\right)^{-d_H-d_W} \\ & \leq C \|f\|_{L^p}^p \sum_{k=1}^{\infty} \int_{\frac{2^{k-1} r}{t^{1/d_W}}}^{\frac{2^k r}{t^{1/d_W}}} s^{d_H} \left(1 + \frac{c_4}{2} s\right)^{-d_H-d_W} \frac{ds}{s} \\ & \leq C \|f\|_{L^p}^p \frac{t^{\alpha p}}{r^{\alpha d_W p}} \int_0^{\infty} s^{\alpha d_W p + d_H - 1} \left(1 + \frac{c_4}{2} s\right)^{-d_H-d_W} ds \\ (6.2) \quad & \leq C \frac{t^{\alpha p}}{r^{\alpha d_W p}} \|f\|_{L^p}^p, \end{aligned}$$

where the last integral is bounded as long as  $\alpha \in [0, 1/p)$ .

<sup>1</sup>Here and after, the letters  $c, C$  denote constants independent of important parameters which may change at each circumstance.

On the other hand, for  $B(t)$ , by (6.1) we have

$$\begin{aligned}
B(t) &\leq \frac{c_3}{t^{d_H/d_W}} \int_X \int_{B(y,r)} \left(1 + c_4 \frac{d(x,y)}{t^{1/d_W}}\right)^{-d_H-d_W} |f(x) - f(y)|^p d\mu(x) d\mu(y) \\
&\leq \frac{c_3}{t^{d_H/d_W}} \sum_{k=1}^{\infty} \int_X \int_{B(y,2^{1-k}r) \setminus B(y,2^{-k}r)} \left(1 + c_4 \frac{d(x,y)}{t^{1/d_W}}\right)^{-d_H-d_W} |f(x) - f(y)|^p d\mu(x) d\mu(y) \\
&\leq \frac{c_3}{t^{d_H/d_W}} \sum_{k=1}^{\infty} \int_X \int_{B(y,2^{1-k}r)} \left(1 + c_4 \frac{2^{-k}r}{t^{1/d_W}}\right)^{-d_H-d_W} |f(x) - f(y)|^p d\mu(x) d\mu(y) \\
&\leq c_3 \sum_{k=1}^{\infty} \frac{(2^{1-k}r)^{\alpha d_W p + d_H}}{t^{d_H/d_W}} \left(1 + c_4 \frac{2^{-k}r}{t^{1/d_W}}\right)^{-d_H-d_W} \frac{1}{(2^{1-k}r)^{\alpha d_W p + d_H}} \int_X \int_{B(y,2^{1-k}r)} |f(x) - f(y)|^p d\mu(x) d\mu(y) \\
&\leq C t^{\alpha p} \sup_{s \in (0,r]} N_p^{\alpha d_W} (f, s)^p \sum_{k=1}^{\infty} \left(\frac{2^{-k}r}{t^{1/d_W}}\right)^{d_H + \alpha d_W p} \left(1 + c_4 \frac{2^{-k}r}{t^{1/d_W}}\right)^{-d_H-d_W} \\
(6.3) \quad &\leq C t^{\alpha p} \sup_{s \in (0,r]} N_p^{\alpha d_W} (f, s)^p,
\end{aligned}$$

where the last series converges for all  $\alpha \in [0, 1/p)$ .

Combining (6.2) and (6.3), we conclude that by taking  $r \rightarrow \infty$ ,

$$\sup_{t>0} \frac{1}{t^{\alpha p}} \int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \leq C \sup_{s>0} N_p^{\alpha d_W} (f, s)^p,$$

and hence  $\|f\|_{p,\alpha} \leq C N_{p,\infty}^{\alpha d_W} (f)$ . The proof is thus complete.  $\square$

We can also establish several inclusions that cover the case when  $\alpha \in [1/p, \infty)$ .

PROPOSITION 6.2. *Let  $p \in [1, \infty)$ . We have*

- (1) if  $\alpha \in [0, 1/p]$ , then  $\mathfrak{B}_{p,p}^{\alpha d_W} (X) \subset \mathbf{B}^{p,\alpha} (X)$ .
- (2) if  $\alpha \in [1/p, \infty)$ , then  $\mathbf{B}^{p,\alpha} (X) \subset \mathfrak{B}_{p,p}^{\alpha d_W} (X)$ .

In particular,  $\mathbf{B}^{p,1/p} (X) = \mathfrak{B}_{p,p}^{d_W/p} (X)$ .

PROOF. We first show (1). Fixing a decreasing geometric sequence  $\{r_k\}_{k \in \mathbb{Z}}$ , we can write

$$\sum_{k \in \mathbb{Z}} N_p^\alpha (f, r_k)^p \simeq \int_0^\infty N_p^\alpha (f, r)^p \frac{dr}{r}.$$

Using the upper bound in (6.1), for any  $\alpha \in [0, 1/p]$  we have that

$$\begin{aligned}
& \int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \\
&= \sum_{k \in \mathbb{Z}} \int_X \int_{B(x, r_k) \setminus B(x, r_{k+1})} |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \\
&\leq \sum_{k \in \mathbb{Z}} c_3 t^{-\frac{d_H}{d_W}} \left(1 + c_4 \frac{r_{k+1}}{t^{1/d_W}}\right)^{-d_H - d_W} \int_X \int_{B(x, r_k) \setminus B(x, r_{k+1})} |f(x) - f(y)|^p d\mu(x) d\mu(y) \\
&= \sum_{k \in \mathbb{Z}} c_3 t \left(t^{1/d_W} + c_4 r_{k+1}\right)^{-d_H - d_W} \int_X \int_{B(x, r_k) \setminus B(x, r_{k+1})} |f(x) - f(y)|^p d\mu(x) d\mu(y) \\
&\leq \sum_{k \in \mathbb{Z}} c_3 t \left(t^{1/d_W}\right)^{-d_W + \alpha p d_W} (c_4 r_{k+1})^{-d_H - \alpha p d_W} \int_X \int_{B(x, r_k) \setminus B(x, r_{k+1})} |f(x) - f(y)|^p d\mu(x) d\mu(y) \\
&\leq C \sum_{k \in \mathbb{Z}} t^{\alpha p} N_p^{\alpha d_W}(f, r_{k+1})^p \leq C t^{\alpha p} N_{p,p}^{\alpha d_W}(f)^p.
\end{aligned}$$

This implies

$$\|f\|_{p,\alpha} \leq C N_{p,p}^{\alpha d_W}(f),$$

and hence  $\mathfrak{B}_{p,p}^{\alpha d_W}(X) \subset \mathbf{B}^{p,\alpha}(X)$ .

Next we prove (2). Notice that for any  $\delta \in (0, 1)$  and  $d'_W = d_W/\delta$ , there exist two constants  $c, C$  such that

$$p_t(x, y) \geq c \int_{t^{1/\delta}}^{\infty} \frac{t}{s^{1+\delta}} s^{-d_H/d'_W} \exp\left(-C \left(\frac{d(x, y)^{d'_W}}{s}\right)^{1/(d'_W-1)}\right) ds.$$

See for instance [Gri03, Section 5.4]. Following the proof in Theorem 5.1, we obtain for  $\alpha \geq 1/p$ ,

$$\begin{aligned}
& \int_X \int_X |f(x) - f(y)|^p p_s(x, y) d\mu(x) d\mu(y) \\
&\geq c \int_{t^{1/\delta}}^{\infty} \frac{t}{s^{1+\delta}} s^{-d_H/d'_W} \int_X \int_{B(y, s^{1/d'_W})} |f(x) - f(y)|^p d\mu(x) d\mu(y) ds \\
&\geq c \int_{t^{1/\delta d'_W}}^{\infty} \frac{t}{r^{\delta d'_W - \alpha d_W p}} N_p^{\alpha d_W}(f, r)^p \frac{dr}{r} \\
&\geq c t^{\alpha p} \int_{t^{1/d_W}}^{\infty} N_p^{\alpha d_W}(f, r)^p \frac{dr}{r},
\end{aligned}$$

where we get the second inequality by changing the variable. It follows that

$$\|f\|_{p,\alpha} \geq c \sup_{t>0} \left( \int_{t^{1/d_W}}^{\infty} N_p^{\alpha d_W}(f, r)^p \frac{dr}{r} \right)^{1/p} = c N_{p,p}^{\alpha d_W}(f),$$

and we conclude (2).  $\square$

We finish this section by giving an alternative, equivalent expression for the seminorm  $\|f\|_{1,1}$  that will turn useful later on.

LEMMA 6.3. *Let  $f \in \mathbf{B}^{1,1}(X)$ . Then the seminorm  $\|f\|_{1,1}$  is equivalent to*

$$W_{1,1}(f) := \int_X \int_X \frac{|f(x) - f(y)|}{d(x, y)^{d_H + d_W}} d\mu(x) d\mu(y).$$

PROOF. We first show that  $\|f\|_{1,1} \leq CW_{1,1}(f)$ , which follows directly from the heat kernel estimate (6.1). Indeed, for every  $t > 0$ ,

$$\begin{aligned} & \frac{1}{t} \int_X \int_X |f(x) - f(y)| p_t(x, y) d\mu(x) d\mu(y) \\ & \leq \frac{c_3}{t} \int_X \int_X t^{-\frac{d_H}{d_W}} \left(1 + c_4 \frac{d(x, y)}{t^{1/d_W}}\right)^{-d_H - d_W} |f(x) - f(y)| d\mu(x) d\mu(y) \\ & = \frac{c_3}{t} \int_X \int_X \frac{t}{(t^{1/d_W} + c_4 d(x, y))^{d_H + d_W}} |f(x) - f(y)| d\mu(x) d\mu(y) \leq CW_{1,1}(f). \end{aligned}$$

Next we show the reverse inequality. For every  $t > 0$ ,

$$\begin{aligned} & \int_X \int_X \frac{c_5}{(t^{1/d_W} + c_6 d(x, y))^{d_H + d_W}} |f(x) - f(y)| d\mu(x) d\mu(y) \\ & \leq \frac{1}{t} \int_X \int_X |f(x) - f(y)| p_t(x, y) d\mu(x) d\mu(y) \leq \|f\|_{1,1}. \end{aligned}$$

By Lebesgue's dominated convergence theorem, we have  $W_{1,1}(f) \leq C\|f\|_{1,1}$ . This completes the proof.  $\square$

## 6.2. Coarea type estimates

From Theorem 6.1 we can derive as in the local case several estimates involving the fractional perimeter in the non-local setting, which in this case include the case when  $d_W < 1$ . As in Chapter 5, given a function  $u \in L^1(X, \mu)$  we denote  $E_s(u) = \{x \in X, u(x) > s\}$ , for  $E \subset K$  denote by  $(E)_r$  the  $r$ -neighborhood of  $E$  and by  $\partial E$  its topological boundary.

THEOREM 6.4. *Let  $X$  be Ahlfors  $d_H$ -regular with a heat kernel that satisfies the upper estimate in (6.1). For  $u \in L^1(X, \mu)$ , assume that there is  $0 < \alpha \leq \min\{d_H/d_W, 1\}$  and  $R > 0$  such that*

$$(6.4) \quad h_R(u, s) = \sup_{r \in (0, R]} \frac{1}{r^{\alpha d_W}} \mu\{x \in E_s(u) : d(x, X \setminus E_s(u)) < r\}$$

is in  $L^1(\mathbb{R}, ds)$ . Then,  $u \in \mathbf{B}^{1, \alpha}(X)$  and there exist constants  $C_1, C_2 > 0$  independent of  $u$  such that

$$(6.5) \quad \|u\|_{1, \alpha} \leq C_1 R^{-\alpha} \|u\|_{L^1(X, \mu)} + C_2 \int_{\mathbb{R}} h(u, s) ds.$$

In particular, for any set  $E \subset X$  we have

$$(6.6) \quad \|\mathbf{1}_E\|_{1, \alpha} \leq C_1 R^{-\alpha} \mu(E) + C_2 \sup_{r \in (0, R]} \frac{1}{r^{\alpha d_W}} \mu\{x \in E : d(x, X \setminus E) < r\}.$$

PROOF. This is analogous to the local case and we only give a sketch of the proof. Applying Theorem 6.4 with  $f = \mathbf{1}_{E_s(u)}$  and  $p = 1$  we have that, for any  $R > 0$  and  $s \in \mathbb{R}$ ,

$$(6.7) \quad \begin{aligned} t^{-\alpha} \int_{X \times X} |\mathbf{1}_{E_s(u)}(y) - \mathbf{1}_{E_s(u)}(x)| p_t(x, y) d\mu(y) d\mu(x) \\ \leq C_1 R^{-\alpha d_W} \|\mathbf{1}_{E_s(u)}\|_{L^1(X, \mu)} + C_2 \sup_{r \in (0, R]} N_1^{\alpha d_W}(\mathbf{1}_{E_s(u)}, r). \end{aligned}$$

Moreover, in view of definition (5.2) and applying the Ahlfors  $d_H$ -regularity of the space  $X$  we get

$$\begin{aligned} N_1^{\alpha d_W}(\mathbf{1}_{E_s(u)}, r) &= \frac{1}{r^{\alpha d_W + d_H}} \mu \otimes \mu(\{(x, y) \in E_s(u) \times (X \setminus E_s(u)) : d(x, y) < r\}) \\ &\leq \frac{1}{r^{\alpha d_W + d_H}} \int_{\{x \in E_s(u) : d(x, X \setminus E_s(u)) < r\}} \mu(B(y, r)) d\mu(x) \\ &\leq \frac{1}{r^{\alpha p}} \mu(\{x \in E_s(u) : d(x, X \setminus E_s(u)) < r\}) = h_R(u, s). \end{aligned}$$

Plugging this into (6.7), integrating both sides of over  $\mathbb{R}$  and noting that  $\int_{\mathbb{R}} \|\mathbf{1}_{E_s(u)}\|_{L^1(X,\mu)} = \|u\|_{L^1(X,\mu)}$  finally yields (6.5). The last assertion follows from the fact that  $E_s(\mathbf{1}_E) = E$  if  $s \in [0, 1]$  and zero otherwise.  $\square$

**REMARK 6.5.** *The previous theorem could be stated with  $0 < \alpha < 1$ . However, if  $\alpha > d_H/d_W$ , there will be no function  $h_R(u, s)$  satisfying the required conditions. Analogous reasons explain this restriction for  $\alpha$  in all subsequent statements.*

We can now derive the same corollaries as in the local case, since these do not involve heat kernel estimates. We restate them for completeness and refer for proof to the previous chapter.

**COROLLARY 6.6.** *Let  $X$  be a uniformly locally connected space with a heat kernel that satisfies the upper heat kernel estimate (6.1). If  $E \subset X$  has finite measure and for some  $0 < \alpha \leq \min\{\frac{d_H}{d_W}, 1\}$*

$$\sup_{r \in (0, R]} \frac{1}{r^{\alpha d_W}} \mu\{x \in E : d(x, \partial E) < r\} < \infty,$$

then  $\mathbf{1}_E \in \mathbf{B}^{1,\alpha}(X)$  and

$$\|\mathbf{1}_E\|_{1,\alpha} \leq C_1 R^{-\alpha d_W} \mu(E) + C_2 \sup_{r \in (0, R]} \frac{1}{r^{\alpha d_W}} \mu\{x \in E : d(x, \partial E) < r\}.$$

**COROLLARY 6.7.** *Let  $X$  be a uniformly locally connected space with a heat kernel that satisfies the upper estimate (6.1) and let  $u \in L^1(X, \mu)$ . Assume that  $\partial E_s(u)$  is finite and  $s \mapsto |\partial E_s(u)|$  is in  $L^1(\mathbb{R})$ , where  $|\partial E_s(u)|$  denotes the cardinality of  $\partial E_s(u)$ . Then, if  $d_H \leq d_W$ , we have  $u \in \mathbf{B}^{1, \frac{d_H}{d_W}}(X)$  and there exists a constant  $C > 0$  independent of  $u$  such that*

$$\|u\|_{1, d_H/d_W} \leq C \int_{\mathbb{R}} |\partial E_s(u)| ds.$$

In particular, if  $E \subset X$  is a set of finite measure whose boundary is finite, then  $\mathbf{1}_E \in \mathbf{B}^{1, \frac{d_H}{d_W}}(X)$  and

$$\|\mathbf{1}_E\|_{1, d_H/d_W} \leq C |\partial E|.$$

In the same fashion, the analogous of Proposition 5.10 will hold when the level sets  $E_s(u)$  are uniformly porous at large scales.

### 6.3. Sets of finite perimeter and fractional content of boundaries

**6.3.1. Characterizations of the sets of finite perimeter.** In the previous sections we have provided the non-local counterpart of all ingredients that lead to the characterization result provided by Theorem 5.11 in the local case. With the corresponding restriction on the parameter  $\alpha$ , the same result will hold in this setting.

**THEOREM 6.8.** *Under the assumptions of Theorem 6.1, for a bounded measurable set  $E \subset X$  and  $0 < \alpha \leq \min\{1, d_H/d_W\}$  we consider the following properties:*

- (1)  $\mathbf{1}_E \in \mathbf{B}^{1,\alpha}(X)$ ;
- (2)  $\sup_{r>0} \frac{1}{r^{\alpha d_W + d_H}} (\mu \otimes \mu)(\{(x, y) \in E \times E^c : d(x, y) < r\}) < +\infty$ ;
- (3)  $\limsup_{r \rightarrow 0^+} \frac{1}{r^{\alpha d_W + d_H}} (\mu \otimes \mu)(\{(x, y) \in E \times E^c : d(x, y) < r\}) < +\infty$ ;
- (4)  $\limsup_{r \rightarrow 0^+} \frac{1}{r^{\alpha d_W}} \mu(\{x \in E : d(x, E^c) < r\}) < +\infty$ ;
- (5)  $\limsup_{r \rightarrow 0^+} \int_{\{x \in E : d(x, E^c) < r\}} \int_{B(x, r) \cap E^c} \frac{1}{d(x, y)^{d_H + \alpha d_W}} d\mu(y) d\mu(x) < +\infty$ ;
- (6)  $\int_E \int_{E^c} \frac{1}{d(x, y)^{d_H + \alpha d_W}} d\mu(y) d\mu(x) < +\infty$ .



Then, the following relations hold:

(i) (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3); moreover there exist constants  $c, C > 0$  independent of  $E$  such that

$$c \sup_{r>0} \frac{1}{r^{\alpha d_W + d_H}} (\mu \otimes \mu) (\{(x, y) \in E \times E^c : d(x, y) < r\}) \\ \leq \|\mathbf{1}_E\|_{1, \alpha} \leq C \sup_{r>0} \frac{1}{r^{\alpha d_W + d_H}} (\mu \otimes \mu) (\{(x, y) \in E \times E^c : d(x, y) < r\});$$

(ii) (4)  $\Rightarrow$  (5)  $\Rightarrow$  (1);

(iii) (6)  $\Rightarrow$  (1);

(iv) If in addition  $E$  is porous, c.f. (5.20), then (1)  $\Rightarrow$  (4);

The proofs follow verbatim, replacing Theorem 5.1 by Theorem 6.1 and Theorem 5.3 by Theorem 6.4. Notice that Proposition 5.16 does not actually involve any heat kernel estimates. Assuming uniformly locally connectedness on the space, which includes the case of  $X$  being geodesic, we obtain as in Corollary 5.5 the corresponding characterization of sets with finite  $\alpha$ -perimeter in terms of inner Minkowski contents.

### 6.3.2. Further results.

**THEOREM 6.9.** *Assume that  $p_t(x, y)$  satisfies the upper heat kernel estimate (6.1) and  $d_H < d_W$ . If there exists  $0 < \alpha < d_H/d_W$  such that for all  $\varepsilon > 0$ ,*

$$(6.8) \quad \mu \otimes \mu \{(x, y) \in E^c \times E : d(x, y) < \varepsilon\} \leq c_7 (\varepsilon^{d_H + \alpha d_W} + \varepsilon^{2d_H}),$$

then for all  $t > 0$  it holds that

$$(6.9) \quad (t^{-\alpha} + t^{-d_H/d_W}) \|P_t \mathbf{1}_E - \mathbf{1}_E\|_{L^1(X, \mu)} \leq c_{11} < \infty,$$

where  $c_{11}$  is a constant given by (6.11) that depends on  $\delta, d_H$  and  $d_W$ .

**PROOF.** In view of (6.1),  $p_t(x, y) > s$  implies

$$(6.10) \quad d(x, y) < t^{1/d_W} c_4^{-1} \left[ \left( c_3^{-1} t^{\frac{d_H}{d_W}} s \right)^{-\frac{1}{d_W + d_H}} - 1 \right] =: F(t, s).$$

Moreover,  $F(t, s) > 0$  if and only if  $s < c_3 t^{-\frac{d_H}{d_W}}$  and the corresponding first term in the expression (5.32) can now be bounded by

$$\int_0^{c_3 t^{-d_H/d_W}} c_7 c_4^{-d_H + \alpha d_W} t^{\frac{d_H + \alpha d_W}{d_W}} \left[ \left( c_3^{-1} t^{\frac{d_H}{d_W}} s \right)^{-\frac{1}{d_W + d_H}} - 1 \right]^{d_H + \alpha d_W} ds \\ = c_7 c_3 c_4^{-(d_H + \alpha d_W)} t^\alpha \int_0^1 \left( u^{-\frac{1}{d_W + d_H}} - 1 \right)^{d_H + \alpha d_W} du$$

which is finite since by assumption  $0 < \delta < d_H < d_W$ . Thus,

$$c_7 \int_0^{c_3 t^{-d_H/d_W}} (F(t, s))^{d_H + \alpha d_W} ds \leq c_7 c_3 c_4^{-(d_H + \alpha d_W)} t^\alpha \frac{d_W + d_H}{d_W(1 - \alpha)}$$

and, since  $d_H < d_W$ , the same estimate for the second term in (5.32) with  $\alpha = d_H/d_W$  yields

$$c_7 \int_0^{c_3 t^{-d_H/d_W}} (F(t, s))^{d_H + d_H} ds \leq c_7 c_3 c_4^{-2d_H} t^{\frac{d_H}{d_W}} \frac{d_W + d_H}{d_W - d_H}$$

so that (6.9) holds with

$$(6.11) \quad c_{11} = c_7 c_3 c_4^{-d_H} \max \left\{ c_4^{\alpha d_W} \frac{d_W + d_H}{d_W(1 - \alpha)}, c_4^{-d_H} \frac{d_W + d_H}{d_W - d_H} \right\}.$$

□

**THEOREM 6.10.** *Assume that  $p_t(x, y)$  satisfies the lower heat kernel estimate in (6.1). If there exist  $\alpha > 0$  and  $\varepsilon > 0$  such that*

$$(6.12) \quad \mu \otimes \mu\{(x, y) \in E^c \times E : d(x, y) < \varepsilon\} \geq c_8 \varepsilon^{d_H + \alpha d_W},$$

*then, for  $t = \varepsilon^{d_W}$ , it holds that*

$$(6.13) \quad t^{-\alpha} \|P_t \mathbf{1}_E - \mathbf{1}_E\|_{L^1(X, \mu)} \geq 2c_5 c_8 e^{-c_6} > 0.$$

**COROLLARY 6.11.** *If there exists  $\alpha > 0$  and  $\varepsilon_0 > 0$  such that (6.12) holds for any  $0 < \varepsilon < \varepsilon_0$ , then (6.13) holds for any  $0 < t < \varepsilon^{d_W}$ .*

**PROOF OF THEOREM 6.10.** The lower estimate in (6.1) implies that

$$p_t(x, y) \geq c_5 t^{-\frac{d_H}{d_W}} c_6^{-(d_H + d_W)}$$

and hence assumption (6.12) yields

$$\begin{aligned} \|\mathbf{1}_{E^c} P_t \mathbf{1}_E\|_{L^1(X, \mu)} &\geq \int_{A_{E, t^{1/d_W}}} p_t(x, y) \mu(dx) \mu(dy) \geq c_5 c_6^{-(d_H + d_W)} t^{-\frac{d_H}{d_W}} \mu \otimes \mu(A_{E, t^{1/d_W}}) \\ &\geq c_5 c_6^{-(d_H + d_W)} t^{-\frac{d_H}{d_W}} c_8 t^{\frac{(d_H + \alpha d_W)}{d_W}} = c_5 c_8 c_6^{-(d_H + d_W)} t^\alpha. \end{aligned}$$

□

Under weaker assumptions, the following localized version of Theorem 6.9 is available.

**THEOREM 6.12.** *Let  $E \subset B \subset X$  and assume that  $p_t(x, y)$  satisfies the upper heat kernel estimate (6.1). If there exists  $0 < \alpha < 1$  such that for all  $\varepsilon > 0$ ,*

$$(6.14) \quad \mu \otimes \mu\{(x, y) \in (B \cap E^c) \times E : d(x, y) < \varepsilon\} \leq c_7 \varepsilon^{d_H + \alpha d_W},$$

*then for all  $t > 0$  it holds that*

$$(6.15) \quad t^{-\alpha} \|P_t \mathbf{1}_E - \mathbf{1}_E\|_{L^1(B, \mu)} \leq c_{11} < \infty,$$

*where  $c_{11}$  is a constant given by (6.16) that depends on  $\alpha, d_H$  and  $d_W$ .*

**PROOF.** Following the proof of Theorem 5.20 verbatim we have

$$\|\mathbf{1}_{E^c} P_t \mathbf{1}_E\|_{L^1(B, \mu)} \leq \int_0^{c_3 t^{-d_H/d_W}} c_7 F(t, s)^{d_H + \alpha d_W} ds.$$

Since  $0 < \alpha < 1$  by assumption, the same computations as in the proof of Theorem 6.9 imply that (6.15) holds with

$$(6.16) \quad c_{11} = c_7 c_3 c_4^{-(d_H + \alpha d_W)} \frac{d_W + d_H}{d_W(1 - \alpha)}.$$

□

#### 6.4. Critical exponent for $\mathbf{B}^{1, \alpha}(X)$

The observation that the space  $\mathbf{B}^{1, \alpha}(X)$  is trivial for any  $\alpha > 1$  is the starting point of the discussion that will be carried out in this section, where we compute

$$(6.17) \quad \alpha^* = \inf\{\alpha > 0 : \mathbf{B}^{1, \alpha}(X) \text{ is trivial}\}.$$

**PROPOSITION 6.13.** *If  $f \in \mathbf{B}^{1, \alpha}(X)$  with  $\alpha > 1$ , then  $f = 0$ .*

PROOF. Notice first that for any  $f \in \mathbf{B}^{1,\alpha}(X)$ ,

$$\|P_t(|f - f(\cdot)|)\|_{L^1(X,\mu)} \leq t^\alpha \|f\|_{1,\alpha}.$$

Now assume that  $f$  is bounded (in fact, one can consider  $f_N$  defined in the proof of Proposition 1.11). Then,

$$\begin{aligned} \|P_t(|f - f(\cdot)|)\|_{L^1(X,\mu)} &\leq \int_X P_t(|f - f(x)|^2)(x) d\mu(x) \\ &\leq 2\|f\|_{L^\infty(X)} \|P_t(|f - f(\cdot)|)\|_{L^1(X,\mu)} \leq 2t^\alpha \|f\|_{L^\infty(X)} \|f\|_{1,\alpha}. \end{aligned}$$

On the one hand, since  $\alpha > 1$ , we have

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_X P_t(|f - f(x)|^2)(x) d\mu(x) = 0.$$

On the other hand, as in Lemma 1.9 and Proposition 1.10 we know that

$$\frac{1}{t} \int_X P_t(|f - f(x)|^2)(x) d\mu(x) = 2\mathcal{E}_t(f),$$

where  $\mathcal{E}_t(f)$  is decreasing. Hence,  $\|P_t(|f - f(\cdot)|)\|_{L^1(X,\mu)} = 0$  for every  $t > 0$  and therefore  $P_t f(x) = f(x)$  for every  $t > 0$ . The upper bound in (6.1) yields

$$|f(x)| = |P_t f(x)| \leq \int_X p_t(x, y) |f(y)| d\mu(y) \leq \frac{C}{t^{d_H/d_W}} \|f\|_{L^1(X,\mu)}$$

and letting  $t \rightarrow \infty$ , we obtain that  $f = 0$ .  $\square$

In order to compute the critical parameter  $\alpha^*$ , we shall make the following assumption on the underlying metric measure space  $(X, d, \mu)$ :

ASSUMPTION 6.14. *There exists a constant  $C > 0$ , such that for every  $f \in \mathfrak{B}_{1,\infty}^1(X)$*

$$\sup_{\varepsilon > 0} \frac{1}{\varepsilon^{d_H+1}} \int_{\Delta_\varepsilon} |f(y) - f(x)| d\mu(x) d\mu(y) \leq C \limsup_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^{d_H+1}} \int_{\Delta_\varepsilon} |f(y) - f(x)| d\mu(x) d\mu(y),$$

where  $\Delta_\varepsilon = \{(x, y) \in X \times X, d(x, y) < \varepsilon\}$ .

This property of the metric space  $(X, d, \mu)$  is for instance satisfied if  $(X, d)$  satisfies the assumptions of section 4.3, see remark 4.12.

One has then the following theorem:

THEOREM 6.15. *Under Assumption 6.14 and if  $d_W \leq 1$ , then  $\alpha^* = 1$  and  $\mathbf{B}^{1,\alpha^*}(X) = \mathfrak{B}_{1,1}^{d_W}(X)$ . If  $d_W > 1$ , then  $\alpha^* = \frac{1}{d_W}$  and  $\mathbf{B}^{1,\alpha^*}(X) = \mathfrak{B}_{1,\infty}^1(X)$ .*

PROOF. If  $d_W \leq 1$ , it follows from Proposition 6.13 that  $\mathbf{B}^{1,\alpha}(X)$  is trivial for any  $\alpha > 1$ . It remains to justify that  $\mathbf{B}^{1,\alpha}(X)$  or equivalently  $\mathfrak{B}_{1,\infty}^{\alpha d_W}(X)$  is not trivial if  $\alpha < 1$ . This follows from the fact 4 below since one has then  $\mathfrak{B}_{1,\infty}^1(X) \subset \mathfrak{B}_{1,\infty}^{\alpha d_W}(X)$ .

If  $d_W > 1$ , the conclusion follows by considering the following four facts.

**Fact 1.**  $\mathbf{B}^{1,\alpha}(X)$  is trivial for  $\alpha > 1$ . This immediately follows from Proposition 6.13.

**Fact 2.**  $\mathbf{B}^{1,1}(X)$  is trivial. Let  $f \in \mathbf{B}^{1,1}(X)$ , then for any  $\varepsilon > 0$ , Lemma 6.3 yields

$$\frac{1}{\varepsilon^{d_H+1}} \int_{\Delta_\varepsilon} |f(y) - f(x)| d\mu(x) d\mu(y) \leq \varepsilon^{d_W-1} \int_{\Delta_\varepsilon} \frac{|f(y) - f(x)|}{d(x, y)^{d_H+d_W}} d\mu(x) d\mu(y) \leq C \varepsilon^{d_W-1} \|f\|_{1,1}.$$

Thus,

$$\limsup_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^{d_H+1}} \int_{\Delta_\varepsilon} |f(y) - f(x)| d\mu(x) d\mu(y) = 0.$$

From assumption (6.14), this implies that  $f$  is constant and thus 0.

**Fact 3.**  $\mathbf{B}^{1,\alpha}(X)$  is trivial for any  $\frac{1}{d_W} < \alpha < 1$ . By Theorem 6.1, we have  $\mathbf{B}^{1,\alpha}(X) = \mathfrak{B}_{1,\infty}^{\alpha d_W}(X)$ . Thus it suffices to show that  $\mathfrak{B}_{1,\infty}^\beta(X)$  is trivial for  $1 < \beta < d_W$ . For any  $\varepsilon > 0$ ,

$$\frac{1}{\varepsilon^{d_H+1}} \int_{\Delta_\varepsilon} |f(y) - f(x)| d\mu(x) d\mu(y) \leq \frac{\varepsilon^{\beta-1}}{\varepsilon^{d_H+\beta}} \int_{\Delta_\varepsilon} |f(y) - f(x)| d\mu(x) d\mu(y) \leq C \varepsilon^{\beta-1} N_{1,\infty}^\beta(f).$$

Taking the limsup when  $\varepsilon$  goes to zero, again we have that  $f$  is a constant.

**Fact 4.** It remains to justify that  $\mathbf{B}^{1,\frac{1}{d_W}}(X)$  is not trivial, or equivalently,  $\mathfrak{B}_{1,\infty}^1(X)$  is not trivial. Fix  $x_0 \in X$  and  $R > 0$ . Set

$$\phi_{x_0,R}(x) = \max\{0, R - d(x_0, x)\}.$$

Then  $\phi_{x_0,R}$  is non-negative,  $L^1$ -integrable and supported in the ball  $B(x_0, R)$  (see also Section 4.1 for these functions). For any  $r > 0$ , we have

$$\begin{aligned} & \iint_{\Delta_r} |\phi_{x_0,R}(x) - \phi_{x_0,R}(y)| d\mu(x) d\mu(y) \\ & \leq 2 \int_{B(x_0,R)} \int_{B(x_0,R)^c \cap B(y,r)} |R - d(x_0, y)| d\mu(x) d\mu(y) \\ & \quad + \int_{B(x_0,R)} \int_{B(x_0,R) \cap B(y,r)} |d(x_0, x) - d(x_0, y)| d\mu(x) d\mu(y) \\ & \leq C r^{d_H+1} R^{d_H}. \end{aligned}$$

The second inequality above holds since  $x \in B(x_0, R)^c$  and  $y \in B(x_0, R)$  imply  $R - d(x_0, y) \leq d(x, y) \leq r$ . Therefore it easily follows from the definition that  $\phi_{x_0,R} \in \mathfrak{B}_{1,\infty}^1(X)$  with semi-norm estimate depending on  $R$ .  $\square$

## 6.5. Sobolev and isoperimetric inequalities

**6.5.1. Discussion on Sobolev inequalities for non-local Dirichlet spaces.** The standard Poincaré type inequalities do not make sense for non-local Dirichlet forms (although there are some forms of Poincaré type inequalities for Dirichlet forms, such as [OSCT08, CW17]); however, the following global Sobolev inequality makes sense: do there exist constants  $C > 0$  and  $\kappa \geq 1$  such that for every  $u \in \mathcal{F}$ ,

$$\left( \int_X |u|^{2\kappa} d\mu \right)^{\frac{1}{2\kappa}} \leq C \sqrt{\mathcal{E}(u, u)}$$

We now explore one condition under which the above inequality will hold. Recall that  $\mathcal{E}$  is *transient* if there is some almost-everywhere positive  $f \in L^1(X)$  such that

$$\int_0^\infty P_t f dt$$

is finite almost everywhere in  $X$ . Here  $P_t f$  is the heat extension of  $f$ , namely,

$$P_t f(x) = \int_X p_t(x, y) f(y) d\mu(y).$$

Thus transience of the form  $\mathcal{E}$  implies that  $P_t f$  decays fast as  $t \rightarrow \infty$  almost everywhere in  $X$ . For the rest of this subsection we will assume that  $\mathcal{E}$  is a transient regular Dirichlet form on  $L^2(X)$ , and that  $(\mathcal{E}, \mathcal{F})$  is strongly continuous. By completing  $(\mathcal{E}, \mathcal{F})$  with respect to the norm  $u \mapsto \|u\|_{L^2(X)} + \sqrt{\mathcal{E}(u, u)}$  if necessary, we may assume also that this pair is a Hilbert space (see [FOT11, Theorem 1.5.2, Lemma 1.5.5]). We now have the following version of Sobolev inequality:

LEMMA 6.16 (Theorem 1.5.3 in [FOT11] or Theorem 2.1.5 in [CF12]). *For a Dirichlet form  $(\mathcal{E}, \mathcal{F})$ , the strongly continuous semigroup  $P_t$  is transient if and only if there exists a bounded function  $g \in L^1(X)$  such that  $g$  is positive almost everywhere in  $X$  such that for all  $u \in \mathcal{F}$  we have*

$$\int_X |u|g \, d\mu \leq \sqrt{\mathcal{E}(u, u)}.$$

While the conclusion of the above lemma looks remarkably like the desired Sobolev inequality, the problem here is that we do not have good control over  $g$ , and in general we would like to have control over the  $L^q(X, \mu)$ -norm of  $u$  for some  $q > 0$ , not just over the  $L^1(X, g \, d\mu)$ -norm.

For a compact set  $K \subset X$  we set the variational capacity of  $K$  to be the number

$$\text{Cap}_0(K) := \inf_u \mathcal{E}(u, u),$$

where the infimum is over all  $u \in \mathcal{F} \cap C_0(X)$  (where  $C_0(X)$  is the collection of all compactly supported continuous functions on  $X$ ) with  $u(x) \geq 1$  for all  $x \in K$ , see [FOT11, Section 2.4]. The following capacity type inequality holds.

LEMMA 6.17 ([FOT11, Lemma 2.4.1]). *For  $u \in \mathcal{F} \cap C_0(X)$  and  $t > 0$  let  $K_t$  denote the set  $\{x \in X : |u(x)| \geq t\}$ . Then we have that*

$$\int_0^\infty 2t \text{Cap}_0(K_t) \, dt \leq 4 \mathcal{E}(u, u).$$

The above lemma indicates that there are plenty of compact sets in  $X$  with finite capacity. Let  $K \subset X$  be a compact set such that  $\text{Cap}_0(K)$  is finite. Then we can find a sequence  $u_k \in \mathcal{F} \cap C_0(X)$  with  $u_k \geq 1$  on  $K$  and  $\lim_k \mathcal{E}(u_k, u_k) = \text{Cap}_0(K) < \infty$ . Since truncations of a function in  $\mathcal{F}$  do not increase its Dirichlet energy, we know that  $\mathcal{E}(\widehat{u}_k, \widehat{u}_k) \leq \mathcal{E}(u_k, u_k)$  where  $\widehat{u}_k = \max\{0, \min\{u_k, 1\}\}$ . Thus without loss of generality we may assume that  $0 \leq u_k \leq 1$  on  $X$ . Therefore  $\{u_k\}_k$  forms a locally bounded sequence in  $L^2(X)$ , and by multiplying each  $u_k$  by a fixed  $\eta \in \mathcal{F} \cap C_0(X)$  with  $0 \leq \eta \leq 1$  on  $X$  and  $\eta = 1$  on the ball  $B(x_0, n)$  for some fixed  $x_0 \in X$  and  $n \in \mathbb{N}$ , we get a bounded sequence in  $\mathcal{F}$ , which, as  $\mathcal{F}$  is a Hilbert space, gives a convex combination subsequence that converges to some function  $u \in \mathcal{F}$  (we do this for each positive integer  $n$  and employ a Cantor-type diagonalization argument to do so) with  $\mathcal{E}(u, u) \leq \lim_k \mathcal{E}(u_k, u_k) = \text{Cap}_0(K)$ . Employing a further subsequence argument, we can ensure that  $u \geq 1$  in  $K$ ; however,  $u$  may no longer be continuous on  $X$ , see also [FOT11, Lemma 2.1.1]. Such a function  $u$  is called a *0-th order equilibrium potential* of  $K$  in [FOT11].

Given the above notion of capacity, the following theorem identifies a property on  $\mathcal{E}$  under which we have the desired Sobolev inequality.

THEOREM 6.18 ([FOT11, Theorem 2.4.1]). *If there is some  $\kappa \geq 1$  and  $\Theta > 0$  such that*

$$(6.18) \quad \mu(K)^{1/\kappa} \leq \Theta \text{Cap}_0(K)$$

*for each compact  $K \subset X$ , then there is some  $C > 0$  with  $0 < C^2 \leq (4\kappa)^\kappa \Theta$  such that*

$$\left( \int_X |u|^{2\kappa} \, d\mu \right)^{1/2\kappa} \leq C \sqrt{\mathcal{E}(u, u)}.$$

The inequality (6.18) is an analog of the *isoperimetric inequality* adapted to the non-local Dirichlet form. The optimal constant  $\Theta$  is called the isoperimetric constant of  $\mathcal{E}$ , and is a non-local analog of the Cheeger constant. Indeed, [FOT11, Theorem 2.4.1] claims even more, that the support of a Sobolev type inequality is equivalent to this isoperimetric inequality. Should  $\mathcal{E}$  support such an inequality, then the total capacity of a compact set  $K$  given by

$$\text{Cap}_1(K) := \inf_u \int_X u^2 \, d\mu + \mathcal{E}(u, u)$$

with infimum over all  $u \in \mathcal{F} \cap C_0(X)$  with  $u \geq 1$  on  $K$  has the same class of null capacity sets as  $\text{Cap}_0$ . We refer the interested reader to [FOT11, Theorem 2.4.3] for examples of Sobolev type inequality for measures in Euclidean spaces.

We point out here that this notion of isoperimetric inequality is weaker than the traditional one where  $\text{Cap}_0(K)$  is replaced with the relative 1-capacity of  $K$ , namely infimum of the numbers  $\int_X |\nabla u| d\mu$  over all  $u \in \mathcal{F} \cap C_0(X)$  with  $u \geq 1$  on  $K$ . In the case of the non-local Dirichlet form the quantity  $\int_X |\nabla u| d\mu$  does not make usual sense (although see [HRT13] for a version of such an integral for arbitrary Dirichlet forms). On the other hand, the results of [FOT11, Theorem 2.1.1] tell us that  $\text{Cap}_0$  has an extension to a Choquet capacity on  $X$ .

Fix  $u \in \mathcal{F} \cap C_0(X)$ , and let  $t > 0$ . For  $\varepsilon > 0$  we set

$$u_t^\varepsilon = \frac{\min\{\varepsilon, \max\{u - t, 0\}\}}{\varepsilon}.$$

Then by the Markov property of  $\mathcal{E}$  we know that  $\mathcal{E}(u_t^\varepsilon, u_t^\varepsilon) \leq \mathcal{E}(u, u)/\varepsilon^2$ . Observe that  $u_t^\varepsilon \rightarrow \chi_{K_t}$  in  $L^1(X)$  as  $\varepsilon \rightarrow 0^+$ . Suppose we know that  $\liminf_{\varepsilon \rightarrow 0^+} \mathcal{E}(u_t^\varepsilon, u_t^\varepsilon) < \infty$ . Then

$$\text{Cap}_0(K_t) \leq \liminf_{\varepsilon \rightarrow 0^+} \mathcal{E}(u_t^\varepsilon, u_t^\varepsilon) < \infty.$$

If  $\mathcal{E}$  is strongly local and supports a 2-Poincaré inequality, then using the fact that distance functions belong to  $\mathcal{F}$  we can obtain a comparison between  $\text{Cap}_0(K_t)$  and the Minkowski co-dimension 1 content of  $\partial K_t$  for almost every  $t > 0$ , see for example the arguments in [CJKS17, JKY14]. For strongly local Dirichlet forms we have shown in Chapter 4 that the correct BV class is  $\mathbf{B}^{1,1/2}(X)$ . For non-local Dirichlet forms (and indeed even for forms that are not strongly local) it is not clear which of the Besov classes  $\mathbf{B}^{1,\alpha}(X)$  should be the correct analog of BV functions, see Theorem 6.8. From this theorem, we see that if  $E \subset X$  is bounded and measurable such that  $\mathbf{1}_E \in \mathbf{B}^{1,\alpha}(X)$ , then by choosing  $r > 2\text{diam}(E)$ ,

$$\frac{c}{r^{\alpha d_W + d_H}} \mu(E) \mu(B(x_0, r/2)) \leq \|\mathbf{1}_E\|_{1,\alpha}$$

where  $x_0 \in X$  such that  $B(x_0, r/2) \cap E$  is empty and  $B(x_0, r/2) \subset \bigcup_{x \in E} B(x, r)$ . One can look on this as the Besov space analog of the isoperimetric inequality.

**6.5.2. Besov spaces embeddings.** Note that by combining the results of Part I, Chapter 2, with the results of this chapter, one immediately obtains the following corollaries. The proofs are similar to the ones in Section 4.7, so will be omitted for concision.

**COROLLARY 6.19.** *Let  $(X, \mu, \mathcal{E}, \mathcal{F})$  be a symmetric non-local Dirichlet space with heat kernel estimates (6.1). Let  $0 < \delta < d_H$ . Let  $1 \leq p < \min\{\frac{d_H}{\delta}, \frac{d_W}{\delta}\}$ . There exists a constant  $C_{p,\delta} > 0$  such that for every  $f \in \mathbf{B}^{p,\delta/d_W}(X)$ ,*

$$\sup_{s \geq 0} s \mu(\{x \in X, |f(x)| \geq s\})^{\frac{1}{q}} \leq C_{p,\delta} \sup_{r > 0} \frac{1}{r^{\delta + d_H/p}} \left( \iint_{\Delta_r} |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p}$$

where  $q = \frac{pd_H}{d_H - p\delta}$ . Furthermore, for every  $0 < \delta < d_H$ , there exists a constant  $C_{\text{iso},\delta}$  such that for every measurable  $E \subset X$ ,  $\mu(E) < +\infty$ ,

$$(6.19) \quad \mu(E)^{\frac{d_H - \delta}{d_H}} \leq C_{\text{iso},\delta} \sup_{r > 0} \frac{1}{r^{\delta + d_H}} (\mu \otimes \mu) \{(x, y) \in E \times E^c : d(x, y) \leq r\}$$

**COROLLARY 6.20.** *Let  $(X, \mu, \mathcal{E}, \mathcal{F})$  be a symmetric non-local Dirichlet space with heat kernel estimates (6.1) and  $d_W \leq d_H$ . Let  $0 < \delta < d_H$ . Let  $p = \frac{d_W}{\delta}$  and assume  $p \geq 1$ . There exists a constant  $C_{p,\delta} > 0$  such that for every  $f \in \mathbf{B}^{p,\delta/d_W}(X)$ ,*

$$\sup_{s \geq 0} s \mu(\{x \in X, |f(x)| \geq s\})^{\frac{1}{q}} \leq C_{p,\delta} \left( \int_0^\infty \left( N_p^{d_W/p}(f, r) \right)^p \frac{dr}{r} \right)^{1/p},$$

where  $q = \frac{pd_H}{d_H - p\delta}$  and

$$N_p^{d_W/p}(f, r) := \frac{1}{r^{d_W/p + d_H/p}} \left( \iint_{\Delta_r} |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p}$$

## Sets of finite perimeter in some infinite-dimensional examples

In this chapter, we study some infinite-dimensional examples of Dirichlet spaces  $X$ , associated sets of finite perimeter and discuss the connection with  $\mathbf{B}^{1,1/2}(X)$ . Our goal is not to study the general theory of Besov spaces related to infinite-dimensional Dirichlet spaces, but rather to look at some specific examples.

In many infinite-dimensional examples, coming back to the original ideas by De Giorgi [DG54] it seems interesting to take advantage of the existence of nice integration by parts formulas to construct BV functions and sets of finite perimeter. To provide the reader with a motivation, we briefly present below those ideas in a general setting, avoiding at first the exact assumptions and technicalities associated with a possible infinite-dimensional setting.

Let  $(X, \mu, \mathcal{E}, \mathcal{F} = \mathbf{dom}(\mathcal{E}))$  be a quasi regular local symmetric Dirichlet space. We assume that  $\mathcal{E}$  can be written as

$$\mathcal{E}(f, f) = \|Df\|_{\mathcal{H}}^2,$$

where  $D$  is a closed unbounded operator from  $L^2(X, \mu)$  to some (separable) Hilbert space  $\mathcal{H}$ . In many situations (see for instance [Ebe99], Theorem 3.9 and the examples below), we are able to think of  $\mathcal{H}$  as a  $L^2$  space of sections of a vector bundle over  $X$ : more precisely,  $\mathcal{H}$  is isometrically isomorphic to some  $\int_X^{\oplus} \mathcal{H}_x d\mu(x)$  and for any  $\eta_1, \eta_2 \in \mathcal{H}$ ,

$$\langle \eta_1, \eta_2 \rangle_{\mathcal{H}} = \int_X \langle \eta_1, \eta_2 \rangle_{\mathcal{H}_x} d\mu(x).$$

In this situation, following De Giorgi in [DG54], for  $f \in L^1(X, \mu)$  it is then natural to define:

$$\mathbf{Var} f = \sup \left\{ \int_X f D^* \eta d\mu : \eta \in \mathbf{dom}(D^*), \|\eta\|_{\mathcal{H}_x} \leq 1, \mu\text{-almost everywhere} \right\},$$

where  $D^*$  is the adjoint operator of  $D$  and one defines then:

$$BV(X) := \{f \in L^1(X, \mu) : \mathbf{Var} f < +\infty\}.$$

This allows one to define the perimeter of a measurable set  $E \subset X$  with  $\mathbf{1}_E \in BV(X)$  by

$$P(E, X) = \mathbf{Var} \mathbf{1}_E.$$

When  $X$  is a complete Riemannian manifold and  $D = \nabla$  is the Riemannian gradient, this construction yields the usual  $BV$  space and sets with finite perimeters are Caccioppoli sets.

In the following examples, the strong Bakry-Émery condition is satisfied

$$\|DP_t f\|_{\mathcal{H}_x} \leq (P_t \|Df\|_{\mathcal{H}})(x).$$

and one shall prove:

**THEOREM 7.1.** *Let  $X$  be one of the Dirichlet spaces studied in Sections 7.1 or 7.2. Let  $E$  be a measurable set with finite measure. The following are equivalent:*

- (1)  $P(E, X) < +\infty$ ;
- (2) The limit  $\lim_{t \rightarrow 0^+} \int_X \|DP_t \mathbf{1}_E\|_{\mathcal{H}_x} d\mu(x)$  exists.

Moreover, if  $E$  is a set satisfying one of the above conditions, then  $\mathbf{1}_E \in \mathbf{B}^{1,1/2}(X)$  and

$$\|\mathbf{1}_E\|_{1,1/2} \leq 2\sqrt{2}P(E, X) = 2\sqrt{2} \lim_{t \rightarrow 0^+} \int_X \|DP_t \mathbf{1}_E\|_{\mathcal{H}_x} d\mu(x).$$

The theorem is stated in a class of specific examples, but the method which is developed is actually relatively general.

### 7.1. Wiener space

In this section, we discuss sets of finite perimeters in abstract Wiener space, The theory of BV functions is already very well established and understood in that case (see Fukushima-Hino [FH01], and also the survey [MNP15] and the references therein).

Let  $(X, \mathcal{H}, \mu)$  be an abstract Wiener space in the sense of L. Gross [Gro67], so  $X$  is a separable real Banach space,  $\mathcal{H}$  is a separable real Hilbert space continuously and densely embedded into  $X$  and  $\mu$  is a Gaussian measure satisfying for  $l \in X^* \subset \mathcal{H}^* \simeq \mathcal{H}$ ,

$$\int_X \exp(i\langle l, x \rangle) \mu(dx) = \exp\left(-\frac{1}{2}\|l\|_{\mathcal{H}}^2\right).$$

We define the Ornstein-Uhlenbeck semigroup on  $L^2(X, \mu)$  by the formula:

$$(7.1) \quad P_t f(x) = \int_X f\left(e^{-t}x + \sqrt{1 - e^{-2t}}y\right) d\mu(y), \quad f \in L^2(X, \mu), x \in X.$$

The associated Dirichlet form may be written in terms of the Malliavin derivative. We now describe this Dirichlet form. We say that a functional  $F : X \rightarrow \mathbb{R}$  is a smooth cylindric functional if, for some  $n$ , it can be written in the form

$$F(x) = f(\langle x, l_1 \rangle, \dots, \langle x, l_n \rangle),$$

where  $f$  is smooth and bounded, and  $l_1, \dots, l_n \in \mathcal{H}^*$ . Here  $\langle x, l \rangle$  stands for the Wiener integral. The set of smooth cylindric functionals will be denoted by  $\mathcal{C}$  and will be used as nice core algebra. Note that  $\mathcal{C}$  is dense in  $L^2(X, \mu)$ . If  $F \in \mathcal{C}$ , one defines its Malliavin derivative by the formula,

$$(7.2) \quad DF(x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\langle x, l_1 \rangle, \dots, \langle x, l_n \rangle) l_i \in \mathcal{H}.$$

We can then consider the pre-Dirichlet form  $(\mathcal{E}, \mathcal{C}, \mu)$  given for  $F, G \in \mathcal{C}$  by

$$\mathcal{E}(F, G) = \int_X \langle DF(x), DG(x) \rangle_{\mathcal{H}} d\mu(x).$$

The closure of this pre-Dirichlet form yields a quasi-regular local symmetric Dirichlet space  $(X, \mathcal{F}, \mu)$  whose semigroup is precisely the Ornstein-Uhlenbeck semigroup (see [Kus82]).

Note that the Dirichlet form  $\mathcal{E}$  admits a carré du champ and that one has for  $f \in \mathcal{F}$ , and  $x \in X$ ,

$$\Gamma(f)(x) = \|Df(x)\|_{\mathcal{H}}^2.$$

where  $D$  stands for the closed extension of the formula (7.2). One can check directly from the formula (7.1) that for  $f \in \mathcal{C}$ ,

$$DP_t f(x) = \vec{P}_t Df(x)$$

where  $\vec{P}_t$  is a semigroup on  $L^2(X, \mathcal{H}, \mu)$  that satisfies for every  $\eta \in L^2(X, \mathcal{H}, \mu)$ ,

$$(7.3) \quad \|\vec{P}_t \eta(x)\|_{\mathcal{H}} \leq e^{-t}(P_t \|\eta\|_{\mathcal{H}})(x)$$

In particular, one has the strong Bakry-Émery curvature condition:

$$\|DP_t f(x)\|_{\mathcal{H}} \leq (P_t \|Df\|_{\mathcal{H}})(x), \quad f \in \mathcal{F}.$$

We are now ready to define sets of finite perimeter, following Fukushima-Hino [FH01].



One can think of  $D$  as a closed unbounded operator  $L^2(X, \mu) \rightarrow L^2(X, \mathcal{H}, \mu)$ . Denote its adjoint by  $D^*$  (this is the so-called Skorohod integral). If  $E \subset X$  is a measurable set,

$$P(E, X) = \sup \left\{ \int_E D^* \eta d\mu : \eta \in \mathbf{dom}(D^*), \|\eta(x)\|_{\mathcal{H}} \leq 1, \mu\text{-almost everywhere} \right\},$$

and we say that  $E$  has a finite perimeter if  $P(E, X) < +\infty$ . As everywhere in the paper, for  $f \in L^p(X, \mu)$ , we define:

$$\|f\|_{p,\alpha} = \sup_{t>0} t^{-\alpha} \left( \int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p}.$$

and

$$\mathbf{B}^{p,\alpha}(X) = \{f \in L^p(X, \mu) : \|f\|_{p,\alpha} < +\infty\}.$$

The theorem is the following:

**THEOREM 7.2.** *Let  $E$  be a measurable set. The following are equivalent:*

- (1)  $P(E, X) < +\infty$ ;
- (2) The limit  $\lim_{t \rightarrow 0^+} \int_X \|DP_t \mathbf{1}_E(x)\|_{\mathcal{H}} d\mu(x)$  exists.

Moreover, if  $E$  is a set satisfying one of the above conditions, then  $\mathbf{1}_E \in \mathbf{B}^{1,1/2}(X)$  and

$$\|\mathbf{1}_E\|_{1,1/2} \leq 2\sqrt{2}P(E, X) = 2\sqrt{2} \lim_{t \rightarrow 0^+} \int_X \|DP_t \mathbf{1}_E\|_{\mathcal{H}_x} d\mu(x).$$

The fact that (1) is equivalent to (2) is actually already essentially known (see [MNP15]), but we give here a proof that immediately adapts to the case considered in the following section. The proof will be divided in several lemmas.

**LEMMA 7.3.** *Let  $E$  be a measurable set. If the limit*

$$\lim_{t \rightarrow 0^+} \int_X \sqrt{\Gamma(P_t \mathbf{1}_E)} d\mu(x)$$

*exists then  $\mathbf{1}_E \in \mathbf{B}^{1,1/2}(X)$  and*

$$\|\mathbf{1}_E\|_{1,1/2} \leq 2\sqrt{2} \lim_{t \rightarrow 0^+} \int_X \sqrt{\Gamma(P_t \mathbf{1}_E)} d\mu(x).$$

**PROOF.** The strong Bakry-Émery estimate implies that for every  $f \in \mathcal{F}$  and  $t \geq 0$ , one has  $\mu$ -almost everywhere

$$\|P_t f - f\|_1 \leq \sqrt{2t} \int_X \sqrt{\Gamma(f)} d\mu.$$

Let  $E$  be a measurable set such that  $\lim_{t \rightarrow 0^+} \int_X \sqrt{\Gamma(P_t \mathbf{1}_E)} d\mu(x)$  exists. One has for every  $s \geq 0$ ,

$$\|P_{t+s} \mathbf{1}_E - P_s \mathbf{1}_E\|_1 \leq \sqrt{2t} \int_X \sqrt{\Gamma(P_s \mathbf{1}_E)} d\mu.$$

Taking the limit when  $s \rightarrow 0^+$  yields,

$$\|P_t \mathbf{1}_E - \mathbf{1}_E\|_1 \leq \sqrt{2t} \lim_{s \rightarrow 0^+} \int_X \sqrt{\Gamma(P_s \mathbf{1}_E)} d\mu$$

and so  $\mathbf{1}_E \in \mathbf{B}^{1,1/2}(X)$  with

$$\|\mathbf{1}_E\|_{1,1/2} \leq 2\|P_t \mathbf{1}_E - \mathbf{1}_E\|_1 \leq 2\sqrt{2} \lim_{t \rightarrow 0^+} \int_X \sqrt{\Gamma(P_t \mathbf{1}_E)} d\mu(x).$$

□

**LEMMA 7.4.** *For every  $\eta \in L^2(X, \mathcal{H}, \mu)$ , and  $t > 0$ ,  $\vec{P}_t \eta \in \mathbf{dom}(D^*)$ .*

PROOF. For  $f \in \mathcal{F}$  and  $\eta \in L^2(X, \mathcal{H}, \mu)$ , one has

$$\int_X \langle Df(x), \vec{P}_t \eta(x) \rangle_{\mathcal{H}} d\mu(x) = \int_X \langle \vec{P}_t Df(x), \eta(x) \rangle_{\mathcal{H}} d\mu(x) = \int_X \langle DP_t f(x), \eta(x) \rangle_{\mathcal{H}} d\mu(x).$$

Since the operator  $DP_t$  is bounded in  $L^2$ , the conclusion follows.  $\square$

LEMMA 7.5. *If  $f \in \mathcal{F}$ , then*

$$\sup \left\{ \int_X f D^* \eta d\mu : \eta \in \mathbf{dom}(D^*), \|\eta(x)\|_{\mathcal{H}} \leq 1, \mu\text{-a.e.} \right\} = \int_X \|Df(x)\|_{\mathcal{H}} d\mu(x).$$

PROOF. Let  $f \in \mathcal{F}$ . It is clear that for  $\eta \in \mathbf{dom}(D^*)$  such that  $\|\eta(x)\|_{\mathcal{H}} \leq 1$ , one has

$$\int_X f D^* \eta d\mu = \int_X \langle Df, \eta \rangle_{\mathcal{H}} d\mu \leq \int_X \|Df\|_{\mathcal{H}} d\mu$$

We now prove the converse inequality. Let  $f \in \mathcal{F}$ . Let  $s, \varepsilon > 0$  and consider

$$\eta = \vec{P}_s \left( \frac{Df}{\|Df\|_{\mathcal{H}} + \varepsilon} \right).$$

From the previous lemma  $\eta \in \mathbf{dom}(D^*)$  and from (7.3), one has

$$\|\eta(x)\|_{\mathcal{H}} \leq 1, \quad \mu\text{-a.e.}$$

Moreover,

$$\int_X f D^* \eta d\mu = \int_X \left\langle Df, \vec{P}_s \left( \frac{Df}{\|Df\|_{\mathcal{H}} + \varepsilon} \right) \right\rangle d\mu.$$

Letting  $s \rightarrow 0^+$  and  $\varepsilon \rightarrow 0$  yields

$$\int_X \|Df\|_{\mathcal{H}} d\mu \leq \sup \left\{ \int_X f D^* \eta d\mu : \eta \in \mathbf{dom}(D^*), \|\eta(x)\|_{\mathcal{H}} \leq 1, \mu\text{-a.e.} \right\}.$$

$\square$

We are now ready to prove Theorem 7.2.

PROOF. It just remains to prove that 1 is equivalent to 2.

Let  $E$  be a measurable set and  $\eta \in \mathbf{dom}(D^*)$ ,  $\|\eta(x)\|_{\mathcal{H}} \leq 1$  a.e.. We have

$$\int_X (P_t \mathbf{1}_E) D^* \eta d\mu = \int_X \langle DP_t \mathbf{1}_E, \eta \rangle_{\mathcal{H}} d\mu \leq \int_X \|DP_t \mathbf{1}_E\|_{\mathcal{H}} d\mu.$$

Since we have in  $L^2$ ,  $\lim_{t \rightarrow 0} P_t \mathbf{1}_E = \mathbf{1}_E$ , one deduces

$$\int_E D^* \eta d\mu \leq \liminf_{t \rightarrow 0^+} \int_X \|DP_t \mathbf{1}_E\|_{\mathcal{H}} d\mu.$$

Thus,

$$\sup \left\{ \int_E D^* \eta d\mu : \eta \in \mathbf{dom}(D^*), \|\eta(x)\|_{\mathcal{H}} \leq 1 \right\} \leq \liminf_{t \rightarrow 0^+} \int_X \|DP_t \mathbf{1}_E\|_{\mathcal{H}} d\mu.$$

On the other hand, again if  $E$  be a measurable set and  $\eta \in \mathbf{dom}(D^*)$ ,  $\|\eta(x)\|_{\mathcal{H}} \leq 1$  a.e., then we have:

$$\int_X (P_t \mathbf{1}_E) D^* \eta d\mu = \int_X \mathbf{1}_E P_t D^* \eta d\mu = \int_X \mathbf{1}_E D^* \vec{P}_t \eta d\mu \leq \sup \left\{ \int_E D^* \eta d\mu : \eta \in \mathbf{dom}(D^*), \|\eta(x)\|_{\mathcal{H}} \leq 1 \right\},$$

where in the last inequality we used the fact that  $\|\vec{P}_t \eta\|_{\mathcal{H}} \leq 1$ . Since from the previous lemma we know that

$$\sup \left\{ \int_X (P_t \mathbf{1}_E) D^* \eta d\mu : \eta \in \mathbf{dom}(D^*), \|\eta(x)\|_{\mathcal{H}} \leq 1 \right\} = \int_X \|DP_t \mathbf{1}_E(x)\|_{\mathcal{H}} d\mu(x),$$

and so,

$$\limsup_{t \rightarrow 0^+} \int_X \|DP_t \mathbf{1}_E(x)\|_{\mathcal{H}} d\mu(x) \leq \sup \left\{ \int_E D^* \eta d\mu : \eta \in \mathbf{dom}(D^*), \|\eta(x)\|_{\mathcal{H}} \leq 1 \right\}.$$

The proof is complete.  $\square$

## 7.2. Path spaces with Gibbs measure

In this section, we present new examples of infinite-dimensional Dirichlet spaces for which one can construct sets of finite perimeter as in the Wiener space. The examples are Dirichlet spaces built on the space of continuous functions  $C(\mathbb{R}, \mathbb{R}^d)$ , where the reference measure is the Gibbs measure associated with the formal Hamiltonian:

$$\mathcal{H}(w) = \frac{1}{2} \int_{\mathbb{R}} \|w'(x)\|^2 dx + \int_{\mathbb{R}} U(w(x)) dx, \quad w \in C(\mathbb{R}, \mathbb{R}^d),$$

where  $U : \mathbb{R}^d \rightarrow \mathbb{R}$  is an interaction potential. The corresponding diffusion processes are defined through the Ginzburg-Landau type stochastic partial differential equations:

$$(7.4) \quad dX_t(x) = \left( \frac{d^2}{dx^2} X_t(x) - \nabla U(X_t(x)) \right) dt + dW_t(x), \quad t > 0, x \in \mathbb{R},$$

where  $(W_t)$  is a white noise process.

In the sequel we will assume that the interaction potential  $U$  satisfies the following conditions:

- (1)  $U$  is two times continuously differentiable and  $\nabla^2 U \geq 0$ ;
- (2) There exists  $A > 0$  and  $p > 0$  such that for every  $x \in \mathbb{R}^d$ :

$$\|\nabla U(x)\| + \|\nabla^2 U(x)\| \leq A(1 + \|x\|^p);$$

- (3)  $\lim_{\|x\| \rightarrow +\infty} U(x) = +\infty$ .

We now briefly sketch, omitting some details, the construction of the relevant Dirichlet space. For the details, we refer to Kawabi [Kaw08] or Kawabi-Röckner [KR07].

We denote  $X = C(\mathbb{R}, \mathbb{R}^d)^1$ . We first define a set of smooth cylindric functionals. We say that a functional  $F : X \rightarrow \mathbb{R}$  is a smooth cylindric functional if, for some  $n$ , it can be written in the form

$$F(w) = f(\langle w, \phi_1 \rangle, \dots, \langle w, \phi_n \rangle),$$

where  $f$  is smooth and bounded, and  $\phi_1, \dots, \phi_n$  are smooth and compactly supported. Here, we use the notation  $\langle w, \phi_i \rangle = \int_{\mathbb{R}} \langle w(x), \phi_i(x) \rangle dx$ . The set of smooth cylindric functionals will be denoted by  $\mathcal{C}$  and will be used as nice core, as in the Wiener space example.

We denote by  $\mathcal{H}$  the space of square integrable functions  $\mathbb{R} \rightarrow \mathbb{R}^d$ . If  $F \in \mathcal{C}$ , one defines its derivative by the formula,

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i} (\langle w, \phi_1 \rangle, \dots, \langle w, \phi_n \rangle) \phi_i \in \mathcal{H}.$$

Let now  $\mu$  be the  $U$ -Gibbs measure defined by (2.1) in [Kaw08]. Note that  $\mathcal{C}$  is dense in  $L^2(X, \mu)$ . We can then consider the pre-Dirichlet form  $(\mathcal{E}, \mathcal{C}, \mu)$  given for  $F, G \in \mathcal{C}$  by

$$\mathcal{E}(F, G) = \int_X \langle DF(w), DG(w) \rangle_{\mathcal{H}} d\mu(w).$$

The closure of this pre-Dirichlet form yields a quasi-regular local symmetric Dirichlet space  $(X, \mathcal{F}, \mu)$ . We shall denote by  $\{P_t, t \geq 0\}$  the semigroup of this Dirichlet form. It is worth noting that  $\{P_t, t \geq 0\}$  is conservative and is indeed the semigroup associated to the spde (7.4) (see [KR07]).

Note that the Dirichlet form  $\mathcal{E}$  admits a carré du champ and that one has for  $f \in \mathcal{F}$ ,

$$\Gamma(f)(w) = \|Df(w)\|_{\mathcal{H}}^2.$$

In this framework, one has the strong Bakry-Émery estimate (see Proposition 2.1 in [Kaw06]):

$$\|DP_t f(w)\|_{\mathcal{H}} \leq (P_t \|Df\|_{\mathcal{H}})(w), \quad f \in \mathcal{F},$$

<sup>1</sup>Strictly speaking, the construction needs to be performed on a subspace  $X$  of  $C(\mathbb{R}, \mathbb{R}^d)$ .

Actually, according to Lemma 3.1. in [Kaw08], as in the Wiener space example, for  $f \in \mathcal{F}$ ,

$$DP_t f(w) = \vec{P}_t Df(w)$$

where  $\vec{P}_t$  is a semigroup on  $L^2(X, \mathcal{H}, \mu)$  that satisfies for every  $\eta \in L^2(X, \mathcal{H}, \mu)$ ,

$$(7.5) \quad \|\vec{P}_t \eta(w)\|_{\mathcal{H}} \leq (P_t \|\eta\|_{\mathcal{H}})(w).$$

From there, the situation is exactly similar to the Wiener space case that was studied in the previous section and, if  $E \subset X$  is a measurable set, we set

$$P(E, X) = \sup \left\{ \int_E D^* \eta d\mu : \eta \in \mathbf{dom}(D^*), \|\eta(x)\|_{\mathcal{H}} \leq 1, \mu\text{-almost everywhere} \right\},$$

and we say that  $E$  has a finite perimeter if  $P(E, X) < +\infty$ . The same proof that we gave in the Wiener space example works and Theorem 7.2 therefore holds.

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