Stat Phys 2009

Test 2

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$$\int_0^\infty \exp\left(-ax^2\right)dx = \frac{\sqrt{\pi}}{2\sqrt{a}} \qquad \int_0^\infty x^2 \exp\left(-ax^2\right)dx = \frac{\sqrt{\pi}}{4a^{3/2}}$$

Problem 1

Consider a zipper with infinite number of links. Each link can be in one of the two states: a state with energy 0 wherein the link is closed, and a state with energy $\varepsilon > 0$ wherein it is open. This zipper can unzip only from one end and a link can open only if all the links before it (from that end) are already open. At temperature T, find the average number of open links and investigate your answer for high and low temperatures.

Solution

Partition function is

$$Z = \sum_{n=0}^{\infty} \exp\left(-\frac{n\varepsilon}{T}\right) = \frac{1}{1-\exp\left(-\frac{\varepsilon}{T}\right)}$$

whereof

$$\langle n \rangle = \frac{1}{Z}\sum_{n=0}^{\infty} n \exp\left(-\frac{n\varepsilon}{T}\right) = -\frac{T}{Z}\frac{\partial}{\partial\varepsilon}\sum_{n=0}^{\infty}\exp\left(-\frac{n\varepsilon}{T}\right) = -\frac{T}{Z}\frac{\partial}{\partial\varepsilon}Z = \frac{1}{\exp\left(\frac{\varepsilon}{T}\right)-1}$$

$$\langle n \rangle \approx \exp\left(-\frac{\varepsilon}{T}\right), \qquad \text{for } T \ll \varepsilon$$

$$\langle n \rangle \approx \frac{T}{\varepsilon}, \qquad \text{for } T \gg \varepsilon$$

Alternatively, energy (average energy) of the system is

$$E = -\frac{\partial \log Z}{\partial\theta} = \frac{\partial}{\partial\theta}\log\left[1-\exp(-\theta\varepsilon)\right] = \frac{\varepsilon}{\exp(\theta\varepsilon)-1} = \frac{\varepsilon}{\exp\left(\frac{\varepsilon}{T}\right)-1}$$

and

$$\langle n \rangle = \frac{E}{\varepsilon}$$

Problem 2

A classical system of N distinguishable (no $N!$), non-interacting particles is placed in a three-dimensional harmonic well

$$U(r) = \frac{x^2 + y^2 + z^2}{2a^2}$$

Find the Helmholtz free energy and the internal energy of the gas.

Solution

Partition function for each particle is found as

$$Z_1 = \frac{1}{(2\pi\hbar)^3} \int_0^\infty \exp\left(-\frac{p^2}{2mT}\right) 4\pi p^2 dp \int_{-\infty}^\infty \exp\left(-\frac{x^2 + y^2 + z^2}{2a^2 T}\right) dxdydz = \frac{T^3 a^3 m^{3/2}}{\hbar^3}$$

$$F = -TN \log Z_1$$

$$E = -N\frac{\partial \log Z_1}{\partial \theta} = N\frac{\partial}{\partial \theta}\log\frac{\hbar^3\theta^3}{a^3 m^{3/2}} = \frac{3N}{\theta} = 3NT$$

Problem 3 (*bonus*)

An ideal gas, $PV = NT$, with the initial pressure P_0, and volume V_0 is heated by a current through a platinum wire. The experiment is done twice: first at a constant volume V_0, with pressure changing from P_0 to P_1, and then at a constant pressure P_0, with volume changing from V_0 to V_1. In both instances, gas receives the same amount of heat (current is applied for the same amount of time). Assuming constant specific heat, find C_P / C_V.

Solution

For constant volume,

$$dQ = dE = C_V dT$$
$$\Delta Q = C_V \Delta T = C_V (T_1 - T_0)$$

For constant pressure,

$$dQ = dW = C_P dT$$
$$\Delta Q = C_P \Delta T = C_P (T_2 - T_0)$$

whereof

$$\frac{C_P}{C_V} = \frac{T_1 / T_0 - 1}{T_2 / T_0 - 1} = \frac{P_1 / P_0 - 1}{V_1 / V_0 - 1}$$