Black-Body Radiation. Solids.

Test 5

Stat Physics

(Dated: 05-20-2008)

1. For a d-dimensional solid, find how the low temperature specific heat scales with temperature.

   Solution

   Energy of the solid

$$E \propto \int_0^\infty \frac{\omega^d d\omega}{\exp\left(\hbar\omega/T\right) - 1} \propto T^{d+1} \int_0^\infty \frac{z^d dz}{\exp z - 1}$$
$$C \propto (d+1)\, T^d \int_0^\infty \frac{z^d dz}{\exp z - 1} \propto T^d$$

2. For the d-dimensional black-body radiation, find entropy and specific heat per photon; express your answers in terms of and the following integrals:

$$I\left(d\right) = \int_0^\infty \frac{z^d dz}{\exp z - 1}$$

   Solution

   Number of photons

$$N \propto \int \frac{\omega^{d-1} d\omega}{\exp\left(\hbar\omega/T\right) - 1} \propto T^d \int \frac{z^{d-1} dz}{\exp z - 1}$$

   Using the expression for C above,

$$\frac{C}{N} = (d+1)\,\frac{I_d}{I_{d-1}}$$

   From

$$F = \Omega = -\frac{E}{d}$$

   entropy

$$S = -\frac{\partial F}{\partial T} = \frac{C}{d}$$

   and

$$\frac{S}{N} = \frac{d+1}{d}\frac{I_d}{I_{d-1}}$$

3. In the Einstein model of a solid, each atom in the lattice is an independent oscillator with frequency ω_E. Find the first correction to the Dulong and Petit's value of the

specific heat per atom. Express your answer in terms of the ratio $\Theta_E/T << 1$, where $\Theta_E = \hbar\omega_E$.

Solution

Energy per oscillator

$$\varepsilon = \frac{\hbar\omega}{\exp\left(\hbar\omega/T\right) - 1}$$

Specific heat

$$c = \frac{\partial\varepsilon}{\partial T} = \frac{x^2 \exp x}{\left(\exp x - 1\right)^2}, \; x = \frac{\Theta_E}{T}$$

Expanding for small x

$$c \approx \frac{x^2\left(1 + x + x^2/2\right)}{\left(x + x^2/2 + x^3/6\right)^2} = \frac{1 + x + x^2/2}{\left(1 + x/2 + x^2/6\right)^2} \approx \frac{1 + x + x^2/2}{1 + x + x^2/3 + x^2/4}$$
$$\approx \left(1 + x + x^2/2\right)\left(1 - x - 7x^2/12 + x^2\right) = 1 - x^2/12$$

4. Omitting numerical coefficients, for a metal with a simple lattice, find the ratio of the Debye temperature to the Fermi temperature in terms of the ratio of the sound velocity to the Fermi velocity.

Solution

Debye temperature

$$T_D \sim \hbar\omega_D \sim \hbar u \left(\frac{N}{V}\right)^{1/3}$$

since

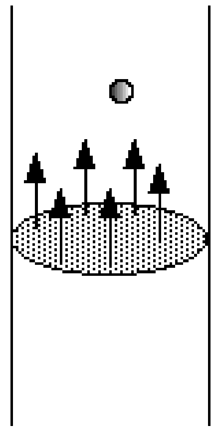
$$V\left(\frac{\omega_D}{u}\right)^3 \sim N$$

Fermi temperature,

$$T_F \sim \frac{\hbar^2}{m}\left(\frac{N}{V}\right)^{2/3} \sim m v_F^2$$
$$v_F \sim \frac{\hbar}{m}\left(\frac{N}{V}\right)^{1/3}$$

Therefore

$$\frac{T_D}{T_F} \sim u\frac{m}{\hbar}\left(\frac{N}{V}\right)^{-1/3} \sim \frac{u}{v_F}$$



5. The total intensity of emission from a flat blackbody at temperature T_0, given by the Stephan-Boltzmann law $J_0 = \sigma T_0^4$, is channeled into a cylindrical column of radius R. A small, spherical blackbody of radius r is placed inside this column of radiation. Find the temperature T of the body (assuming that it is the uniform throughout the body).

   Bonus: Find the force exerted on the small blackbody by the radiation.

   Solution

   The equilibrium condition

$$\sigma T_0^4 \pi r^2 = \sigma T^4 4\pi r^2$$
$$T = 2^{-1/2} T_0$$

   The pressure and the force are found from

$$P = \frac{\triangle p}{\triangle t \triangle A} = \frac{\triangle \varepsilon}{c \triangle t \triangle A} = \frac{J_0 \triangle t \triangle A}{c \triangle t \triangle A} = \frac{J_0}{c}, \qquad F = P\pi r^2$$

   Notice that for the black-body radiation in a cavity,

$$F = -\frac{VT_0^4}{3\pi^2\hbar^3c^3}\int_0^\infty \frac{z^3dz}{\exp z - 1} = -\frac{4V\sigma T_0^4}{3c}$$

   whereof the pressure is

$$P = -\frac{\partial F}{\partial V} = \frac{4\sigma T_0^4}{3c} = \frac{4\sigma J_0}{3c}$$

   where J_0 is the total radiative flux from the walls.

   Note:

$$\sigma = \frac{\pi^2}{60\hbar^3c^2}, \int_0^\infty \frac{z^3dz}{\exp z - 1} = \frac{\pi^4}{15}$$