Quantum Gases. BE Condensation.

Test 4

Stat Physics

(Dated: 04-29-2008)

1. For a non relativistic, degenerate Bose gas, show that, in three and four dimensions, $(\partial N/\partial\mu)_{V,T}$ diverges at the temperature of Bose-Einstein condensation, T_0.

   Bonus: Close to the temperature of BE condensation, $T \to T_0+0$, find the relationship between $\partial N/\partial\mu$ and $N_0 - N$, where

$$N_0 = \int \mathcal{N}(\varepsilon)\left[\exp\left(\frac{\varepsilon}{T}\right) - 1\right]^{-1} d\varepsilon$$

   and $\mathcal{N}(\varepsilon)$ is the density of levels. Using the latter, find the dependence of μ on $(T - T_0)$.

   Solution

   In 3D,

$$N \propto \int_0^\infty \varepsilon^{1/2}\left[\exp\left(\frac{\varepsilon-\mu}{T}\right) - 1\right]^{-1} d\varepsilon$$

$$\left(\frac{\partial N}{\partial\mu}\right)_{T=T_0} \propto \int_0^\infty \varepsilon^{1/2}\exp\left(\frac{\varepsilon}{T_0}\right)\left[\exp\left(\frac{\varepsilon}{T_0}\right) - 1\right]^{-2} d\varepsilon \propto \int_0^\infty x^{1/2}\left[\exp x - 1\right]^{-2} dx \propto x^{-1/2}\,|_{x=0}$$

   In 4D,

$$N \propto \int_0^\infty \varepsilon\left[\exp\left(\frac{\varepsilon-\mu}{T}\right) - 1\right]^{-1} d\varepsilon$$

$$\left(\frac{\partial N}{\partial\mu}\right)_{T=T_0} \propto \int_0^\infty \varepsilon\exp\left(\frac{\varepsilon}{T_0}\right)\left[\exp\left(\frac{\varepsilon}{T_0}\right) - 1\right]^{-2} d\varepsilon \propto \int_0^\infty x\left[\exp x - 1\right]^{-2} dx \propto \log x\,|_{x=0}$$

   Bonus:

$$N_0 - N = \int \mathcal{N}(\varepsilon)\left\{\left[\exp\left(\frac{\varepsilon}{T}\right) - 1\right]^{-1} - \left[\exp\left(\frac{\varepsilon-\mu}{T}\right) - 1\right]^{-1}\right\} d\varepsilon$$
$$\simeq -\frac{\mu}{T}\int \mathcal{N}(\varepsilon)\exp\left(\frac{\varepsilon}{T}\right)\left[\exp\left(\frac{\varepsilon}{T}\right) - 1\right]^{-2} d\varepsilon$$

   while

$$\frac{\partial N}{\partial\mu} \simeq \frac{1}{T}\int \mathcal{N}(\varepsilon)\exp\left(\frac{\varepsilon}{T}\right)\left[\exp\left(\frac{\varepsilon}{T}\right) - 1\right]^{-2} d\varepsilon$$

   so that

$$\mu\frac{\partial N}{\partial\mu} \sim -\left(N_0 - N\right)$$

   (this can be also obtained by a direct expansion of $N - N_0$ in μ) and

$$|\mu| \propto \left(\frac{\partial N}{\partial\mu}\right)^{-1}\left(N_0 - N\right)$$

where $(N_0 - N) \propto (T - T_0)$. From the form of divergence of $\partial N/\partial\mu$ above,

$$\frac{\partial N}{\partial \mu} \propto \begin{array}{cc} |\mu|^{-1/2}, & 3D \\ \log|\mu| \propto \log(T - T_0), & 4D \end{array} \text{ and } \mu \propto \begin{array}{l} (N_0 - N)^2 \propto (T - T_0)^2, 3D \\ (T - T_0)\log^{-1}(T - T_0), 4D \end{array}$$

2. For a non-relativistic quantum gas in four dimensions, derive the relationship between E and Ω.

   Solution

   Compare E

$$E = \int_0^\infty \varepsilon \mathcal{N}(\varepsilon)\left[\exp\left(\frac{\varepsilon - \mu}{T}\right) \pm 1\right]^{-1} d\varepsilon = C\int_0^\infty \varepsilon^2 \left[\exp\left(\frac{\varepsilon - \mu}{T}\right) \pm 1\right]^{-1} d\varepsilon$$

   with Ω

$$\Omega = \mp T\int_0^\infty \mathcal{N}(\varepsilon)\log\left[1 \pm \exp\left(\frac{\mu - \varepsilon}{T}\right)\right] d\varepsilon = \mp CT\int_0^\infty \varepsilon \log\left[1 \pm \exp\left(\frac{\mu - \varepsilon}{T}\right)\right] d\varepsilon$$
$$= \frac{\mp CT}{2}\int_0^\infty \log\left[1 \pm \exp\left(\frac{\mu - \varepsilon}{T}\right)\right] d\varepsilon^2 = -\frac{C}{2}\int_0^\infty \varepsilon^2 \left[\exp\left(\frac{\varepsilon - \mu}{T}\right) \pm 1\right]^{-1} d\varepsilon = -\frac{E}{2}$$

3. Derive the adiabatic process law for a non-relativistic quantum gas in four dimensions.

   Solution

   From

$$\Omega = -\frac{CT^3}{2}\int_0^\infty z^2\left[\exp\left(z - \frac{\mu}{T}\right) \pm 1\right]^{-1} dz = VT^3 f\left(\frac{\mu}{T}\right)$$

   where V is the four-dimensional volume. Consequently,

$$S = -\frac{\partial \Omega}{\partial T},\ N = -\frac{\partial \Omega}{\partial \mu} \Longrightarrow \frac{S}{N} = \phi\left(\frac{\mu}{T}\right)$$

   and since N is fixed, we find

$$\frac{\mu}{T} = \text{const}$$

   in an adiabatic process, $S = \text{const}$. Since also

$$N = VT^2\varphi\left(\frac{\mu}{T}\right)$$

   we find

$$VT^2 = \text{const}$$

   From $\Omega = -PV$, it also follows that

$$\frac{P}{T^3} = \text{const} \ \text{ and } PV^{3/2} = \text{const}$$

   in an adiabatic process.