Degenerate Electron Gas

Test 3

Stat Physics

(Dated: 04-22-2008)

1. For a relativistic, two-dimensional, completely degenerate electron gas, evaluate the dependence of the Fermi energy on the areal density of electrons. Evaluate the mean energy per electron and investigate your answer in the non-relativistic and ultra-relativistic limits.

Hints:

- Relationship between the energy and momentum of a relativistic particle is given by

$$\varepsilon = c\sqrt{p^2 + m^2c^2}$$

- Small x expansion

$$(1+x)^{3/2} \approx 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

- In the non-relativistic limit, subtract the rest mass mc^2

Solution

(a) N is the total number of electrons, A is the area to which gas is confined:

$$\frac{N}{A} = 2\int_0^{p_F} \frac{2\pi p dp}{(2\pi\hbar)^2} = \frac{p_F^2}{2\pi\hbar^2}$$

Fermi energy

$$\varepsilon_F = c\sqrt{p_F^2 + m^2c^2} = c\sqrt{2\pi\hbar^2 (N/A) + m^2c^2}$$

Total energy

$$E = 2A\int_0^{p_F} c\sqrt{p^2 + m^2c^2}\frac{2\pi p dp}{(2\pi\hbar)^2} = \frac{m^3c^4A}{\pi\hbar^2}\int_0^{p_F/mc} \sqrt{x^2+1}x dx$$
$$= \frac{2m^3c^4N}{3p_F^2}\left\{\left[\left(\frac{p_F}{mc}\right)^2 + 1\right]^{3/2} - 1\right\}$$

Energy per electron

$$\frac{E}{N} = \frac{2mc^2}{3}\left\{\left[\left(\frac{\varepsilon_F}{mc^2}\right)^2 + 1\right]^{3/2} - 1\right\}\left[\left(\frac{\varepsilon_F}{mc^2}\right)^2 - 1\right]^{-1}$$

In the ultra-relativistic limit

$$\frac{E}{N} \approx \frac{2}{3}\varepsilon_F$$

In the non-relativistic limit, ,

$$\frac{E}{N} \approx \frac{2m^3c^4}{3p_F^2}\left\{\left[\frac{3}{8}\left(\frac{p_F}{mc}\right)^4 + \frac{3}{2}\left(\frac{p_F}{mc}\right)^2 + 1\right] - 1\right\} = mc^2 + \frac{p_F^2}{4m}$$

Subtracting the rest energy mc^2,

$$\frac{E}{N} \approx \frac{1}{2}\varepsilon_F$$

2. For a ultra-relativistic, $\varepsilon = cp$, three-dimensional, degenerate electron gas, calculate the temperature correction of the internal energy per electron relative to a completely degenerate electron gas.

Hint:

$$\int_0^\infty f(\varepsilon)\left[\exp\left(\frac{\varepsilon-\mu}{T}\right)+1\right]^{-1} d\varepsilon = \int_0^\mu f(\varepsilon)\, d\varepsilon + \frac{\pi^2}{6}T^2 f'(\mu) + \ldots$$

Solution

Using

$$N = \int_0^{\varepsilon_F} \mathcal{N}(\varepsilon)\, d\varepsilon = \int_0^\infty \mathcal{N}(\varepsilon)\left[\exp\left(\frac{\varepsilon-\mu}{T}\right)+1\right]^{-1} d\varepsilon$$

where $\mathcal{N}(\varepsilon)$ is the density of levels, find

$$E - E_0 = \frac{\pi^2}{6}T^2\mathcal{N}(\varepsilon_F)$$

In this case,

$$\mathcal{N}(\varepsilon)\, d\varepsilon = 2\frac{4\pi p^2 dp V}{(2\pi\hbar)^3} = \frac{V\varepsilon^2 d\varepsilon}{\pi^2(\hbar c)^3}$$

Consequently,

$$\varepsilon_F = \hbar c\left(\frac{3\pi^2 N}{V}\right)^{1/3}$$

and

$$\frac{E}{N} - \frac{E_0}{N} = \frac{\pi^2}{6}\frac{T^2}{\varepsilon_F}\frac{\varepsilon_F^3 V}{\pi^2(\hbar c)^3 N} = \frac{\pi^2}{2}\frac{T^2}{\varepsilon_F}$$