Quiz 4: Bose Gas

1. For a 2-dimensional gas of ultra-relativistic spinless particles find

   (a) Temperature of Bose-Einstein condensation T_0 as a function of particle concentration $n = N/A$

   For $T \leq T_0$, find

   (b) Number of particles in the ground state in terms of N, T, and T_0

   (c) Total energy E in terms of T, T_0, and N

   (d) Specific heat and entropy in terms of E and T

   (e) Relationship between E and PV

   The following integral may be useful:

$$\int_0^\infty \frac{x dx}{\exp x - 1} = \varsigma(2)$$

$$\int_0^\infty \frac{x^2 dx}{\exp x - 1} = 2\varsigma(3)$$

Solution

$$n = \frac{1}{2\pi\hbar^2 c^2} \int_0^\infty \frac{\varepsilon d\varepsilon}{\exp(\varepsilon/T_0) - 1} = \frac{T_0^2}{2\pi\hbar^2 c^2} \int_0^\infty \frac{x dx}{\exp x - 1}$$

$$T_0 = \hbar c \sqrt{2\pi \varsigma^{-1}(2)\, n}$$

$$N_{\varepsilon=0} = N - N_{\varepsilon>0} = N - \frac{A}{2\pi\hbar^2 c^2} \int_0^\infty \frac{\varepsilon d\varepsilon}{\exp(\varepsilon/T) - 1} = N\left(1 - \frac{T^2}{T_0^2}\right)$$

$$E=\frac{A}{2\pi\hbar^2c^2}\int_0^\infty\frac{\varepsilon^2d\varepsilon}{\exp\left(\varepsilon/T\right)-1}=\frac{T^3}{2\pi\hbar^2c^2}\int_0^\infty\frac{x^2dx}{\exp x-1}=\frac{2\varsigma\left(3\right)}{\varsigma\left(2\right)}\frac{NT^3}{T_0^2}$$

$$C=\frac{3E}{T}$$

$$dS=\frac{dE}{T}\left(=\frac{CdT}{T}\right)\propto 3TdT\Rightarrow S=\frac{3E}{2T}$$

$$F=E-TS=-\frac{E}{2}$$
$$P=-\left(\frac{\partial F}{\partial V}\right)_T\Rightarrow PV=\frac{E}{2}$$