Final Spring

1. Show that in d-dimensions and for $\varepsilon = \alpha p^s$, $PV = sE/d$ regardless of the statistics of particles in an ideal gas.

   Solution

$$
\begin{aligned}
PV &= -\Omega \\
&= \pm \frac{gVT}{(2\pi\hbar)^d} \int_0^\infty \ln\left(1 \pm \exp\frac{\mu - \varepsilon}{T}\right) d^d p \\
&= \pm \frac{gVT}{(2\pi\hbar)^d} \frac{\alpha^{-d/s}}{s} \int_0^\infty \ln\left(1 \pm \exp\frac{\mu - \varepsilon}{T}\right) \varepsilon^{d/s-1} d\varepsilon \\
&= \pm \frac{gVT}{(2\pi\hbar)^d} \frac{\alpha^{-d/s}}{d} \int_0^\infty \ln\left(1 \pm \exp\frac{\mu - \varepsilon}{T}\right) d\varepsilon^{d/s} \\
&= \pm \frac{gVT}{(2\pi\hbar)^d} \frac{\alpha^{-d/s}}{d} \int_0^\infty \frac{\varepsilon^{d/s}\left[\mp T^{-1} \exp\left(\mu - \varepsilon\right)/T\right]}{1 \pm \exp\left(\mu - \varepsilon\right)/T} d\varepsilon \\
&= \frac{gV}{(2\pi\hbar)^d} \frac{\alpha^{-d/s}}{d} \int_0^\infty \frac{\varepsilon^{d/s}}{\exp\left(\varepsilon - \mu\right)/T \pm 1} d\varepsilon
\end{aligned}
$$

$$
\begin{aligned}
E &= \frac{gV}{(2\pi\hbar)^d} \int_0^\infty \frac{\varepsilon}{\exp\left(\varepsilon - \mu\right)/T \pm 1} d^d p \\
&= \frac{gV}{(2\pi\hbar)^d} \frac{\alpha^{-d/s}}{s} \int_0^\infty \frac{\varepsilon^{d/s}}{\exp\left(\varepsilon - \mu\right)/T \pm 1} d\varepsilon
\end{aligned}
$$

$$PV = \frac{s}{d} E$$

2. Evaluate the isothermal compressibility of an ideal, spinless, 3D, non-relativistic Bose gas as $T \to T_0$, the temperature of Bose-Einstein condensation, from above.

   Hint: Evaluate the inverse compressibility $\kappa_T^{-1} = -V\left(\partial P/\partial V\right)_{N,T}$ from the Gibbs free energy and relate the latter to the chemical potential.

Solution

$$d\Phi = -SdT + VdP + \mu dN$$

$$\kappa_T^{-1} = -V\left(\frac{\partial P}{\partial V}\right)_{N,T} = -\left(\frac{\partial \Phi}{\partial V}\right)_{N,T} = -N\left(\frac{\partial \mu}{\partial V}\right)_{N,T}$$

But as $T \to T_0$

$$\mu = -\frac{9I_0^2}{4\pi^2}\frac{(T-T_0)^2}{T_0}$$
$$I_0 \equiv \int_0^\infty \frac{\sqrt{z}dz}{\exp z - 1} = \frac{\sqrt{\pi}}{2}\varsigma\left(\frac{3}{2}\right), \varsigma\left(\frac{3}{2}\right) = 2.612$$

whereof, using $T_0 \propto V^{-2/3}$, we find

$$\kappa_T^{-1} \cong -\frac{9I_0^2 N}{2\pi^2}\frac{T-T_0}{T_0}\frac{\partial T_0}{\partial V} = -\frac{9I_0^2 N}{2\pi^2}(T-T_0)\frac{\partial \log T_0}{\partial V} = \frac{3I_0^2 N}{\pi^2 V}(T-T_0)$$

leading to a divergent compressibility κ_T.

3. Electron gas is confined, at zero temperature, to a compartment of volume V inside a thermally isolated container of volume $V + \Delta V$, $\Delta V \ll 1$. The other compartment (of volume ΔV) is initially empty. Subsequently, the partition separating the two compartments, V and ΔV, is removed. Find the temperature of the electron gas once the equilibrium is reached.

Solution

By energy conservation

$$\frac{E_i}{N} = \frac{3\pi^2}{10\beta}\left(\frac{N}{V}\right)^{2/3} = \frac{3\pi^2}{10\beta}\left(\frac{N}{V+\Delta V}\right)^{2/3} + \frac{\beta T^2}{2}\left(\frac{V+\Delta V}{N}\right)^{2/3} = \frac{E_f}{N}$$

$$\left(\frac{V+\Delta V}{V}\right)^{2/3} = 1 + \frac{5\beta^2 T^2}{3\pi^2}\left(\frac{V+\Delta V}{N}\right)^{4/3} \approx 1 + \frac{5\beta^2 T^2}{3\pi^2}\left(\frac{V}{N}\right)^{4/3}$$

$$\frac{2\Delta V}{3V} \approx \frac{5\beta^2 T^2}{3\pi^2}\left(\frac{V}{N}\right)^{4/3} = \frac{5\beta T^2}{6\varepsilon_F}\left(\frac{V}{N}\right)^{2/3} = \frac{5\pi^2 T^2}{12\varepsilon_F^2}$$

$$T \approx \varepsilon_F \sqrt{\frac{8}{5\pi^2}\frac{\Delta V}{V}}$$

4. Find the relationship between the isothermal compressibility and the particle number fluctuation in the grand canonical ensemble.

   Hint: use the hint in Problem 2 and make a conversion from a fixed number of particles to a fixed volume, similar to p. 342 in LL.

   Solution

$$\kappa_T^{-1} = -N\left(\frac{\partial \mu}{\partial V}\right)_{N,T} = \frac{N^2}{V}\left(\frac{\partial \mu}{\partial N}\right)_{V,T}$$

$$\kappa_T = \frac{V}{N^2}\left(\frac{\partial N}{\partial \mu}\right)_{V,T} = \frac{V}{TN^2}\left\langle (\Delta N)^2\right\rangle$$

   where eq. (112.14) from LL was used.

5. Derive eq. (144.9) in the strong field limit and evaluate the heat capacity in this limit.