Quiz 5: Blackbody Radiation. Debye's Theory.

1. Apply Debye's theory to a 2D solid. Specifically, derive the temperature dependence of the heat capacity in the limits of high and low temperature. For simplicity, assume a simple lattice, $v = 1$, with N unit cells.

   Hint:

$$\int_0^\infty \frac{z^2 dz}{\exp z - 1} = 2\zeta(3)$$

   Solution

$$\frac{A}{\pi \overline{u}^2} \int_0^{\omega_m} \omega d\omega = 2N$$

$$\omega_m = 2\sqrt{\pi N/A}\,\overline{u},\ k_m = 2\sqrt{\pi N/A},\ \Theta = \hbar\omega_m$$

$$\begin{aligned} F &= F_0 + T\frac{4N}{\omega_m^2} \int_0^{\omega_m} \log\left[1 - \exp\left(-\frac{\hbar\omega}{T}\right)\right] \omega d\omega \\ &= F_0 + 4NT\left(\frac{T}{\Theta}\right)^2 \int_0^{\Theta/T} \log\left[1 - \exp(-z)\right] z dz \end{aligned}$$

   (a) $\Theta/T \gg 1$

$$\begin{aligned} F &\simeq F_0 + 4NT\left(\frac{T}{\Theta}\right)^2 \int_0^\infty \log\left[1 - \exp(-z)\right] z dz \\ &= F_0 - 2NT\left(\frac{T}{\Theta}\right)^2 \int_0^\infty \frac{z^2}{\exp z - 1} dz = F_0 - 4\zeta(3) NT\left(\frac{T}{\Theta}\right)^2 \end{aligned}$$

$$\begin{aligned} E &= F - T\frac{\partial F}{\partial T} \simeq 8\zeta(3) NT\left(\frac{T}{\Theta}\right)^2 \\ C &\simeq 24\zeta(3) N\left(\frac{T}{\Theta}\right)^2 \end{aligned}$$

   (b) $\Theta/T \ll 1$

$$F \simeq F_0 + 4NT\left(\frac{T}{\Theta}\right)^2 \int_0^{\Theta/T} (\log z) z dz \simeq F_0 + 2NT\log\frac{\Theta}{T}$$

$$E \simeq E_0 + 2NT,\ C \simeq 2N$$

2. Ignoring numerical coefficients, derive the functional dependence of the spectral energy distribution of blackbody radiation in D dimensions. Investigate it in the limits of low frequencies (Rayleigh-Jeans limit) and high frequencies (Wien's limit).

   Solution

$$dE_\omega \propto \frac{V\hbar}{c^D} \frac{\omega^D}{\exp\left(\hbar\omega/T\right) - 1} d\omega$$

$$dE_\omega \propto \frac{V\hbar}{c^D} \omega^D \exp\left(-\frac{\hbar\omega}{T}\right), \, \hbar\omega \gg T$$

$$dE_\omega \propto \frac{V}{c^D} T \omega^{D-1} d\omega, \, \hbar\omega \ll T$$

3. Knowing the temperature of the Earth T_E, the radius of the Sun R_S and the distance d between the Earth and the Sun, find the temperature of the Sun. Assume that both the Earth and the Sun are blackbodies with uniform temperatures

   Solution

$$T_S = T_E \sqrt{\frac{2d}{R_S}}$$