Quiz 4: Quantum Gases

1. For a concentration $n = N/A$ of extreme relativistic ($\varepsilon = pc$) bosons in 2D, find the temperature of Bose-Einstein condensation.

   Hint:

$$\int_0^\infty \frac{zdz}{\exp z - 1} = \frac{\pi^2}{6}$$

   Solution

$$N = \int \frac{gd\tau}{\exp\left[\left(\varepsilon - \mu\right)/T\right] - 1}$$

$$d\tau = \frac{A\left(2\pi pdp\right)}{\left(2\pi\hbar\right)^2} = \frac{A\left(2\pi\varepsilon d\varepsilon\right)}{\left(2\pi\hbar c\right)^2}$$

$$n = \frac{g}{2\pi\hbar^2c^2}\int_0^\infty \frac{\varepsilon d\varepsilon}{\exp\left(\varepsilon/T_0\right) - 1} = \frac{gT_0^2}{2\pi\hbar^2c^2}\int_0^\infty \frac{zdz}{\exp z - 1} = \frac{\pi gT_0^2}{12\hbar^2c^2}$$

$$T_0 = 2\hbar c\sqrt{\frac{3n}{\pi g}}$$

2. Consider volume V of an ideal gas of N Bose particles above the Bose-Einstein condensation temperature T_0. Find the dependence of the chemical potential on temperature as the latter approaches T_0, $T \to T_0$.

Hint:

$$\int_0^\infty \frac{dz}{(z+a)\sqrt{z}} = \frac{\pi}{\sqrt{a}}$$

$$\int_0^\infty \frac{\sqrt{z}dz}{\exp z - 1} = \frac{\sqrt{\pi}}{2}\varsigma\left(\frac{3}{2}\right), \varsigma\left(\frac{3}{2}\right) = 2.612$$

Solution

$$N = \int \frac{gd\tau}{\exp\left[(\varepsilon-\mu)/T\right]-1} = \int \frac{gd\tau}{\exp(\varepsilon/T_0)-1}$$

$$\int \frac{\sqrt{\varepsilon}d\varepsilon}{\exp\left[(\varepsilon-\mu)/T\right]-1} - \int \frac{\sqrt{\varepsilon}d\varepsilon}{\exp(\varepsilon/T)-1} = \int \frac{\sqrt{\varepsilon}d\varepsilon}{\exp(\varepsilon/T_0)-1} - \int \frac{\sqrt{\varepsilon}d\varepsilon}{\exp(\varepsilon/T)-1}$$

$$-\pi T_0\sqrt{|\mu|} \approx \frac{\sqrt{\pi}}{2}\varsigma\left(\frac{3}{2}\right)\left(T_0^{3/2}-T^{3/2}\right)$$

$$\sqrt{|\mu|} \approx \frac{1}{2\sqrt{\pi}}T_0^{1/2}\left[\left(\frac{T}{T_0}\right)^{3/2}-1\right] \approx \frac{3}{4\sqrt{\pi}}\varsigma\left(\frac{3}{2}\right)\frac{T-T_0}{T_0^{1/2}}$$

$$\mu \approx -\frac{9}{16\pi}\varsigma^2\left(\frac{3}{2}\right)\frac{(T-T_0)^2}{T_0}$$