Quiz 3: Quantum Gases

1. Determine the entropy and the heat capacity of a degenerate, extreme relativistic ($\varepsilon = pc$) electron gas.

   Hint:

$$\int_0^\infty \frac{f(\varepsilon)\,d\varepsilon}{\exp[(\varepsilon-\mu)/T]+1} \simeq \int_0^\mu f(\varepsilon)\,d\varepsilon + \frac{\pi^2}{6}T^2 f'(\mu)$$

$$\int_0^\infty \frac{\mathcal{N}(\varepsilon)\,h(\varepsilon)\,d\varepsilon}{\exp[(\varepsilon-\mu)/T]+1} \simeq \int_0^{\varepsilon_F} \mathcal{N}(\varepsilon)\,h(\varepsilon)\,d\varepsilon + \frac{\pi^2}{6}T^2 \mathcal{N}(\varepsilon_F)\,h'(\varepsilon_F)$$

   Solution

   Low T expansion for E

$$E = \int \frac{\varepsilon\,(g d\tau)}{\exp[(\varepsilon-\mu)/T]+1} = V\int \frac{\varepsilon\mathcal{N}(\varepsilon)\,d\varepsilon}{\exp[(\varepsilon-\mu)/T]+1},\ \mathcal{N}(\varepsilon) = \frac{\varepsilon^2}{\pi^2\hbar^3c^3}$$

$$E \simeq V\int_0^{\varepsilon_F} \frac{\varepsilon^3}{\pi^2\hbar^3c^3}d\varepsilon + \frac{VT^2\varepsilon_F^2}{6\hbar^3c^3} = E_0 + \frac{VT^2\varepsilon_F^2}{6\hbar^3c^3}$$

   Relate N and ε_F

$$N = V\int_0^{\varepsilon_F} \mathcal{N}(\varepsilon)\,d\varepsilon = \frac{V\varepsilon_F^3}{3\pi^2\hbar^3c^3}$$

   Substitute in expression for E

$$E \simeq E_0 + \frac{VT^2}{6\hbar^3c^3}\left(\frac{3\pi^2\hbar^3c^3N}{V}\right)^{2/3}$$

   For quadratic dependence on T

$$F = F_0 - aT^2$$

$$E = F + TS = F_0 + aT^2$$

$$S = -\frac{\partial F}{\partial T} = 2aT = \frac{\partial E}{\partial T} = C_v$$

   Finally

$$S = C_v = 2\frac{VT}{6\hbar^3c^3}\left(\frac{3\pi^2\hbar^3c^3N}{V}\right)^{2/3} = NT\frac{(3\pi^2)^{2/3}}{3\hbar c}\left(\frac{V}{N}\right)^{1/3}$$

2. Repeat the previous problem in 2D.

Solution

$$\mathcal{N}\left(\varepsilon\right) = \frac{\varepsilon}{\pi\hbar^2 c^2}$$

$$N = A\int_0^{\varepsilon_F} \mathcal{N}\left(\varepsilon\right) d\varepsilon = \frac{A\varepsilon_F^2}{2\pi\hbar^2 c^2}$$

$$E \simeq E_0 + \frac{\pi A T^2 \varepsilon_F}{6\hbar^2 c^2} = \frac{\pi A T^2}{6\hbar^2 c^2}\left(\frac{2\pi\hbar^2 c^2 N}{A}\right)^{1/2}$$

$$S = C_v = 2\frac{\pi A T}{6\hbar^2 c^2}\left(\frac{2\pi\hbar^2 c^2 N}{A}\right)^{1/2} = NT\frac{\sqrt{2}\pi^{3/2}}{3\hbar c}\left(\frac{A}{N}\right)^{1/2}$$